Potential Analysis (2024) 60:45–77
https://doi.org/10.1007/s11118-022-10044-7

Volume Growth and On-diagonal Heat Kernel Bounds on Riemannian Manifolds with an End

Alexander Grigor'yan[1] **· Philipp Sürig**[1]

Received: 1 February 2022 / Accepted: 17 September 2022 / Published online: 30 September 2022
© The Author(s) 2022

Abstract
We investigate heat kernel estimates of the form $p_t(x,x) \geq c_x t^{-\alpha}$, for large enough t, where α and c_x are positive reals and c_x may depend on x, on manifolds having at least one end with a polynomial volume growth.

Keywords Manifolds with ends · Heat kernel · Isoperimetric inequality

Mathematics Subject Classification (2010) 58J35 · 60J65 · 31B05

1 Introduction

Let M be a complete connected non-compact Riemannian manifold and $p_t(x,y)$ be the *heat kernel* on M, that is, the minimal positive fundamental solution of the heat equation $\partial_t u = \Delta u$, where Δ is the Laplace-Beltrami operator on M. In this paper, we investigate the long time behaviour of $p_t(x,x)$ for $t \to +\infty$, $x \in M$. Especially, we are interested in lower bounds for large enough t of the form

$$p_t(x,x) \geq c_x t^{-\alpha}, \tag{1.1}$$

where α and c_x are positive reals and c_x may depend on x.

Let $V(x,r) = \mu(B(x,r))$ be the volume function of M where $B(x,r)$ denotes the geodesic balls in M and μ the Riemannian measure on M. It was proved by A. Grigor'yan and T. Coulhon in [7], that if for some $x_0 \in M$ and all large enough r,

$$V(x_0,r) \leq Cr^N \tag{1.2}$$

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB 1283/2 2021 – 317210226

✉ Alexander Grigor'yan
grigor@math.uni-bielefeld.de

Philipp Sürig
philipp.suerig@uni-bielefeld.de

1 Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

where C and N are positive constants, then

$$p_t(x,x) \geq \frac{c_x}{(t\log t)^{N/2}}, \tag{1.3}$$

which obviously implies (1.1).

It is rather surprising that such a weak hypothesis as (1.2) implies a pointwise lower bound (1.3) of the heat kernel. In this paper we obtain heat kernel bounds assuming even weaker hypotheses about M. We say that an open connected proper subset Ω of M is an *end* of M if $\partial\Omega$ is compact but $\overline{\Omega}$ is non-compact (see also Section 2). One of our aims here is to obtain lower bounds for the heat kernel assuming only hypotheses about the intrinsic geometry of Ω, although a priori it was not obvious at all that such results can exist.

One of the motivations was the following question asked by A. Boulanger in [1] (although for a more restricted class of manifolds). Considering the volume function in Ω given by

$$V_\Omega(x,r) = \mu(B(x,r)\cap\Omega),$$

Boulanger asked if the heat kernel satisfies (1.1) provided it is known that

$$V_\Omega(x_0,r) \leq Cr^N, \tag{1.4}$$

for some $x_0 \in \Omega$ and all r large enough.

A first partial answer to this question was given by A. Grigor'yan, who showed in [10], that if (1.4) holds and $\overline{\Omega}$, considered as a manifold with boundary, is *non-parabolic*, (and hence, $N > 2$ in Eq. 1.4 by [5]) then Eq. 1.3 is satisfied. More precisely, denoting by $p_t^\Omega(x,y)$ the heat kernel in Ω with the Dirichlet boundary condition on $\partial\Omega$, it was proved in [10] that, for all $x \in \Omega$ and large enough t,

$$p_t^\Omega(x,x) \geq \frac{c_x}{(t\log t)^{N/2}}, \tag{1.5}$$

which implies (1.3) by the comparison principle.

From a probabilistic point of view, the estimate (1.5) for non-parabolic $\overline{\Omega}$ is very natural if one compares it with (1.3), since the non-parabolicity of $\overline{\Omega}$ implies that the probability that Brownian motion started in Ω never hits the boundary $\partial\Omega$ is positive (see [[12], Corollary 4.6]). Hence, one expects that the heat kernel in $\overline{\Omega}$ and the heat kernel in Ω with Dirichlet boundary condition are comparable.

The main direction of research in this paper is the validity of the estimate (1.1) in the case when $\overline{\Omega}$ is parabolic and the volume function of Ω satisfies (1.4). We prove (1.1) for a certain class of manifolds M when $\overline{\Omega}$ is parabolic as well as construct a class of manifolds M with parabolic ends where (1.1) does not hold.

In Section 2 we are concerned with positive results. One of our main results -Theorem 2.6, ensures the estimate (1.1) when $\overline{\Omega}$ is a *locally Harnack* manifold (see Section 2.2 for the definition). In order to handle difficulties that come from the parabolicity of the end, we use the method of *h-transform* (see Section 2.1). For that we construct a positive harmonic function h in Ω and define a new measure $\widetilde{\mu}$ by $d\widetilde{\mu} = h^2 d\mu$. Thus, we obtain a *weighted manifold* $\left(\overline{\Omega}, \widetilde{\mu}\right)$. We prove that this manifold is non-parabolic, satisfies the polynomial volume growth and, hence, the heat kernel $\widetilde{p}_t^\Omega$ of $(\Omega, \widetilde{\mu})$ satisfies the lower bound (1.5). Then a similar lower bound for p_t^Ω and, hence, for p_t, follows from the identity

$$p_t^\Omega(x,x) = h^2(x)\widetilde{p}_t^\Omega(x,x)$$

(see Lemma 2.3). Note that the techniques of h-transform for obtaining heat kernel estimates was used in [17] and [16] although in different settings (see also [[15], Section 9.2.4] and [22]).

In Section 3 we present a technique for obtaining *isoperimetric inequalities* on *warped products* of weighted manifolds. We say that a function J on $[0, +\infty)$ is a *lower isoperimetric function* for (M, μ) if, for any precompact open set $U \subset M$ with smooth boundary,

$$\mu^+(U) \geq J(\mu(U)), \tag{1.6}$$

where μ^+ denotes the perimeter with respect to the measure μ (see Section 3 for more details).

The isoperimetric inequality on *Riemannian products* was proved in [19]. We develop further the method of [19] to deal with warped products. The main result here is stated in Theorem 3.3. Given two weighted manifolds (M_1, μ_1) and (M_2, μ_2) consider the weighted manifold (M, μ) such that $M = M_1 \times M_2$ as topological spaces, the Riemannian metric ds^2 on M is defined by

$$ds^2 = dx^2 + \psi^2(x)dy^2,$$

with ψ being a smooth positive function on M_1 and dx^2 and dy^2 denoting the Riemannian metrics on M_1 and M_2, respectively and measure μ on M is defined by $\mu = \mu_1 \times \mu_2$. Assume that the function ψ is bounded and (M_1, μ_1) and (M_2, μ_2) admit continuous lower isoperimetric functions J_1 and J_2, respectively. Then we prove in Theorem 3.3 that (M, μ) admits a lower isoperimetric function

$$J(v) = c \inf_{\varphi, \phi} \left(\int_0^\infty J_1(\varphi(t))dt + \int_0^\infty J_2(\phi(s))ds. \right),$$

for some positive constant $c > 0$ and where φ and ϕ are *generalized mutually inverse* functions such that

$$v = \int_0^\infty \varphi(t)dt = \int_0^\infty \phi(s)ds.$$

In Theorem 3.6 we construct a *weighted model manifold with boundary* $(M_0, \widetilde{\mu})$ (see Section 3.2 for the definition of this term), where M_0 topologically coincides with $[0, +\infty) \times \mathbb{S}^{n-1}$, $n \geq 2$, while the Riemannian metric on M_0 is given by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2, \tag{1.7}$$

where $d\theta^2$ is a standard Riemannian metric on $\mathbb{S}^{n-1}$ and

$$\psi(r) = e^{-\frac{1}{n-1}r^\alpha}, \tag{1.8}$$

with $0 < \alpha \leq 1$, and obtain as a consequence of Theorem 3.3, that $(M_0, \widetilde{\mu})$ admits a lower isoperimetric function J such that for large enough v,

$$J(v) = \frac{cv}{(\log v)^{\frac{2-2\alpha}{\alpha}}}, \tag{1.9}$$

for some positive constant $c > 0$.

In Section 4 we construct examples of manifolds M having a parabolic end Ω with finite volume (in particular, satisfying (1.4)) but such that the heat kernel $p_t(x, x)$ decays *super-polynomially* as $t \to \infty$. In fact, the end Ω is constructed by means of the aforementioned model manifold M_0, particularly, $\overline{\Omega}$ topologically coincides with M_0. Our fourth main result -Theorem 4.3, says that for a certain manifold M with this end Ω the following heat kernel estimate holds:

$$p_t(x, x) \leq C_x \exp\left(-Ct^{\frac{\alpha}{2-\alpha}}\right), \tag{1.10}$$

for all $x \in M$ and large enough t. The estimate (1.10) follows from Theorem 4.2 where we obtain the upper bound of the heat kernel $\widetilde{p}_t$ of a weighted manifold $(M, \widetilde{\mu})$ after an appropriate h-transform. In this theorem we prove that

$$\widetilde{p}_t(x,x) \le C \exp\left(-C_1 t^{\frac{\alpha}{2-\alpha}}\right). \tag{1.11}$$

In fact, this decay is sharp, meaning that we have a matching lower bound

$$\sup_{x\in M} \widetilde{p}_t(x,x) \ge c \exp\left(-C_2 t^{\frac{\alpha}{2-\alpha}}\right)$$

(see the remark after Theorem 4.2). The key ingredient in the proof of Theorem 4.2 is utilizing the lower isoperimetric function J on $(\overline{\Omega}, \widetilde{\mu})$ given by Eq. 1.9, which then yields the heat kernel upper bound (1.11) by a well-known technique based on *Faber-Krahn inequalities* (see [[13], Proposition 7.1] and Proposition 4.1).

Even though we managed to give both positive and negative results for manifolds with parabolic end concerning the estimate (1.1), a gap still remains. Closing this gap seems to be interesting for future work, for example, it might be desirable to construct a manifold with parabolic end of infinite volume for which (1.1) does not hold.

Notation For any nonnegative functions f, g, we write $f \simeq g$ if there exists a constant $C > 1$ such that

$$C^{-1} f \le g \le Cf.$$

2 On-diagonal Heat Kernel Lower Bounds

Let M be a non-compact Riemannian manifold with boundary δM (which may be empty). Given a smooth positive function ω on M, let μ be the measure defined by

$$d\mu = \omega^2 d\mathrm{vol},$$

where $d$vol denotes the Riemannian measure on M. Similarly. we define μ' as the measure with density ω^2 with respect to the Riemannian measure of codimension 1 on any smooth hypersurface. The pair (M, μ) is called *weighted manifold*.

The Riemannian metric induces the Riemannian distance $d(x, y),\ x, y \in M$. Let $B(x, r)$ denote the geodesic ball of radius r centered at x, that is

$$B(x,r) = \{x \in M : d(x,y) < r\}$$

and $V(x, r)$ its volume on (M, μ) given by

$$V(x,r) = \mu(B(x,r)).$$

We say that M is complete if the metric space (M, d) is complete. It is known that M is complete, if and only if, all balls $B(x, r)$ are precompact sets. In this case, $V(x, r)$ is finite.

The Laplace operator Δ_μ is the second order differential operator defined by

$$\Delta_\mu f = \mathrm{div}_\mu(\nabla f) = \omega^{-2}\mathrm{div}(\omega^2 \nabla f).$$

If $\omega \equiv 1$, then Δ_μ coincides with the Laplace-Beltrami operator $\Delta = \mathrm{div} \circ \nabla$.

Consider the Dirichlet form

$$\mathcal{E}(u,v) = \int_M (\nabla u, \nabla v) d\mu,$$

defined on the space $C_0^\infty(M)$ of smooth functions with compact support. The form $\mathcal{E}$ is closable in $L^2(M,\mu)$ and positive definite. Let us denote by $\overline{\Delta_\mu}$ its infinitesimal generator. By integration by parts, we obtain for all $u, v \in C_0^\infty(M)$,

$$\mathcal{E}(u,v) = \int_M (\nabla u, \nabla v) d\mu = -\int_M v\Delta_\mu u d\mu + \int_{\delta M} v\frac{\partial u}{\partial \nu} d\mu', \tag{2.1}$$

where ν denotes the outward unit normal vector field on δM. If $u \in C^2 \cap \mathrm{dom}(\overline{\Delta_\mu})$ then $\frac{\partial u}{\partial \nu} = 0$ on δM and $\Delta_\mu u = \overline{\Delta_\mu} u$, so that $\overline{\Delta_\mu}$ can be considered as an extension of $\overline{\Delta_\mu}$ with Neumann boundary condition on δM.

A function u is called *harmonic* in M if $u \in C^2(M)$, $\Delta_\mu u = 0$ in $M \setminus \delta M$ and $\frac{\partial u}{\partial \nu} = 0$ on δM. We call a function $u \in C^2(M)$ *superharmonic* if $\Delta_\mu u \le 0$ in $M \setminus \delta M$ and $\frac{\partial u}{\partial \nu} \ge 0$ on δM. A *subharmonic* function $u \in C^2(M)$ satisfies the opposite inequalities.

The operator $\overline{\Delta_\mu}$ generates the heat semi-group $P_t := e^{t\overline{\Delta_\mu}}$ which possesses a positive smooth, symmetric kernel $p_t(x,y)$.

Let Ω be an open subset of M and denote $\delta\Omega := \delta M \cap \Omega$. Then we can consider Ω as a manifold with boundary $\delta\Omega$. Hence, using the same constructions as above for Ω instead of M, we obtain the heat semigroup P_t^Ω with the heat kernel $p_t^\Omega(x,y)$, which satisfies the Dirichlet boundary condition on $\partial\Omega$ and the Neumann boundary condition on $\delta\Omega$.

Definition Let M be a complete non-compact manifold. Then we call Ω an *end* of M, if Ω is an open connected proper subset of M such that $\overline{\Omega}$ is non-compact but $\partial\Omega$ is compact (in particular, when $\partial\Omega$ is a smooth closed hypersurface).

If $\delta\Omega$ is nonempty, we will assume that $\delta\Omega \cap \partial\Omega = \emptyset$.

In many cases, the end Ω can be considered as an exterior of a compact set of another manifold M_0, that means, Ω is $M_0 \setminus K_0$ for some compact set $K_0 \subset M_0$. If (M,μ) and (M_0,μ_0) are weighted manifolds, with ω^2 being the smooth density of measure μ and the measure μ_0 having smooth density ω_0^2, then, in particular, we have $\omega_0 = \omega$ on Ω.

Definition We say that a weighted manifold (M,μ) is *parabolic* if any positive superharmonic function on M is constant, and *non-parabolic* otherwise.

Definition Let (M,μ) be a weighted manifold and Ω be a subset of M. Then we define *the volume function* of Ω, for all $x \in M$ and $r > 0$, by

$$V_\Omega(x,r) = \mu(B_\Omega(x,r)),$$

where $B_\Omega(x,r) = B(x,r) \cap \Omega$.

Definition Let (M,μ) be a weighted manifold. We say that $\Omega \subset M$ satisfies the *polynomial volume growth condition*, if there exist $x_0 \in \Omega$ and $r_0 > 0$ such that for all $r \ge r_0$,

$$V_\Omega(x_0,r) \le Cr^N, \tag{2.2}$$

where N and C are positive constants.

Theorem 2.1 ([10], Theorem 8.3) *Let M be a complete non-compact manifold with end Ω. Assume that $\left(\overline{\Omega},\mu\right)$ is a weighted manifold such that*

- *$\left(\overline{\Omega},\mu\right)$ is non-parabolic as a manifold with boundary $\partial\Omega \cup \delta\Omega$.*
- *Ω satisfies the polynomial volume growth condition (2.2) with $N > 2$.*

Then for any $x \in \Omega$ there exist $c_x > 0$ and $t_x > 0$ such that for all $t \geq t_x$,

$$p_t^\Omega(x,x) \geq \frac{c_x}{(t \log t)^{N/2}}, \tag{2.3}$$

where c_x and t_x depend on x.

Consequently, if (M, μ) is a complete non-compact weighted manifold with end Ω such that the above conditions are satisfied, we have for any $x \in M$ and all $t \geq t_x$,

$$p_t(x,x) \geq \frac{c_x}{(t \log t)^{N/2}}. \tag{2.4}$$

2.1 *h*-transform

Recall that any smooth positive function h induces a new weighted manifold $(M, \widetilde{\mu})$, where the measure $\widetilde{\mu}$ is defined by

$$d\widetilde{\mu} = h^2 d\mu = h^2 \omega^2 d\mathrm{vol} \tag{2.5}$$

and we denote, for all $r > 0$ and $x \in M$, by $\widetilde{V}(x,r)$ the volume function of measure $\widetilde{\mu}$. The Laplace operator $\Delta_{\widetilde{\mu}}$ on $(M, \widetilde{\mu})$ is then given by

$$\Delta_{\widetilde{\mu}} f = h^{-2} \mathrm{div}_\mu (h^2 \nabla f) = (h\omega)^{-2} \mathrm{div}((h\omega)^2 \nabla f).$$

Lemma 2.2 ([16], Lemma 4.1) *Assume that $\Omega \subset M$ is open and $\Delta_\mu h = 0$ in Ω. Then for any smooth function f in Ω, we have*

$$\Delta_{\widetilde{\mu}} f = h^{-1} \Delta_\mu (hf). \tag{2.6}$$

Lemma 2.3 ([16], Proposition 4.2) *Assume that h is a harmonic function in an open set $\Omega \subset M$. Then the Dirichlet heat kernels p_t^Ω and $\widetilde{p}_t^\Omega$ in Ω, associated with the corresponding Laplace operators Δ_μ and $\Delta_{\widetilde{\mu}}$, are related by*

$$p_t^\Omega(x,y) = h(x)h(y)\widetilde{p}_t^\Omega(x,y), \tag{2.7}$$

for all $t > 0$ and $x, y \in \Omega$.

Remark In particular, if we assume that h is harmonic in M, we get that the heat kernels are related by

$$\widetilde{p}_t(x,y) = \frac{p_t(x,y)}{h(x)h(y)} \tag{2.8}$$

for all $t > 0$ and $x, y \in M$.

Definition Let Ω be an open set in M and K be a compact set in Ω. Then we call the pair (K, Ω) a *capacitor* and define the capacity $\mathrm{cap}(K, \Omega)$ by

$$\mathrm{cap}(K,\Omega) = \inf_{\phi \in \mathcal{T}(K,\Omega)} \int_\Omega |\nabla \phi|^2 d\mu, \tag{2.9}$$

where $\mathcal{T}(K, \Omega)$ is the set of test functions defined by

$$\mathcal{T}(K,\Omega) = \{\phi \in C_0^\infty(\Omega) : \phi|_K = 1\}.$$

Let Ω be precompact. Then it is known that the Dirichlet integral in Eq. 2.9 is minimized by a harmonic function φ, so that the infimum is attained by the weak solution to the Dirichlet problem in $\Omega \setminus K$:

$$\begin{cases} \Delta\varphi = 0 \\ \varphi|_{\partial K} = 1 \\ \varphi|_{\partial\Omega} = 0. \\ \frac{\partial\varphi}{\partial\nu}|_{\delta(\Omega\setminus K)} = 0 \end{cases}$$

The function φ is called the *equilibrium potential* of the capacitor (K, Ω).

We always have the following identity:

$$\mathrm{cap}(K, \Omega) = \int_\Omega |\nabla\varphi|^2 d\mu = \int_{\Omega\setminus K} |\nabla\varphi|^2 d\mu = -\mathrm{flux}(\varphi), \tag{2.10}$$

where $\mathrm{flux}(\varphi)$ is defined by

$$\mathrm{flux}(\varphi) := \int_{\partial W} \frac{\partial\varphi}{\partial\nu} d\mu',$$

where W is any open region in the domain of φ with smooth precompact boundary such that $K \subset W$ and ν is the outward normal unit vector field on ∂W. By the Green formula (2.1) and the harmonicity of φ, $\mathrm{flux}(\varphi)$ does not depend on the choice of W.

Definition We say that a compact set $K \subset M$ has locally positive capacity, if there exists a precompact open set Ω such that $K \subset \Omega$ and $\mathrm{cap}(K, \Omega) > 0$.

It is a consequence of the local Poincaré inequality, that if $\mathrm{cap}(K, \Omega) > 0$ for some precompact open Ω, then this is true for all precompact open Ω containing K.

Lemma 2.4 *Let (M, μ) be a complete, non-compact weighted manifold and K be a compact set in M with locally positive capacity and smooth boundary ∂K. Fix some $x_0 \in M$ and set $B_r := B(x_0, r)$ for all $r > 0$ and assume that K is contained in a ball B_{r_0} for some $r_0 > 0$. Let us also set $\Omega = M \setminus K$, so that $\left(\overline{\Omega}, \mu\right)$ becomes a weighted manifold with boundary. Then there exists a positive smooth function h in $\overline{\Omega}$ that is harmonic in Ω and satisfies for all $r \geq r_0$,*

$$\min_{\partial B_r} h \leq C\, \mathrm{cap}(K, B_r)^{-1}, \tag{2.11}$$

for some constant $C > 0$. Moreover, the weighted manifold $\left(\overline{\Omega}, \widetilde{\mu}\right)$ is non-parabolic, where measure $\widetilde{\mu}$ on $\overline{\Omega}$ is defined by Eq. 2.5.

Proof For any $R > r_0$, let φ_R be the equilibrium potential of the capacitor (K, B_R). It follows from Eq. 2.10, that

$$\mathrm{cap}(K, B_R) = -\mathrm{flux}(\varphi_R). \tag{2.12}$$

Note that $\partial\Omega = \partial K$. By our assumption on K, we have for all $R > r_0$,

$$\mathrm{cap}(K, B_R) > 0,$$

whence we can consider the sequence

$$v_R = \frac{1 - \varphi_R}{\mathrm{cap}(K, B_R)}.$$

By Eq. 2.12 this sequence satisfies

$$\mathrm{flux}(v_R) = 1. \tag{2.13}$$

Let us extend all v_R to K by setting $v_R \equiv 0$ on K. We claim that for all $R > r > r_0$,

$$\min_{\partial B_r} v_R \leq \operatorname{cap}(K, B_r)^{-1}. \tag{2.14}$$

For $R > r > r_0$, denote $m_r = \min_{\partial B_r} v_R$. It follows from the minimum principle and the fact that $v_R \equiv 0$ on K, that the set

$$U_r := \{x \in B_R : v_R(x) < m_r\}$$

is inside B_r and contains K. Then observe that the function $1 - \frac{v_R}{m_r}$ is the equilibrium potential for the capacitor (K, U_r), whence

$$\operatorname{cap}(K, B_r) \leq \operatorname{cap}(K, U_r) = \operatorname{flux}\left(\frac{v_R}{m_r}\right) = \frac{1}{m_r},$$

which proves (2.14).

Since v_R vanishes on $\partial\Omega$, the maximum principle implies that, for all $R > r > r_0$,

$$\sup_{B_r \setminus K} v_R = \max_{\partial B_r} v_R. \tag{2.15}$$

Hence, we obtain from Eq. 2.15, the local elliptic Harnack inequality, and Eq. 2.14, that for every $R > r > r_0$,

$$\sup_{B_r \setminus K} v_R \leq C(r) \min_{\partial B_r} v_R \leq C(r)\operatorname{cap}(K, B_r)^{-1}, \tag{2.16}$$

where the constant $C(r)$ depends only on r. Let us choose an increasing sequence $\{R_k\}$ such that $R_k > r_0$ and $R_k \to \infty$. Then $\{v_{R_k}\}$ is a sequence of non-negative harmonic functions that by Eq. 2.16 is uniformly bounded in $\overline{B_r \setminus K}$ for each fixed r. By the local properties of harmonic functions, the sequence $\{v_{R_k}\}$ is also equicontinuous in $\overline{B_r \setminus K}$ and, hence, has a subsequence that converges uniformly in $\overline{B_r \setminus K}$. Using a standard diagonal process with $r = r_l \to \infty$, we obtain a subsequence of $\{v_{R_k}\}$ that converges locally uniformly in $\overline{\Omega}$. Denoting the limit by v, we see that v is non-negative and continuous in $\overline{\Omega}$, harmonic in Ω, and $v|_{\partial\Omega} = 0$. It follows that v is, in fact, smooth in $\overline{\Omega}$.

By renaming the sequence $\{R_k\}$, we can assume that $v_{R_k} \to v$ as $k \to \infty$. By the local properties of convergence of harmonic functions, we have $\nabla v_{R_k} \to \nabla v$ where the convergence is also locally uniform in Ω. It follows that, for any $r > r_0$,

$$\int_{\partial B_r} \frac{\partial v}{\partial \nu} d\mu' = \lim_{k\to\infty} \int_{\partial B_r} \frac{\partial v_{R_k}}{\partial \nu} d\mu',$$

which together with Eq. 2.13 implies

$$\operatorname{flux}(v) = 1.$$

Let us define the function $h = 1 + v$ so that h is smooth and positive in $\overline{\Omega}$ and is harmonic in Ω. It follows from Eq. 2.14, that for all $r > r_0$,

$$\min_{\partial B_r} h \leq 1 + \operatorname{cap}(K, B_r)^{-1} \leq (1 + \operatorname{cap}(K, B_{r_0}))\operatorname{cap}(K, B_r)^{-1},$$

which proves (2.11) with $C = 1 + \operatorname{cap}(K, B_{r_0})$.

Let us now show that the weighted manifold $(\overline{\Omega}, \widetilde{\mu})$ is non-parabolic. For that purpose, consider in $\overline{\Omega}$ the positive smooth function $w = \frac{1}{h}$. Then we have by Lemma 2.2, that function w satisfies in Ω,

$$\Delta_{\widetilde{\mu}}(w) = \Delta_{\widetilde{\mu}}\left(\frac{1}{h}\right) = \frac{1}{h}\Delta_{\mu} 1 = 0.$$

so that the function w is $\Delta_{\widetilde{\mu}}$-harmonic in Ω. Observe that

$$\frac{\partial w}{\partial \nu} = -\frac{\partial h}{\partial \nu}\frac{1}{h^2}, \tag{2.17}$$

where ν denotes the outward normal unit vector field on $\partial\Omega$. Since v is non-negative in Ω and $v = 0$ on $\partial\Omega$, we have $\frac{\partial h}{\partial \nu} \leq 0$ on $\partial\Omega$, whence we get by Eq. 2.17,

$$\frac{\partial w}{\partial \nu} \geq 0 \quad \text{on } \partial\Omega.$$

Hence, we conclude that w is $\Delta_{\widetilde{\mu}}$-superharmonic in $\overline{\Omega}$, positive and non-constant, which implies that $(\overline{\Omega}, \widetilde{\mu})$ is non-parabolic. □

Remark Note that the function h constructed in Lemma 2.4 is Δ_μ-subharmonic in $\overline{\Omega}$. If we assume that the weighted manifold $(\overline{\Omega}, \mu)$ is parabolic, we obtain that h is unbounded since a non-constant bounded subharmonic function can only exist on non-parabolic manifolds.

2.2 Locally Harnack Case

Definition The weighted manifold (M, μ) is said to be a *locally Harnack manifold* if there is $\rho > 0$, called the *Harnack radius*, such that for any point $x \in M$ the following is true:

(1) for any positive numbers $r < R < \rho$

$$\frac{V(x,R)}{V(x,r)} \leq a\left(\frac{R}{r}\right)^n \tag{2.18}$$

(2) Poincaré inequality: for any Lipschitz function f in the ball $B(x, R)$ of a radius $R < \rho$ we have

$$\int_{B(x,R)} |\nabla f|^2 d\mu \geq \frac{b}{R^2}\int_{B(x,R/2)} (f - \overline{f})^2 d\mu, \tag{2.19}$$

where we denote

$$\overline{f} := \fint_{B(x,R/2)} f d\mu := \frac{1}{V(x, R/2)}\int_{B(x,R/2)} f d\mu$$

and a, b and n are positive constants and $V(x, r)$ denotes the volume function of (M, μ).

For example, the conditions (1) and (2) are true in the case when the manifold M has Ricci curvature bounded below by a (negative) constant $-\kappa$ (see [3]).

Definition For any open set $\Omega \subset M$, define

$$\lambda_1(\Omega) = \inf_u \frac{\int_\Omega |\nabla u|^2 d\mu}{\int_\Omega u^2 d\mu}, \tag{2.20}$$

where the infimum is taken over all nonzero Lipschitz functions u compactly supported in Ω.

Lemma 2.5 ([11], Theorem 2.1) *Let (M, μ) be a locally Harnack manifold. Then we have, for any precompact open set $U \subset M$,*

$$\lambda_1(U) \geq \frac{c}{\rho^2}\min\left(\left(\frac{V_0}{\mu(U)}\right)^2, \left(\frac{V_0}{\mu(U)}\right)^{2/n}\right), \tag{2.21}$$

where

$$V_0 = \inf_{x \in M} \{V(x, \rho) : B(x, \rho) \cap U \neq \emptyset\}$$

and the constant c depends on a, b, n *from* Eqs. 2.18 *and* 2.19.

Definition We say that a manifold M satisfies the *spherical Harnack inequality* if there exist $x_0 \in M$ and constants $r_0 > 0$, $C_H > 0$, $N_H > 0$ and $A > 1$, so that for any positive harmonic function u in $M \setminus \overline{B(x_0, A^{-1}r)}$ with $r \geq r_0$,

$$\sup_{\partial B(x_0,r)} u \leq C_H r^{N_H} \inf_{\partial B(x_0,r)} u. \tag{2.22}$$

Assumption In this section, when considering an end Ω of a complete non-compact weighted manifold (M, μ), we always assume that there exists a complete weighted manifold (M_0, μ_0) and a compact set $K_0 \subset M_0$ that is the closure of a non-empty open set, such that Ω is $M_0 \setminus K_0$ in the sense of weighted manifolds. For simplicity and since we only use the intrinsic geometry of M_0, we denote by $B(x, r)$ the geodesic balls in M_0 and by $V(x, r)$ the volume function of M_0.

Theorem 2.6 *Let* Ω *be an end of a complete non-compact weighted manifold* (M, μ). *Assume that* M_0 *is a locally Harnack manifold with Harnack radius* $\rho > 0$, *where* M_0 *is defined as above, and that there exists* $x_0 \in M_0$ *so that*

- M_0 *satisfies the spherical Harnack inequality* (2.22).
- M_0 *satisfies the polynomial volume growth condition* (2.2).
- *There are constants* $v_0 > 0$ *and* $\theta \geq 0$ *so that for any* $x \in M_0$, *if* $d(x, x_0) \leq R$ *for some* $R > \rho$, *it holds that*

$$V(x, \rho) \geq v_0 R^{-\theta}. \tag{2.23}$$

Then, for any $x \in M$, *there exist* $\alpha > 0$, $t_x > 0$ *and* $c_x > 0$ *such that for all* $t \geq t_x$,

$$p_t(x, x) \geq \frac{c_x}{t^\alpha}, \tag{2.24}$$

where $\alpha = \alpha(N, \theta, n, N_H)$ *and* n *is as in* Eq. 2.18.

Proof Let us set $B_r = B(x_0, r)$ and $V(r) = V(x_0, r)$ and K_0 be contained in a ball B_δ for some $\delta > 0$. It follows from [20, Theorem 2.25] that K_0 has locally positive capacity. Then by Lemma 2.4 there exists a positive smooth function h in $\overline{\Omega}$ that is harmonic in Ω and such that the weighted manifold $(\overline{\Omega}, \widetilde{\mu})$ is non-parabolic, where measure $\widetilde{\mu}$ is defined by Eq. 2.5. Now, our aim is to apply the estimate (2.3) in Theorem 2.1 to the weighted manifold $(\overline{\Omega}, \widetilde{\mu})$. For that purpose, it is sufficient to show that there are positive constants $\widetilde{r_0}$, $\widetilde{C}$ and $\widetilde{N} > 2$ such that for all $r \geq \widetilde{r_0}$,

$$\widetilde{V}_\Omega(r) = \int_{B_r \cap \Omega} h^2 d\mu \leq \widetilde{C} r^{\widetilde{N}}. \tag{2.25}$$

Firstly, by Eq. 2.11, there is a constant $C_\delta > 0$ such that for all $r \geq \delta$,

$$\min_{\partial B_r} h \leq C_\delta \mathrm{cap}(K_0, B_r)^{-1}. \tag{2.26}$$

As h is harmonic in $M_0 \setminus \overline{B_\delta}$, the hypothesis (2.22) implies that there exists a constant $C_H > 0$, so that for every $r \geq \max(r_0, A\delta)$,

$$\max_{\partial B_r} h \leq C_H r^{N_H} \min_{\partial B_r} h.$$

Combining this with Eq. 2.26, we obtain for all $r \geq \max(r_0, A\delta)$ with $C_0 = C_H C_\delta$,

$$\max_{\partial B_r} h \leq C_0 r^{N_H} \operatorname{cap}(K_0, B_r)^{-1}. \tag{2.27}$$

For any $r \geq \delta$, let φ_r be the equilibrium potential of the capacitor (K_0, B_r). Since

$$\int_{B_r} |\nabla \varphi_r|^2 d\mu_0 = \operatorname{cap}(K_0, B_r)$$

and

$$\int_{B_r} \varphi_r^2 d\mu_0 \geq \mu_0(K_0),$$

we obtain

$$\lambda_1(B_r) \leq \frac{\int_{B_r} |\nabla \varphi_r|^2 d\mu_0}{\int_{B_r} \varphi_r^2 d\mu_0} \leq \frac{\operatorname{cap}(K_0, B_r)}{\mu(K_0)},$$

whence, together with Eq. 2.27, we deduce

$$\max_{\partial B_r} h \leq C_0 \mu(K_0)^{-1} r^{N_H} \lambda_1(B_r)^{-1}. \tag{2.28}$$

Since M_0 is a locally Harnack manifold, we can apply Lemma 2.5 and obtain from Eq. 2.21, that for all $r \geq \delta$,

$$\lambda_1(B_r) \geq \frac{c}{\rho^2} \min\left(\left(\frac{V_0}{V(r)}\right)^2, \left(\frac{V_0}{V(r)}\right)^{2/n}\right), \tag{2.29}$$

where

$$V_0 = \inf_{x \in M_0} \{V(x, \rho) : B(x, \rho) \cap B_r \neq \emptyset\}.$$

Note that the condition $B(x, \rho) \cap B_r \neq \emptyset$ implies that $d(x_0, x) \leq r + \rho$. Thus, we obtain from the hypothesis (2.23), assuming $r \geq \rho$,

$$V(x, \rho) \geq v_0 (r + \rho)^{-\theta} \geq v_0 2^{-\theta} r^{-\theta}.$$

Therefore, we have for all $r \geq \rho$,

$$V_0 \geq C_\theta r^{-\theta},$$

with $C_\theta = v_0 2^{-\theta}$. Hence, using the polynomial volume growth condition (2.2), we obtain from Eq. 2.29, that for all $r \geq \max(r_0, \rho, A\delta)$,

$$\lambda_1(B_r) \geq C_1 \min\left(r^{-2(N+\theta)}, r^{-2(N+\theta)/n}\right),$$

where

$$C_1 = \frac{c}{\rho^2} \min\left(\left(\frac{C_\theta}{C}\right)^2, \left(\frac{C_\theta}{C}\right)^{2/n}\right),$$

so that by setting

$$\beta = 2 \max\left(N + \theta, \frac{N + \theta}{n}\right), \tag{2.30}$$

we deduce for $r \geq \max(r_0, \rho, A\delta, 1)$,

$$\lambda_1(B_r) \geq C_1 r^{-\beta}.$$

Combining this with Eq. 2.28, we obtain for every $r \geq \max(r_0, \rho, A\delta, 1)$,

$$\max_{\partial B_r} h \leq C_2 r^{\beta + N_H}, \tag{2.31}$$

where

$$C_2 = C_0 C_1^{-1} \mu_0(K_0)^{-1}.$$

Hence, Eq. 2.31, the polynomial volume growth condition (2.2) and the maximum principle imply that for all $r \geq \max(r_0, \rho, A\delta, 1)$,

$$\widetilde{V}_\Omega(r) = \int_{B_r \cap \Omega} h^2 d\mu \leq V(r) \max_{\partial B_r} h^2 \leq C_2^2 C r^{N+2(\beta+N_H)},$$

which proves (2.25) with $\widetilde{r}_0 = \max(r_0, \rho, A\delta, 1)$, $\widetilde{N} = 2(\beta + N_H) + N$ and $\widetilde{C} = C_2^2 C$, and implies that the weighted manifold $(\Omega, \widetilde{\mu})$ has polynomial volume growth. Thus, the hypotheses of Theorem 2.1 are fulfilled and we obtain by Eq. 2.3, that for any $x \in \Omega$, there exist $\widetilde{t}_x > 0$ and $\widetilde{c}_x > 0$, such that for all $t \geq \widetilde{t}_x$,

$$\widetilde{p}_t^\Omega(x, x) \geq \frac{\widetilde{c}_x}{(t \log t)^{\beta+N_H+N/2}},$$

where β is defined by Eq. 2.30. Since h is harmonic in Ω, we therefore conclude by Eq. 2.7 that for any $x \in \Omega$ and all $t \geq \widetilde{t}_x$,

$$p_t^\Omega(x, x) = h^2(x)\widetilde{p}_t^\Omega(x, x) \geq \frac{\widetilde{c}_x h^2(x)}{(t \log t)^{\beta+N_H+N/2}},$$

which yields (2.24) for all $x \in M$ by using $p_t^\Omega \leq p_t$ and by means of a local parabolic Harnack inequality (cf. [21]) □

Remark Note that it follows from the non-parabolicity of $\left(\overline{\Omega}, \widetilde{\mu}\right)$, that $4 \max\left(N + \theta, \frac{N+\theta}{n}\right) + 2N_H + N > 2$.

2.3 End with Relatively Connected Annuli

Definition We say that a manifold M with fixed point $x_0 \in M$ satisfies *the relatively connected annuli condition (RCA)* if there exists $A > 1$ such that, for any $r > A^2$ and all x, y with $d(x_0, x) = d(x_0, y) = r$, there exists a continuous path $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$, whose image is contained in $B(x_0, Ar) \setminus B(x_0, A^{-1}r)$.

Remark Note that, even though the condition (RCA) is formulated for the specific point x_0, it is equivalent to the (RCA) condition with respect to any other point x_1 with possibly a different constant A.

Example Any *Riemannian model with a pole* (see Section 2.4) with dimension $n \geq 2$ has relatively connected annuli.

Corollary 2.7 *Let Ω be an end of a complete non-compact weighted manifold (M, μ) and assume that M_0 is a locally Harnack manifold with Harnack radius $\rho > 0$, where M_0 is defined as above. Also assume that there exists $x_0 \in M_0$ so that*

- *M_0 satisfies (RCA) with some constant $A > 1$.*
- *There exist constants $L > 0$ and $C > 0$ so that for all $r \geq L$,*

$$V(Ar) - V(A^{-1}r) \leq C \log r, \tag{2.32}$$

  where we denote $V(r) = V(x_0, r)$.
- *There exists a constant $v_0 > 0$ such that for any $y \in M_0$,*

$$V(y, \rho/3) \geq v_0. \tag{2.33}$$

Then, for any $x \in M$, there exist $\alpha > 0$, $t_x > 0$ and $c_x > 0$ such that for all $t \geq t_x$,

$$p_t(x, x) \geq \frac{c_x}{t^\alpha},$$

where $\alpha = \alpha(n, v_0, \rho, C)$.

Proof As before, we denote $B_r = B(x_0, r)$. Obviously, the hypothesis (2.33) implies the condition (2.23) with $\theta = 0$. Hence, to apply Theorem 2.6, it remains to show that M_0 has polynomial volume growth as in Eq. 2.2 and M_0 satisfies the spherical Harnack inequality (2.22). The polynomial volume growth condition (2.2) follows from Eq. 2.32.

Let us now prove that the spherical Harnack inequality (2.22) holds in M_0. Assume that $r \geq L$ and cover the set $B_{Ar} \setminus B_{A^{-1}r}$, with balls $B(x_i, \rho/3)$ where $x_i \in M_0$ and $A > 1$ is as in (RCA). By applying a standard covering argument, there exists a number $\tau(r)$ and a subsequence of disjoint balls $\{B(x_{i_k}, \rho/3)\}_{k=1}^{\tau(r)}$ such that the union of the balls $\{B(x_{i_k}, \rho)\}_{k=1}^{\tau(r)}$ cover the set $B_{Ar} \setminus B_{A^{-1}r}$. Hence, it follows from Eq. 2.32, that

$$\sum_{i=1}^{\tau(r)} V(x_i, \rho/3) \leq V(Ar) - V(A^{-1}r) \leq C \log r. \tag{2.34}$$

Then the hypothesis (2.33), combined with Eq. 2.34, implies that

$$\tau(r) \leq \frac{C \log r}{v_0}. \tag{2.35}$$

For all $r > A^2$, let y_1, y_2 be two points on ∂B_r and γ be a continuous path connecting them in $B_{Ar} \setminus B_{A^{-1}r}$ as it is ensured by (RCA). Now select out of the sequence $\{B(x_{i_k}, \rho)\}_{k=1}^{\tau(r)}$ those balls that intersect γ. In this way, we obtain a chain of at most $\tau(r)$ balls, which connect y_1 and y_2. Now let u be a positive harmonic function in $M_0 \setminus \overline{B_{A_0^{-1}r}}$, where $A_0 \geq A$ is such that any ball of this chain lies in $M_0 \setminus \overline{B_{A_0^{-1}r}}$ for all $1 \leq i \leq \tau(r)$ and $r > A_0^2$. Applying the local elliptic Harnack inequality to u repeatedly in the balls of this chain and letting y_1, y_2 such that $\min_{\partial B_r} u = u(y_1)$ and $\max_{\partial B_r} u = u(y_2)$, we obtain

$$\max_{\partial B_r} u = u(y_2) \leq (C_\rho)^\tau u(y_1) = (C_\rho)^\tau \min_{\partial B_r} u,$$

where C_ρ is the Harnack constant in all $B(x_{i_k}, \rho)$. Together with Eq. 2.35, this yields

$$\max_{\partial B_r} u \leq r^{\frac{c}{v_0} \log C_\rho} \min_{\partial B_r} u,$$

which proves the spherical Harnack inequality (2.22) with $N_H = \frac{C}{v_0} \log C_\rho$. Thus the hypotheses of Theorem 2.6 are fulfilled and we obtain from Eq. 2.24, that for any $x \in M$, there exist $t_x > 0$, $c_x > 0$ and $\alpha > 0$ such that for all $t \geq t_x$,

$$p_t(x, x) \geq \frac{c_x}{t^\alpha},$$

where $\alpha = \alpha(n, N_H)$, which finishes the proof. □

Definition As usual, for any piecewise C^1 path $\gamma : I \to M$, where I is an interval in $\mathbb{R}$, denote by $l(\gamma)$ the length of γ defined by

$$l(\gamma) = \int_I |\dot{\gamma}(t)| dt,$$

where $\dot{\gamma}$ is the velocity of γ, given by $\dot{\gamma}(t)(f) = \frac{d}{dt} f(\gamma(t))$ for any $f \in C^\infty(M)$.

Corollary 2.8 *Let Ω be an end of a complete non-compact weighted manifold (M, μ) and assume that for some $\kappa \geq 0$, we have*

$$Ric(M_0) \geq -\kappa, \tag{2.36}$$

where M_0 is defined as above. Suppose that there exists $x_0 \in M_0$ so that

- *M_0 satisfies (RCA) with $A > 1$ and piecewise C^1 path γ so that there is some constant $c > 0$ such that for all $r > A^2$,*
$$l(\gamma) \leq c \log r. \tag{2.37}$$
- *There are constants $v_0 > 0$ and $\theta \geq 0$ so that for any $y \in M_0$, if $d(y, x_0) \leq R$ for some $R > 1$, it holds that*
$$V(y, \rho) \geq v_0 R^{-\theta}.$$

Then, for any $x \in M$, there exist $\alpha > 0$, $t_x > 0$ and $c_x > 0$ such that for all $t \geq t_x$,

$$p_t(x, x) \geq \frac{c_x}{t^\alpha},$$

where $\alpha = \alpha(c, \theta, \kappa)$.

Proof The assumption (2.36) implies that M_0 is a locally Harnack manifold. Hence we are left to show that M_0 has polynomial volume growth as in Eq. 2.2 and satisfies the spherical Harnack inequality (2.22) to apply Theorem 2.6. Again we denote $B_r = B(x_0, r)$ and $V(r) = V(x_0, r)$. By the Bishop-Gromov theorem, the hypothesis (2.36) implies that there exists a constant $C_\kappa > 1$, so that for any $y \in M_0$ and $R > 1$,

$$V(y, R) \leq e^{C_\kappa R}.$$

Together with the assumption (2.37), this yields that the polynomial volume growth condition (2.2) holds in M_0.

Let us now show that M_0 satisfies the spherical Harnack inequality (2.22). Let $A > 1$ be as above and assume that $r > A^2$. Fix two points y_1, y_2 on ∂B_r and let γ be a continuous path connecting them in $B_{Ar} \setminus B_{A^{-1}r}$ as is it ensured by (RCA). Then cover the path γ with balls $\{B(x_i, \rho)\}_{i=1}^{\tau(r)}$, where $x_i \in M_0$ and $\rho > 0$. Now let u be a positive harmonic function in $M_0 \setminus \overline{B_{A_0^{-1}r}}$, where $A_0 \geq A$ is such that $B(x_i, \rho) \subset M_0 \setminus \overline{B_{A_0^{-1}r}}$ for all $1 \leq i \leq \tau(r)$ and $r > A_0^2$. In this way, we obtain a chain of at most $\tau(r)$ balls $B(x_i, \rho)$, which connect y_1 and y_2. By Eq. 2.37, we deduce that

$$\tau(r) \leq \frac{c}{\rho} \log(r). \tag{2.38}$$

Applyig the local elliptic Harnack inequality to u repeatedly in the balls of this chain and letting y_1, y_2 such that $\min_{\partial B_r} u = u(y_1)$ and $\max_{\partial B_r} u = u(y_2)$, we obtain

$$\max_{\partial B_r} u = u(y_2) \leq (C_\rho)^\tau u(y_1) = (C_\rho)^\tau \min_{\partial B_r} u,$$

where C_ρ is the Harnack constant in all $B(x_i, \rho)$. Together with Eq. 2.38, this yields

$$\max_{\partial B_r} u \leq r^{\frac{c}{\rho} \log C_\rho} \min_{\partial B_r} u,$$

which proves (2.22) with $N_H = \frac{c}{\rho} \log C_\rho$. Thus the hypotheses of Theorem 2.6 are fulfilled and we obtain by Eq. 2.24, that for any $x \in M$, there exist $t_x > 0$, $c_x > 0$ and $\alpha > 0$ such that for all $t \geq t_x$,

$$p_t(x, x) \geq \frac{c_x}{t^\alpha},$$

which finishes the proof. □

2.4 An Example in Dimension Two

Consider the topological space $M = (0, +\infty) \times \mathbb{S}^1$, that is, any point $x \in M$ can be represented in the polar coordinates $x = (r, \theta)$ with $r > 0$ and $\theta \in \mathbb{S}^1$. Equip M with the Riemannian metric ds^2 given by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2,$$

where $\psi(r)$ is a smooth positive function on $(0, +\infty)$ and $d\theta^2$ is the normalized Riemannian metric on $\mathbb{S}^1$. In this case, M is called a *two-dimensional Riemannian model with a pole.*

Remark A sufficient and necessary condition, for the existence of this manifold is that ψ satisfies the conditions $\psi(0) = 0$ and $\psi'(0) = 1$. This ensures that the metric ds^2 can be smoothly extended to the origin $r = 0$ (see [9]).

We define the area function S on $(0, +\infty)$ by

$$S(r) = \psi(r).$$

Proposition 2.9 *Let M be a two-dimensional Riemannian model with a pole. Suppose that for any $A > 1$, there exists a constant $c > 0$, so that for all large enough r,*

$$\sup_{t\in(A^{-1}r,Ar)} \frac{S''_+(t)}{S(t)} \leq c\frac{S''_+(r)}{S(r)}. \tag{2.39}$$

Also assume that there exists a constant $N > 0$ such that, for every large enough r,

$$\frac{S(r)}{r} + \sqrt{S''_+(r)S(r)} \leq N\log(r). \tag{2.40}$$

Then the spherical Harnack inequality (2.22) holds in M.

Proof Fix some $x_0 \in M$ and denote $B_r = B(x_0, r)$. Since any model manifold of dimension $n \geq 2$ satisfies the (RCA) condition, there exists $A_0 > 1$ such that for all $r > A_0^2$ and any $x_1, x_2 \in \partial B_r$, there exists $T > 0$ and a continuous path $\gamma : [0, T] \to M$ such that $\gamma(0) = x_1$ and $\gamma(T) = x_2$, whose image is contained in $B_{A_0r} \setminus B_{A_0^{-1}r}$. Let us choose $A > A_0$ so that there exists a constant $\epsilon > 0$, such that $B(x, R) \subset B_{Ar} \setminus \overline{B_{A^{-1}r}}$, for any $x \in \gamma([0, T])$, where $R = \epsilon r$. Let u be a positive harmonic function in $M \setminus \overline{B_{A^{-1}r}}$ and $x_1, x_2 \in \partial B_r$ such that $\max_{\partial B_r} u = u(x_1)$ and $\min_{\partial B_r} u = u(x_2)$. Thus, we have to show that there are constants $N_H > 0$ and $C_H > 0$, so that if r is large enough, then

$$u(x_1) \leq C_H r^{N_H} u(x_2). \tag{2.41}$$

Let $x \in \gamma([0, T])$. Recall from [15, Exercise 3.31], that the Ricci curvature Ric on M is given by

$$Ric = -\frac{S''}{S}. \tag{2.42}$$

Hence, we obtain from Eq. 2.42,

$$Ric(x) \geq \inf_{t\in(A^{-1}r,Ar)} \left(-\frac{S''(t)}{S(t)}\right) \geq -\sup_{t\in(A^{-1}r,Ar)} \left(\frac{S''_+(t)}{S(t)}\right).$$

By Eq. 2.39, we get, assuming that r is large enough,

$$Ric(x) \geq -c\frac{S''_+(r)}{S(r)} =: -\kappa(r). \tag{2.43}$$

Clearly, we can assume that $|\gamma'(t)| = 1$. We have

$$\int_0^T \frac{|\nabla u(\gamma(t))|}{u(\gamma(t))} dt \leq \sup_{0\leq t\leq T} \frac{|\nabla u(\gamma(t))|}{u(\gamma(t))} \int_0^T dt \leq \sup_{0\leq t\leq T} \frac{|\nabla u(\gamma(t))|}{u(\gamma(t))} d(x_1, x_2).$$

Again, since M has dimension $n = 2$, and as $x_1, x_2 \in \partial B_r$, we see that

$$d(x_1, x_2) \leq S(r),$$

whence

$$\int_0^T \frac{|\nabla u(\gamma(t))|}{u(\gamma(t))} dt \leq \sup_{0\leq t\leq T} \frac{|\nabla u(\gamma(t))|}{u(\gamma(t))} S(r).$$

Applying the well-known gradient estimate (cf. [6]) to the harmonic function u in all balls $B(x, R)$, we obtain,

$$\sup_{0\leq t\leq T} \frac{|\nabla u(\gamma(t))|}{u(\gamma(t))} \leq C_n \left(\frac{1 + R\sqrt{\kappa(r)}}{R} \right),$$

where $\kappa(r)$ is given by Eq. 2.43 and $C_n > 0$ is a constant depending only on n. Therefore, we deduce

$$\begin{aligned}
\log u(x_1) - \log u(x_2) = \left| \int_0^T \frac{d \log u(\gamma(t))}{dt} \right| &\leq \int_0^T \frac{|du(\gamma(t))|}{u(\gamma(t))} \\
&= \int_0^T \frac{|\langle \nabla u, \gamma'(t)\rangle|}{u(\gamma(t))} dt \\
&\leq \int_0^T \frac{|\nabla u(\gamma(t))|}{u(\gamma(t))} dt \\
&\leq C_n \left(\frac{1}{\epsilon r} + \sqrt{\kappa(r)} \right) S(r),
\end{aligned}$$

which is equivalent to

$$u(x_1) \leq \exp\left(C_n \left(\frac{S(r)}{\epsilon r} + S(r)\sqrt{\kappa(r)} \right) \right) u(x_2).$$

Hence, we get by Eq. 2.43,

$$u(x_1) \leq \exp\left(C_n \left(\frac{S(r)}{\epsilon r} + \sqrt{c S_+''(r) S(r)} \right) \right) u(x_2).$$

Finally, by Eq. 2.40, we deduce for large enough r,

$$u(x_1) \leq r^{C_n \max\left\{\sqrt{c}, \frac{1}{\epsilon}\right\} N} u(x_2),$$

which proves (2.41) with $C_H = 1$ and $N_H = C_n \max\left\{\sqrt{c}, \frac{1}{\epsilon}\right\} N$ and finishes the proof. □

Example Let (M, μ) be a two-dimensional weighted manifold with end Ω and, following the notation in Theorem 2.6, suppose that M_0 is a Riemannian model with a pole such that

$$S_0(r) = \begin{cases} r \log r, & r \geq 2 \\ r, & r \leq 1. \end{cases}$$

Let us show that M_0 satisfies the hypotheses of Theorem 2.6 so that for any $x \in M$, there exist $t_x > 0$, $c_x > 0$ and $\alpha > 0$ such that for all $t \geq t_x$,

$$p_t(x, x) \geq \frac{c_x}{t^\alpha}. \tag{2.44}$$

Since $S_0''(r) = \frac{1}{r}$ for $r \geq 2$, the inequality (2.39) is satisfied and also

$$\frac{S_0(r)}{r} + \sqrt{\max\{(S_0'')(r), 0\} S_0(r)} = \log r + \sqrt{\log r} \leq 2 \log r,$$

whence (2.40) holds and we get that M_0 satisfies the spherical Harnack inequality (2.22). On the other hand, we have for $r \geq 2$, $-\frac{S_0''(r)}{S_0(r)} = -\frac{1}{r^2 \log r}$ so that it follows from Eq. 2.42 that M_0 has non-positive bounded below sectional curvature. Hence, M_0 is a locally Harnack manifold and, as it is simply connected, is a Cartan-Hadamard manifold which yields that the balls in M_0 of have at least euclidean volume. Therefore, condition (2.23) holds as well and we conclude from Theorem 2.6 that (M, μ) admits the estimate (2.44).

3 Isoperimetric Inequalities for Warped Products

Definition For any Borel set $A \subset M$, define its perimeter $\mu^+(A)$ by

$$\mu^+(A) = \liminf_{r \to 0^+} \frac{\mu(A^r) - \mu(A)}{r},$$

where A^r is the r-neighborhood of A with respect to the Riemannian metric of M.

Definition We say that (M, μ) admits the *lower isoperimetric function* J if, for any precompact open set $U \subset M$ with smooth boundary,

$$\mu^+(U) \geq J(\mu(U)). \tag{3.1}$$

For example, the euclidean space $\mathbb{R}^n$ with the Lebesgue measure satisfies the inequality in Eq. 3.1 with the function $J(v) = c_n v^{\frac{n-1}{n}}$.

3.1 Setting and Main Theorem

Let (M_1, μ_1) and (M_2, μ_2) be weighted manifolds and let $M = M_1 \times M_2$ be the direct product of M_1 and M_2 as topological spaces. This means that any point $z \in M$ can be written as $z = (x, y)$ with $x \in M_1$ and $y \in M_2$. Then we define the Riemannian metric ds^2 on M by

$$ds^2 = dx^2 + \psi^2(x) dy^2, \tag{3.2}$$

where ψ is a smooth positive function on M_1 and dx^2 and dy^2 denote the Riemannian metrics on M_1 and M_2, respectively. Let us define the measure μ on M by

$$\mu = \mu_1 \times \mu_2 \tag{3.3}$$

and note that then (M, μ) becomes a weighted manifold with respect to the metric in Eq. 3.2 (see Section 3.2 for an example).

Denote by ∇ the gradient on M and with ∇_x and ∇_y the gradients on M_1 and M_2, respectively. It follows from Eq. 3.2, that we have the identity

$$|\nabla u|^2 = |\nabla_x u|^2 + \frac{1}{\psi^2(x)} |\nabla_y u|^2, \tag{3.4}$$

for any smooth function u on M.

Definition Let $\varphi : (0, +\infty) \to (0, +\infty)$ be a monotone decreasing function. Then we define the generalized inverse function ϕ of φ on $(0, +\infty)$ by

$$\phi(s) = \sup\{t > 0 : \varphi(t) > s\}. \tag{3.5}$$

We will use the convention that the supremum of the empty set is zero.

One can easily prove the following

Lemma 3.1 *The generalized inverse ϕ of φ has the following properties:*

(1) *ϕ is monotone decreasing, right continuous and* $\lim_{s\to\infty} \phi(s) = 0$*;*

(2) *φ is right continuous if and only if φ itself is the generalized function of ϕ, that is*

$$\varphi(t) = \sup\{s > 0 : \phi(s) > t\}; \tag{3.6}$$

(3) *we have the identity*

$$\int_0^\infty \varphi(t)dt = \int_0^\infty \phi(s)ds. \tag{3.7}$$

The following lemma is well-known.

Lemma 3.2 *Let U be a precompact open subset of a weighted manifold (M, μ) with smooth boundary. Then*

$$\mu^+(U) = \inf_{\{u_n\}} \limsup_{n\to\infty} \int_M |\nabla u_n| d\mu = \sup_{\{u_n\}} \liminf_{n\to\infty} \int_M |\nabla u_n| d\mu,$$

where $\{u_n\}_{n\in\mathbb{M}}$ is a monotone increasing sequence of smooth non-negative functions with compact support, converging pointwise to the characteristic function of the set U.

The proof of the following theorem follows the ideas of Theorem 1 in [19], where an isoperimetric inequality is obtained for Riemannian products $M = M_1 \times M_2$ of two Riemannian manifolds M_1 and M_2.

Theorem 3.3 *Let (M_1, μ_1) and (M_2, μ_2) be weighted manifolds and let the weighted manifold (M, μ) be defined as above, that is, the Riemannian metric on M is defined by* Eq. 3.2 *and measure μ is defined by* Eq. 3.3. *Assume that there exists a constant $C_0 > 0$, such that for all $x \in M_1$,*

$$\psi(x) \le C_0. \tag{3.8}$$

Suppose that (M_1, μ_1) and (M_2, μ_2) have the lower isoperimetric functions J_1 and J_2, which are continuous on the intervals $(0, \mu_1(M_1))$ and $(0, \mu_2(M_2))$, respectively. Then (M, μ) admits the lower isoperimetric function J, defined by

$$J(v) = c \inf_{\varphi,\phi} \left(\int_0^\infty J_1(\varphi(t))dt + \int_0^\infty J_2(\phi(s))ds. \right),$$

where $c = \frac{1}{2} \min\left\{1, \frac{1}{C_0}\right\}$ and φ and ϕ are generalized mutually inverse functions such that

$$\varphi \le \mu_1(M_1), \quad \phi \le \mu_2(M_2), \tag{3.9}$$

and

$$v = \int_0^\infty \varphi(t)dt = \int_0^\infty \phi(s)ds. \tag{3.10}$$

Proof Let U be an open precompact set in M with smooth boundary such that $\mu(U) = v$. Let us define the function

$$I(v) = \inf_{\varphi,\phi}\left(\int_0^\infty J_1(\varphi(t))dt + \int_0^\infty J_2(\phi(s))ds.\right), \tag{3.11}$$

where φ and ϕ are generalized mutually inverse functions satisfying (3.9) and (3.10). We need to prove that

$$\mu^+(U) \geq cI(v),$$

where I is defined by Eq. 3.11 and c is defined as above. Let $\{f_n\}_{n\in\mathbb{N}}$ be a monotone increasing sequence of smooth non-negative functions on M with compact support such that $f_n \to 1_U$ as $n \to \infty$. Note that by Lemma 3.2, it suffices to show that

$$\liminf_{n\to\infty} \int_M |\nabla f_n| d\mu \geq cI(v). \tag{3.12}$$

By the identity (3.4) and using (3.8), we have

$$|\nabla f_n|^2 = |\nabla_x f_n|^2 + \frac{1}{\psi(x)^2}|\nabla_y f_n|^2 \geq \frac{1}{2}\min\left\{1, \frac{1}{C_0}\right\}^2 \left(|\nabla_x f_n| + |\nabla_y f_n|\right)^2.$$

Together with Eq. 3.12, it therefore suffices to prove that

$$\liminf_{n\to\infty} \int_M |\nabla_x f_n| d\mu + \liminf_{n\to\infty} \int_M |\nabla_y f_n| d\mu \geq I(v). \tag{3.13}$$

Let us first estimate the second summand on the left-hand side of Eq. 3.13. For that purpose, consider for every $x \in M_1$, the section

$$U_x = \{y \in M_2 : (x, y) \in U\}.$$

By Sard's theorem, the set U_x has smooth boundary for allmost all x. Considering the function $f_n(x, y)$ as a function on M_2 with fixed $x \in M_1$, we obtain by Lemma 3.2 for allmost all x,

$$\liminf_{n\to\infty} \int_{M_2} |\nabla_y f_n(x, y)| d\mu_2(y) \geq \mu_2^+(U_x).$$

Integrating this over M_1 and using Fatou's lemma, we deduce

$$\liminf_{n\to\infty} \int_M |\nabla_y f_n| d\mu \geq \int_{M_1} \mu_2^+(U_x) d\mu_1(x). \tag{3.14}$$

The first summand on the left-hand side of Eq. 3.13 could be estimated analogously, but instead, we will estimate it using the assumption that (M_1, μ_1) and (M_2, μ_2) admit lower isoperimetric functions J_1 and J_2, respectively. First, by Fubini's formula, we have

$$\int_M |\nabla_x f_n| d\mu = \int_{M_1}\int_{M_2} |\nabla_x f_n| d\mu_2 d\mu_1 \geq \int_{M_1} \left|\nabla_x \int_{M_2} f_n(x, y) d\mu_2(y)\right| d\mu_1(x). \tag{3.15}$$

Now let us consider on M_1 the function

$$F_n(x) = \int_{M_2} f_n(x, y) d\mu_2(y).$$

Note that $F_n(x)$ is a monotone increasing sequence of non-negative smooth functions on M_1, such that

$$F(x) := \lim_{n\to\infty} F_n(x) = \mu_2(U_x). \tag{3.16}$$

Since F_n is smooth for all n, we deduce that the sets $\{F_n > t\}$ have smooth boundary, so that we can apply the isoperimetric inequality on M_1, that is,

$$\mu_1^+\{F_n > t\} \geq J_1(\mu_1\{F_n > t\}).$$

Hence, we obtain, using (3.15) and the co-area formula,

$$\begin{aligned}\int_M |\nabla_x f_n| d\mu \geq \int_{M_1} |\nabla_x F_n| d\mu_1 &= \int_0^\infty \mu_1'\{F_n = t\} dt \\ &= \int_0^\infty \mu_1^+\{F_n > t\} dt \\ &\geq \int_0^\infty J_1(\mu_1\{F_n > t\}) dt.\end{aligned}$$

Passing to the limit as $n \to \infty$, we get by Fatou's lemma, using the continuiuty of J_1,

$$\limsup_{n\to\infty} \int_M |\nabla_x f_n| d\mu \geq \int_0^\infty J_1(\mu_1\{F > t\}) dt. \tag{3.17}$$

By the isoperimetric inequality on M_2 with function J_2 and by Eq. 3.16,

$$\mu_2^+(U_x) \geq J_2(\mu_2(U_x)) = J_2(F(x)),$$

whence combining this with Eqs. 3.14 and 3.17, we get

$$\limsup_{n\to\infty} \int_M |\nabla_x f_n| d\mu + \limsup_{n\to\infty} \int_M |\nabla_y f_n| d\mu \geq \int_0^\infty J_1(\mu_1\{F > t\}) dt + \int_{M_1} J_2(F(x)) d\mu_1(x). \tag{3.18}$$

Let us set

$$\varphi(t) = \mu_1\{F > t\}$$

and note that φ is monotone decreasing and right-continuous. Let ϕ be the generalized inverse function to φ defined by Eq. 3.5. Then we obtain by Eq. 3.6,

$$\sup\{s > 0 : \phi(s) > t\} = \mu_1\{F > t\}, \tag{3.19}$$

which means that ϕ and F are equimeasurable. Clearly, $\varphi \leq \mu_1(M_1)$. Since $F \leq \mu_2(M_2)$, which implies $\varphi(t) = 0$ for all $t > \mu_2(M_2)$, we also obtain $\phi \leq \mu_2(M_2)$ by Eq. 3.5. By Eq. 3.7, the definition of φ and Fubini's formula,

$$\int_0^\infty \phi(t) dt = \int_0^\infty \varphi(t) dt = \int_{M_1} F d\mu_1 = \mu(U) = v.$$

Hence, the pair φ, ϕ satisfies the condition in Eq. 3.10. Note that by Eq. 3.19,

$$\int_{M_1} J_2(F(x)) d\mu_1(x) = \int_0^\infty J_2(\phi(t)) dt,$$

whence we obtain for the right-hand side of Eq. 3.18,

$$\int_{M_1} J_2(F(x)) d\mu_1(x) + \int_0^\infty J_1(\mu_1\{F > t\}) dt = \int_0^\infty J_2(\phi(t)) dt + \int_0^\infty J_1(\varphi(t)) dt \geq I(v),$$

which proves (3.13) and thus, finishes the proof (Fig. 1). □

Let $P > 0$. Given two non-negative functions f on $(0, +\infty)$ and g on $(0, P)$ define the function h on $(0, +\infty)$ by

$$h(v) = \inf_{\varphi,\phi} \left(\int_0^\infty f(\varphi(t)) dt + \int_0^\infty g(\phi(s)) ds. \right),$$

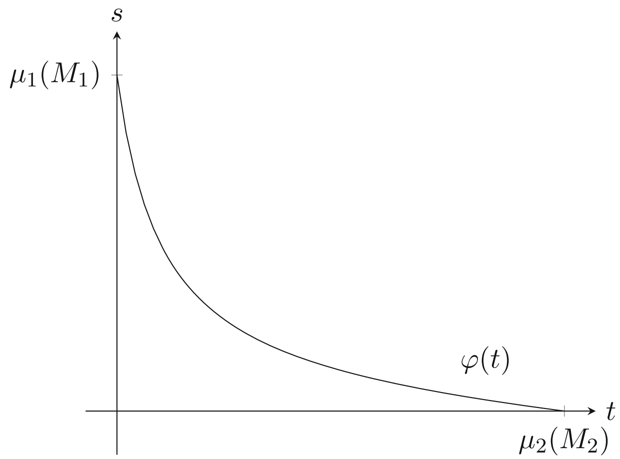

Fig. 1 Function $\varphi(t)$

where φ and ϕ are generalized mutually inverse functions on $(0,+\infty)$ such that

$$\int_0^\infty \varphi(t)dt = \int_0^\infty \phi(s)ds = v. \tag{3.20}$$

and with the condition that $\phi < P$. For fixed φ, ϕ, where φ, ϕ are as above, let us denote

$$S = \int_0^\infty f(\varphi(t))dt + \int_0^\infty g(\phi(s))ds. \tag{3.21}$$

Lemma 3.4 *Let f and g be continuous functions on the intervals $(0,+\infty)$ and $(0,P)$, respectively and suppose that g is symmetric with respect to $\frac{1}{2}P$. Also, assume that the functions $\frac{f(x)}{x}$ and $\frac{g(y)}{y}$ are monotone decreasing while the functions f and g are monotone increasing on the intervals $(0,+\infty)$ and $\left(0,\frac{P}{2}\right)$, respectively. Then, for any $v>0$,*

$$h(v) \geq \min\left(\frac{1}{6}h_0(v), \frac{1}{8}f\left(\frac{v}{P}\right)P\right),$$

where the function h_0 is defined for all $v>0$, by

$$h_0(v) = \inf_{\substack{xy=v \\ x>0,\ 0<y\leq \frac{1}{2}P}} (f(x)y + g(y)x). \tag{3.22}$$

Remark A similar functional inequality was stated in [19, Theorem 2a] without proof.

In the following we denote by $|A|$ the Lebesgue measure of a domain $A \subset \mathbb{R}^2$.

Proof Let φ be decreasing and right-continuous and ϕ be its generalized inverse function satisfying (3.20) and let S be defined as in Eq. 3.21. We need to prove that

$$S \geq \min\left(\frac{1}{6}h_0(v), \frac{1}{8}f\left(\frac{v}{P}\right)P\right). \tag{3.23}$$

Let us first suppose that φ is strictly monotone decreasing and continuous on an interval $(0,T) \subset (0,P)$ such that $\lim_{t\to T}\varphi(t) = 0$ and $\varphi((0,T)) = (0,+\infty)$. Denote by ϕ the inverse function of φ on $(0, +\infty)$ and note that ϕ is then also strictly monotone decreasing and continuous and satisfies $\phi < T$. Let us show that

$$S \geq \min\left(\frac{1}{6}h_T(v), \frac{1}{8}f\left(\frac{v}{T}\right)T\right), \tag{3.24}$$

where

$$h_T(v) = \inf_{\substack{xy=v \\ x>0,\ 0<y\leq\frac{1}{2}T}} (f(x)y + g(y)x),$$

which will then imply (3.23) by an approximation argument.

For any $p \in (0, T)$, consider the domain

$$\Phi_p = \{(t,s) \in \mathbb{R}^2 : p \leq t < T,\ 0 \leq s \leq \varphi(t)\}$$

and for any $q > 0$ the domain

$$\Psi_q = \{(t,s) \in \mathbb{R}^2 : s \geq q,\ 0 \leq t \leq \phi(s)\}.$$

Since ϕ is strictly monotone decreasing and continuous, there exists $q > 0$ such that $|\Psi_q| = \frac{1}{3}v$. Let us set $p = \phi(q)$ and note that

$$v = \int_0^\infty \phi(s)ds = |\Phi_p| + |\Psi_q| + pq. \tag{3.25}$$

The proof will be split into two main cases.

Case 1. Let us assume that

$$|\Phi_p| \geq \frac{1}{3}v.$$

Then we obtain by Eq. 3.25 that $p \leq \frac{1}{3q}v$. By the monotonicity of $\frac{g(y)}{y}$, we therefore get

$$\int_0^\infty g(\phi(s))ds \geq \frac{1}{3}xg(y),$$

where $x = 3q$ and $y = \frac{1}{3q}v$ and similarly,

$$\int_0^\infty f(\varphi(t))dt \geq \frac{1}{3}f(x)y.$$

Hence, we obtain that

$$S \geq \frac{1}{3}h_0(v).$$

Case 2. Let us now assume that

$$|\Phi_p| < \frac{1}{3}v.$$

Then we can decrease p to p' such that $|\Phi_{p'}| = \frac{1}{3}v$. Set $q' = \varphi(p')$ and note that this q' is larger than the q from Case 1, whence

$$|\Psi_{q'}| \leq \frac{1}{3}v,$$

so that (3.25) implies

$$\frac{1}{3}v \leq p'q' \leq \frac{2}{3}v.$$

Case 2a. Assume further that $p' \geq \frac{1}{4}T$. It follows that

$$\int_0^\infty f(\varphi(t))dt \geq \frac{1}{3}\frac{f(q')}{q'}v$$

and since f is monotone increasing, we conclude

$$S \geq \frac{T}{8} f\left(\frac{v}{T}\right),$$

which proves (3.24).

Case 2b. Assume now that $p' < \frac{1}{4}T$ and set $q_0 = \varphi\left(\frac{1}{2}T\right)$.

Case 2b(i). Let us first consider the case when $q_0 \leq \frac{1}{2}q'$. Using that $g(y)$ is monotone increasing on $\left(0, \frac{T}{2}\right)$, we obtain,

$$\int_0^\infty g(\phi(s))ds \geq \frac{1}{2}g(p')q'.$$

Together with

$$\int_0^\infty f(\varphi(t))dt \geq f(q')p',$$

we deduce

$$S \geq \frac{1}{2}g(p')q' + f(q')p',$$

so that setting $x = \frac{v}{p'}$ and $y = p'$, yields

$$S \geq \frac{1}{6}\left(f(x)y + g(y)x\right) \geq \frac{1}{6}h_T(v).$$

Case 2b(ii). Finally, let us consider the case when $q_0 > \frac{1}{2}q'$. Note that the condition that $\frac{f(x)}{x}$ is monotone decreasing, implies that for any $\lambda \in (0, 1)$,

$$f(\lambda x) \geq \lambda f(x).$$

Together with the monotonicity of f, we therefore obtain

$$\int_0^{T/2} f(\varphi(t))dt \geq f(q')\frac{T}{4},$$

which yields

$$S \geq f\left(\frac{v}{T}\right)\frac{T}{4},$$

and thus, proves (3.24) also in this case.

Now let us consider the general case, when φ is monotone decreasing and right-continuous and ϕ being its generalized inverse function satisfying (3.20). Then consider an increasing sequence $\{\varphi_n\}_n$ of functions which are positive, continuous, strictly decreasing functions on an interval $(0, T_n) \subset (0, P)$ such that $T_n \to P$, $\varphi_n(t) \to \varphi(t)$ and $v_n := \int_0^\infty \varphi_n(t)dt \to v$ for $n \to +\infty$. Letting ϕ_n be the inverse function of φ_n on $(0, T_n)$ for all n, we get by [8, Lemma 1.1.1], that for every continuity point $s \in (0, +\infty)$ of ϕ,

$$\phi_n(s) \to \phi(s) \quad \text{as } n \to +\infty.$$

By the former case, we have the inequality (3.24) for all φ_n, that is,

$$\int_0^\infty f(\varphi_n(t))dt + \int_0^\infty g(\phi_n(s))ds \geq \min\left(\frac{1}{6}h_{T_n}(v_n), \frac{1}{8}f\left(\frac{v_n}{T_n}\right)T_n\right). \tag{3.26}$$

Now let $q_1 = \varphi\left(\frac{P}{2}\right)$ and note that $\phi_n(s) \leq \frac{P}{2}$ for all n and $s \geq q_1$, whence using that g is monotone increasing on $\left(0, \frac{P}{2}\right)$, we obtain for all $s \geq q_1$,

$$g(\phi_n(s)) \leq g(\phi_{n+1}(s)).$$

Hence, we obtain by the dominated convergence theorem, the monotone convergence theorem and the continuity of g,

$$\lim_{n\to\infty}\int_0^\infty g(\phi_n(s))ds = \lim_{n\to\infty}\left(\int_0^{q_1} g(\phi_n(s))ds + \int_{q_1}^\infty g(\phi_n(s))ds\right) = \int_0^\infty g(\phi(s))ds.$$

Using the monotonicity and the continuity of f, we get by the monotone convergence theorem,

$$\lim_{n\to\infty}\int_0^\infty f(\varphi_n(t))dt = \int_0^\infty f(\varphi(t))dt.$$

Hence, passing to the limit as $n \to +\infty$ in Eq. 3.26, we conclude by the continuity of the right-hand side of Eq. 3.26, that inequality (3.23) holds, which finishes the proof. □

Corollary 3.5 *In the situation of Theorem 3.3 suppose that*

$$\mu_1(M_1) = \infty \quad \textit{and} \quad \mu_2(M_2) < \infty$$

and assume that $\frac{J_1(x)}{x}$ *and* $\frac{J_2(y)}{y}$ *are monotone decreasing while the functions* J_1 *and* J_2 *are monotone increasing on the intervals* $(0, +\infty)$ *and* $\left(0, \frac{1}{2}\mu_2(M_2)\right)$*, respectively. Then the manifold* (M, μ) *admits the lower isoperimetric function*

$$J(v) = c\min\left(\frac{1}{6}J_0(v), \frac{1}{8}J_1\left(\frac{v}{\mu_2(M_2)}\right)\mu_2(M_2)\right), \tag{3.27}$$

where function J_0 *is defined for all* $v > 0$*, by*

$$J_0(v) = \inf_{\substack{xy=v \\ x>0,\ 0<y\le\frac{1}{2}\mu_2(M_2)}} (J_1(x)y + J_2(y)x), \tag{3.28}$$

and the constant c *is defined as in Theorem 3.3.*

Proof From Theorem 3.3, we know that (M, μ) has the lower isoperimetric function cI, where I is defined by

$$I(v) = \inf_{\varphi,\phi}\left(\int_0^\infty J_1(\varphi(t))dt + \int_0^\infty J_2(\phi(s))ds.\right),$$

where φ and ϕ are generalized mutually inverse functions satisfying $\phi \le \mu_2(M_2)$ and the condition in Eq. 3.20. Since $\mu_2(M_2)$ is finite, we can assume that the isoperimetric function J_2 is symmetric with respect to $\frac{1}{2}\mu_2(M_2)$, because the topological boundaries of an open set and its complement coincide. Applying Lemma 3.4 to I with $f = J_1$, $g = J_2$ and $P = \mu_2(M_2)$, we obtain

$$I(v) \ge \min\left(\frac{1}{6}J_0(v), \frac{1}{8}J_1\left(\frac{v}{\mu_2(M_2)}\right)\mu_2(M_2)\right),$$

where function J_0 is defined by Eq. 3.28, which implies that function J given by Eq. 3.27 is a lower isoperimetric function for (M, μ). □

3.2 Weighted Models with Boundary

Let us also consider the topological space $M = \mathbb{R}_+ \times \mathbb{S}^{n-1}$, $n \ge 2$, where $\mathbb{R}_+ = [0, +\infty)$, so that any point $x \in M$ can be written in the polar form $x = (r, \theta)$ with $r \in \mathbb{R}_+$ and

$\theta \in \mathbb{S}^{n-1}$. We equip M with the Riemannian metric ds^2 that is defined in polar coordinates (r, θ) by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2$$

with $\psi(r)$ being a smooth positive function on $\mathbb{R}_+$ and $d\theta^2$ being the Riemannian metric on $\mathbb{S}^{n-1}$. Note that M with this metric becomes a manifold with boundary

$$\delta M = \{(r, \theta) \in M : r = 0\}$$

and we call M in this case a *Riemannian model with boundary*. The Riemannian measure μ on M with respect to this metric is given by

$$d\mu = \psi^{n-1}(r)drd\sigma(\theta),$$

where dr denotes the Lebesgue measure on $\mathbb{R}_+$ and $d\sigma$ denotes the Riemannian measure on $\mathbb{S}^{n-1}$. Let us normalize the metric $d\theta^2$ on $\mathbb{S}^{n-1}$ so that $\sigma(\mathbb{S}^{n-1}) = 1$ and define the area function S on $\mathbb{R}_+$ by

$$S(r) = \psi^{n-1}(r).$$

Given a smooth positive function h on M, that only depends on the polar radius r, and a measure $\widetilde{\mu}$ on M defined by $d\widetilde{\mu} = h^2 d\mu$, we obtain that the weighted manifold $(M, \widetilde{\mu})$ has the area function

$$\widetilde{S}(r) = h^2(r)S(r).$$

Then the weighted manifold $(M, \widetilde{\mu})$ is called a *weighted model* and we get that

$$d\widetilde{\mu} = \widetilde{S}(r)drd\sigma(\theta). \tag{3.29}$$

Theorem 3.6 *Let (M_0, μ_0) be a model manifold with boundary. Assume that there exists a constant $C_0 > 0$ such that for all $r \geq 0$,*

$$\psi_0(r) \leq C_0. \tag{3.30}$$

Assume also, that

$$\widetilde{S_0}(r) \simeq \begin{cases} r^\delta e^{r^\alpha}, & r \geq 1, \\ 1, & r < 1, \end{cases} \tag{3.31}$$

where $\delta \in \mathbb{R}$ and $\alpha \in (0, 1]$. Then the weighted model $(M_0, \widetilde{\mu_0})$ with area function $\widetilde{S_0}$ admits the lower isoperimetric function J defined by

$$J(w) = \widetilde{c} \begin{cases} \frac{w}{(\log w)^{\frac{1-\alpha}{\alpha}}}, & w \geq 2, \\ c' w^{\frac{n-1}{n}}, & w < 2, \end{cases} \tag{3.32}$$

where $\widetilde{c}$ is a small enough constant and c' is a positive constant chosen such that J is continuous.

Proof Let ν be the measure on $\mathbb{R}_+$ defined by $d\nu(r) = \widetilde{S_0}(r)dr$. Then (3.29) implies that measure $\widetilde{\mu_0}$ has the representation $\widetilde{\mu_0} = \nu \times \sigma$, where σ is the normalized Riemannian measure on the sphere $\mathbb{S}^{n-1}$. Obviously, we have by Eq. 3.31, that

$$\nu(\mathbb{R}_+) = \int_0^\infty \widetilde{S_0}(r)dr = +\infty.$$

Since $\widetilde{S_0}$ is a positive, continuous and non-decreasing function on $\mathbb{R}_+$, we obtain from [2, Proposition 3.1], that $(\mathbb{R}_+, \nu)$ has a lower isoperimetric function $J_\nu(v)$ given by

$$J_\nu(v) = \widetilde{S_0}(r),$$

where $v = \nu([0, r))$. Clearly, for small R, we have $J_\nu(v) \simeq 1$. For large enough R, we obtain

$$v = \int_0^R \widetilde{S}_0(r)dr \simeq R^{\delta+1-\alpha}e^{R^\alpha}.$$

This implies that for large v,

$$\log v \simeq R^\alpha + (\delta + 1 - \alpha)\log R \simeq R^\alpha,$$

and thus,

$$J_\nu(v) = \widetilde{S}_0(R) \simeq R^\delta e^{R^\alpha} = R^{\alpha-1}R^{\delta+1-\alpha}e^{R^\alpha} \simeq \frac{v}{(\log v)^{\frac{1-\alpha}{\alpha}}},$$

which proves that

$$J_\nu(v) = c_0 \begin{cases} \frac{v}{(\log v)^{\frac{1-\alpha}{\alpha}}}, & v \geq 2, \\ c_1, & v < 2, \end{cases}$$

is a lower isoperimetric function of $(\mathbb{R}_+, \nu)$ if $c_0 > 0$ is a small enough constant and continuous for an appropriate choice of constant $c_1 > 0$. Note J_ν is monotone increasing on $\mathbb{R}_+$ and, since $\alpha \in (0, 1]$, the function $\frac{J_\nu(v)}{v}$ is monotone decreasing. Let J_σ be the function defined by

$$J_\sigma(v) = c_n \begin{cases} v^{\frac{n-2}{n-1}}, & \text{if } 0 \leq v \leq \frac{1}{2}, \\ (1-v)^{\frac{n-2}{n-1}}, & \text{if } \frac{1}{2} < v \leq 1. \end{cases}$$

It is a well-known fact that J_σ is a lower isoperimetric function for $(\mathbb{S}^{n-1}, \sigma)$ assuming that the constant $c_n > 0$ is sufficiently small. Since we assume that ψ_0 satisfies the condition in Eq. 3.30, we can apply Corollary 3.5 and deduce that a lower isoperimetric function J of $(M_0, \widetilde{\mu_0})$ is given by

$$J(w) = c\min\left(\frac{1}{6}J_0(w), \frac{1}{8}J_\nu(w)\right), \tag{3.33}$$

where J_0 is defined by

$$J_0(w) = \inf_{\substack{uv=w \\ u>0,\ 0<v\leq\frac{1}{2}}} (J_\nu(u)v + J_\sigma(v)u)$$

and the constant $c > 0$ is defined as in Theorem 3.3.

In order to estimate J in this case, let us consider the function K, defined for all $w > 0$, by

$$K(w) = \frac{J(w)}{w} = c\min\left(\frac{1}{6}K_0(w), \frac{1}{8}K_\nu(w)\right), \tag{3.34}$$

where K_0 is given by

$$K_0(w) = \inf_{\substack{uv=w \\ u>0,\ 0<v\leq\frac{1}{2}}} (K_\nu(u) + K_\sigma(v)), \tag{3.35}$$

where $K_\nu(u) = \frac{J_\nu(u)}{u}$ and $K_\sigma(v) = \frac{J_\sigma(v)}{v}$. Observe that, since K_σ is monotone decreasing,

$$K_0(w) \geq \inf_{0<v\leq\frac{1}{2}} K_\sigma(v) \geq K_\sigma\left(\frac{1}{2}\right).$$

Note that if $w \geq 2$ and $v \leq \frac{1}{2}$, then $u = \frac{w}{v} \geq 4$. Hence, we obtain that for $w \geq 2$,

$$K_0(w) \simeq \text{const}.$$

Substituting this into Eq. 3.34, we get, using that K_ν is monotone decreasing, $K(w) \simeq K_\nu(w)$ for $w \geq 2$, and whence

$$J(w) \simeq J_\nu(w) \simeq \frac{w}{(\log w)^{\frac{1-\alpha}{\alpha}}}, \quad w \geq 2. \tag{3.36}$$

Note that if $w \leq 2$, the infimum is attained when $u \leq 2$ and the summands in Eq. 3.35 are comparable. Observe that this holds true when

$$v \simeq w^{\frac{1}{2-\frac{n-2}{n-1}}},$$

so that substituting this into Eq. 3.35, we deduce for $w \leq 2$,

$$K_0(w) \simeq w^{-\frac{1}{n}}.$$

Hence, we obtain that for all $w \leq 2$,

$$J_0(w) \simeq w^{\frac{n-1}{n}},$$

and therefore by Eq. 3.33,

$$J(w) \simeq w^{\frac{n-1}{n}}, \quad w \leq 2.$$

Combining this with Eq. 3.36, we conclude that the function $J(w)$ defined by Eq. 3.32 is a lower isoperimetric function for the weighted model $(M_0, \widetilde{\mu_0})$. □

4 On-diagonal Heat Kernel Upper Bounds

Recall from Eq. 2.20, that for any open set $\Omega \subset M$, we define

$$\lambda_1(\Omega) = \inf_u \frac{\int_\Omega |\nabla u|^2 d\mu}{\int_\Omega u^2 d\mu},$$

where the infimum is taken over all nonzero Lipschitz functions u compactly supported in Ω.

Definition We say that (M, μ) satisfies a *Faber-Krahn inequality* with a function $\Lambda : (0, +\infty) \to (0, +\infty)$ if, for any non-empty precompact open set $\Omega \subset M$,

$$\lambda_1(\Omega) \geq \Lambda(\mu(\Omega)). \tag{4.1}$$

It is well-known that a Faber-Krahn inequality (4.1) implies certain heat kernel upper bounds of the heat kernel (see [4] and [14]).

Proposition 4.1 ([14], Theorem 5.1) *Suppose that a weighted manifold (M, μ) satisfies a Faber-Krahn inequality (4.1) with Λ being a continuous and decreasing function such that*

$$\int_0^1 \frac{dv}{v\Lambda(v)} < \infty. \tag{4.2}$$

Then for all $t > 0$,

$$\sup_{x \in M} p_t(x, x) \leq \frac{4}{\gamma(t/2)}, \tag{4.3}$$

where the function γ is defined by

$$t = \int_0^{\gamma(t)} \frac{dv}{v\Lambda(v)}. \tag{4.4}$$

Definition Let $\{M_i\}_{i=0}^k$ be a finite family of non-compact Riemannian manifolds. We say that a manifold M is a *connected sum* of the manifolds M_i and write

$$M = \bigsqcup_{i=0}^{k} M_i \tag{4.5}$$

if, for some non-empty compact set $K \subset M$ the exterior $M \setminus K$ is a disjoint union of open sets $E_0, \dots, E_k$ such that each E_i is isometric to $M_i \setminus K_i$ for some compact set $K_i \subset M_i$.

Conversely, we have the following definition.

Definition Let M be a non-compact manifold and $K \subset M$ be a compact set with smooth boundary such that $M \setminus K$ is a disjoint union of finitely many ends $E_0, \dots, E_k$. Then M is called a *manifold with ends*.

Remark Let M be a manifold with ends $E_0, \dots, E_k$. Considering each end E_i as an exterior of another manifold M_i, then M can be written as in Eq. 4.5.

Let $(M = \bigsqcup_{i=0}^k M_i, \mu)$ be a connected sum of complete non-compact weighted manifolds (M_i, μ_i) and h be a positive smooth function on M. As before, let us consider the weighted manifold $(M, \widetilde{\mu})$, where $\widetilde{\mu}$ is defined by $d\widetilde{\mu} = h^2 d\mu$. By restricting h to the end $E_i = M_i \setminus K_i$ and then extending this restriction smoothly to a function h_i on M_i, we obtain weighted manifolds $(M_i, \widetilde{\mu}_i)$, where $\widetilde{\mu}_i$ is given by $d\widetilde{\mu}_i = h_i^2 d\mu$.

From now on, we always have $\dim(M) = n$.

Theorem 4.2 *Let $(M, \widetilde{\mu}) = \left(\bigsqcup_{i=0}^k M_i, \widetilde{\mu}\right)$ be a weighted manifold with ends where M_0 is a model manifold with boundary so that for all $r \geq 0$,*

$$\psi_0(r) \leq C_0$$

and

$$\widetilde{S_0}(r) \simeq \begin{cases} r^\delta e^{r^\alpha}, & r \geq 1, \\ 1, & r < 1, \end{cases}$$

where $0 < \alpha \leq 1$, $\delta \in \mathbb{R}$ and $\widetilde{S_0}$ denotes the area function of a weighted model $(M_0, \widetilde{\mu_0})$. Assume also that all $(M_i, \widetilde{\mu_i})$, $i = 1, \dots k$, have Faber-Krahn functions $\widetilde{\Lambda_i}$ such that

$$\widetilde{\Lambda_i}(v) \geq c_i \begin{cases} \frac{1}{(\log v)^{\frac{2-2\alpha}{\alpha}}}, & v \geq 2, \\ v^{-\frac{2}{n}}, & v < 2, \end{cases}$$

for constants $c_i > 0$. Then there exist constants $C > 0$ and $C_1 > 0$ depending on α and n so that the heat kernel $\widetilde{p_t}$ of $(M, \widetilde{\mu})$ satisfies

$$\sup_{x \in M} \widetilde{p_t}(x, x) \leq C \begin{cases} \exp\left(-C_1 t^{\frac{\alpha}{2-\alpha}}\right), & t \geq 1, \\ t^{-\frac{n}{2}}, & 0 < t < 1. \end{cases} \tag{4.6}$$

Proof It follows from Theorem 3.6, that $(M_0, \widetilde{\mu_0})$ has the lower isoperimetric function J given by Eq. 3.32, that is

$$J(v) = \widetilde{c} \begin{cases} \frac{v}{(\log v)^{\frac{1-\alpha}{\alpha}}}, & v \geq 2, \\ c' v^{\frac{n-1}{n}}, & v < 2, \end{cases}$$

where $\widetilde{c} > 0$ is a small enough constant and c' is a positive constant chosen such that J is continuous. Since J is continuous and the function $\frac{J(v)}{v}$ is non-increasing, we obtain from [13, Proposition 7.1], that $(M_0, \widetilde{\mu_0})$ admits a Faber-Krahn function $\widetilde{\Lambda}_0$ given by

$$\widetilde{\Lambda}_0(v) = \frac{1}{4}\left(\frac{J(v)}{v}\right)^2 \simeq \begin{cases} \frac{1}{(\log v)^{\frac{2-2\alpha}{\alpha}}}, & v \geq 2, \\ v^{-\frac{2}{n}}, & v < 2. \end{cases}$$

We obtain from [18, Theorem 3.4] that there exist constants $c > 0$ and $Q > 1$ such that $(M, \widetilde{\mu})$ admits the Faber-Krahn function

$$\widetilde{\Lambda}(v) = c \min_{0 \leq i \leq k} \widetilde{\Lambda}_i(Qv).$$

Hence $(M, \widetilde{\mu})$ has a Faber-Krahn function $\widetilde{\Lambda}$, satisfying

$$\widetilde{\Lambda}(v) \simeq \begin{cases} \frac{1}{(\log v)^{\frac{2-2\alpha}{\alpha}}}, & v \geq 2, \\ v^{-\frac{2}{n}}, & v < 2. \end{cases} \tag{4.7}$$

Observe that the Faber-Krahn function $\widetilde{\Lambda}$ satisfies condition (4.2). Thus, we can apply Proposition 4.1, which yields the heat kernel upper bound in Eq. 4.3. Hence, it remains to estimate the function γ from the right hand side of Eq. 4.3 by using (4.4). In the case when $t > 0$ is small enough, we get by Eqs. 4.4 and 4.7,

$$t = \int_0^{\gamma(t)} \frac{dv}{v\widetilde{\Lambda}(v)} = C' \int_0^{\gamma(t)} \frac{dv}{v^{1-\frac{2}{n}}} = C'\gamma(t)^{\frac{2}{n}},$$

which implies for some constant $C'' > 0$,

$$\gamma(t) = C'' t^{\frac{n}{2}}.$$

For large enough t on the other hand, we deduce

$$t = \int_0^{\gamma(t)} \frac{dv}{v\widetilde{\Lambda}(v)} \simeq \int_2^{\log(\gamma(t))} u^{\frac{2-2\alpha}{\alpha}} du \simeq \log(\gamma(t))^{\frac{2-\alpha}{\alpha}}.$$

Therefore,

$$\gamma(t) \simeq \exp\left(\text{const}\, t^{\frac{\alpha}{2-\alpha}}\right),$$

where const is a positive constant depending on α and n. Substituting these estimates for $\gamma(t)$ into (4.3), we obtain the upper bound (4.6) for the heat kernel $\widetilde{p}_t$ of $(M, \widetilde{\mu})$ for small and large values of t. For the intermediate values of t, we deduce the upper bound (4.6) from the fact that the function $t \mapsto \sup_{x \in M} \widetilde{p}_t(x, x)$ is continuous. □

Example In Theorem 4.2 one can take $(M_i, \widetilde{\mu_i}) = (\mathbb{H}^n, \mu_i)$, $i = 1, \ldots k$, where μ_i is the Riemannian measure on the hyperbolic space $\mathbb{H}^n$ since for all $0 < \alpha \leq 1$, we have

$$\Lambda_{\mathbb{H}^n}(v) \simeq \begin{cases} 1, & v \geq 2, \\ v^{-\frac{2}{n}}, & v < 2 \end{cases} \geq c \begin{cases} \frac{1}{(\log v)^{\frac{2-2\alpha}{\alpha}}}, & v \geq 2, \\ v^{-\frac{2}{n}}, & v < 2. \end{cases}$$

Remark Let $(M, \widetilde{\mu})$ be the weighted manifold with ends, defined as in Theorem 4.2, so that $\widetilde{S}_0(r) \simeq e^{r^\alpha} r^\delta$ for $r > 1$ and hence, for $R > 1$,

$$\widetilde{V}_0(R) = \int_0^R \widetilde{S}_0(r) dr \simeq \int_0^R e^{r^\alpha} r^\delta dr \simeq e^{R^\alpha} R^{\delta+1-\alpha}.$$

Then, we obtain from [7, Proposition 3.4] for large enough R,

$$\widetilde{\lambda_1}(\Omega_R) \leq 4\left(\frac{\widetilde{S_0}(R)}{\widetilde{V_0}(R)}\right)^2 \leq \frac{C}{R^{2-2\alpha}},$$

where $\Omega_R = \{(r,\theta) \in M_0 : 0 < r < R\}$. Hence, setting $R = R(t) = t^{\frac{1}{2-\alpha}}$, [7, Proposition 2.3] yields the following lower bound for the heat kernel $\widetilde{p_t}$ in $(M, \widetilde{\mu})$ for large enough t:

$$\sup_x \widetilde{p_t}(x,x) \geq \frac{1}{\widetilde{\mu}(\Omega_R)} \exp\left(-\widetilde{\lambda_1}(\Omega_R)t\right) \geq \frac{C_1}{e^{R^\alpha(t)}R^{\delta+1-\alpha}(t)} \exp\left(-\frac{Ct}{R^{2-2\alpha}(t)}\right) \geq \frac{C_1}{e^{C_2 t^{\frac{\alpha}{2-\alpha}}}},$$

which shows that the exponential decay in the upper bound given in Eq. 4.6 is sharp.

4.1 Weighted Models with Two Ends

Let M be the topological space $M = \mathbb{R} \times \mathbb{S}^{n-1}$, $n \geq 2$, that is, any point $x \in M$ can be written in the polar form $x = (r, \theta)$ with $r \in \mathbb{R}$ and $\theta \in \mathbb{S}^{n-1}$. For a fixed smooth positive function ψ on $\mathbb{R}$ consider on M the Riemannian metric ds^2 given by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2,$$

where $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$. The Riemannian measure μ on M with respect to this metric is given by

$$d\mu = \psi^{n-1}(r)drd\sigma(\theta),$$

where dr denotes the Lebesgue measure on $\mathbb{R}$ and $d\sigma$ the Riemannian measure on $\mathbb{S}^{n-1}$. As before, we normalize the metric $d\theta^2$ on $\mathbb{S}^{n-1}$ so that $\sigma(\mathbb{S}^{n-1}) = 1$. Then we define the area function S on $\mathbb{R}$ by

$$S(r) = \psi^{n-1}(r).$$

Given a smooth positive function h on M, that only depends on the polar radius $r \in \mathbb{R}$, and considering the measure $\widetilde{\mu}$ on M defined by $d\widetilde{\mu} = h^2 d\mu$, we get that the weighted model $(M, \widetilde{\mu})$, has the area function

$$\widetilde{S}(r) = h^2(r)S(r).$$

The Laplace-Beltrami operator Δ_μ on M can be represented in the polar coordinates (r, θ) as follows:

$$\Delta_\mu = \frac{\partial^2}{\partial r^2} + \frac{S'(r)}{S(r)}\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_\theta, \tag{4.8}$$

where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. If we assume that u is a radial function, that is, u depends only on the polar radius r, we obtain from Eq. 4.8, that u is harmonic in M if and only if

$$u(r) = c_1 + c_2 \int_{r_1}^{r} \frac{dt}{S(t)}, \tag{4.9}$$

where $r_1 \in [-\infty, +\infty]$ so that the integral converges and c_1, c_2 are arbitrary reals.

Theorem 4.3 *Let $(M, \mu) = (M_0 \sqcup M_1, \mu)$ be a Riemannian model with two ends, where $M_0 = \{(r, \theta) \in M : r \geq 0\}$ is a model manifold with boundary such that for all $r \geq 0$,*

$$\psi_0(r) = e^{-\frac{1}{n-1}r^\alpha}.$$

Also assume that (M_1, μ_1) is a Riemannian model with

$$\int_1^\infty \frac{dt}{S_1(t)} < \infty, \tag{4.10}$$

and Faber-Krahn function Λ_1*, so that*

$$\Lambda_1(v) \geq c_1 \begin{cases} \frac{1}{(\log v)^{\frac{2-2\alpha}{\alpha}}}, & v \geq 2, \\ v^{-\frac{2}{n}}, & v < 2, \end{cases} \tag{4.11}$$

for some constant $c_1 > 0$*. Then there exist positive constants* $C_x = C_x(x, \alpha, n)$ *and* $C_1 = C_1(\alpha, n)$ *such that the heat kernel of* (M, μ) *satisfies, for all* $x \in M$*, the inequality*

$$p_t(x,x) \leq C_x \begin{cases} \exp\left(-C_1 t^{\frac{\alpha}{2-\alpha}}\right), & t \geq 1, \\ t^{-\frac{n}{2}}, & 0 < t < 1. \end{cases} \tag{4.12}$$

Proof Observe that the assumption (4.10) yields that we can choose positive constants κ_1 and κ_2 so that the smooth function h on M defined by

$$h(r) = \kappa_1 + \kappa_2 \int_1^r \frac{dt}{S(t)},$$

is positive in M and satisfies $h \simeq 1$ in $\{r \leq 0\}$. Consider the weighted model with two ends $(M, \widetilde{\mu})$, where $\widetilde{\mu}$ is defined by $d\widetilde{\mu} = h^2 d\mu$. It follows from Eq. 4.11 that the weighted model $(M_1, \widetilde{\mu_1})$ has the Faber-Krahn function $\widetilde{\Lambda}_1$ satisfying

$$\widetilde{\Lambda}_1(v) \geq \widetilde{c}_1 \begin{cases} \frac{1}{(\log v)^{\frac{2-2\alpha}{\alpha}}}, & v \geq 2, \\ v^{-\frac{2}{n}}, & v < 2, \end{cases}$$

for some constant $\widetilde{c}_1 > 0$. Further, note that

$$h|_{M_0}(r) \simeq \begin{cases} r^{1-\alpha} e^{r^\alpha}, & r \geq 1, \\ 1, & 0 \leq r < 1, \end{cases}$$

whence the area function $\widetilde{S}_0$ of the weighted model with boundary $(M_0, \widetilde{\mu_0})$ admits the estimate

$$\widetilde{S}_0(r) \simeq \begin{cases} r^{2-2\alpha} e^{r^\alpha}, & r \geq 1, \\ 1, & 0 \leq r < 1. \end{cases}$$

Since also $\psi_0 \leq 1$, we can apply Theorem 4.2 and obtain that there exist constants $C > 0$ and $C_1 > 0$ depending on α and n so that the heat kernel $\widetilde{p}_t$ of $(M, \widetilde{\mu})$ satisfies

$$\sup_{x \in M} \widetilde{p}_t(x,x) \leq C \begin{cases} \exp\left(-C_1 t^{\frac{\alpha}{2-\alpha}}\right), & t \geq 1, \\ t^{-\frac{n}{2}}, & 0 < t < 1. \end{cases} \tag{4.13}$$

Using that h is harmonic in M, we have by Eq. 2.8, for all $t > 0$ and $x \in M$, the identity

$$\widetilde{p}_t(x,x) = \frac{p_t(x,x)}{h^2(x)},$$

which together with Eq. 4.13 implies the upper bound (4.12) and thus, finishes the proof. □

Remark Consider the end $\Omega := \{r > 0\}$ of the Riemannian model (M, μ) from Theorem 4.3 and note that $\left(\overline{\Omega} = \{r \geq 0\}, \mu|_{\{r \geq 0\}}\right)$ is parabolic by [12, Proposition 3.1], whence the estimate (4.12) implies that we cannot get a polynomial decay of the heat kernel in M as it follows from Eq. 2.4 in Theorem 2.1, just by assuming the polynmial volume growth condition (2.2).

Remark Consider again the end $\Omega := \{r > 0\}$ of the Riemannian model (M, μ) from Theorem 4.3 and assume for simplicity that $n = 2$. Let M_0 be defined as in Theorem 2.6, that is, there exists a compact set $K_0 \subset M_0$ that is the closure of a non-empty open set, such that Ω is isometric to $M_0 \setminus K_0$. Let us check which conditions from Theorem 2.6 are not satisfied in M_0. A simple computation shows that the area function S_0 of the manifold M_0 satisfies $S_0''(r) \sim \alpha^2 e^{-r^\alpha} r^{2\alpha-2}$ as $r \to +\infty$, so that $-\frac{S_0''(r)}{S_0(r)} \to 0$ as $r \to +\infty$. Together with the fact that on a compact set, the Gaussian curvature is non-negative, it then follows from Eq. 2.42 that the curvature on M_0 is bounded below, which implies that M_0 is a locally Harnack manifold. Obviously, S_0 also satisfies the conditions (2.39) and (2.40) from Proposition 2.9, whence we obtain that on M_0 the spherical Harnack inequality (2.22) holds. On the other hand, condition (2.23) in M_0 fails, since for fixed $\rho > 0$, the volume $V(x, \rho)$ decreaeses exponentially when $r \to +\infty$ where $x = (r, \theta) \in \Omega$. Hence, we have that in general, we can not drop the condition (2.23) in Theorem 2.6 to get the polynomial decay (2.24) of the heat kernel in M.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability This article has no associated data.

Declarations

Conflict of Interests The authors confirm that they do not have actual or potential conflict of interest in relation to this publication.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Boulanger, A.: Counting problems of geometrically infinite Kleinian groups. arXiv:1902.06580v3 (2020)
2. Brock, F., Chiacchio, F., Mercaldo, A.: Weighted isoperimetric inequalities in cones and applications. Nonlinear Analysis: Theory Methods & Applications **75**(15), 5737–5755 (2012)
3. Buser, P.: A note on the isoperimetric constant. Annales scientifiques de l'Ecole normale superieure **15**(2), 213–230 (1982)
4. Carron, G.: Inégalités isopérimétriques de Faber-Krahn et conséquences. In: Actes de la table ronde de géométrie différentielle (Luminy, 1992), Collection SMF Séminaires et Congres, vol. 1, pp. 205–232 (1996)
5. Cheng, S.Y., Yau, S.T.: Differential equations on riemannian manifolds and their geometric applications. Commun. Pure Appl. Math. **28**(3), 333–354 (1975)
6. Cheng, S.Y., Yau, S.-T.: Differential equations on Riemannian manifolds and their geometric applications. Commun. Pure Appl. Math. **28**(3), 333–354 (1975)
7. Coulhon, T., Grigor'yan, A.: On-diagonal lower bounds for heat kernels and Markov chains. Duke Math. J. **89**, 133–199 (1997)
8. De Haan, L., Ferreira, A., Ferreira, A.: Extreme Value Theory: An Introduction, vol. 21. Springer (2006)
9. Greene, R.E., Wu, H.-H.: Function Theory on Manifolds Which Possess a Pole, vol. 699. Springer (2006)

10. Grigor'yan, A.: Analysis on manifolds and volume growth. Adv. Anal. Geom. **3**, 299–324 (2021)
11. Grigor'yan, A.: Heat kernel on a manifold with a local Harnack inequality. Commun. Anal. Geom. **2**(1), 111–138 (1994)
12. Grigor'yan, A.: Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull. Amer. Math. Soc. **36**(02), 135–250 (1999)
13. Grigor'yan, A.: Estimates of heat kernels on Riemannian manifolds. London Math. Soc. Lecture Note Ser **273**, 140–225 (1999)
14. Grigor'yan, A.: Heat kernels on weighted manifolds and applications. Cont. Math **398**(2006), 93–191 (2006)
15. Grigor'yan, A.: Heat Kernel and Analysis on Manifolds. American Mathematical Society (2012)
16. Grigor'yan, A., Saloff-Coste, L.: Dirichlet heat kernel in the exterior of a compact set. Commun. Pure Appl. Math. **55**(1), 93–133 (2001)
17. Grigor'yan, A., Saloff-Coste, L.: Heat kernel on manifolds with ends. In: Annales de l'institut Fourier, vol. 59, pp. 1917–1997 (2009)
18. Grigor'yan, A., Saloff-Coste, L.: Surgery of the Faber–Krahn inequality and applications to heat kernel bounds. Nonlinear Anal. **131**, 243–272 (2016)
19. Grigor'yan, A.A.: Isoperimetric inequalities for Riemannian products. Math. Notes Acad. Sci. USSR **38**(4), 849–854 (1985)
20. Heinonen, J., Kipelainen, T., Martio, O.: Nonlinear Potential Theory of Degenerate Elliptic Equations. Courier Dover Publications (2018)
21. Saloff-Coste, L.: A note on Poincaré, Sobolev, and Harnack inequalities. Int. Math. Res. Not. **1992**, 27–38 (1992)
22. Saloff-Coste, L., Gyrya, P.: Neumann and Dirichlet Heat Kernels in Inner Uniform Domains (2011)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.