Potential Analysis (2023) 59:1425–1448
https://doi.org/10.1007/s11118-022-10007-y

Off-Diagonal Heat Kernel Estimates for Symmetric Diffusions in a Degenerate Ergodic Environment

Peter A. Taylor[1]

Received: 14 May 2021 / Accepted: 24 April 2022 / Published online: 18 June 2022

Abstract
We study a symmetric diffusion process on $\mathbb{R}^d$, $d \geq 2$, in divergence form in a stationary and ergodic random environment. The coefficients are assumed to be degenerate and unbounded but satisfy a moment condition. We derive upper off-diagonal estimates on the heat kernel of this process for general speed measure. Lower off-diagonal estimates are also shown for a natural choice of speed measure, under an additional mixing assumption on the environment. Using these estimates, a scaling limit for the Green function is proven.

Keywords Heat kernel estimates · Diffusions in random environment · Moser iteration · Intrinsic metric

Mathematics Subject Classification (2010) 60J60 · 60K37 · 60J35 · 31B05

1 Introduction

We study a diffusion process on $\mathbb{R}^d$, formally associated with the following generator

$$\mathcal{L}^\omega u(x) = \frac{1}{\theta^\omega(x)} \nabla \cdot (a^\omega(x) \nabla u(x)), \quad x \in \mathbb{R}^d, \tag{1.1}$$

where the random field $\{a^\omega(x)\}_{x\in\mathbb{R}^d}$ is a symmetric d-dimensional matrix for each $x \in \mathbb{R}^d$, and θ^ω is a positive speed measure which may also depend on the random environment ω. Firstly, we set out the precise assumptions on the random environment. Let $\left(\Omega, \mathcal{G}, \mathbb{P}, \{\tau_x\}_{x\in\mathbb{R}^d}\right)$ be a probability space together with a measurable group of translations. $\mathbb{E}$ will denote the expectation under this probability measure. To construct the random field let $a : \Omega \to \mathbb{R}^{d\times d}$ be a $\mathcal{G}$-measurable random variable and define $a^\omega(x) := a(\tau_x\omega)$. The speed measure is defined similarly, take a $\mathcal{G}$-measurable random variable $\theta : \Omega \to (0, \infty)$ and let $\theta^\omega(x) := \theta(\tau_x\omega)$. We refer to this function as the speed measure because the process with general θ^ω can be obtained from the process with $\theta^\omega \equiv 1$ via a time-change. As made

✉ Peter A. Taylor
peter.taylor876@gmail.com; pat47@cam.ac.uk

[1] Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB, UK

precise in the following, we assume throughout that the random environment is stationary, ergodic and satisfies a non-uniform ellipticity condition.

Assumption 1.1 The probability space satisfies:

(i) $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$ for all $A \in \mathcal{G}$ and any $x \in \mathbb{R}^d$.
(ii) If $\tau_x A = A$ for all $x \in \mathbb{R}^d$ then $\mathbb{P}(A) \in \{0, 1\}$.
(iii) The mapping $(x, \omega) \mapsto \tau_x\, \omega$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$ measurable.

Furthermore for each $x \in \mathbb{R}^d$, $a^\omega(x)$ is symmetric and there exist positive, $\mathcal{G}$-measurable $\lambda,\ \Lambda : \Omega \to (0, \infty)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $\xi \in \mathbb{R}^d$, $x \in \mathbb{R}^d$,

$$\lambda(\tau_x\omega)|\xi|^2 \leq \xi \cdot (a^\omega(x)\xi) \leq \Lambda(\tau_x\omega)|\xi|^2. \tag{1.2}$$

Also, defining $\Lambda^\omega(x) := \Lambda(\tau_x\omega)$ and $\lambda^\omega(x) := \lambda(\tau_x\omega)$ for $x \in \mathbb{R}^d$, assume that $\mathbb{P}$-a.s.

$$\Lambda^\omega,\ (\lambda^\omega)^{-1},\ \theta^\omega,\ (\theta^\omega)^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d). \tag{1.3}$$

The final assumption of local boundedness will allow us to pass from estimates on the semigroup of the diffusion process to pointwise bounds on the heat kernel. Rather than assuming these functions are uniformly bounded, we work with moment conditions given in terms of the following, for $p, q, r \in (0, \infty]$ define

$$\begin{aligned} M_1(p, q, r) &:= \mathbb{E}[\theta^\omega(0)^r] + \mathbb{E}[\lambda^\omega(0)^{-q}] + \mathbb{E}[\Lambda^\omega(0)^p\theta^\omega(0)^{1-p}], \\ M_2(p, q) &:= \mathbb{E}[\lambda^\omega(0)^{-q}] + \mathbb{E}[\Lambda^\omega(0)^p]. \end{aligned} \tag{1.4}$$

By the ergodic theorem, these conditions together with Assumption 1.1 allow us to control average values of the functions on large balls. For instance, denoting $B(x, r)$ the closed Euclidean ball of radius r centred at x, $\bar{\Lambda}_p := \mathbb{E}[\Lambda^\omega(0)^p]$ and $\bar{\lambda}_q := \mathbb{E}[\lambda^\omega(0)^q]$, then $M_2(p, q) < \infty$ implies that for $\mathbb{P}$-a.e. ω, there exists $N_1^\omega(x) > 0$ such that for all $r \geq N_1^\omega(x)$,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \Lambda^\omega(u)^p\, du < 2\,\bar{\Lambda}_p, \qquad \frac{1}{|B(x,r)|}\int_{B(x,r)} \lambda^\omega(u)^q\, du < 2\,\bar{\lambda}_q. \tag{1.5}$$

In the uniformly elliptic case, where $\Lambda^\omega(x)$ and $\lambda^\omega(x)$ are bounded above and below respectively, uniformly in ω, the model we are considering has been extensively studied. A quenched invariance principle is established in [31] for differentiable, periodic coefficients. Further results for smooth, periodic, uniformly elliptic coefficients are given in [29]. The quenched invariance principle was extended to a random environment with a uniformly elliptic symmetric part and differentiable skew-symmetric part satisfying a growth condition in [26]. Outside the uniformly elliptic regime and more recently, [9] proved this homogenization result for operators taking a specific, periodic form, with measurable and locally integrable coefficients. Without assuming differentiability of the random field, some work is required to construct the process associated with (1.1) in a general ergodic environment. The diffusion is constructed using the theory of Dirichlet forms, with the corresponding form being

$$\mathcal{E}^\omega(u, v) := \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^\omega_{ij}(x)\partial_i u(x)\partial_j v(x)\, dx, \tag{1.6}$$

for $u,\ v$ in a proper class of functions $\mathcal{F}^\theta \subset L^2(\mathbb{R}^d, \theta^\omega\, dx)$, defined precisely in Section 2. The construction of a diffusion process $(X_t^\theta)_{t\geq 0}$ associated to (1.1) is a recent result of [19]. This is done under Assumption 1.1 together with the moment condition $M_2(p, q) < \infty$

for some $p,\ q \in (1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$. The main result in [19, Theorem 1.1] is the quenched invariance principle, that is for $\mathbb{P}$-a.e. ω the law of the process $\left(\frac{1}{n} X_{n^2 t}\right)_{t \geq 0}$ on $C([0, \infty), \mathbb{R}^d)$ converges weakly as $n \to \infty$ to that of a Brownian motion. This is first proven for $\theta^\omega \equiv 1$ and then for general speed measure satisfying $\mathbb{E}[\theta^\omega(0)] < \infty$ and $\mathbb{E}[\theta^\omega(0)^{-1}] < \infty$, after showing that the general speed process can be obtained via a time change.

Regarding the heat kernel of the operator $\mathcal{L}^\omega$, it is also shown therein that the semigroup P_t of the above diffusion process has a transition kernel $p_\theta^\omega(t, x, y)$ with respect to $\theta^\omega(x)\, dx$, furthermore this is jointly continuous in x and y. Explicitly, for continuous, bounded $f : \mathbb{R}^d \to \mathbb{R}$,

$$P_t f(x) = \int_{y \in \mathbb{R}^d} f(y) p_\theta^\omega(t, x, y) \theta^\omega(y)\, dy, \qquad \forall x \in \mathbb{R}^d,\ t > 0. \tag{1.7}$$

A second, stronger result that has recently been established under Assumption 1.1 and moment condition $M_1(p, q, r) < \infty$ for some $p, q, r \in (1, \infty]$ satisfying $\frac{1}{r} + \frac{1}{q} + \frac{1}{p-1}\frac{r-1}{r} < \frac{2}{d}$ is the quenched local central limit theorem [18, Theorem 1.1]. This states that the rescaled transition kernel $p_\theta^\omega(n^2 t, 0, nx)$ converges as $n \to \infty$ to the heat kernel of a Brownian motion $k_t^\Sigma(0, x)$ with some deterministic, positive definite covariance matrix Σ implicitly depending on the law $\mathbb{P}$. Namely, for $t > 0$ and $x,\ y \in \mathbb{R}^d$,

$$k_t^\Sigma(x, y) := \frac{1}{\sqrt{(2\pi t)^d \det \Sigma}} \exp\Big(- \frac{(y - x) \cdot \Sigma^{-1}(y - x)}{2t}\Big). \tag{1.8}$$

The convergence is uniform on compact sets in t and x and the key step is to apply a parabolic Harnack inequality to obtain Hölder regularity of the heat kernel; this is achieved via Moser iteration which will also play an important role in our analysis. Many of the techniques take inspiration from the random conductance model (RCM) setting, cf. [1, 3, 8, 14] for recent RCM local limit theorems in a degenerate, ergodic setting. The diffusion studied in this paper is a continuum analogue of that model, where a random walk moves on a lattice, usually $\mathbb{Z}^d$ equipped with nearest-neighbour edges. Importantly, the RCM literature indicates that moment conditions are indeed necessary for a general ergodic environment, for instance [3] proves a local limit theorem under a moment assumption equivalent to $M_2(p, q) < \infty$ for $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ and shows that this condition is optimal for the canonical choice of speed measure (known as the constant speed random walk). Another recent result under moment conditions is a Liouville theorem for the elliptic equation associated to (1.1) in [12], cf. also [11] for a related result on the parabolic equation associated to a time-dynamic, uniformly elliptic version of (1.1). Local boundedness and a Harnack inequality for solutions to the elliptic equation were recently proven in [13] under moment conditions.

A local limit theorem quantifies the limiting behaviour of the heat kernel and is known to provide near-diagonal estimates on the kernel prior to rescaling – see Proposition 3.1. In this paper our aim is to derive full Gaussian estimates on the heat kernel $p_\theta^\omega(t, x, y)$ for all x and y, also known as off-diagonal estimates. For general speed measure, it is known that these bounds should be governed by the intrinsic metric, cf. [17, 20, 34]. In the random environment setting, this is a metric on $\mathbb{R}^d$ dependent on a^ω and θ^ω, defined as

$$d_\theta^\omega(x, y) := \sup\Big\{\phi(y) - \phi(x) \,:\, \phi \in C(\mathbb{R}^d) \cap \mathcal{F}_{\text{loc}}^\theta,\ \operatorname{ess\,sup}_{z \in \mathbb{R}^d} \frac{(\nabla\phi \cdot a^\omega \nabla\phi)(z)}{\theta^\omega(z)} \leq 1\Big\}.$$

In the above, $\mathcal{F}_{\text{loc}}^\theta$ is the local domain of the Dirichlet form $\mathcal{E}^\omega$, defined precisely in Section 2. Outside of the uniformly elliptic case it is clear that the above is not in general comparable to the Euclidean metric, which we denote $d(\cdot\,,\cdot)$. A natural follow-up question to this work would be to find the minimal conditions on an ergodic environment for which these two metrics are comparable. However here we require some regularity of the intrinsic metric in order to derive off-diagonal heat kernel estimates in terms of it. Specifically we must show it is strictly local, meaning it generates the Euclidean topology on $\mathbb{R}^d$. We therefore make the following additional assumption.

Assumption 1.2 (Continuity of the Environment) For $\mathbb{P}$-a.e. ω, the functions $a^\omega : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $\theta^\omega : \mathbb{R}^d \to (0,\infty)$ are continuous.

Our first main result is an upper off-diagonal heat kernel estimate for the symmetric diffusion process with general speed measure in an ergodic, degenerate environment, and is proven in Section 2.

Theorem 1.3 *Suppose Assumption 1.1 and Assumption 1.2 hold. Let $d \geq 2$ and assume $M_1(p,q,r) < \infty$ for some $p,q,r \in (1,\infty]$ satisfying $\frac{1}{r} + \frac{1}{q} + \frac{1}{p-1}\frac{r-1}{r} < \frac{2}{d}$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $x \in \mathbb{R}^d$, there exist $N_2^\omega(x) > 0$, $c_1(d,p,q,r) > 0$ and $\gamma(d,p,q,r) > 0$ such that the following holds for all $y \in \mathbb{R}^d$ and $\sqrt{t} > N_2^\omega(x)$,*

$$p_\theta^\omega(t,x,y) \leq c_1\, t^{-\frac{d}{2}}\Big(1 + \frac{d(x,y)}{\sqrt{t}}\Big)^\gamma \exp\Big(-\frac{d_\theta^\omega(x,y)^2}{8t}\Big). \tag{1.9}$$

Remark 1.4 (i) In the 'constant speed' setting of $\theta^\omega \equiv \Lambda^\omega$ (and more generally whenever $\theta^\omega \geq c\,\Lambda^\omega$), we obtain off-diagonal heat kernel estimates in terms of the Euclidean metric d, without the need for Assumption 1.2. We get a full Gaussian upper estimate here because the polynomial prefactor in (1.9) can be absorbed into the exponential when the two metrics are comparable.

(ii) We restrict to $d \geq 2$ because the technique for deriving the maximal inequality in Section 2.1 breaks down in $d = 1$ due to issues with the Sobolev inequality (Lemma 2.4). Similarly, the lower estimate (Theorem 1.6) relies on the parabolic Harnack inequality in [18] which is also derived using Moser iteration for $d \geq 2$ only.

To prove the above estimate we use Davies' perturbation method, a technique for deriving upper off-diagonal estimates, well-established in the elliptic and parabolic equations literature for uniformly elliptic operators cf. [17, 20, 21, 28, 34]. The idea also translates to heat kernels on graphs [22, 23] and recently the RCM in a degenerate, ergodic environment [4, 5]. The first step of Davies' method is to consider the Cauchy problem associated to the perturbed operator $\mathcal{L}_\psi^\omega := \mathrm{e}^\psi \mathcal{L}^\omega \mathrm{e}^{-\psi}$ where ψ is an arbitrary test function, and use a maximal inequality to bound the fundamental solution. In [36], off-diagonal estimates are derived for solutions of a parabolic equation in a uniformly elliptic setting with degenerate, locally integrable weight; this was useful inspiration for the Cauchy problem we consider in Section 2.1. To derive the maximal inequality we use a Moser iteration scheme adapted to the perturbed operator, similar to the method used to derive the parabolic Harnack inequality for the original operator $\mathcal{L}^\omega$ in [18]. Ergodic theory plays a key role here in controlling constants which depend on the random environment. Moser iteration has previously been applied to prove the corresponding RCM results – the quenched invariance principle in [2], the Harnack inequality in [3] and off-diagonal estimates in [4, 5].

The second part of the argument is to optimise over the test function ψ. In the uniformly elliptic case this is straightforward as one can work with the Euclidean metric, however in our general setting of degenerate coefficient matrix and speed measure the off-diagonal estimate is governed by the intrinsic metric defined above. Utilising a test function related to this metric requires certain regularity properties, for instance that it generates the Euclidean topology on $\mathbb{R}^d$. In Section 2.3 we first relate the intrinsic metric to a Riemannian metric, then apply a recent result from geometric analysis [16] to prove the necessary regularity properties under Assumption 1.2.

As a counterpart to the preceding upper estimate, we also present a lower off-diagonal estimate for the heat kernel, in the 'constant speed' case of $\theta^\omega \equiv \Lambda^\omega$. Whilst Assumption 1.2 for regularity of the intrinsic metric is no longer required, we need stronger control on the environment than given in Assumption 1.1. In particular, a decorrelation assumption is necessary for the proof and we assume finite-range dependence of the environment.

Assumption 1.5 Suppose there exists a positive constant $\mathcal{R} > 0$ such that for all $x \in \mathbb{R}^d$ and $\mathbb{P}$-a.e. ω, $\tau_x\omega$ is independent of $\{\tau_y\omega : y \in B(x, \mathcal{R})^c\}$.

To prove the lower estimate we adapt the established chaining argument to the diffusion in a degenerate random environment, the method originated in [25] using the ideas of Nash. It was adapted to the weighted graph setting in [23], to random walks on percolation clusters in [10], and it was recently applied to the RCM [6]. The strategy is to repeatedly apply lower near-diagonal estimates, derived from the parabolic Harnack inequality established in [19], along a sequence of balls. The form of the constant in the Harnack inequality means that averages of the functions $\lambda^\omega(\cdot)$, $\Lambda^\omega(\cdot)$ on balls with varying centre-points must be controlled simultaneously to derive the lower off-diagonal estimate. Something stronger than the classical ergodic theorem is required to do this, so given Assumption 1.5 we establish a specific form of concentration inequality (Proposition 3.3) for this purpose. By an argument similar to [6] this inequality is then used to control the environment-dependent terms arising from the Harnack inequality, see Proposition 3.4. The statement is given below and proven in Section 3.

Theorem 1.6 *Suppose $d \geq 2$ and Assumptions 1.1 and 1.5 hold. There exist $p_0, q_0 \in (1, \infty)$ such that if $M_2(p_0, q_0) < \infty$ then for $\mathbb{P}$-a.e. ω and every $x \in \mathbb{R}^d$, there exist $c_i(d) > 0$ and a random constant $N_3^\omega(x) > 0$ satisfying*

$$\mathbb{P}(N_3^\omega(x) > n) \leq c_2\, n^{-\alpha} \qquad \forall n > 0, \tag{1.10}$$

for some $\alpha > d(d-1) - 1$, such that the following holds. For all $y \in \mathbb{R}^d$ and $t \geq N_3^\omega(x)\big(1 \vee d(x, y)\big)$,

$$p_\Lambda^\omega(t, x, y) \geq c_3\, t^{-d/2} \exp\Big(-c_4 \frac{d(x, y)^2}{t}\Big). \tag{1.11}$$

For the moment condition, it suffices to take $p_0 > 2d\kappa p$ and $q_0 > 2d\kappa q$, where

$$\kappa(d, p, q) := \frac{(2 + d)pq - (p + 2q)d}{2pq - (p + q)d},$$

is the constant in Proposition 3.1.

Remark 1.7 (i) In [6], three other assumptions such as an FKG inequality or a spectral gap inequality are offered as alternatives to finite-range dependence. Some of these are specific to the discrete setting and Assumption 1.5 is the most natural for our context, but it may be possible to replace it with other similar conditions.

(ii) We state the above only for $\theta^\omega \equiv \Lambda^\omega$ because for general speed measure the intrinsic metric is not necessarily comparable to the Euclidean metric which is used for the chaining argument. It may be possible to adapt the argument to general speed however and it is unclear whether this would require further assumptions in order to compare the two metrics.

Our final result is a scaling limit for the Green function of the diffusion process, defined as

$$g^\omega(x, y) := \int_0^\infty p_\theta^\omega(t, x, y)\, dt.$$

As already noted, the diffusion with general speed measure may be obtained from the process with speed measure $\theta^\omega \equiv 1$ via a time change [18, Theorem 2.4]. Therefore the Green function, which exists in dimension $d \geq 3$ due to the upper off-diagonal heat kernel estimate above, is independent of the speed measure θ^ω. Applying Theorem 1.3 together with a long-range bound obtained in Section 4, we obtain sufficient bounds to apply the local limit theorem [18, Theorem 1.1] and show that an appropriately rescaled version of the Green function converges to that of a Brownian motion,

$$g_{\mathrm{BM}}(x, y) := \int_0^\infty k_t^\Sigma(x, y)\, dt. \tag{1.12}$$

For the purposes of the Green function scaling limit result, we make the additional assumption that the intrinsic metric, which we denote $d^\omega(\cdot\,,\cdot)$ when $\theta^\omega \equiv 1$, is bounded below by the Euclidean metric. This allows us to get an upper off-diagonal estimate in terms of the Euclidean metric. This choice of speed measure, $\theta^\omega \equiv 1$, is sufficient for proving the long-range bound, Proposition 4.2. Other choices may require uniform boundedness of the speed measure for the proof of that proposition, which would be quite a restrictive assumption.

Assumption 1.8 There exists $c_5 > 0$ such that, for $\mathbb{P}$-a.e. ω,

$$d^\omega(x, y) \geq c_5\, d(x, y), \qquad \forall\, x, y \in \mathbb{R}^d. \tag{1.13}$$

Theorem 1.9 *Let $d \geq 3$, suppose Assumptions 1.1, 1.2 and 1.8 hold. Also, assume there exist $p, q \in (1, \infty]$ satisfying $\frac{1}{p-1} + \frac{1}{q} < \frac{2}{d}$ such that $M_2(p, q) < \infty$. Then for $x_0 \in \mathbb{R}^d$, $0 < r_1 < r_2$ and the annulus $A := \{x \in \mathbb{R}^d \,:\, 0 < r_1 \leq d(x_0, x) \leq r_2\}$,*

$$\lim_{n\to\infty} \sup_{x\in A} |n^{d-2} g^\omega(x_0, nx) - g_{\mathrm{BM}}(x_0, x)| = 0 \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega. \tag{1.14}$$

Remark 1.10 (i) Analogous results have been proven for the RCM in [3, Theorem 1.14] and [27, Theorem 5.3]. See also [6, Theorem 1.6] for further estimates on the Green function which can be derived from off-diagonal heat kernel estimates.

(ii) Assumption 1.8 holds, for instance, in the setting of lower uniform ellipticity, $\lambda^\omega(x) \geq c_- > 0$ for all $x \in \mathbb{R}^d$ and $\mathbb{P}$-a.e. ω. It is possible that the intrinsic metric can be bounded by the Euclidean metric outside of the uniformly elliptic regime; for recent results in this direction in the discrete setting see [7].

A key ingredient to prove the above is a local limit theorem. For the RCM, this was recently proven under a less restrictive inequality on p and q in [13], for the variable speed random walk, which is analogous to $\theta^\omega \equiv 1$ in our setting. As such, it is possible that by leveraging on the improved Moser iteration in [13], the moment condition for the local limit theorem in [18, Theorem 1.1] could be relaxed. As a consequence, the inequality $\frac{1}{p-1}+\frac{1}{q}<\frac{2}{d}$ in Theorem 1.9 could then be improved.

Notation and Structure of the Paper. For $x \in \mathbb{R}^d$, $|x|$ denotes the standard Euclidean norm. For vectors $u,\ v \in \mathbb{R}^d$, the canonical scalar product is given by $u \cdot v$ and gradient ∇u. We write c to denote a positive, finite constant which may change on each appearance. Constants denoted by c_i will remain the same. For $\alpha,\ \beta \in \mathbb{R}$, we write $\alpha \simeq \beta$ to mean there exist constants $c,\ \tilde{c} > 0$ such that $c\,\alpha \le \beta \le \tilde{c}\,\alpha$. For a countable set A, its cardinality is denoted $|A|$. Otherwise if $A \subset \mathbb{R}^d$, $|A|$ is the Lebesgue measure. For any $p \in (1,\infty)$, the Hölder conjugate is written $p_* := \frac{p}{p-1}$. We will work with inner products as follows, for functions $f,\ g : \mathbb{R}^d \to \mathbb{R}$ and positive weight $\nu : \mathbb{R}^d \to \mathbb{R}$,

$$(f,g) := \int_{\mathbb{R}^d} f(x)g(x)\,dx, \qquad (f,g)_\nu := \int_{\mathbb{R}^d} f(x)g(x)\nu(x)\,dx.$$

Furthermore, for $p \in (0,\infty)$ and bounded $B \subset \mathbb{R}^d$, define norms

$$\|f\|_p := \Big(\int_{\mathbb{R}^d} |f(x)|^p\,dx\Big)^{1/p}, \qquad \|f\|_{p,\nu} := \Big(\int_{\mathbb{R}^d} |f(x)|^p \nu(x)\,dx\Big)^{1/p},$$
$$\|f\|_{p,B} := \Big(\frac{1}{|B|}\int_B |f(x)|^p\,dx\Big)^{1/p}, \qquad \|f\|_{p,B,\nu} := \Big(\frac{1}{|B|}\int_B |f(x)|^p \nu(x)\,dx\Big)^{1/p}.$$

For $q \in (0,\infty)$, $I \subset \mathbb{R}$, $B \subset \mathbb{R}^d$, $Q = I \times B$ and $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, let

$$\|u\|_{p,q,Q} := \Big(\frac{1}{|I|}\int_I \|u_t\|_{p,B}^q\,dt\Big)^{1/q}, \qquad \|u\|_{p,q,Q,\nu} := \Big(\frac{1}{|I|}\int_I \|u_t\|_{p,B,\nu}^q\,dt\Big)^{1/q},$$
$$\|u\|_{p,\infty,Q} := \operatorname{ess\,sup}_{t\in I} \|u_t\|_{p,B}, \qquad \|u\|_{p,\infty,Q,\nu} := \operatorname{ess\,sup}_{t\in I} \|u_t\|_{p,B,\nu},$$
$$\|u\|_{\infty,\infty,Q} := \operatorname{ess\,sup}_{(t,x)\in Q} u(t,x).$$

Finally for $f : \mathbb{R}^d \to \mathbb{R}^m$, let

$$\|f\|_\infty := \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|.$$

All of the results herein will be quenched, in that they hold for $\mathbb{P}$-a.e. instance of the environment ω unless stated otherwise. Regarding the structure of the paper, Section 2 is devoted to the proof of the upper off-diagonal heat kernel estimate Theorem 1.3. The lower estimate, Theorem 1.6, is then proven in Section 3. Finally, Section 4 concerns the proof of the Green function scaling limit.

2 Davies' Method

Throughout this section assume $d \ge 2$, Assumption 1.1 holds and let $p,\ q,\ r \in (1,\infty]$ satisfy $\frac{1}{r}+\frac{1}{q}+\frac{1}{p-1}\frac{r-1}{r} < \frac{2}{d}$. One important space we will work with is $\mathcal{F}_G^\theta$ which, for open $G \subseteq \mathbb{R}^d$, is the closure of $C_0^\infty(G)$ in $L^2(G,\theta^\omega dx)$ with respect to $\mathcal{E}^\omega + (\cdot\,,\cdot)_\theta$. We write $\mathcal{F}^\theta$ in the case $G = \mathbb{R}^d$ and if $\theta^\omega \equiv 1$ also we simply write $\mathcal{F}$. Define $\mathcal{F}_{\mathrm{loc}}^\theta$ by $u \in \mathcal{F}_{\mathrm{loc}}^\theta$ if for all balls $B \subset \mathbb{R}^d$ there exists $u_B \in \mathcal{F}_B^\theta$ such that $u = u_B$ $\mathbb{P}$-a.s. In the case $\theta^\omega \equiv 1$ this space is denoted $\mathcal{F}_{\mathrm{loc}}$. We write $W^{1,\infty}(\mathbb{R}^d)$ for the Sobolev space of $L^\infty(\mathbb{R}^d)$ functions with a weak derivative in $L^\infty(\mathbb{R}^d)$, also we denote $W_{\mathrm{loc}}^{1,\infty}(\mathbb{R}^d)$ the corresponding local Sobolev space.

Following [18], we define the weak parabolic equation satisfied by the heat kernel.

Definition 2.1 (Caloric function) Let $I \subseteq \mathbb{R}$ and $G \subseteq \mathbb{R}^d$ be open sets. A function $u : I \to \mathcal{F}_G^\theta$ is caloric if the map $t \mapsto (u(t,\cdot), \phi)_\theta$ is differentiable for any $\phi \in L^2(G, \theta^\omega dx)$ and

$$\frac{d}{dt}(u_t, \phi)_\theta + \mathcal{E}^\omega(u_t, \phi) = 0, \tag{2.1}$$

for all $\phi \in \mathcal{F}_G^\theta$ and for all $t \in I$.

2.1 Maximal Inequality for the Perturbed Cauchy Equation

The first step in applying Davies' method is to establish a bound on solutions to the following Cauchy problem.

Lemma 2.2 (Cauchy Problem) *Let u be caloric on $\mathbb{R} \times \mathbb{R}^d$ and $u(0,\cdot) = f(\cdot)$ for some $f \in L^2(\mathbb{R}^d, \theta^\omega dx)$. Let $\psi \in W^{1,\infty}_{loc}(\mathbb{R}^d)$ satisfy $\|\psi\|_\infty < \infty$ and*

$$h^\omega(\psi)^2 := \operatorname{ess\,sup}_{x\in\mathbb{R}^d} \frac{(\nabla\psi \cdot a^\omega \nabla\psi)(x)}{\theta^\omega(x)} < \infty.$$

Then writing $v(t,x) := \mathrm{e}^{\psi(x)}u(t,x)$, we have for all $t > 0$,

$$\|v_t\|_{2,\theta}^2 \leq \mathrm{e}^{2h^\omega(\psi)^2 t}\|\mathrm{e}^\psi f\|_{2,\theta}^2.$$

Proof Formally, $v_t = v(t,\cdot)$ solves the caloric equation

$$\frac{d}{dt}(v_t, \phi)_\theta + J^\omega(v_t, \phi) = 0, \tag{2.2}$$

where we have defined an operator

$$J^\omega(v,\phi) := \int_{\mathbb{R}^d} (a^\omega\nabla v)\cdot\nabla\phi + \phi(a^\omega\nabla v)\cdot\nabla\psi - v(a^\omega\nabla\psi)\cdot\nabla\phi - v\phi(a^\omega\nabla\psi)\cdot\nabla\psi\, dx.$$

The caloric equation for v_t can be formulated properly, using a suitable space of test functions, akin to $\mathcal{F}^\theta$, but for our purposes it suffices to study one specific instance of this equation, which we derive directly from (2.1) for brevity.

More precisely, for $t > 0$, $u_t = u(t,\cdot) \in \mathcal{F}^\theta$ by Definition 2.1 and the supposed properties of ψ guarantee that $\mathrm{e}^{2\psi}u_t \in \mathcal{F}^\theta$ also. Therefore, setting $\phi = \mathrm{e}^{2\psi}u_t$ in (2.1) and rearranging gives

$$\frac{d}{dt}(v_t, v_t)_\theta + 2\,J^\omega(v_t, v_t) = 0.$$

Since a^ω is symmetric,

$$\begin{aligned} J^\omega(v_t, v_t) &= \int_{\mathbb{R}^d} (a^\omega\nabla v_t)\cdot\nabla v_t - v_t^2(a^\omega\nabla\psi)\cdot\nabla\psi\, dx \\ &\geq -\int_{\mathbb{R}^d} v_t^2(a^\omega\nabla\psi)\cdot\nabla\psi\, dx \\ &\geq -h^\omega(\psi)^2\,\|v_t\|_{2,\theta}^2. \end{aligned}$$

Therefore,

$$\frac{d}{dt}\|v_t\|_{2,\theta}^2 \leq 2\,h^\omega(\psi)^2\|v_t\|_{2,\theta}^2,$$

from which the result follows. □

We now establish an energy estimate which we will go on to apply iteratively in order to derive a maximal inequality for v.

Lemma 2.3 *Let $I = (t_1, t_2) \subseteq \mathbb{R}_+$ and $B \subseteq \mathbb{R}^d$ be any Euclidean ball, $Q := I \times B$. Let u be a locally bounded positive caloric function on $I \times B$, and $v(t,x) := \mathrm{e}^{\psi(x)} u(t,x)$ for $\psi \in W^{1,\infty}_{loc}(\mathbb{R}^d)$ with $\|\psi\|_\infty < \infty$ and $h^\omega(\psi)^2 < \infty$. Define cut-off functions $\eta \in C_0^\infty(B)$ such that $0 \le \eta \le 1$ and $\xi : \mathbb{R} \to [0,1]$ differentiable with $\xi \equiv 0$ on $(-\infty, t_1]$. Then, there exists $c_6 > 1$ such that for any $\alpha \ge 1$,*

$$\frac{1}{|I|}\big\|\xi(\eta v^\alpha)^2\big\|_{1,\infty,Q,\theta} + \frac{1}{|I|}\int_I \xi(t)\frac{\mathcal{E}^\omega(\eta v_t^\alpha, \eta v_t^\alpha)}{|B|}\,dt$$
$$\le c_6\Big(\|\nabla\eta\|_\infty^2 \big\|\Lambda^\omega/\theta^\omega\big\|_{p,B,\theta}\big\|v^{2\alpha}\big\|_{p_*,1,Q,\theta} + \big(\alpha^2\, h^\omega(\psi)^2 + \|\xi'\|_\infty\big)\big\|(\eta v^\alpha)^2\big\|_{1,1,Q,\theta}\Big).$$

Proof One can show using (2.2) and the same argument as [18, Lemma B.3] that

$$\frac{d}{dt}(v_t^{2\alpha}, \eta^2)_\theta + 2\alpha\, J^\omega(v_t, \eta^2 v_t^{2\alpha-1}) \le 0 \quad \forall t \ge 0. \tag{2.3}$$

Then

$$\begin{aligned}\alpha\, J^\omega(v_t, \eta^2 v_t^{2\alpha-1}) \quad &= \textstyle\int_{\mathbb{R}^d} \alpha\,(a^\omega\nabla v_t)\cdot\nabla(\eta^2 v_t^{2\alpha-1}) + \alpha\,\eta^2 v_t^{2\alpha-1}(a^\omega\nabla v_t)\cdot\nabla\psi \\ &\quad -\alpha\, v_t(a^\omega\nabla\psi)\cdot\nabla(\eta^2 v_t^{2\alpha-1}) - \alpha\,\eta^2 v_t^{2\alpha}(a^\omega\nabla\psi)\cdot\nabla\psi\, dx.\end{aligned}$$

We label these integrands $J_1, \ldots, J_4$ in order.

$$\begin{aligned}J_1 &= \alpha\,(a^\omega\nabla v_t)\cdot\nabla(\eta^2 v_t^{2\alpha-1}) \\ &= \eta^2(a^\omega\nabla v_t)\cdot\nabla(\alpha v_t^{2\alpha-1}) + 2\alpha\,\eta\, v_t^{2\alpha-1}(a^\omega\nabla v_t)\cdot\nabla\eta.\end{aligned}$$

By algebraic manipulation,

$$J_1 = \frac{2\alpha-1}{\alpha}(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla(\eta v_t^\alpha) - \frac{\alpha-1}{\alpha}v_t^\alpha(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla\eta - \frac{1}{\alpha}(a^\omega\nabla\eta)\cdot\nabla(\eta v_t^{2\alpha}).$$

Then since $\alpha \ge 1$,

$$\begin{aligned}J_1 &\ge (a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla(\eta v_t^\alpha) - v_t^\alpha|(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla\eta| - v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta \\ &\ge \frac{1}{2}(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla(\eta v_t^\alpha) - 2\,v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta.\end{aligned} \tag{2.4}$$

Similarly,

$$|J_2| \le \frac{1}{8}(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla(\eta v_t^\alpha) + 3\eta v_t^{2\alpha}(a^\omega\nabla\psi)\cdot\nabla\psi + v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta. \tag{2.5}$$

$$\begin{aligned}|J_3| &\le \frac{1}{8}(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla(\eta v_t^\alpha) + 8\alpha^2\eta^2 v^{2\alpha}(a^\omega\nabla\psi)\cdot\nabla\psi \\ &\quad +\eta^2 v_t^{2\alpha}(a^\omega\nabla\psi)\cdot\nabla\psi + v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta.\end{aligned} \tag{2.6}$$

Substituting the above estimates into (2.3),

$$\begin{aligned}\frac{d}{dt}\big\|\eta^2 v_t^{2\alpha}\big\|_{1,\theta} &\le \int_{\mathbb{R}^d} 4\,v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta - \frac{1}{4}(a^\omega\nabla(\eta\, v_t^\alpha))\cdot\nabla(\eta\, v_t^\alpha)\,dx \\ &\quad +(9\alpha^2+4)\int_{\mathbb{R}^d}\eta^2 v_t^{2\alpha}(a^\omega\nabla\psi)\cdot\nabla\psi\, dx.\end{aligned} \tag{2.7}$$

Therefore,

$$\frac{d}{dt}\left\|\eta^2 v_t^{2\alpha}\right\|_{1,B,\theta}+\quad\frac{\mathcal{E}^\omega(\eta v_t^\alpha,\eta v_t^\alpha)}{4|B|}$$
$$\leq\frac{1}{|B|}\quad\left(4\int_{\mathbb{R}^d}v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta\,dx+(9\alpha^2+4)\int_{\mathbb{R}^d}\eta^2v_t^{2\alpha}(a^\omega\nabla\psi)\cdot\nabla\psi\,dx\right).$$

We can then bound these terms as follows

$$\begin{aligned}\int_{\mathbb{R}^d}v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta\,dx &\leq \left\|\nabla\eta\right\|_\infty^2\int_B v_t^{2\alpha}\Lambda^\omega\,dx\leq\left\|\nabla\eta\right\|_\infty^2|B|\left\|v_t^{2\alpha}\right\|_{1,B,\Lambda}\\ &\leq \left\|\nabla\eta\right\|_\infty^2|B|\left\|\Lambda^\omega/\theta^\omega\right\|_{p,B,\theta}\left\|v_t^{2\alpha}\right\|_{p_*,B,\theta}.\\ \int_{\mathbb{R}^d}\eta^2v_t^{2\alpha}(a^\omega\nabla\psi)\cdot\nabla\psi\,dx &\leq h^\omega(\psi)^2\int_{\mathbb{R}^d}\eta^2v_t^{2\alpha}\theta^\omega\,dx\\ &= h^\omega(\psi)^2|B|\left\|\eta^2v_t^{2\alpha}\right\|_{1,B,\theta},\end{aligned}$$

where we used Hölder's inequality on the first term. So,

$$\begin{aligned}&\frac{d}{dt}\left\|\eta^2v_t^{2\alpha}\right\|_{1,B,\theta}+\frac{\mathcal{E}^\omega(\eta v_t^\alpha,\eta v_t^\alpha)}{4|B|}\\ &\leq 4\left\|\nabla\eta\right\|_\infty^2\left\|\Lambda^\omega/\theta^\omega\right\|_{p,B,\theta}\left\|v_t^{2\alpha}\right\|_{p_*,B,\theta}+(9\alpha^2+4)h^\omega(\psi)^2\left\|\eta^2v_t^{2\alpha}\right\|_{1,B,\theta}.\end{aligned}\tag{2.8}$$

Now let $t\in(t_1,t_2)$, multiply the above by $\xi(s)$ and integrate from $s=t_1$ to $s=t$,

$$\begin{aligned}&\frac{1}{|I|}\left(\xi(t)\left\|\eta^2v_t^{2\alpha}\right\|_{1,B,\theta}+\frac{1}{4}\int_{t_1}^t\frac{\mathcal{E}^\omega(\eta v_s^\alpha,\eta v_s^\alpha)}{|B|}\,ds\right)\\ &\leq 4\left\|\nabla\eta\right\|_\infty^2\left\|\Lambda^\omega/\theta^\omega\right\|_{p,B,\theta}\left\|v^{2\alpha}\right\|_{p_*,1,I\times B,\theta}+(9\alpha^2+4)h^\omega(\psi)^2\left\|\eta^2v^{2\alpha}\right\|_{1,1,I\times B,\theta}\\ &\quad+\sup_{s\in I}|\xi'(s)|\left\|\eta^2v^{2\alpha}\right\|_{1,1,I\times B,\theta}.\end{aligned}\tag{2.9}$$

Note that the final term on the right-hand side appears by integration by parts with the first term on the left-hand side. Finally, take supremum over $t\in I$ on the left-hand side. □

The following Sobolev inequality is another component in deriving the maximal inequality in Proposition 2.5.

Lemma 2.4 (Sobolev Inequality) *Let $B\subseteq\mathbb{R}^d$ be a Euclidean ball and $\eta\in C_0^\infty(B)$ a cut-off function. Then there exists $c_7(d,q)>0$ such that for all $u\in\mathcal{F}_{loc}^\theta\cup\mathcal{F}_{loc}$,*

$$\left\|\eta^2u^2\right\|_{\rho/r_*,B,\theta}\leq c_7\,|B|^{\frac{2}{d}}\left\|(\lambda^\omega)^{-1}\right\|_{q,B}\left\|\theta^\omega\right\|_{r,B}^{r_*/\rho}\frac{\mathcal{E}^\omega(\eta u,\eta u)}{|B|},\tag{2.10}$$

where $\rho:=qd/(q(d-2)+d)$.

Proof Firstly, by Hölder's inequality,

$$\left\|\eta^2u^2\right\|_{\rho/r_*,B,\theta}\leq\left\|\theta^\omega\right\|_{r,B}^{r_*/\rho}\left\|\eta^2u^2\right\|_{\rho,B}.\tag{2.11}$$

Also by [18, Proposition 2.3],

$$\left\|\eta^2u^2\right\|_\rho\leq\left\|\mathbb{1}_B(\lambda^\omega)^{-1}\right\|_q\mathcal{E}^\omega(\eta u,\eta u).$$

After averaging over B this yields

$$\left\|\eta^2 u^2\right\|_{\rho,B} \leq c\,|B|^{2/d}\left\|(\lambda^\omega)^{-1}\right\|_{q,B}\frac{\mathcal{E}^\omega(\eta u,\eta u)}{|B|}, \tag{2.12}$$

for some $c = c(d,q) > 0$. The result then follows from (2.11) and (2.12). □

We now derive the maximal inequality for v using Moser iteration. For $x_0 \in \mathbb{R}^d$, $\delta \in (0,1]$ and $n \in \mathbb{R}_+$ we denote a space-time cylinder $Q_\delta(n) := [0,\delta n^2] \times B(x_0,n)$. Furthermore, for $\sigma \in (0,1]$ and $\epsilon \in (0,1]$, let $s' = \epsilon\delta n^2$, $s'' = (1-\epsilon)\delta n^2$ and define

$$Q_{\delta,\sigma}(n) := [(1-\sigma)s', (1-\sigma)s'' + \sigma\delta n^2] \times B(x_0,\sigma n). \tag{2.13}$$

Proposition 2.5 *Let $x_0 \in \mathbb{R}^d$, $\delta \in (0,1]$, $\epsilon \in (0,1/4)$, $1/2 \leq \sigma' < \sigma \leq 1$ and $n \in [1,\infty)$. Let v be as in Lemma 2.3. Then there exist constants $c_8(d,p,q,r)$ and $\kappa(d,p,q,r)$ such that*

$$\|v\|_{\infty,\infty,Q_{\delta,1/2}(n)} \leq c_8\left((1+\delta n^2 h^\omega(\psi)^2)\frac{\mathcal{A}^\omega(n)}{\epsilon(\sigma-\sigma')^2}\right)^{\frac{\kappa}{p_*}}\|v\|_{2p_*,2,Q_{\delta,\sigma}(n),\theta}. \tag{2.14}$$

In the above, $\mathcal{A}^\omega(n) := \left\|1\vee(\Lambda^\omega/\theta^\omega)\right\|_{p,B(x_0,n),\theta}\left\|1\vee(\lambda^\omega)^{-1}\right\|_{q,B(x_0,n)}\left\|1\vee\theta^\omega\right\|_{r,B(x_0,n)}$ and

$$\kappa(d,p,q,r) = \frac{p_*\big((p_*+1)qd - r_*(q(d-2)+d)\big)}{2\big(qd - p_*r_*(q(d-2)+d)\big)}.$$

Proof Define $\alpha := 1 + \frac{1}{p_*} - \frac{r_*}{\rho} > 1$ and write $\alpha_k := \alpha^k$ for $k \in \mathbb{N}$. Let $\sigma_k := \sigma' + 2^{-k}(\sigma - \sigma')$ and $\tau_k := 2^{-k-1}(\sigma-\sigma')$. Also introduce shorthand $I_k = [(1-\sigma_k)s', (1-\sigma_k)s'' + \sigma_k\delta n^2]$, $B_k := B(x_0,\sigma_k n)$ and $Q_k = I_k \times B_k = Q_{\delta,\sigma_k}(n)$. Note that $|I_k|/|I_{k+1}| \leq 2$ and $|B_k\|/|B_{k+1}| \leq c\,2^d$. We begin by applying Hölder's and Young's inequalities,

$$\left\|v^{2\alpha_k}\right\|_{\alpha p_*,\alpha,Q_{k+1},\theta} \leq \left\|v^{2\alpha_k}\right\|_{1,\infty,Q_{k+1},\theta} + \left\|v^{2\alpha_k}\right\|_{\rho/r_*,1,Q_{k+1},\theta}, \tag{2.15}$$

with ρ as in Lemma 2.4. Now let $k \in \mathbb{N}$ and define a sequence of cut-off functions in space, $\eta_k : \mathbb{R}^d \to [0,1]$ such that supp $\eta_k \subseteq B_k$, $\eta_k \equiv 1$ on B_{k+1} and $\left\|\nabla\eta_k\right\|_\infty \leq \frac{2}{\tau_k n}$. Similarly, let $\xi_k : \mathbb{R} \to [0,1]$ be time cut-offs such that $\xi_k \equiv 1$ on I_{k+1}, $\xi_k \equiv 0$ on $(-\infty,(1-\sigma_k)s']$ and $\left\|\xi'\right\|_\infty \leq \frac{2}{\tau_k\delta n^2}$. Then by (2.15),

$$\left\|v^{2\alpha_k}\right\|_{\alpha p_*,\alpha,Q_{k+1},\theta} \leq c\left(\left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{1,\infty,Q_k,\theta} + \left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{\rho/r_*,1,Q_k,\theta}\right). \tag{2.16}$$

We will bound both terms on the right-hand side. By the Sobolev inequality (2.10),

$$\begin{aligned}&\left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{\rho/r_*,1,Q_k,\theta}\\ &\quad\leq c\,n^2\left\|(\lambda^\omega)^{-1}\right\|_{q,B_k}\left\|\theta^\omega\right\|_{r,B_k}^{r_*/\rho}\frac{1}{|I_k|}\int_{I_k}\xi_k(t)\frac{\mathcal{E}^\omega(\eta_k v_t^{\alpha_k},\eta_k v_t^{\alpha_k})}{|B_k|}\,dt.\end{aligned} \tag{2.17}$$

So,

$$\begin{aligned}&\left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{1,\infty,Q_k,\theta} + \left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{\rho/r_*,1,Q_k,\theta} \leq \frac{c\,n^2}{|I_k|}\left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{1,\infty,Q_k,\theta}\\ &+\frac{c\,n^2}{|I_k|}\left\|(\lambda^\omega)^{-1}\right\|_{q,B_k}\left\|\theta^\omega\right\|_{r,B_k}^{r_*/\rho}\int_{I_k}\xi_k(t)\frac{\mathcal{E}^\omega(\eta_k v_t^{\alpha_k},\eta_k v_t^{\alpha_k})}{|B_k|}\,dt.\end{aligned} \tag{2.18}$$

By Lemma 2.3 and Hölder's inequality,

$$(2.18) \le c\,\alpha_k\,\mathcal{A}^\omega(n)\Big(\frac{1}{\delta\tau_k^2}+n^2h^\omega(\psi)^2\Big)\big\|v^{2\alpha_k}\big\|_{p_*,1,Q_k,\theta}.$$

Returning to (2.15),

$$\begin{aligned}\|v\|_{2\alpha_{k+1}p_*,2\alpha_{k+1},Q_{k+1},\theta} &= \big\|v^{2\alpha_k}\big\|^{1/(2\alpha_k)}_{\alpha p_*,\alpha,Q_{k+1},\theta}\\ &\le \Big(c\,\alpha_k\,2^{2k}\frac{(1+\delta n^2h^\omega(\psi)^2)}{\delta(\sigma-\sigma')^2}\mathcal{A}^\omega(n)\Big)^{\frac{1}{2\alpha_k}}\|v\|_{2\alpha_k p_*,2\alpha_k,Q_k,\theta}.\end{aligned}$$

Iterating the above, for any $K\in\mathbb{Z}_+$,

$$\|v\|_{2\alpha_K p_*,2\alpha_K,Q_K,\theta} \le c\prod_{k=0}^{K-1}\Big(\alpha_k\,2^{2k}\frac{(1+\delta n^2h^\omega(\psi)^2)}{\delta(\sigma-\sigma')^2}\mathcal{A}^\omega(n)\Big)^{\frac{1}{2\alpha_k}}\|v\|_{2p_*,2,Q_{\delta,\sigma}(n),\theta}.$$

Sending $K\to\infty$, observing that $Q_K\downarrow Q_{\delta,\frac12}(n)$ and $\prod_{k=0}^{K-1}(\alpha_k 2^{2k})^{\frac{1}{2\alpha_k}}$ is uniformly bounded in K, we have

$$\|v\|_{\infty,\infty,Q_{\delta,1/2}(n)} \le c\Big((1+\delta n^2h^\omega(\psi)^2)\frac{\mathcal{A}^\omega(n)}{\epsilon(\sigma-\sigma')^2}\Big)^{\frac{\kappa}{p_*}}\|v\|_{2p_*,2,Q_{\delta,\sigma}(n),\theta}, \tag{2.19}$$

where $\kappa := \frac{p_*}{2}\sum_{k=0}^{\infty}\frac{1}{\alpha_k}<\infty$. □

Corollary 2.6 *In the same setting as Proposition 2.5, there exists $c_9(d,p,q,r)>0$ such that*

$$\|v\|_{\infty,\infty,Q_{\delta,1/2}(n)} \le c_9\Big((1+\delta n^2h^\omega(\psi)^2)\frac{\mathcal{A}^\omega(n)}{\epsilon(\sigma-\sigma')^2}\Big)^{\kappa}\|v\|_{2,\infty,Q_\delta(n),\theta}. \tag{2.20}$$

Proof This is derived from Proposition 2.5, in a similar fashion to [24, Theorem 2.2.3]. □

2.2 Heat Kernel Bound

We first conglomerate the two results of the preceding section – the Cauchy problem estimate and the maximal inequality.

Proposition 2.7 *In the same setting as Proposition 2.5, there exists $c_{10}(d,p,q,r,\epsilon)>0$ such that*

$$\|v\|_{\infty,\infty,Q_{\delta,1/2}(n)} \le \frac{c_{10}}{n^{d/2}}\Big(\frac{\mathcal{A}^\omega(n)}{\epsilon\delta}\Big)^{\kappa}\mathrm{e}^{2(1-\epsilon)h^\omega(\psi)^2\delta n^2}\big\|\mathrm{e}^\psi f\big\|_{2,\theta}. \tag{2.21}$$

Proof By combining Corollary 2.6 with Lemma 2.2, we obtain

$$\|v\|_{\infty,\infty,Q_{\delta,1/2}(n)} \le \frac{c}{n^{d/2}}\Big((1+\delta n^2h^\omega(\psi)^2)\frac{\mathcal{A}^\omega(n)}{\epsilon\delta}\Big)^{\kappa}\mathrm{e}^{h^\omega(\psi)^2\delta n^2}\big\|\mathrm{e}^\psi f\big\|_{2,\theta}.$$

The result follows since for any $\epsilon\in(0,1/2)$ there exists $c(\epsilon)<\infty$ such that

$$(1+\delta n^2h^\omega(\psi)^2)^\kappa \le c(\epsilon)\,\mathrm{e}^{(1-2\epsilon)h^\omega(\psi)^2\delta n^2},$$

for all $n\ge1,\ \delta\in(0,1]$. □

Proposition 2.8 (Heat Kernel Bound) *Suppose* $M_1(p,q,r) < \infty$ *and let* $x_0 \in \mathbb{R}^d$*. Then* $\mathbb{P}$*-a.s. there exist* $c_{11}(d,p,q,r)$, $\gamma(d,p,q,r) > 0$ *such that for all* $\sqrt{t} \geq N_2^\omega(x_0)$ *and* $x,\ y \in \mathbb{R}^d$,

$$p_\theta^\omega(t,x,y) \leq c_{11}\, t^{-\frac{d}{2}} \Big(1 + \frac{d(x_0,x)}{\sqrt{t}}\Big)^\gamma \Big(1 + \frac{d(x_0,y)}{\sqrt{t}}\Big)^\gamma \mathrm{e}^{2h^\omega(\psi)^2 t - \psi(x) + \psi(y)}. \tag{2.22}$$

The exponent $\gamma = 2\kappa - \frac{d}{2}$*, with* κ *as in Proposition 2.5.*

Proof Fix $\epsilon = \frac{1}{8}$. By the ergodic theorem there exists $N_2^\omega(x_0) > 0$ such that

$$\mathcal{A}^\omega(n) \leq c\left(1 + \mathbb{E}[\Lambda^\omega(0)^p \theta^\omega(0)^{1-p}]\right)\left(1 + \mathbb{E}[\lambda^\omega(0)^{-q}]\right)\left(1 + \mathbb{E}[\theta^\omega(0)^r]\right) =: \bar{A} < \infty,$$

for all $n \geq N_2^\omega(x_0)$. For given $x \in \mathbb{R}^d$ and $\sqrt{t} > N_2^\omega(x_0)$, we choose δ, n such that $(t,x) \in Q_{\delta,\frac{1}{2}}(n)$, for example by setting $n = 2d(x_0,x) + \sqrt{8t/7}$ and $\delta := 8t/(7n^2)$. Then considering the caloric function $u(t,x) := P_t f(x)$ for $f \in \mathcal{F}^\theta$, by Proposition 2.7,

$$\begin{aligned} \mathrm{e}^{\psi(x)} u(t,x) &\leq c\, n^{-d/2} (n^2/t)^\kappa \mathrm{e}^{2h^\omega(\psi)^2 t} \left\|\mathrm{e}^\psi f\right\|_{2,\theta} \\ &\leq c\, n^\gamma t^{-\kappa} \mathrm{e}^{2h^\omega(\psi)^2 t} \left\|\mathrm{e}^\psi f\right\|_{2,\theta}, \end{aligned} \tag{2.23}$$

for some $c = c(\epsilon, d, p, q, r, \bar{A})$, where $\gamma = 2\kappa - \frac{d}{2}$. Write $r(t) := c\, t^{-\kappa} \mathrm{e}^{2h^\omega(\psi)^2 t}$ and $b_t(x) := \left(2d(x_0,x) + \sqrt{8t/7}\right)^\gamma$. Since the above holds for all $x \in \mathbb{R}^d$ and $\sqrt{t} > N_2^\omega(x_0)$, we have

$$\mathrm{e}^{\psi(x)} P_t f(x) \leq b_t(x) r(t) \left\|\mathrm{e}^\psi f\right\|_{2,\theta}.$$

That is,

$$\left\|b_t^{-1} \mathrm{e}^\psi P_t f\right\|_\infty \leq r(t) \left\|\mathrm{e}^\psi f\right\|_{2,\theta}. \tag{2.24}$$

Now define an operator $P_t^\psi(g) := \mathrm{e}^\psi P_t(\mathrm{e}^{-\psi} g)$ for $\mathrm{e}^{-\psi} g \in \mathcal{F}^\theta$. Then we can bound the operator norm

$$\left\|b_t^{-1} P_t^\psi\right\|_{L^2(\mathbb{R}^d, \theta^\omega dx)} \leq r(t).$$

The above also holds with ψ replaced by $-\psi$. Since the dual of P_t^ψ is $P_t^{-\psi}$, the dual of $b_t^{-1} P_t^{-\psi}(\cdot)$ is $P_t^\psi(b_t^{-1} \cdot)$. So by duality,

$$\left\|P_t^\psi(b_t^{-1} g)\right\|_{2,\theta} \leq r(t) \left\|g\right\|_{1,\theta}. \tag{2.25}$$

Since $b_{\frac{t}{2}}(x) \leq b_t(x)$, we have

$$\begin{aligned} \left\|b_t^{-1} \mathrm{e}^\psi P_t f\right\|_\infty &\leq \left\|b_{\frac{t}{2}}^{-1} \mathrm{e}^\psi P_{\frac{t}{2}} P_{\frac{t}{2}} f\right\|_\infty \\ &\leq r(t/2) \left\|\mathrm{e}^\psi P_{\frac{t}{2}} f\right\|_{2,\theta} \quad \text{by (2.24)}, \\ &\leq r(t/2)^2 \left\|\mathrm{e}^\psi b_{\frac{t}{2}} f\right\|_{1,\theta} \quad \text{by (2.25)}. \end{aligned} \tag{2.26}$$

That is, for all $x \in \mathbb{R}^d$ and $\sqrt{t} \geq N_2^\omega(x_0)$, we have

$$P_t f(x) \leq \frac{c}{t^{2\kappa}} \mathrm{e}^{2h^\omega(\psi)^2 t - \psi(x)} (d(x_0,x) + \sqrt{t})^\gamma \int_{\mathbb{R}^d} (d(x_0,y) + \sqrt{t})^\gamma \mathrm{e}^{\psi(y)} |f(y)| \theta^\omega(y)\, dy.$$

It is standard that the above implies the heat kernel estimate (2.22) for almost all $x,\ y \in \mathbb{R}^d$. Furthermore, local boundedness in Assumption 1.1 allows us to pass to all $x,\ y \in \mathbb{R}^d$.

Finally, we show that $\gamma = 2\kappa - \frac{d}{2} > 0$. It suffices to bound $\kappa = \frac{p_*}{2}\left(1 - \frac{1}{\alpha}\right)^{-1}$ below. Recall that $\alpha = 1 + \frac{1}{p_*} - \frac{r_*}{\rho}$, where $\rho = qd/(q(d-2)+d)$ is defined in Lemma 2.4. We need to bound α above; by definition, $p_*, r_* \geq 1$ and since ρ is monotonically increasing in q, we have $\rho \leq \frac{d}{d-2}$. Therefore,

$$\alpha \leq 2 - \frac{d-2}{d} = \frac{d+2}{d}.$$

So,

$$\kappa \geq \frac{1}{2}\Big(1 - \frac{d}{d+2}\Big)^{-1} = \frac{d+2}{4} > \frac{d}{4}.$$

□

2.3 Properties of the Intrinsic Metric

In order to prove the off-diagonal estimate in Theorem 1.3 from Proposition 2.8, we aim to set the function $\psi(\cdot) = \beta\, d_\theta^\omega(x, \cdot)$ in (2.22), then optimise over the constant β. This requires checking that this function ψ satisfies the necessary regularity assumptions for the proofs in Section 2.1. Recall that the intrinsic metric is defined as follows,

$$d_\theta^\omega(x, y) := \sup\Big\{\phi(y) - \phi(x) \,:\, \phi \in C(\mathbb{R}^d) \cap \mathcal{F}_{\text{loc}}^\theta,\ h^\omega(\phi)^2 \leq 1\Big\}.$$

In deriving the required regularity of d_θ^ω, we first show that it is equal to D_θ^ω, the Riemannian distance computed with respect to $(\frac{a^\omega}{\theta^\omega})^{-1}$. This Riemannian metric is defined via the following path relation. Consider the following Hilbert space

$$\mathcal{H} := \big\{f \in C\big([0,\infty), \mathbb{R}^d\big) \,:\, f(0) = 0,\ \dot{f} \in L^2\big([0,\infty), \mathbb{R}^d\big)\big\},$$

where $\dot{f}$ denotes the weak derivative of f, together with the following norm

$$\|f\|_{\mathcal{H}} := \|\dot{f}\|_{L^2([0,\infty),\mathbb{R}^d)}.$$

Given $f \in \mathcal{H}$, define $\Phi(t, x; f) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ via

$$\frac{d}{dt}\Phi(t, x; f) = \Big(\frac{a^\omega(\Phi(t, x; f))}{\theta^\omega(\Phi(t, x; f))}\Big)^{1/2} \dot{f}(t),$$

with initial condition $\Phi(0, x; f) = x$. The Riemannian distance is then given by

$$D_\theta^\omega(x, y) := t^{1/2} \inf\Big\{\|f\|_{\mathcal{H}} \,:\, f \in \mathcal{H},\ \Phi(t, x; f) = y\Big\},$$

for any $t > 0$.

Lemma 2.9 (Riemannian Distance Representation) *For all* $x, y \in \mathbb{R}^d$, $d_\theta^\omega(x, y) = D_\theta^\omega(x, y)$.

Proof This follows by the proof of [34, Lemma I.1.24]. □

Next we will apply the additional Assumption 1.2 on the environment to derive the regularity we require of d_θ^ω. Our objective is to pass a function resembling $\rho_x(\cdot) := d_\theta^\omega(x, \cdot)$ into (2.22). In order to do this we must show some conditions such as $\rho_x \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d)$ and $h^\omega(\rho_x)^2 \leq 1$. The requisite property is that the metric d_θ^ω is strictly local i.e. that d_θ^ω induces the original topology on $\mathbb{R}^d$. For further discussion of the properties of such intrinsic metrics and the distance function ρ_x see [33], [15, Appendix A] and [35]. In the following proposition, we invoke a recent result from geometric analysis to directly deduce strict locality of the intrinsic metric d_θ^ω under Assumption 1.2.

Proposition 2.10 *If Assumption 1.2 holds then the intrinsic metric d_θ^ω is strictly local for $\mathbb{P}$-a.e. ω.*

Proof Given Assumption 1.2 and Lemma 2.9, this follows directly from Proposition 4.1ii) or Theorem 4.5 in [16], noting that the Euclidean metric corresponds to the Riemannian metric given by the identity matrix [33, Proposition 3.3]. □

2.4 Upper Off-Diagonal Estimate

Having proven the necessary regularity of the intrinsic metric in the preceding subsection, we are now in a position to optimise over the test function in Proposition 2.8 and derive the upper off-diagonal estimate.

Proof of Theorem 1.3 As a corollary to Proposition 2.10, we have for example by [35, Lemma 1] that for any $x \in \mathbb{R}^d$, $\rho_x \in C(\mathbb{R}^d) \cap L^2_{\text{loc}}(\mathbb{R}^d, \theta)$ and $h^\omega(\rho_x)^2 \le 1$ almost surely. Furthermore ρ_x has a weak derivative and [33, Theorem 5.1] implies that $\operatorname{ess\,sup}_{z\in\mathbb{R}^d} |\nabla \rho_x(z)| < \infty$. The final property to check is that our test function is essentially bounded, whilst ρ_x may be unbounded we can take a bounded version with the desired properties. In accordance with [15, Eqn. (2)], consider $\eta_x = \xi \circ \rho_x$ for a continuously differentiable cut-off function ξ to construct a function such that $\eta_x(x) = d_\theta^\omega(x, x) = 0$, $\eta_x(y) = d_\theta^\omega(x, y)$, η_x is essentially bounded and η_x satisfies the aforementioned properties, including $h^\omega(\eta_x)^2 \le 1$. This is another consequence of Proposition 2.10. Therefore we are justified in setting $\psi(\cdot) = -\beta\, \eta_x(\cdot)$ in (2.22) for $\beta \in \mathbb{R}$, and $h^\omega(\psi)^2 \le \beta^2$. Then by choosing the constant $\beta = d_\theta^\omega(x, y)/(4t)$ and setting $x_0 = x$ in (2.22) we have for $\mathbb{P}$-a.e. ω, all $x, y \in \mathbb{R}^d$ and $\sqrt{t} \ge N_2^\omega(x)$,

$$p_\theta^\omega(t, x, y) \le c\, t^{-\frac{d}{2}} \Big(1 + \frac{d(x, y)}{\sqrt{t}}\Big)^\gamma \exp\Big(-\frac{d_\theta^\omega(x, y)^2}{8t}\Big), \tag{2.27}$$

which completes the proof. □

3 Lower Off-Diagonal Estimate

The starting point for proving the lower off-diagonal estimate of Theorem 1.6 is the following near-diagonal estimate. Throughout this section suppose Assumptions 1.1 and 1.5 hold. Also let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$.

Proposition 3.1 *Let $t > 0$ and $x \in \mathbb{R}^d$, then for all $y \in B\big(x, \frac{\sqrt{t}}{2}\big)$ we have*

$$p_\Lambda^\omega(t, x, y) \ge \frac{t^{-d/2}}{C_{\text{PH}}\big(\|\Lambda^\omega\|_{p,B(x,\sqrt{t})}, \|\lambda^\omega\|_{q,B(x,\sqrt{t})}\big)}. \tag{3.1}$$

The constant C_{PH} is given explicitly by

$$C_{\text{PH}} = c_{12} \exp\Big(c_{13}\Big(\big(1 \vee \|\Lambda^\omega\|_{p,B(x,\sqrt{t})}\big)\big(1 \vee \|\lambda^\omega\|_{q,B(x,\sqrt{t})}\big)\Big)^\kappa\Big), \tag{3.2}$$

for $c_i(d, p, q) > 0$ and $\kappa(d, p, q) := \frac{(2+d)pq-(p+2q)d}{2pq-(p+q)d} > 0$.

Proof A parabolic Harnack inequality with constant C_{PH} is established in [18, Theorem 3.9] and this is a standard consequence of it, see for instance [3, Proposition 4.7] or [23, Proposition 3.1]. □

The chaining method is to apply Proposition 3.1 along a sequence of balls. Let $x \in \mathbb{R}^d$, a radius $0 < r \le 4d(0,x)$ and $k \in \mathbb{N}$ satisfying $\frac{12d(0,x)}{r} \le k \le \frac{16d(0,x)}{r}$. Consider the sequence of points $x_j = \frac{j}{k}x$ for $j = 0, \dots, k$ that interpolates between 0 and x. Let $B_{x_j} = B(x_j, \frac{r}{48})$ and $s := \frac{r\,d(0,x)}{k}$, noting $\frac{r^2}{16} \le s \le \frac{r^2}{12}$.

To apply estimate (3.1) along a sequence we will need to control the ergodic average terms in (3.2) simultaneously for balls with different centre-points. To this end we establish a moment bound in Proposition 3.3 which employs finite range dependence to get better control than in the general ergodic setting. First, a prerequisite lemma.

Lemma 3.2 *For any $k > 2$ and independent random variables $Y_1, ..., Y_n \in L^k(\mathbb{P})$ with $\mathbb{E}[Y_i] = 0$ for all i, there exists $c_{14}(k) > 0$ such that*

$$\mathbb{E}\left[\left|\sum_{i=1}^n Y_i\right|^k\right] \le c_{14} \max\left\{\sum_{i=1}^n \mathbb{E}\big[|Y_i|^k\big],\ \left(\sum_{i=1}^n \mathbb{E}\big[|Y_i|^2\big]\right)^{\frac{k}{2}}\right\}. \tag{3.3}$$

Proof This follows from [32, Theorem 3]. □

For $u \in \mathbb{R}^d$, $p, q > 0$ we write $\Delta\Lambda_p^\omega(u) := \Lambda^\omega(u)^p - \mathbb{E}[\Lambda^\omega(0)^p]$ and $\Delta\lambda_q^\omega(u) := \lambda^\omega(u)^q - \mathbb{E}[\lambda^\omega(0)^q]$ for the deviation of these moments from their respective means.

Proposition 3.3 *Let $\xi > 1$ and assume $M_2(2\xi p, 2\xi q) < \infty$. Recall that $\mathcal{R}$ is as in Assumption 1.5. Let $R \subset \mathbb{R}^d$ be any region which can be covered by a disjoint partition of K balls of radius $\mathcal{R}$ in the maximum norm, i.e. $R \subset \bigcup_{i=1}^K \{z_i + [0,\mathcal{R})^d\}$ for some $z_1, \dots, z_K \in \mathbb{R}^d$. Then there exists $c_{15}(d, \mathcal{R}, \xi) > 0$ such that*

$$\mathbb{E}\Big[\Big|\int_R \Lambda^\omega(u)^p - \mathbb{E}[\Lambda^\omega(0)^p]\,du\Big|^{2\xi}\Big] \le c_{15}\, K^\xi, \tag{3.4}$$

$$\mathbb{E}\Big[\Big|\int_R \lambda^\omega(u)^q - \mathbb{E}[\lambda^\omega(0)^q]\,du\Big|^{2\xi}\Big] \le c_{15}\, K^\xi. \tag{3.5}$$

Proof We prove the statement only for Λ^ω, since the one for λ^ω is analogous. Denote $f(u) := \Delta\Lambda_p^\omega(u)\mathbb{1}_{u\in R}$. Then by Jensen's inequality and Fubini's theorem,

$$\begin{aligned}
\mathbb{E}\Big[\Big|\int_R \Delta\Lambda_p^\omega(u)\,du\Big|^{2\xi}\Big] &= \mathbb{E}\Big[\Big|\int_{[0,\mathcal{R})^d} \sum_{i=1}^K f(z_i+u)\,du\Big|^{2\xi}\Big] \\
&\le \mathcal{R}^{d(2\xi-1)}\mathbb{E}\Big[\int_{[0,\mathcal{R})^d} \Big|\sum_{i=1}^K f(z_i+u)\Big|^{2\xi}\,du\Big] \\
&= c\int_{[0,\mathcal{R})^d} \mathbb{E}\Big[\Big|\sum_{i=1}^K f(z_i+u)\Big|^{2\xi}\Big]\,du.
\end{aligned} \tag{3.6}$$

For fixed $u \in [0, \mathcal{R})^d$ the sequence $\big(f(z_i + u)\big)_{i=1}^K$ has mean zero and is independent by Assumption 1.5. So we have by Lemma 3.2 and shift-invariance of the environment,

$$\mathbb{E}\Big[\Big|\sum_{i=1}^K f(z_i+u)\Big|^{2\xi}\Big] \le c_{14} \max\left\{\sum_{i=1}^K \mathbb{E}\big[\big|f(z_i+u)\big|^{2\xi}\big],\ \left(\sum_{i=1}^K \mathbb{E}\big[\big|f(z_i+u)\big|^2\big]\right)^{\xi}\right\} \le c\,K^{\xi}. \tag{3.7}$$

Combining (3.6) and (3.7) gives the result. □

Proposition 3.4 *Let $\xi > d$ and assume $M_2(2\xi p, 2\xi q) < \infty$. For $\mathbb{P}$-a.e. ω, there exists $N_4(\omega) \in \mathbb{N}$ such that for all $r > 0$ and $x \in \mathbb{R}^d$ with $N_4(\omega) < r \le 4\,d(0,x)$, for any sequence $y_0, \dots, y_k$ where $y_0 = 0$, $y_k = x$ and $y_j \in B_{x_j}$ for $1 \le j \le k-1$, we have $c_{16}(d, \mathcal{R}, \xi) > 0$ such that*

$$\sum_{j=0}^{k-1} \Big(1 \vee \big\|\Lambda^\omega\big\|_{p,B(y_j,\sqrt{s})}\Big)\Big(1 \vee \big\|\lambda^\omega\big\|_{q,B(y_j,\sqrt{s})}\Big) \le c_{16}\,k. \tag{3.8}$$

Furthermore, we have the following estimate on $N_4(\omega)$, there exists $c_{17}(d, \mathcal{R}, \xi) > 0$ such that

$$\mathbb{P}(N_4(\omega) > n) \le c_{17}\,n^{1-d(\xi-1)} \qquad \forall n \in \mathbb{N}. \tag{3.9}$$

Proof Let x and r be as in the statement and denote $z = \lfloor x \rfloor \in \mathbb{Z}^d$, $r_0 = \lceil r \rceil \in \mathbb{Z}$. We will work with these discrete approximations of the variables x and r in order to apply countable union bounds and the Borel-Cantelli lemma. Note that $x \in C_z := z + [0,1]^d$ and $r \in I_{r_0} := [r_0 - 1, r_0]$. Assuming w.l.o.g. that $r > 1$ and $d(0,x) > d$ we have $r \simeq r_0$ and $|x| \simeq |z|$. We define a region that covers the union of balls of interest

$$\bigcup_{j=0}^k B(y_j, \sqrt{s}) \subset R_{z,r_0} := \Big\{\tau z + \big[-2r_0, 2r_0\big]^d \ :\ \tau \in \big[0,2\big]\Big\}.$$

This region has volume $|R_{z,r_0}| \le c\,r_0^{d-1}|z| \le c\,r^d k$ and can be covered by at most $K \le c\,r_0^{d-1}|z|/\mathcal{R}^d$ non-intersecting balls of radius $\mathcal{R}$ in the maximal norm. Also there exists $c_{18}(d)$ such that for all $w \in \mathbb{R}^d$, $|\{j \in \{0,\dots,k\} : w \in B(y_j, \sqrt{s})\}| \le c_{18}$, therefore

$$\begin{aligned}\sum_{j=0}^{k-1} \big\|\Lambda^\omega\big\|^p_{p,B(y_j,\sqrt{s})} &\le c_{18}\,r^{-d} \int_{\bigcup_{j=0}^k B(y_j,\sqrt{s})} \Lambda^\omega(u)^p\,du \le c\,r^{-d} \int_{R_{z,r_0}} \Lambda^\omega(u)^p\,du \\ &\le c\,r^{-d}|R_{z,r_0}|\mathbb{E}\big[\Lambda^\omega(0)^p\big] + c\,r^{-d}\int_{R_{z,r_0}} \Delta\Lambda^\omega_p(u)\,du \\ &\le c\,k + c\,r^{-d}\int_{R_{z,r_0}} \Delta\Lambda^\omega_p(u)\,du. \end{aligned} \tag{3.10}$$

By Markov's inequality and Proposition 3.3 we have

$$\begin{aligned}\mathbb{P}\Big(\int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\,du > kr^d\Big) &\le \mathbb{P}\Big(\Big|\int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\,du\Big| > c|z|r_0^{d-1}\Big)\\ &\le c\,\mathbb{E}\Big[\Big|\int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\,du\Big|^{2\xi}\Big]/\big(|z|r_0^{d-1}\big)^{2\xi}\\ &\le c\,\big(|z|r_0^{d-1}\big)^{-\xi}. \end{aligned}\tag{3.11}$$

Now let $\rho,\ l \in \mathbb{N}$ with $\rho \le l$. By (3.11) and a union bound, summing over $\{z \in \mathbb{Z}^d : |z| = l\}$ and $r_0 \ge \rho$,

$$\begin{aligned}&\mathbb{P}\Big(\exists\, z \in \mathbb{Z}^d,\ r_0 \in \mathbb{N} : |z| = l, r_0 \in [\rho, 4|z|], \int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\,du > kr^d\Big)\\ &\le c\, l^{d-1-\xi}\rho^{-\xi(d-1)+1}.\end{aligned}\tag{3.12}$$

Now consider the event

$$E_\rho := \Big\{\exists\, z \in \mathbb{Z}^d,\ r_0 \in \mathbb{N} : |z| \ge \rho, r_0 \in [\rho, 4|z|], \int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\,du > kr^d\Big\}.$$

Since $\xi > d$, we can take a countable union bound over l in (3.12) to obtain

$$\mathbb{P}\big(E_\rho\big) \le c\,\rho^{d(1-\xi)+1}.\tag{3.13}$$

Also $d(1-\xi)+1 < -1$ so by the Borel-Cantelli lemma there exists $\tilde{N}(\omega) \in \mathbb{N}$ such that for all $z \in \mathbb{Z}^d,\ r_0 \in \mathbb{N}$ with $\tilde{N}(\omega) < r_0 < 4|z|$ we have

$$\int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\,du \le kr^d.$$

Together with (3.10), this implies the existence of $N_4(\omega) \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$ and $r > 1$ with $N_4(\omega) < r \le 4\,d(0,x)$ we have for $y_0, \dots, y_k$ defined as in the statement,

$$\sum_{j=0}^{k-1} \big\|\Lambda^\omega\big\|_{p,B(y_j,\sqrt{s})}^p \le c\,k.\tag{3.14}$$

By the exact same reasoning, one can show the corresponding inequality for λ^ω. Moreover by Hölder's inequality,

$$\begin{aligned}&\sum_{j=0}^{k-1}\Big(1 \vee \big\|\Lambda^\omega\big\|_{p,B(y_j,\sqrt{s})}\Big)\Big(1 \vee \big\|\lambda^\omega\big\|_{q,B(y_j,\sqrt{s})}\Big)\\ &\le k^{1-\frac{1}{p}-\frac{1}{q}}\left(\sum_{j=0}^{k-1}\Big(1 \vee \big\|\Lambda^\omega\big\|_{p,B(y_j,\sqrt{s})}^p\Big)\right)^{\frac{1}{p}}\left(\sum_{j=0}^{k-1}\Big(1 \vee \big\|\lambda^\omega\big\|_{q,B(y_j,\sqrt{s})}^q\Big)\right)^{\frac{1}{q}}.\end{aligned}\tag{3.15}$$

This, together with (3.14) and the equivalent bound for λ^ω, gives the result.

The stated decay of $N_4(\omega)$ follows from $N_4(\omega) < c\,\tilde{N}(\omega)$ and the following bound on $\tilde{N}(\omega)$. For $n \in \mathbb{N}$, we have

$$\{\tilde{N}(\omega) \le n\} = \bigcup_{m=1}^{n}\bigcap_{\rho=m}^{\infty} E_\rho^c.$$

Note that, by the definition, $E_{\rho+1} \subseteq E_\rho$ for all $\rho \in \mathbb{N}$, so the events E_ρ^c are increasing, $E_\rho^c \subseteq E_{\rho+1}^c$ for all $\rho \in \mathbb{N}$. In particular, for all $m \in \mathbb{N}$,

$$\bigcap_{\rho=m}^{\infty} E_\rho^c = E_m^c.$$

Therefore, using (3.13),

$$\begin{aligned}\mathbb{P}(\tilde{N}(\omega) \le n) &= \mathbb{P}\Big(\bigcup_{m=1}^{n} E_m^c\Big) \ge \mathbb{P}(E_n^c)\\ &= 1 - \mathbb{P}(E_n) \ge 1 - c\, n^{1-d(\xi-1)}.\end{aligned}$$

Since $\tilde{N}(\omega)$ is finite a.s., we conclude that

$$\mathbb{P}(\tilde{N}(\omega) > n) \le c\, n^{1-d(\xi-1)}.$$

□

Corollary 3.5 *Let $\xi > d$ and assume $M_2(2\xi\kappa p, 2\xi\kappa q) < \infty$. In the same setting as Proposition 3.4 there exists $N_5(\omega) \in \mathbb{N}$ with decay as in (3.9) and $c_{19}(d, p, q, \mathcal{R}, \xi) > 0$ such that $\mathbb{P}$-a.s. for all $r > 0$, $x \in \mathbb{R}^d$ with $N_5(\omega) < r \le 4\, d(0, x)$ we have*

$$\sum_{j=0}^{k-1} \Big(1 \vee \|\Lambda^\omega\|_{p,B(y_j,\sqrt{s})}\Big)^\kappa \Big(1 \vee \|\lambda^\omega\|_{q,B(y_j,\sqrt{s})}\Big)^\kappa \le c_{19}\, k. \tag{3.16}$$

Proof By Jensen's inequality $\|\Lambda^\omega\|^\kappa_{p,B(y_j,\sqrt{s})} \le \|(\Lambda^\omega)^\kappa\|_{p,B(y_j,\sqrt{s})}$ and similarly for the λ^ω terms. Then proceed as for Proposition 3.4 to prove the result, with Λ^ω replaced by $(\Lambda^\omega)^\kappa$. □

Proof of Theorem 1.6 By shift-invariance of the environment it suffices to prove the estimate for $p_\Lambda^\omega(t, 0, x)$. Fix $\xi > d$ and for the moment assumption $M_2(p_0, q_0) < \infty$ choose $p_0 = 2\xi\kappa p$, $q_0 = 2\xi\kappa q$, in order to apply Corollary 3.5. Let $N_3^\omega(0) := N_1^\omega(0)^2 \vee N_4^\omega \vee N_5^\omega$ and assume as in the statement that $t \ge N_3^\omega(0)\big(1 \vee d(0, x)\big)$. We split the proof into two cases.

Firstly in the case $|x|^2/t < 1/4$ we have $x \in B\big(0, \sqrt{t}/2\big)$ so we may apply the near-diagonal lower estimate of Proposition 3.1,

$$p_\Lambda^\omega(t, 0, x) \ge \frac{t^{-d/2}}{C_{\mathrm{PH}}\big(\|\Lambda^\omega\|_{p,B(0,\sqrt{t})}, \|\lambda^\omega\|_{q,B(0,\sqrt{t})}\big)}.$$

Since $\sqrt{t} \ge N_1^\omega(0)$, recalling the form of C_{PH} we apply the ergodic theorem to bound

$$C_{\mathrm{PH}}\big(\|\Lambda^\omega\|_{p,B(0,\sqrt{t})}, \|\lambda^\omega\|_{q,B(0,\sqrt{t})}\big) \le c_{12} \exp\Big(c\,\big((1 \vee \bar{\Lambda}_p)(1 \vee \bar{\lambda}_q)\big)^\kappa\Big). \tag{3.17}$$

Therefore,

$$p_\Lambda^\omega(t, 0, x) \ge c\, t^{-d/2}.$$

Secondly, consider the case $|x|^2/t \ge 1/4$. Since Λ^ω and λ^ω are locally bounded, it follows from the semigroup property that for any $0 < \tau < t$,

$$p_\Lambda^\omega(t, 0, x) = \int_{\mathbb{R}^d} p_\Lambda^\omega(\tau, 0, u)\, p_\Lambda^\omega(t - \tau, u, x)\Lambda^\omega(u)\, du. \tag{3.18}$$

We will employ the chaining argument over the sequence of balls introduced below Proposition 3.1, set $r = t/|x| \geq N_3^{\omega}(0)$ which gives $s = t/k$. Iterating the above relation $k-1$ times gives

$$p_{\Lambda}^{\omega}(t,0,x) \geq \int_{B_{x_1}} \cdots \int_{B_{x_{k-1}}} p_{\Lambda}^{\omega}(s,0,y_1)\dots p_{\Lambda}^{\omega}(s,y_{k-1},x)\Lambda^{\omega}(y_1)\dots\Lambda^{\omega}(y_{k-1})\,dy_1\dots dy_{k-1}.$$

We have by Proposition 3.1, for all $y_j \in B_{x_j}$,

$$\prod_{j=0}^{k-1} p_{\Lambda}^{\omega}(s,y_j,y_{j+1}) \geq \frac{c\,s^{-dk/2}}{\exp\Big(c\sum_{j=0}^{k-1}\Big(\big(1\vee\big\|\Lambda^{\omega}\big\|_{p,B(y_j,\sqrt{s})}\big)\big(1\vee\big\|\lambda^{\omega}\big\|_{q,B(y_j,\sqrt{s})}\big)\Big)^{\kappa}\Big)} \geq \frac{c\,s^{-dk/2}}{\exp(c\,k)}, \tag{3.19}$$

where the second step is due to Corollary 3.5. Therefore,

$$\begin{aligned} p_{\Lambda}^{\omega}(t,0,x) &\geq \frac{c\,s^{-dk/2}\prod_{j=1}^{k-1}|B_{x_j}|\big\|\Lambda^{\omega}\big\|_{1,B_{x_j}}}{\exp(c\,k)} \\ &\geq \frac{c\,r^{-dk}r^{d(k-1)}\prod_{j=1}^{k-1}\big\|\Lambda^{\omega}\big\|_{1,B_{x_j}}}{c^k}. \end{aligned} \tag{3.20}$$

To bound the remaining stochastic term in the numerator we apply the harmonic-geometric mean inequality,

$$\Big(\prod_{j=1}^{k-1}\big\|\Lambda^{\omega}\big\|_{1,B_{x_j}}\Big)^{\frac{1}{k-1}} \geq \frac{k-1}{\sum_{j=1}^{k-1}\big\|\Lambda^{\omega}\big\|_{1,B_{x_j}}^{-1}} \geq \frac{c(k-1)}{\sum_{j=1}^{k-1}\big\|\lambda^{\omega}\big\|_{1,B_{x_j}}}. \tag{3.21}$$

Since $r > N_4^{\omega}$, it follows from Proposition 3.4 with the choice $y_j = x_j$, that $\sum_{j=1}^{k-1}\big\|\lambda^{\omega}\big\|_{1,B_{x_j}} \leq c\,k$. Therefore,

$$\prod_{j=1}^{k-1}\big\|\Lambda^{\omega}\big\|_{1,B_{x_j}} \geq c^k. \tag{3.22}$$

Combining (3.20) and (3.22) gives for some $c_{20} > 0$, $c_{21} \in (0,1)$,

$$p_{\Lambda}^{\omega}(t,0,x) \geq c_{20}\,r^{-d}\,c_{21}^k. \tag{3.23}$$

Finally, since $\frac{|x|^2}{t} \geq \frac{1}{4}$ we have $r \leq 2\,t^{1/2}$. Also $k \simeq \frac{|x|}{r} = \frac{|x|^2}{t}$ so we arrive at

$$p_{\Lambda}^{\omega}(t,0,x) \geq c_2\,t^{-\frac{d}{2}}\exp\Big(-\frac{c_3\,d(0,x)^2}{t}\Big), \tag{3.24}$$

which completes the proof. □

4 The Green Function Scaling Limit

We shall now prove the Green function scaling limit in Theorem 1.9. The strategy is to apply the local limit theorem [18, Theorem 1.1], then control remainder terms using the off-diagonal estimate of Theorem 1.3 and the long range bound established below in Proposition 4.2. Throughout this section, suppose Assumptions 1.1, 1.2 and 1.8 hold. Also, let $d \geq 3$ so that the Green function exists and let $p,\ q \in (1,\infty]$ satisfying $\frac{1}{p-1} + \frac{1}{q} < \frac{2}{d}$.

Herein, since the Green function is independent of the choice of speed measure, we specify the case $\theta^\omega \equiv 1$. We denote the corresponding heat kernel $p^\omega(\cdot,\cdot,\cdot)$, and the intrinsic metric $d^\omega(\cdot,\cdot)$. This choice is analogous to the variable speed random walk in the random conductance model setting, cf. for example [2]. The benefit of this speed measure is that it is clearly uniformly bounded, which assists with deriving the pointwise estimate in Proposition 4.2.

Corollary 4.1 *Suppose $M_2(p,q) < \infty$. For $\mathbb{P}$-a.e. ω, there exist $N_6^\omega(x) > 0$ and $c_{22}(d,p,q)$, $c_{23}(d,p,q) > 0$ such that for all $x,\ y \in \mathbb{R}^d$, $\sqrt{t} > N_6^\omega(x)$,*

$$p^\omega(t,x,y) \le c_{22}\, t^{-\frac{d}{2}} \exp\Big(-c_{23}\frac{d(x,y)^2}{t}\Big). \tag{4.1}$$

Proof Note that $M_2(p,q) < \infty$ implies $M_1(p,q,r) < \infty$ with $r = \infty$ if $\theta^\omega \equiv 1$. So by Theorem 1.3, we have that $\mathbb{P}$-a.s., there exists $N_6^\omega(x) > 0$ such that if $\sqrt{t} > N_6^\omega(x)$, then for all $x, y \in \mathbb{R}^d$,

$$p^\omega(t,x,y) \le c\, t^{-\frac{d}{2}}\Big(1+\frac{d(x,y)}{\sqrt{t}}\Big)^\gamma \exp\Big(-\frac{d^\omega(x,y)^2}{8t}\Big).$$

Assumption 1.8 gives that $d^\omega(x,y) \ge c\, d(x,y)$ for all $x,\ y \in \mathbb{R}^d$, we then conclude by absorbing the polynomial pre-factor into the exponential term. □

Whilst the above off-diagonal estimate provides optimal bounds on the heat kernel for large enough time t, it is clear that to control the convergence in (1.14) we also require a bound on the rescaled heat kernel that holds for small $t > 0$. We obtain this from the following long range bound, derived in a similar fashion to results in the graph setting such as [22, Theorem 10]. Interestingly, we obtain stronger decay in the present diffusion setting than for the aforementioned random walks on graphs [22, 30], where a logarithm appears in the exponent. See also [4, Theorem 1.6(ii)] for the degenerate environment.

Proposition 4.2 *Suppose $M_2(p,q) < \infty$. There exists $c_{24} > 0$ such that for all $t \ge 0$, $n \ge 1$ and $x \in \mathbb{R}^d$ with $|x| \le 2$, we have*

$$p^\omega(t,0,nx) \le \exp\Big(-c_{24}\frac{n^2|x|^2}{t}\Big), \qquad \mathbb{P}\text{-a.s.} \tag{4.2}$$

Proof Firstly note that by Lemma 2.2, for any $f \in L^2(\mathbb{R}^d)$ and suitable ψ,

$$\big\|e^\psi P_t f\big\|_2^2 \le e^{2h^\omega(\psi)^2 t}\big\|e^\psi f\big\|_2^2. \tag{4.3}$$

By the local boundedness in Assumption 1.1, this implies the pointwise estimate

$$e^{2\,\psi(x)}p^\omega(t,x,y)^2 \le e^{2h^\omega(\psi)^2 t+2\,\psi(y)}, \tag{4.4}$$

for all $t \ge 0$ and $x,\ y \in \mathbb{R}^d$. Rearranging,

$$p^\omega(t,x,y) \le \exp\big(h^\omega(\psi)^2 t + \psi(y) - \psi(x)\big). \tag{4.5}$$

Arguing as in Section 2.4, then applying the bound in Assumption 1.8 gives

$$p^{\omega}(t,x,y)\le \exp\Big(-\frac{c\,d(x,y)^2}{t}\Big).$$

Setting $x=0$ and re-labelling $y=nx$ with $|x|\le 2$ gives the result

$$p^{\omega}(t,0,nx)\le \exp\Big(-\frac{c\,n^2|x|^2}{t}\Big). \tag{4.6}$$

□

Proof of Theorem 1.9 By shift-invariance of the environment it suffices to prove the result for $x_0=0$. For simplicity we set $r_1=1$, $r_2=2$, and in a slight abuse of notation we write $k_t^{\Sigma}(x)=k_t^{\Sigma}(0,x)$. For $1\le|x|\le 2$, $T_1,\ T_2>0$ and $n>0$ we have

$$|n^{d-2}g^{\omega}ega(0,nx)-g_{BM}(0,x)|=\Big|n^d\int_0^{\infty}p^{\omega}(n^2t,0,nx)\,dt-\int_0^{\infty}k_t^{\Sigma}(x)\,dt\Big|$$
$$\le n^d\int_0^{T_1}p^{\omega}(n^2t,0,nx)\,dt+\int_0^{T_1}k_t^{\Sigma}(x)\,dt+\int_{T_1}^{T_2}\big|n^dp^{\omega}(n^2t,0,nx)-k_t^{\Sigma}(x)\big|\,dt$$
$$+n^d\int_{T_2}^{\infty}p^{\omega}(n^2t,0,nx)\,dt+\int_{T_2}^{\infty}k_t^{\Sigma}(x)\,dt. \tag{4.7}$$

In controlling these terms we first employ the main result of this paper; the off-diagonal estimate in Corollary 4.1 gives

$$n^d\int_{T_2}^{\infty}p^{\omega}(n^2t,0,nx)\,dt\le c_{22}\int_{T_2}^{\infty}t^{-d/2}\mathrm{e}^{-c_{23}/t}\,dt, \tag{4.8}$$

provided $n>\sqrt{N_6^{\omega}(0)/T_2}$. Similarly, for the Gaussian heat kernel there exists $c>0$ such that for all $t\ge 0$ and $1\le|x|\le 2$,

$$k_t^{\Sigma}(x)\le c\,t^{-d/2}\mathrm{e}^{-c/t}. \tag{4.9}$$

For the first term in (4.7) we apply both the off-diagonal estimate and the long range bound of Proposition 4.2. Provided $n>N_6^{\omega}(0)/\sqrt{T_1}$, we have

$$n^d\int_0^{T_1}p^{\omega}(n^2t,0,nx)\,dt\le n^d\int_0^{\frac{N_6^{\omega}(0)^2}{n^2}}p^{\omega}(n^2t,0,nx)\,dt+n^d\int_{\frac{N_6^{\omega}(0)^2}{n^2}}^{T_1}p^{\omega}(n^2t,0,nx)\,dt$$
$$\le\ n^d\int_0^{\frac{N_6^{\omega}(0)^2}{n^2}}\mathrm{e}^{-\frac{c}{t}}\,dt+n^d\int_{\frac{N_6^{\omega}(0)^2}{n^2}}^{T_1}c\,n^{-d}t^{-\frac{d}{2}}\mathrm{e}^{-\frac{c}{t}}\,dt\quad\text{(by (4.2) and (4.1) resp.)}$$
$$\le\ N_6^{\omega}(0)^2n^{d-2}\exp\big(-c\,n^2/N_6^{\omega}(0)^2\big)+c\int_0^{T_1}t^{-\frac{d}{2}}\mathrm{e}^{-\frac{c}{t}}\,dt. \tag{4.10}$$

Let $\epsilon>0$. Combining the above we have that for suitably large n,

$$|n^{d-2}g^{\omega}ega(0,nx)-g_{BM}(0,x)|\le c\Big(\int_0^{T_1}t^{-d/2}\mathrm{e}^{-c/t}\,dt+\int_{T_2}^{\infty}t^{-d/2}\mathrm{e}^{-c/t}\,dt$$
$$+N_6^{\omega}(0)^2n^{d-2}\exp\big(-c\,n^2/N_6^{\omega}(0)^2\big)+\int_{T_1}^{T_2}\big|n^dp^{\omega}(n^2t,0,nx)-k_t^{\Sigma}(x)\big|\,dt\Big). \tag{4.11}$$

Now, $t^{-d/2}e^{-c/t}$ is integrable on $(0, \infty)$, so we may fix T_1, T_2 such that

$$\int_0^{T_1} t^{-d/2}e^{-c_2/t}\,dt + \int_{T_2}^{\infty} t^{-d/2}e^{-c_2/t}\,dt < \epsilon.$$

For large enough n,

$$N_6^{\omega}(0)^2 n^{d-2} \exp\big(-c\,n^2/N_6^{\omega}(0)^2\big) < \epsilon.$$

Furthermore, by the local limit theorem, see [18, Theorem 1.1] and [18, Remark 1.2],

$$\int_{T_1}^{T_2} \big|n^d p^{\omega}(n^2 t, 0, nx) - k_t^{\Sigma}(x)\big|\,dt < \epsilon,$$

for large enough n, uniformly over $x \in A$. This gives the claim. □

Acknowledgements With many thanks to Sebastian Andres for valuable discussions and guidance on this project.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Competing interests The author has no competing interests to declare that are relevant to the content of this article.

References

1. Andres, S., Chiarini, A., Slowik, M.: Quenched local limit theorem for random walks among time-dependent ergodic degenerate weights. Probab. Theory Related Fields **179**(3-4), 1145–1181 (2021)
2. Andres, S., Deuschel, J.-D., Slowik, M.: Invariance principle for the random conductance model in a degenerate ergodic environment. Ann. Probab. **43**(4), 1866–1891 (2015)
3. Andres, S., Deuschel, J.-D., Slowik, M.: Harnack inequalities on weighted graphs and some applications to the random conductance model. Probab. Theory Related Fields **164**(3-4), 931–977 (2016)
4. Andres, S., Deuschel, J.-D., Slowik, M.: Heat kernel estimates for random walks with degenerate weights. Electron. J. Probab. **21**(33), 21 (2016)
5. Andres, S., Deuschel, J.-D., Slowik, M.: Heat kernel estimates and intrinsic metric for random walks with general speed measure under degenerate conductances. Electron. Commun. Probab. **24**(5), 17 (2019)
6. Andres, S., Halberstam, N.: Lower Gaussian heat kernel bounds for the random conductance model in a degenerate ergodic environment. Stochastic Process. Appl. **139**, 212–228 (2021)
7. Andres, S., Prévost, A.: First passage percolation with long-range correlations and applications to random schrödinger operators. arXiv:2112.12096 (2021)
8. Andres, S., Taylor, P.A.: Local limit theorems for the random conductance model and applications to the Ginzburg-Landau $\nabla\phi$ interface model. J. Stat. Phys. **182**(2), 35 (2021)
9. Ba, M., Mathieu, P.: A Sobolev inequality and the individual invariance principle for diffusions in a periodic potential. SIAM J. Math. Anal. **47**(3), 2022–2043 (2015)

10. Barlow, M.T.: Random walks on supercritical percolation clusters. Ann. Probab. **32**(4), 3024–3084 (2004)
11. Bella, P., Chiarini, A., Fehrman, B.: A Liouville theorem for stationary and ergodic ensembles of parabolic systems. Probab. Theory Related Fields **173**(3-4), 759–812 (2019)
12. Bella, P., Fehrman, B., Otto, F.: A Liouville theorem for elliptic systems with degenerate ergodic coefficients. Ann. Appl Probab. **28**(3), 1379–1422 (2018)
13. Bella, P., Schäffner, M.: Local boundedness and Harnack inequality for solutions of linear nonuniformly elliptic equations. Comm. Pure Appl. Math. **74**(3), 453–477 (2021)
14. Bella, P., Schäffner, M.: Non-uniformly parabolic equations and applications to the random conductance model. Probab. Theory Related Fields **182**(1-2), 353–397 (2022)
15. Boutet de Monvel, A., Lenz, D., Stollmann, P.: Sch'nol's theorem for strongly local forms. Israel J. Math. **173**, 189–211 (2009)
16. Burtscher, A.Y.: Length structures on manifolds with continuous Riemannian metrics. New York J. Math. **21**, 273–296 (2015)
17. Carlen, E.A., Kusuoka, S.: D. W. Stroock. Upper bounds for symmetric Markov transition functions. Ann. Inst. H. Poincaré Probab. Statist. **23**(2), 245–287 (1987)
18. Chiarini, A., Deuschel, J.-D.: Local central limit theorem for diffusions in a degenerate and unbounded random medium. Electron. J. Probab. **20**(112), 30 (2015)
19. Chiarini, A., Deuschel, J.-D.: Invariance principle for symmetric diffusions in a degenerate and unbounded stationary and ergodic random medium. Ann. Inst. Henri Poincaré, Probab. Stat. **52**(4), 1535–1563 (2016)
20. Davies, E.B.: Explicit constants for Gaussian upper bounds on heat kernels. Amer. J. Math. **109**(2), 319–333 (1987)
21. Davies, E.B.: Heat volume 92 of cambridge tracts in mathematics. Cambridge University Press, Cambridge (1989)
22. Davies, E.B.: Large deviations for heat kernels on graphs. J. London Math. Soc. (2) **47**(1), 65–72 (1993)
23. Delmotte, T.: Parabolic Harnack inequality estimates of Markov chains on graphs. Rev. Mat Iberoamericana **15**(1), 181–232 (1999)
24. Diaconis, P., Saloff-Coste, L.: Logarithmic Sobolev inequalities for finite Markov chains. Ann. Appl. Probab. **6**(3), 695–750 (1996)
25. Fabes, E.B., Stroock, D.W.: A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. Arch Rational Mech. Anal. **96**(4), 327–338 (1986)
26. Fannjiang, A., Komorowski, T.: A martingale approach to homogenization of unbounded random flows. Ann Probab. **25**(4), 1872–1894 (1997)
27. Gerard, T.: Representations of the vertex reinforced jump process as a mixture of Markov processes on $\mathbb{Z}^d$ and infinite trees. Electron. J. Probab. **25**(108), 45 (2020)
28. Grigor'yan, A., Telcs, A.: Two-sided estimates of heat kernels on metric measure spaces. Ann. Probab. **40**(3), 1212–1284 (2012)
29. Osada, H.: Homogenization of diffusion processes with random stationary coefficients. In: Probability theory and mathematical statistics (Tbilisi, 1982), volume 1021 of Lecture Notes in Math, pp. 507–517. Springer, Berlin (1983)
30. Pang, M.M.H.: Heat kernels of graphs. J. London Math. Soc. (2) **47**(1), 50–64 (1993)
31. Papanicolaou, G.C., Varadhan, S.R.S.: Boundary value problems with rapidly oscillating random coefficients. In: Random fields, Vol. I, II (Esztergom, 1979), volume 27 of Colloq. Math. Soc. János Bolyai, pp. 835–873. North-Holland, Amsterdam-New York (1981)
32. Rosenthal, H.P.: On the subspaces of $L^p (p > 2)$ spanned by sequences of independent random variables. Israel J. Math. **8**, 273–303 (1970)
33. Stollmann, P.: A dual characterization of length spaces with application to Dirichlet metric spaces. Studia Math. **198**(3), 221–233 (2010)
34. Stroock, D.W.: Diffusion semigroups corresponding to uniformly elliptic divergence form operators. In: Séminaire de Probabilités, XXII, volume 1321 of Lecture Notes in Math, pp. 316–347. Springer, Berlin (1988)
35. Sturm, K.-T.: On the geometry defined by Dirichlet forms. In: Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993), volume 36 of Progr. Probab., pp. 231–242. Birkhäuser, Basel (1995)
36. Zhikov, V.V.: Estimates of nash-Aronson type for degenerate parabolic equations. Sovrem. Mat. Fundam. Napravl. **39**, 66–78 (2011)