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Some convergences results on the stochastic Cahn-Hilliard-Navier-Stokes equations with multiplicative noise

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Abstract

In this paper, we prove that the sequence (u_n, ϕ_n) of the Galerkin approximation of the solution (u, ϕ) to a stochastic 2D Cahn-Hilliard-Navier-Stokes model verifies the following convergence result

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \tilde{\psi}\left(\|(u_n(t)), \phi_n(t) - (u(t), \phi(t))\|_{\mathbb{V}}^2\right)\right] = 0$$

for any deterministic time $T > 0$ and for a specified moment function $\tilde{\psi}(x)$. Also, we provide a result on uniform boundedness of the moment

$$\mathbb{E} \sup_{t\in[0,T]} \psi(\|(u(t), \phi(t))\|_{\mathbb{V}}^2)$$

where ψ grows as a single logarithm at infinity and furthermore, we establih the results on convergence of the Galerkin approximation up to a deterministic time T when the $\mathbb{V}$-norm is replaced by the $\mathbb{H}$-norm.

Keywords Stochastic · Cahn-Hilliard equations · Navier-Stokes equations · Strong solution · Galerkin approximation

Mathematics Subject Classification (2010) 35R60 · 35Q35 · 60H15 · 76M35 · 86A05

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1 Introduction

In a series of recent papers (see [6–8, 15, 18]) the following nonlinear evolution system has been analyzed

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = \mathcal{K}\mu\nabla\phi + g_1(t,u) + g_2(t,u)\dot{W}_t, \\ \nabla\cdot u = 0, \\ \partial_t\phi + u\cdot\nabla\phi - \varrho_0\Delta\mu = 0, \\ \mu = -\epsilon\Delta\phi + \alpha f(\phi), \\ u = 0,\ \frac{\partial\mu}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}\times(0,T), \\ u(0) = u_0,\ \phi(0) = \phi_0 \text{ in } \mathcal{M}, \end{cases} \tag{1.1}$$

on a bounded domain $\mathcal{M} \subset \mathbb{R}^d$, $d = 2, 3$, for $t \in (0, T)$, and where η is the unit outward normal to the boundary $\partial\mathcal{M}$ and $T > 0$ is a fixed time. This system is the stochastic version of the well-known Cahn-Hilliard-Navier-Stokes system, which is based on a well-known diffuse interface model and which describes the evolution of an incompressible isothermal mixture of binary fluids. The system consits of the Navier-Stokes equations (NSE) for the fluid velocity u coupled with a convective Cahn-Hilliard (CH) equation for the order (phase) parameter ϕ (i.e. the relative concentration of one fluid or the difference of the two concentrations). Here p, $u = (u_1, u_2)$ and ϕ denote the pressure, the velocity and the order (phase) parameter, respectively; $g_1(t, u)$ is an external volume force applied to the binary mixture fluid and $g_2(t, u)\dot{W}_t$ represents random external forces depending eventually on u, where $\dot{W}_t$ denotes the time derivative of a cylindrical Wiener process; ν, ϱ_0 and $\mathcal{K}$ are positive constants that correspond to the kinematic viscosity of fluid, mobility constant and capillarity (stress) coefficient, respectively, and we assume the density equal to one. Also, the quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_{\mathcal{M}} \left[\frac{\epsilon}{2}|\nabla\phi|^2 + \alpha F(\phi)\right] dx,$$

where, e.g., $F(r) = \int_0^r f(\zeta)d\zeta$, is a suitable double-well potential, and $\epsilon, \alpha > 0$ are two positive parameters describing the interactions between the two phases. In particular, ϵ is related to the thickness of the interface separating the two fluids. We note that the no flux boundary condition for the chemical potential μ yields the conservation of the following quantity

$$\langle\phi(t)\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} \phi(t,x)dx, \tag{1.2}$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. More precisely, we get from Eq. 1.1_3 that

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle \text{ for all } t \geq 0.$$

The addition of the white noise driven terms to the basic governing equations is natural for both practical and theoretical applications. Such stochastically forced terms are used to account for numerical and empirical uncertainties and have been proposed as a model for turbulence.

As far as the stochastic CH-NSE (1.1) is concerned, there are very few work about his solvability (see for instance [6–8, 15, 18, 19] and the references therein). In [18], the third author of the present paper proved the existence and uniqueness of the probabilistic strong solution to a stochastic 2D CH-NSE. The proof is based on Galerkin approximation and the principle of weak convergence in functional analysis. Moreover, he showed that the Galerkin approximation converges in means square to the probabilistic strong solution. The paper [15] concerns the existence of a random attractor for the stochastic 2D CH-NSE. The authors in [7, 8] have also studied the stochastic 3D globally modified CH-NSE. In [7], we proved the existence of the unique strong solution (in probability and partial differential equations senses), and used the limiting argument to derive the existence of a global weak martingale solution for the stochastic CH-NSE. In [8], we studied the stability of the solution of the stochastic 3D globally modified CH-NSE. In particular, we proved that under some conditions on the forcing terms, the solution converges exponentially in the mean square and almost surely exponentially to the stationary solution. In [6] recently, we prove the existence and uniqueness of a local maximal strong solution of problem (1.1) when the initial data (u_0, ϕ_0) takes values in $H^1 \times H^2$ and particularly in the two-dimensional case, we prove global existence of the solutions. The proof is based on a finite dimensional approximations, decomposition into high low modes and pairwise comparison techniques for the solvability of the stochastic NSE.

When $\phi = 0$, Eq. 1.1 becomes the stochastic system of compressible or incompressible NSE. Since the work of Bensoussan and Temam [1], there have been numeros studies on the existence and uniqueness of solutions for the the stochastic NSE in the literature. Here we only mention some of them related to our study of the convergence properties of the Galerkin approximation to the stochastic CH-NSE. In [3], the author proved that the solutions u of the stochastic NSE can be approximated by solutions u_n of the corresponding Galerkin systems. More precisely, she proved that for all $t > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\|u_n(t) - u(t)\|^2_{H_1} + \int_0^t \|u_n(s) - u(s)\|^2_{V_1} ds\right] = 0,$$

where the space H_1 and V_1 are defined as in Eq. 2.3 below. Her results can be extended to the case of stronger norms, in the absence of boundaries conditions. Namely, using the cancellation property

$$(B_0(u, u), A_0 u) = 0 \tag{1.3}$$

where A_0 is the Stokes operator and B_0 is the bilinear form defined in Eqs. 2.4 and 2.9, respectively below, which is valid in the case of periodic boundary conditions, one can obtain a stronger convergence result

$$\lim_{n\to\infty} \mathbb{E}\left[\|u_n(t) - u(t)\|^2_{V_1} + \int_0^t \|u_n(s) - u(s)\|^2_{H^2} ds\right] = 0,$$

under suitable assumptions on the noise. However, the finiteness of the expected value of the second moment of the norm $\|u(t)\|^2_{V_1}$ for any fixed non-random time t is an open problem. By the same token, it is not known whether the expected value of the supremum of $\|u_n(t) - u(t)\|^2_{V_1}$ up to a deterministic time convergenges to 0 as $n \to \infty$. A positive result in this direction was obtained in [13], where the authors proved that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \psi\left(\|u(t)\|^2_{V_1}\right)\right] < \infty \tag{1.4}$$

where

$$\psi(\tau) = \log\left(1 + \log(1 + \tau)\right), \ \tau \in (0, \infty). \tag{1.5}$$

In [14], the authors proved that Eq. 1.4 holds with $\tilde{\psi}(\tau) = \log(1 + \tau)$ instead of ψ and they also obtained the convergence of the Galerkin approximation pointwise in time for the V_1-

norm in the case of the Dirichlet boundary conditions when the cancellation property (1.3) does not hold. More precisely, they proved that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \tilde{\psi}\left(\|u_n(t)-u(t)\|_{V_1}^2\right)^{1-\delta}\right] = 0, \text{ for all } \delta \in (0,1). \tag{1.6}$$

Motivated by the above work, we prove the convergence properties of the Galerkin approximation to the stochastic CH-NSE and obtain new estimates on the convergence in the strong norm. Namely, we prove that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \tilde{\psi}\left(\|(u(t),\phi(t))\|_{\mathbb{V}}^2\right)\right] < \infty$$

where $\tilde{\psi} = \log(1+\tau)$, $\tau \in (0,\infty)$ and $\mathbb{V}$ is defined as in Eq. 2.14 below; and that the following convergence holds

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \tilde{\psi}\left(\|(u_n(t),\phi_n(t))-(u(t),\phi(t))\|_{\mathbb{V}}^2\right)^{1-\delta}\right] = 0, \text{ for all } \delta \in (0,1).$$

The exposition is organized as follows. In Section 2, we present the mathematical setting of our model, the stochastic framework and various notions of solutions. In Section 3, we present our main results on the convergence of the Galerkin approximations in the $\mathbb{V}$-norm and on the finiteness of the logarithmic moment functions. In this section, we also gived (cf. Proposition 3.1 below) some generalized estimates on the $\mathbb{H}$-norm for the Galerkin sequence (u_n,ϕ_n) and anothers generalized estimates on the $\mathbb{H}$-norm for the solutions (u,ϕ) of problem (1.1) instead of the estimates obtain in [6, Lemma 4.2]. In Proposition 3.2 on the same section, we summarize results on convergence when the $\mathbb{V}$-norm is replaced by the $\mathbb{H}$-norm defined in Eq. 2.12 below. Finally, Section 4 is devoted to the proof of the convergence of the Galerkin approximation to the original solutions.

2 Functional Setting of the Equation

2.1 Deterministic Framework

Without loss of generality, we set $\varrho_0 = 1$. Hereafter, we assume that $\mathcal{M}$ is a bounded open connected subset of $\mathbb{R}^2$ with a smooth boundary $\partial\mathcal{M}$ (e.g. of class $\mathcal{C}^2$). We also assume that $f \in \mathcal{C}^2(\mathbb{R})$ satisfies

$$\begin{cases} \liminf_{|r|\to+\infty} f'(r) > 0, \\ |f^{(i)}(r)| \le c_f(1+|r|^{\kappa+1-i}), \ \forall r \in \mathbb{R}, \ i = 0,1,2, \end{cases} \tag{2.1}$$

where c_f is some positive constant and $\kappa \in [2,+\infty)$ is fixed. Thanks to Eq. 2.1, we have

$$|f'(r)| \le c_f(1+|r|^{\kappa}), \ |f(r)| \le c_f(1+|r|^{\kappa+1}), \ \forall r \in \mathbb{R}. \tag{2.2}$$

Note that the derivative of the typical double-well potential f satisfies (2.1) with $\kappa = 2$.

Let us now introduce the functional setup of Eq. 1.1. If X is real Hilbert space, we denote by X' its dual.

We consider the Hilbert spaces

$$
\begin{aligned}
H_1 &:= \overline{\left\{u \in \mathcal{C}_c^\infty((\mathcal{M}))^2 : \mathrm{div} u = 0 \text{ in } \mathcal{M}\right\}}^{\mathbb{L}^2}, \\
V_1 &:= \overline{\left\{u \in \mathcal{C}_c^\infty((\mathcal{M}))^2 : \mathrm{div} u = 0 \text{ in } \mathcal{M}\right\}}^{\mathbb{H}_0^1},
\end{aligned}
\tag{2.3}
$$

where $\mathbb{L}^2(\mathcal{M}) := (L^2(\mathcal{M}))^2$ and $\mathbb{H}_0^1(\mathcal{M}) := (H_0^1(\mathcal{M}))^2$. On H_1 we take the L^2 inner product and norm

$$
(u, v) := \int_{\mathcal{M}} u(x).v(x)\, dx, \ |u| := (u, u)^{1/2}.
$$

Moreover, the space V_1 is endowed with the scalar product and norm

$$
((u, v)) := \sum_{i=1}^{d} (\partial_{x_i} u, \partial_{x_i} v), \ \|u\| = ((u, u))^{1/2}.
$$

The norm in V_1 is equivalent to the $\mathbb{H}^1(\mathcal{M})$-norm (due to Poincaré's inequality). We refer the reader to [20] for more details on these spaces.

We now define the operator A_0 by

$$
A_0 u = -\mathcal{P}\Delta u, \ \forall u \in D(A_0) = \mathbb{H}^2(\mathcal{M}) \cap V_1,
\tag{2.4}
$$

where $\mathcal{P}$ is the Leray-Helmholtz projector in $\mathbb{L}^2(\mathcal{M})$ onto H_1. It is well-known that, see e.g. Constantin [5, p.33] or Temam [21, p.36], that A_0 is a non-negative self adjoint operator in H_1. Moreover, see [21, p.57], $V_1 = D(A_0^{1/2})$. Furthermore, A_0^{-1} is a compact linear operator on H_1 and by the classical spectral theorem, there exists a sequence λ_j with $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_n \leq \lambda_{n+1} \leq ...$ and a family $w_j \in D(A_0)$ which is an orthonormal basis in H_1 and such that $A_0 w_j = \lambda_j w_j$.

We introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$

$$
A_1 \varphi = -\Delta\varphi, \ \forall \varphi \in D(A_1) = \left\{\varphi \in H^2(\mathcal{M}), \ \partial_\eta \varphi = 0, \text{ on } \partial\mathcal{M}\right\},
\tag{2.5}
$$

and we endow $D(A_1)$ with the norm $|A_1 \cdot| + |\langle\cdot\rangle|$, which is equivalent to the H^2-norm. We also define the linear positive unbounded operator on the Hilbert space $L^2_{(0)}(\mathcal{M})$ of the L^2-functions with null mean

$$
B_n \varphi = -\Delta\varphi, \ \forall \varphi \in D(B_n) = D(A_1) \cap L^2_{(0)}(\mathcal{M}).
\tag{2.6}
$$

Note that B_n^{-1} is a compact linear operator on $L^2_{(0)}(\mathcal{M})$. More generally, we can define B_n^s, for any $s \in \mathbb{R}$, noting that $|B_n^{s/2} \cdot|$, $s > 0$, is an equivalent norm to the canonical H^s-norm on $D(B_n^{s/2}) \subset H^s(\mathcal{M}) \cap L^2_{(0)}(\mathcal{M})$. Also note that $A_1 = B_n$ on $D(B_n)$. If φ is such that $\varphi - \langle\varphi\rangle \in D(B_n^{s/2})$, we have that $|B_n^{s/2}(\varphi - \langle\varphi\rangle)| + |\varphi - \langle\varphi\rangle|$ is equivalent to the H^s-norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))'$, whenever $s < 0$.

We note that

$$
D(A_1) = \left\{\phi \in H^2(\mathcal{M}) : \frac{\partial\phi}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}\right\}, A_1\phi = -\sum_{l=1}^{2} \frac{\partial^2\phi}{\partial x_j^2}, \ \phi \in D(A_1).
\tag{2.7}
$$

Classically, there exists a sequence β_j with $0 < \beta_1 < \beta_2 \leq \cdots \leq \beta_n \leq \beta_{n+1} \leq \cdots$ and a family $\psi_j \in D(A_1)$ which is an orthonormal basis in $H_2 = L^2(\mathcal{M})$ and such that $A_1\psi_j = \beta_j\psi_j$.

Hereafter, we set

$$
H_2 = L^2(\mathcal{M}), \ V_2 = H^1(\mathcal{M}), \ H = H_1 \times H_2, \ V = V_1 \times V_2.
\tag{2.8}
$$

In order to define the variational setting for the Cahn-Hilliard-Navier-Stokes equation (1.1), we introduce the following bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0 which are defined, from $D(A_0) \times D(A_0)$ into H_1, $D(A_0) \times D(A_1)$ into H_2 and $H_2 \times D(A_1^{3/2})$ into H_1, respectively. More precisely, we set

$$
\begin{aligned}
(B_0(u,v),w) &= \int_{\mathcal{M}} [(u\cdot\nabla)v]w\,dx \\
&= b_0(u,v,w),\ \forall u,v,w \in D(A_0),\\
(B_1(u,\varphi),\psi) &= \int_{\mathcal{M}} [(u\cdot\nabla)\varphi]\psi\,dx \\
&= b_1(u,\varphi,\psi),\ \forall u \in D(A_0), \varphi,\psi \in D(A_1),\\
(R_0(\mu,\varphi),w) &= \int_{\mathcal{M}} [\mu\nabla\varphi]w\,dx \\
&= b_1(w,\varphi,\mu),\ \forall w \in D(A_0),\ (\mu,\varphi) \in H_2 \times D(A_1^{3/2}).
\end{aligned}
\tag{2.9}
$$

Note that

$$
\begin{aligned}
R_0(\mu,\varphi) &= \mathcal{P}\mu\nabla\varphi,\ \forall(\mu,\varphi) \in H_2 \times D(A_1^{3/2}),\\
b_0(u,v,v) &= 0,\ \forall u,v \in V_1,\\
b_1(v,\phi,\phi) &= 0,\ \forall v \in V_1,\ \phi \in V_2,\\
\langle R_0(\epsilon A_1\phi,\phi),v\rangle &= \langle B_1(v,\phi),\mu\rangle = \langle B_1(v,\phi),\epsilon A_1\phi\rangle\\
&= b_1(v,\phi,\epsilon A_1\phi),\ \forall(v,\phi) \in V_1 \times D(A_1).
\end{aligned}
$$

Here $\mu = \epsilon A_1\phi + \alpha f(\phi)$.

We recall that (due to the mass conservation) we have

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle =: M_0,\ \forall t \geq 0. \tag{2.10}$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of ϕ is zero a the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle = 0,\ \forall t > 0. \tag{2.11}$$

We set

$$\mathbb{H} = H_1 \times D(A_1^{1/2}). \tag{2.12}$$

The norm in $D(A_1^{1/2})$ is denoted by $\|\cdot\|$, where $\|\psi\|^2 = |A_1^{1/2}\psi|^2$. The space $\mathbb{H}$ is a complete metric space with respect to the metric associated with the norm

$$|(v,\psi)|_{\mathbb{H}}^2 = \mathcal{K}^{-1}|v|^2 + \epsilon\|\psi\|^2. \tag{2.13}$$

We define the Hilbert space $\mathbb{V}$ by

$$\mathbb{V} = V_1 \times D(A_1), \tag{2.14}$$

endowed with the scalar product whose associated norm is

$$\|(v,\psi)\|_{\mathbb{V}}^2 = \|v\|^2 + |A_1\psi|^2. \tag{2.15}$$

Hereafter, for any $(w,\psi) \in \mathbb{H}$, we set

$$\mathcal{E}_{tot}(w,\psi) = \mathcal{K}^{-1}|u|^2 + \epsilon\|\psi\|^2 + 2\alpha\,\langle F(\psi),1\rangle + c_1, \tag{2.16}$$

where $c_1 > 0$ is a constant large enough and independent on (w,ψ) such that $\mathcal{E}_{tot}(w,\psi)$ is non-negative.

We can check that (see [11]) there exists a monotone non-decreasing function $\tilde{\mathcal{Q}}_0$ (independent on time and the initial condition) such that

$$|(w,\psi)|_{\mathbb{H}}^2 \leq \mathcal{E}_{tot}(w,\psi) \leq \tilde{\mathcal{Q}}_0(|(w,\psi)|_{\mathbb{H}}^2),\ \forall(w,\psi) \in \mathbb{H}. \tag{2.17}$$

Using the notations above, we rewrite problem (1.1) as:

$$\begin{cases} \frac{du}{dt} + \nu A_0 u + B_0(u,u) - \mathcal{K} R_0(\epsilon A_1\phi, \phi) = g_1(t,u) + g_2(t,u)\dot{W}_t, \text{ in } V_1', \\ \frac{d\phi}{dt} + A_1\mu + B_1(u,\phi) = 0, \text{ in } V_2', \\ \mu = \epsilon A_1\phi + \alpha f(\phi), \\ (u,\phi)(0) = (u_0,\phi_0). \end{cases} \tag{2.18}$$

Remark 2.1 In the weak formulation (2.18), the term $\mu\nabla\phi$ is replaced by $\epsilon A_1\phi\nabla\phi$. This is justified since $f(\phi)\nabla\phi$ is the gradient of $F(\phi)$ and can be incorporated into the pressure gradient, see [11] for details.

2.2 Stochastic Framework and Notions of a Solution

In order to define the stochastic terms in Eq. 2.18, that is $g_2(t,u)\dot{W}_t$, we first recall some basic notations and notions from stochastic analysis in Hilbert spaces, used here and after. For an extended treatment of this topic, we refer to [9, 12, 16]. Fix a stochastic basis

$$S = \left\{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{\beta_t^k, t \geq 0, k = 1, 2, ...\})\right\},$$

which consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a complete, right-continuous filtration, namely $\mathbb{P}(A) = 0 \Rightarrow A \in \mathcal{F}_0, \mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ and a sequence of mutually independent one dimensional Brownian motions β_t^k adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

Let $\mathcal{U}$ be an auxiliary separable real Hilbert space endowed with a Hilbert basis $\{e_k\}_{k\geq 1}$. We denote by $\{W_t : t \geq 0\}$ the cylindrical Wiener process with values in $\mathcal{U}$ defined formally as

$$W(t, ., \omega) := W_t(., \omega) = \sum_{j=1}^{\infty} \beta_t^j(\omega) e_j. \tag{2.19}$$

It is well known that this series does not converge in $\mathcal{U}$, but rather in any Hilbert space $\tilde{\mathcal{U}}$ such that $\mathcal{U} \subset \tilde{\mathcal{U}}$, being the injection of $\mathcal{U}$ in $\tilde{\mathcal{U}}$ Hilbert-Schmidt, see [9] for more details. Given a separable Hilbert space X, we denote by $L_2(\mathcal{U}, X)$ the space of Hilbert-Schmidt operators from $\mathcal{U}$ to X, endowed with the following inner product and norm

$$((R, Z))_{L_2(\mathcal{U},X)} = \sum_{k=1}^{\infty} \langle Re_k, Ze_k\rangle_X \text{ and } \|R\|^2_{L_2(\mathcal{U},X)} = \sum_{k=1}^{\infty} |Re_k|^2_X. \tag{2.20}$$

It is well-known that the definition of a Hilbert-Schmidt operator is independent of the choice of the orthonormal basis e_k.

Given an X-valued predictable process $\rho \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2_{loc}([0,\infty); L_2(\mathcal{U}, X)))$, the stochastic integral of ρ with respect to the cylindrical Wiener process W_t is denoted $M_t := \int_0^t \rho(s)dW_s$, and is defined as the unique continuous X-valued $\mathcal{F}_t$-martingale such that for all $z \in X$, we have

$$\left\langle \int_0^t \rho(s)dW_s, z\right\rangle_X = \sum_{k=1}^{\infty} \int_0^t \langle \rho(s)e_k, z\rangle_X \, d\beta_s^k,$$

where the integral with respect to $d\beta_s^k$ is understood in the sense of Itô, and the series converges in $L^2(\Omega, \mathcal{C}([0, T]))$. For more details on the general theory of infinite dimensional stochastic integration, the reader is referred to classical textbook such as [9, 16].

We recall the following definition of Lipschitz continuous functions.

Definition 2.1 Let X and Y be two Banach spaces. We say that a continuous function $h : [0, \infty) \times X \to Y$ is Lipschitz if

$$\|h(t, u_1) - h(t, u_2)\|_Y \le L_X \|u_1 - u_2\|_X, \text{ for all } t \ge 0, u_1, u_2 \in X, \tag{2.21}$$

for some positive constant L_X and independent of t.

We denote the collection of all such mappings $\mathrm{Lip}_u(X, Y)$.

For the analysis below we shall assume that

$$\begin{aligned} &g_1 : \Omega \times [0, \infty) \times V_1 \to H_1, \\ &g_1 \in Lip_u(H_1, V_1') \cap Lip_u(V_1, H_1) \\ &\text{and } g_1(., 0) \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2_{loc}([0, \infty); H_1)). \end{aligned} \tag{2.22}$$

Concerning the hypotheses for g_2, we assume that

$$\begin{aligned} &g_2 : \Omega \times [0, \infty) \times H_1 \to L_2(\mathcal{U}, H_1), \\ &g_2 \in Lip_u(H_1, L_2(\mathcal{U}, H_1)) \cap Lip_u(V_1, L_2(\mathcal{U}, V_1)) \cap Lip_u(D(A_0), L_2(\mathcal{U}, D(A_0))), \\ &g_2(., 0) \in \tilde{\mathcal{Z}}_0 \end{aligned} \tag{2.23}$$

with

$$\tilde{\mathcal{Z}}_0 = L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2_{loc}([0, \infty); L_2(\mathcal{U}, H_1))) \cap L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2_{loc}([0, \infty); L_2(\mathcal{U}, V_1))). \tag{2.24}$$

Remark 2.2 An example of operator g_2 satisfying conditions (2.23)–(2.24) is the operator defined as $g_2(., u)e_j = \tilde{g}_j(u)$, with $(\tilde{g}_j)_{j\in\mathbb{N}} \subset W^{1,\infty}(\mathbb{R})$ and such that $\sum_{j=1}^{\infty} \|\tilde{g}_j\|^2_{W^{1,\infty}(\mathbb{R})} < +\infty$. It is widely employed in literature (see for example [17] and referein therein). Here for example, $\{e_j\}_{j\ge 1}$ is an orthonormal basis in separable Hilbert space $\mathcal{U}$ having the following regularity $e_j \in H^1(\mathcal{M}) \cap L^\infty(\mathcal{M})$. The following external force $g_1(., u) = u$ satisfy the assumption (2.22).

Next, we introduce the concept of strong solutions of the stochastic Cahn-Hilliard-Navier-Stokes equations. Here the word "strong" should be understood in the PDE and probabilistic sense.

Definition 2.2 Let $f \in \mathcal{C}^2(\mathbb{R})$ satisfy (2.1). We assume that g_1 and g_2 are V_1' and $L_2(\mathcal{U}, H_1)$ valued, predictable processes respectively with g_1 and g_2 satisfying (2.22)–(2.23). We also assume that $g_1(t, 0) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; L^{(8\kappa+10)(\kappa+1)}(0, T; V_1'))$ and $g_2(t, 0) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; L^{(8\kappa+10)(\kappa+1)}(0, T; L_2(\mathcal{U}, H_1)))$. Assume that the initial data $(u_0, \phi_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{H}) \cap L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V})$ satisfies $\mathbb{E}[\mathcal{E}_{tot}(u_0, \phi_0)]^{(4\kappa+5)(\kappa+1)} < \infty$. The pair $\{(u, \phi), \tau\}$ is called a local strong solution of the system if τ is a strictly positive stopping time and $\{u(\cdot \wedge \tau), \phi(\cdot \wedge \tau)\}$ is a predictable process in $\mathbb{H}$ with

$$\begin{aligned} &(u(\cdot \wedge \tau), \phi(\cdot \wedge \tau)) \in L^2(\Omega, \mathcal{C}([0, \infty); V_1 \times D(A_1^{1/2}))), \\ &(u\mathbb{I}_{t\le\tau}, \phi\mathbb{I}_{t\le\tau}) \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2_{loc}([0, \infty); D(A_0) \times D(A_1^2)), \end{aligned} \tag{2.25}$$

and such that (u, ϕ) satisfies

$$
\begin{aligned}
&(u(t\wedge\tau), v) + \int_0^{t\wedge\tau} \langle \nu A_0 u + B_0(u,u) - \mathcal{K}R_0(\epsilon A_1\phi, \phi), v\rangle\, ds = (u_0, v) \\
&+ \int_0^{t\wedge\tau} \langle g_1(s,u), v\rangle\, ds + \sum_{k=1}^{\infty} \int_0^{t\wedge\tau} \langle g_2(s,u)e_k, v\rangle\, d\beta_s^k, \\
&\langle \phi(t\wedge\tau), \psi\rangle + \int_0^{t\wedge\tau} \langle A_1\mu + B_1(u,\phi), \psi\rangle\, ds = \langle \phi_0, \psi\rangle\,, \\
\mu \ &= \epsilon A_1\phi + \alpha f(\phi),
\end{aligned}
\tag{2.26}
$$

for all $(v, \phi) \in H_1 \times H_2$. Moreover, $\{(u, \phi), \xi\}$ is called a local maximal strong solution if ξ is a strictly positive stopping time and there exists a non-decreasing sequence of stopping times τ_n such that $\tau_n \to \xi$ and $\{(u, \phi), \tau_n\}$ is a local strong solution and

$$
\sup_{s\in[0,\tau_n]} \|(u,\phi)(s)\|_{\mathbb{V}}^2 + \int_0^{\tau_n} (|A_0 u(s)|^2 + |A_1^2\phi(s)|^2) ds \geq n \tag{2.27}
$$

on the set $\{\xi < \infty\}$. Such a solution is called global if

$$
\mathbb{P}(\xi < \infty) = 0. \tag{2.28}
$$

Now we introduce our Galerkin system.

Let $\{(w_i, \psi_i), i = 1, 2, 3, ...\} \subset \mathbb{V}$ be an orthonormal basis in $\mathbb{H}$, where $\{w_i, i = 1, 2, ...\}$, $\{\psi_i, i = 1, 2, ...\}$ are eigenvectors of A_0 and A_1, respectively. We set $\mathbb{V}_n = \mathbb{H}_n = \operatorname{span}\{(w_1, \psi_1), ..., (w_n, \psi_n)\}$.

We look for $(u_n, \phi_n) \in \mathbb{H}_n$ solution to

$$
\begin{cases}
d\,\langle u_n, w_i\rangle + \langle \nu A_0 u_n + B_0(u_n,u_n) - \mathcal{K}R_0(\epsilon A_1\phi_n, \phi_n), w_i\rangle\, dt \\
= \langle g_1(t,u_n), w_i\rangle\, dt + \sum_{k=1}^{\infty} \langle g_2(t,u_n)e_k, w_i\rangle\, d\beta_t^k, \\
d\,\langle \phi_n, A_1\psi_i\rangle + \langle A_1\mu_n + B_1(u_n,\phi_n), A_1\psi_i\rangle\, dt = 0, \\
\mu_n = \epsilon A_1\phi_n + \alpha f(\phi_n), \\
\langle u_n(0), w_i\rangle = \langle u_0, w_i\rangle\,,\ \ \langle \phi_n(0), \psi_i\rangle = \langle \phi_0, \psi_i\rangle\,,
\end{cases}
\tag{2.29}
$$

$1 \leq i \leq n$.

We can also write (2.29) as an system of equations in $\mathbb{H}_n$ $(\cong \mathbb{R}^n)$

$$
\begin{cases}
du_n + \nu A_0 u_n + \mathcal{P}_n^1\left[B_0(u_n,u_n) - \mathcal{K}\mathcal{P}_n^1 R_0(\epsilon A_1\phi_n, \phi_n) - g_1(t,u_n)\right] dt \\
= \sum_{k=1}^{\infty} \mathcal{P}_n^1 g_2(t,u_n)e_k d\beta_t^k, \\
d\phi_n + \mathcal{P}_n^2(A_1\mu_n + B_1(u_n,\phi_n))dt = 0, \\
\mu_n = \epsilon A_1\phi_n + \alpha f(\phi_n), \\
(u_n(0), \phi_n(0)) = \mathcal{P}_n(u_0, \phi_0) := (u_{0n}, \phi_{0n}),
\end{cases}
\tag{2.30}
$$

where $\mathcal{P}_n = (\mathcal{P}_n^1, \mathcal{P}_n^2) : H_1 \times H_2 \to \mathbb{H}_n$ is the orthogonal projection.

As in the proof of Theorem 1.2.1 in [4], we can obtain the existence and uniqueness of a solution $(u_n, \phi_n) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{V}_n)$ of Eq. 2.29 with continuous trajectories.

3 Our Mains Results

Here we estabished the $\mathbb{V}$-norm convergence of the sequence (u_n, ϕ_n) of the Galerkin approximation up to any deterministic time T.

Theorem 3.1 *Let $\delta \in (0,1)$ and let $T > 0$ be arbitrary. Suppose that (u, ϕ) is a solution to the* Eq. 2.18*, and let (u_n, ϕ_n) be the corresponding Galerkin approximation. It follows that*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \tilde{\psi}\left(\|(u_n(t), \phi_n(t)) - (u(t), \phi(t))\|_{\mathbb{V}}^2\right)\right] = 0 \tag{3.1}$$

where $\tilde{\psi}(x) = [\log(1+x)]^{1-\delta}$.

For the proof, we will drew our inspiration to the main result in [13].

Theorem 3.2 *We assume that g_1, g_2, u_0 and ϕ_0 satisfy the same hypotheses as in* Definition 2.2 *and we suppose that (u, ϕ) is the solution to the problem* (2.18)*. Then we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \psi\left(\|(u(t), \phi(t))\|_{\mathbb{V}}^2\right)\right] \le C(u_0, \phi_0, g_1, g_2, T), \tag{3.2}$$

with $\psi(x) = \log(1+x)$ and $C(u_0, \phi_0, g_1, g_2, T)$ is defined as in Eq. 4.18 *below.*

Proposition 3.1 *Let $p \ge 2$ be fixed. We assume that $\partial_\eta \phi = \partial_\eta \Delta\phi = 0$, on $(0, +\infty) \times \partial\mathcal{M}$ and $(u_0, \phi_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{H})$ is such that $\mathbb{E}[\mathcal{E}_{tot}(u_0, \phi_0)]^{p/2} < \infty$. We also suppose that $g_1(t, 0) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; L^p(0, T; V_1'))$ and $g_2(t, 0) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; L^p(0, T; L_2(\mathcal{U}, H_1)))$. Then the sequence (u_n, ϕ_n) of the Galerkin approximation satisfies*

$$\begin{aligned} &\mathbb{E}\left(\sup_{[0,T]} [\mathcal{E}_{tot}(u_n, \phi_n)]^{p/2}\right) + \mathbb{E}\left[\int_0^T (\|u_n\|^2 + \|\bar{\mu}_n\|^2) ds\right]^{p/2} \\ &\le c\mathbb{E}[\mathcal{E}_{tot}(u_0, \phi_0)]^{p/2} + c\mathbb{E}\left[\int_0^T \|g_1(s,0)\|_{V_1'}^p ds\right] + c\left[\int_0^T \|g_2(s,0)\|_{L_2(\mathcal{U}, H_1)}^p ds\right] \\ &= \mathcal{Z}_p := \mathcal{Z}_p(u_0, \phi_0, g_1, g_2), \end{aligned} \tag{3.3}$$

where $\bar{\mu}_n := \mu_n - \langle \mu_n \rangle = \epsilon A_1 \phi_n + \alpha f(\phi_n) - \alpha \langle f(\phi_n) \rangle$, c is a positive constant depending only on $\nu, \epsilon, \alpha, L_{H_1}, p, T$.

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]} |(u_n, \phi_n)|_{\mathbb{H}}^p\right] + \mathbb{E}\left[\int_0^T \|(u_n, \phi_n)\|_{\mathbb{V}}^2 |(u_n, \phi_n)|_{\mathbb{H}}^{p-2} ds\right] \\ &\le \mathcal{Z}_p + C\left(\mathcal{Z}_p\right)^{\dfrac{p-2}{2}} \left(1 + \mathcal{Z}_{p(\kappa+1)} + \mathcal{Z}_p\right)^{\frac{2}{p}}, \end{aligned} \tag{3.4}$$

where C is a positive constant depending only on $\mathcal{M}, \alpha, \epsilon, f, \kappa, T$ and p, and $\mathcal{Z}_p$ is defined as in Eq. 3.3.

Proof Reasoning similarly as in [18, inequality (3.27)], we derive (3.3).

Let us move to the proof of Eq. 3.4.

From Eqs. 2.17 and 3.3 we easily derive that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |(u_n,\phi_n)|_{\mathbb{H}}^p\right] \leq \mathcal{Z}_p(u_0,\phi_0,g_1,g_2). \tag{3.5}$$

Now fix $p \geq 4$. Thanks to the Hölder inequality, Eqs. 3.5 and 3.3, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \|u_n\|^2 |(u_n,\phi_n)|_{\mathbb{H}}^{p-2} ds \\
&\leq E\left[\sup_{t\in[0,T]} |(u_n,\phi_n)|_{\mathbb{H}}^{p-2} \int_0^T \|u_n\|^2 ds\right] \\
&\leq \left[\mathbb{E}\sup_{t\in[0,T]} |(u_n,\phi_n)|_{\mathbb{H}}^{p}\right]^{\frac{p-2}{p}} \left(\mathbb{E}\left[\int_0^T \|u_n\|^2 ds\right]^{\frac{p}{2}}\right)^{\frac{2}{p}} \\
&\leq c\mathbb{E}[\mathcal{E}_{tot}(u_0,\phi_0)]^{p/2} + c\mathbb{E}\left[\int_0^T \|g_1(s,0)\|_{V_1'}^p ds\right] + c\left[\int_0^T \|g_2(s,0)\|_{L_2(\mathcal{U},H_1)}^p ds\right].
\end{aligned} \tag{3.6}$$

Taking the average over $\mathcal{M}$ of the third equation of Eq. 2.30 and notice that, due to $\partial_\eta \Delta\phi = 0$ and assumption (2.1), we can deduce as in [11, Page 8] the following estimate for the average of μ_n over $\mathcal{M}$; that is

$$\begin{aligned}
\langle\mu_n\rangle^2 &= \alpha^2 \langle f(\phi_n)\rangle^2 \\
&\leq \alpha^2 c_f (1 + |\phi_n|_{L^{2\kappa+2}}^{2\kappa+2}) \\
&\leq \alpha^2 C_{f,\kappa}\left[1 + \|\phi_n\|^{2\kappa+2}\right].
\end{aligned} \tag{3.7}$$

In Eq. 3.7, we have also used the embedding of $H^1(\mathcal{M})$ in $L^{2\kappa+2}(\mathcal{M})$, $\kappa \in [2,+\infty)$, the Poincaré-Wirtinger inequality and the fact that $\langle\phi_n\rangle = 0$ due to the mass conservation. Here, $C_{f,\kappa}$ is a positive constant depending on c_f and κ.

Arguing similarly as in [11, Page 10], we obtain

$$\begin{aligned}
|A_1\phi_n|^2 &\leq \epsilon^{-2}|\mu_n|^2 + \alpha^2\epsilon^{-2}c_f(1 + |\phi_n|_{L^{2\kappa+2}}^{2\kappa+2}) \\
&\leq \epsilon^{-2}|\mu_n|^2 + \alpha^2\epsilon^{-2}c_f + \alpha^2 c_f''\epsilon^{-2}\|\phi_n\|^{2\kappa+2},
\end{aligned} \tag{3.8}$$

for some positive constant c_f'' depending on c_f.

Now, owing to Eq. 3.7, Eq. 3.8 in conjunction with the Poincaré-Wirtinger inequality (see [2]), we obtain

$$\begin{aligned}
|A_1\phi_n|^2 &\leq \epsilon^{-2}|\mu_n|^2 + \alpha^2\epsilon^{-2}c_f + \alpha^2 c_f''\epsilon^{-2}\|\phi_n\|^{2\kappa+2} \\
&\leq c\epsilon^{-2}\left(\|\bar{\mu}_n\|^2 + \langle\mu_n\rangle^2\right) + \alpha^2\epsilon^{-2}c_f + \alpha^2 c_f''\epsilon^{-2}\|\phi_n\|^{2\kappa+2} \\
&\leq c\epsilon^{-2}\|\bar{\mu}_n\|^2 + \mathcal{K}^1_{\alpha,\epsilon,f,\kappa} + \mathcal{K}^2_{\alpha,\epsilon,f,\kappa}\|\phi_n\|^{2\kappa+2},
\end{aligned} \tag{3.9}$$

where $\mathcal{K}^1_{\alpha,\epsilon,f,\kappa} = C_{f,\kappa}\alpha^2\epsilon^{-2} + \alpha^2\epsilon^{-2}c_f$ and $\mathcal{K}^2_{\alpha,\epsilon,f,\kappa} = \alpha^2\epsilon^{-2}C_{f,\kappa} + \alpha^2 c_f''\epsilon^{-2}$.

Thanks to Eq. 3.5, Eq. 3.9 and the Hölder inequality, we have for $p \geq 4$.

$$\begin{aligned}
&\mathbb{E}\int_0^T |A_1\phi_n|^2 |(u_n,\phi_n)|_{\mathbb{H}}^{p-2} ds \\
&\leq E\left[\sup_{[0,T]} |(u_n,\phi_n)|_{\mathbb{H}}^{p-2} \int_0^T |A_1\phi_n|^2 ds\right] \\
&\leq \left[\mathbb{E}\sup_{[0,T]} |(u_n,\phi_n)|_{\mathbb{H}}^{p}\right]^{\frac{p-2}{p}} \left(\mathbb{E}\left[\int_0^T |A_1\phi_n|^2 ds\right]^{\frac{p}{2}}\right)^{\frac{2}{p}} \\
&\leq C\left[\mathbb{E}\sup_{[0,T]} |(u_n,\phi_n)|_{\mathbb{H}}^{p}\right]^{\frac{p-2}{p}} \left(1+\mathbb{E}\sup_{[0,T]}\|\phi_n\|^{p(\kappa+1)} + \mathbb{E}\left[\int_0^T \|\bar{\mu}_n\|^2 ds\right]^{\frac{p}{2}}\right)^{\frac{2}{p}} \\
&\leq C\left(\mathcal{Z}_p\right)^{\frac{p-2}{2}} \left(1+\mathcal{Z}_{p(\kappa+1)}+\mathcal{Z}_p\right)^{\frac{2}{p}},
\end{aligned} \tag{3.10}$$

where C is a positive large constant depending only on $\mathcal{M}, \alpha, \epsilon, f, \kappa, T$ and p and $\mathcal{Z}_p$ is defined as in Eq. 3.3.

Combining now these estimates (3.5), (3.6) and (3.10), we get (3.4) for $p \geq 4$. With this being proved for any $p \geq 4$, it is subsequently true for any $p \geq 2$. □

Remark 3.1 Note that (see [11, page 403]) the natural no-flux conditions $\partial_\eta\phi = \partial_\eta\Delta\phi = 0$ ensure the mass conservation, since it implies that $\partial_\eta\mu = 0$ on $(0,\infty)\times\partial\mathcal{M}$, which yields the conservation of the following quantity $\langle\phi(t)\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} \phi(t,x)dx$.

The following proposition gives a stronger results concerning the convergence of the Gakerkin approximations (u_n,ϕ_n) in $\mathbb{H}$.

Proposition 3.2 *Let the assumptions* (2.22)–(2.23) *be satisfied. Let* $\delta \in (0,1)$. *We fix* $p \geq \frac{2}{1-\delta}$ *such,* $g_1(t,0) \in L^p(\Omega,\mathcal{F},\mathbb{P}; L^p_{loc}([0,\infty); V_1'))$, $g_2(t,0) \in L^p(\Omega,\mathcal{F},\mathbb{P}; L^p_{loc}([0,\infty); L_2(\mathcal{U},H_1)))$ *and* $(u_0,\phi_0) \in L^2(\Omega;\mathbb{H})\cap L^2(\Omega;\mathbb{V})$ *satisfies* $\mathbb{E}[\mathcal{E}_{tot}(u_0,\phi_0)]^{\frac{p}{2}} < \infty$. *Let* (u,ϕ) *be the solution to the problem* (1.1)*, and let* (u_n,ϕ_n) *be the corresponding Galerkin approximation. Then we have the following convergence*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |(u_n(t),\phi_n(t)) - (u(t),\phi(t))|_{\mathbb{H}}^{p(1-\delta)}\right) \to 0 \tag{3.11}$$

as $n \to +\infty$*, for any deterministic time* $T > 0$.

Proof Let $\delta \in (0,1)$ and $p \geq \frac{2}{1-\delta}$. Arguing similarly as in Eq. 3.4, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |(u,\phi)|_{\mathbb{H}}^{p}\right] + \mathbb{E}\left[\int_0^T \|(u,\phi)\|_{\mathbb{V}}^2 |(u,\phi)|_{\mathbb{H}}^{p-2} ds\right] \\
&\leq \mathcal{Z}_p + C\left(\mathcal{Z}_p\right)^{\frac{p-2}{2}} \left(1+\mathcal{Z}_{p(\kappa+1)}+\mathcal{Z}_p\right)^{\frac{2}{p}},
\end{aligned} \tag{3.12}$$

where C is a positive constant depending only on $\mathcal{M}, \alpha, \epsilon, f, \kappa, T$ and p, and $\mathcal{Z}_p$ is defined as in Eq. 3.3. For an amelioreted estimate of Eq. 3.12 we refer the reader to [6, Lemma 4.2].

By Eqs. 3.4 and 3.12, we deduce that the sequence (u_n, ϕ_n) is uniformly integrable in $L^{p(1-\delta)}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{H})$ with $(u, \phi) \in L^{p(1-\delta)}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{H})$. Also thanks to [18, see the first equality after Lemma 3.4], this sequence converges to $(u, \phi) \in L^{p(1-\delta)}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{H})$ in probability; i.e., we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} |(u_n(t), \phi_n(t)) - (u(t), \phi(t))|_{\mathbb{H}} \geq k\right) \to 0. \tag{3.13}$$

Hence, it follows from the uniform integrability principle or Vitali's convergence theorem that

$$\mathbb{E} \sup_{t\in[0,T]} |(u_n(t), \phi_n(t)) - (u(t), \phi(t))|_{\mathbb{H}}^{p(1-\delta)} \to 0$$

as $n \to \infty$, for every $\delta \in (0, 1)$, any deterministic time $T > 0$, and Eq. 3.11 is proven. □

4 Galerkin Convergence in $\mathbb{V}$

This section is devoted to the proof of our main result, Theorem 3.1. We begin by recalling the existence result from [6].

Theorem 4.1 *Let (u_n, ϕ_n) be the sequence of solutions of* Eq. 2.29*, and let (u, ϕ) be the solution to the* Eq. 2.18 *with g_1, g_2, and (u_0, ϕ_0) as in Definition 2.2. Then there exists a global, maximal strong solution $\{(u, \phi), \xi\}$. More precisely, there exists an increasing sequence of strictly positive stopping times $\{\tau_k\}_{k\geq 1}$ converging to ξ, for which $\mathbb{P}(\xi < \infty) = 0$. Moreover, there exists an increasing sequence of measurable subsets $\{\Omega_\iota\}_{\iota\geq 1}$ with $\Omega_\iota \uparrow \Omega$ as $\iota \to \infty$ so that on any Ω_ι we have*

$$\lim_{n\to\infty} \mathbb{E}\left[1_{\Omega_\iota}\left(\sup_{t\in[0,\tau_k]} \|(u_n, \phi_n) - (u, \phi)\|_{\mathbb{V}}^2 + 2\int_0^{\tau_k} \left(\nu|A_0(u_n - u)|^2 + \epsilon|A_1^2(\phi_n - \phi)|^2\right) ds\right)\right] = 0 \tag{4.1}$$

for any τ_k.

Lemma 4.1 *Let (u, ϕ) and (u_n, ϕ_n) be defined as in Definition 2.2, and as in* Eqs. 2.29–2.30*, respectively. Let $T > 0$ be a fix deterministic time. Then, the sequence (u_n, ϕ_n) converges in probability with respect to the $\mathbb{V}$-norm to the solution (u, ϕ) of the problem* (1.1)*, i.e., for any $\zeta > 0$ we have*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|(u_n(t), \phi_n(t)) - (u(t), \phi(t))\|_{\mathbb{V}}^2 \geq \zeta\right) \to 0 \text{ as } n \to \infty. \tag{4.2}$$

Proof Let $\delta > 0$ and $\{\tau_k\}_{k\geq 1}$ be the stopping time as in Theorem 4.1. We denote by $\tilde{\tau}_k = \tau_k \wedge T$. Then there exists k_0 such that $\mathbb{P}(\tilde{\tau}_{k_0} < T) \leq \delta/4$. Now, choose an ι such that $\mathbb{P}(\Omega_\iota) > 1 - \frac{\delta}{2}$, where ι is an in Theorem 4.1. From Eq. 4.1 in Theorem 4.1, we have

$$\lim_{n\to\infty} \mathbb{E}\left[1_{\Omega_\iota} \sup_{t\in[0,\tilde{\tau}_{k_0}]} \|(u_n(t),\phi_n(t)) - (u(t),\phi(t))\|_{\mathbb{V}}^2 \right] = 0, \tag{4.3}$$

from which we deduce the following convergence in probability:

$$\lim_{n\to\infty} \mathbb{P}\left(1_{\Omega_\iota} \sup_{t\in[0,\tilde{\tau}_{k_0}]} \|(u_n(t),\phi_n(t)) - (u(t),\phi(t))\|_{\mathbb{V}}^2 \geq \zeta \right) = 0, \text{ for any } \zeta > 0. \tag{4.4}$$

Hence, we have

$$\begin{aligned} &\mathbb{P}\left(1_{\Omega_\iota} \sup_{t\in[0,T]} \|(u_n,\phi_n) - (u,\phi)\|_{\mathbb{V}}^2 \geq \zeta \right) \\ &= \mathbb{P}\left(\left\{ \sup_{t\in[0,T]} \|(u_n,\phi_n) - (u,\phi)\|_{\mathbb{V}}^2 \geq \zeta \right\} \cap \left\{ \tilde{\tau}_{k_0} < T \right\} \cap \{\omega \in \Omega_\iota\} \right) \\ &\quad + \mathbb{P}\left(\left\{ \sup_{t\in[0,T]} \|(u_n,\phi_n) - (u,\phi)\|_{\mathbb{V}}^2 \geq \zeta \right\} \cap \left\{ \tilde{\tau}_{k_0} = T \right\} \cap \{\omega \in \Omega_\iota\} \right) \\ &\leq \mathbb{P}(\tilde{\tau}_{k_0} < T) + \mathbb{P}\left(1_{\Omega_\iota} \sup_{t\in[0,\tilde{\tau}_{k_0}]} \|(u_n,\phi_n) - (u,\phi)\|_{\mathbb{V}}^2 \geq \zeta \right). \end{aligned} \tag{4.5}$$

So thanks to Eqs. 4.5 and 4.3, we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in[0,T]} \|(u_n,\phi_n)(t) - (u,\phi)(t)\|_{\mathbb{V}}^2 \geq \zeta \right) \\ &\leq \mathbb{P}(\tilde{\tau}_{k_0} < T) + \mathbb{P}\left(1_{\Omega_\iota} \sup_{t\in[0,\tilde{\tau}_{k_0}]} \|(u_n,\phi_n)(t) - (u,\phi)(t)\|_{\mathbb{V}}^2 \geq \zeta \right) + \mathbb{P}(\Omega_\iota^c) \\ &\leq \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta \end{aligned} \tag{4.6}$$

for n sufficiently large, and the proof of Lemma 4.1 is completed. □

We now give the proof of Theorem 3.2.

On the proof of Theorem 3.2, we will denote by c, a generic positive constant (possibly depending on $\mathcal{K}, \nu, \epsilon, \alpha, c_f, \kappa, \mathcal{M}, L_{V_1},$), which can vary even within the same line.

Proof From the infinite dimensional version of Itô's lemma applied to the process $\|u\|^2$, taking the inner product in $L^2(\mathcal{M})$ of Eq. 2.18_2 with $2A_1^2\phi$, using Eq. 2.18_3 and adding the resulting equalities, we derive an evolution system for the $\mathbb{V}$-norm of (u,ϕ)

$$\begin{aligned} d\|(u,\phi)\|_{\mathbb{V}}^2 &= -2[\nu|A_0u|^2 + \epsilon|A_1^2\phi|^2]dt + 2\,(g_1(t,u), A_0u)\,dt - 2\,(B_0(u,u), A_0u)\,dt \\ &\quad + \|g_2(t,u)\|_{L_2(\mathcal{U},V_1)}^2 dt + 2\mathcal{K}(R_0(\epsilon A_1\phi,\phi), A_0u)\,dt - 2\alpha\left(A_1 f(\phi), A_1^2\phi\right)dt \\ &\quad - 2\left(B_1(u,\phi), A_1^2\phi\right)dt + 2\sum_{j=1}^{\infty}\left(g_2(t,u)e_j, A_0u\right)d\beta_t^j. \end{aligned} \tag{4.7}$$

Also from the infinite dimensional version of Itô's lemma and Eq. 4.7, we have

$$
\begin{aligned}
&d\left[\psi\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right]+2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)[\nu|A_0u|^2+\epsilon|A_1^2\phi|^2]dt\\
=\;&2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)(g_1(t,u),A_0u)\,dt-2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)(B_0(u,u),A_0u)\,dt\\
&+\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\|g_2(t,u)\|_{L_2(\mathcal{U},V_1)}^2dt+2\mathcal{K}\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)(R_0(\epsilon A_1\phi,\phi),A_0u)\,dt\\
&-2\alpha\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left(A_1f(\phi),A_1^2\phi\right)dt-2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left(B_1(u,\phi),A_1^2\phi\right)dt\\
&+2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\sum_{j=1}^{\infty}\left(g_2(t,u)e_j,A_0u\right)d\beta_t^j\\
&+2\psi''\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\sum_{j=1}^{\infty}\left(g_2(t,u)e_j,A_0u\right)^2dt.
\end{aligned}
\tag{4.8}
$$

Thanks to Cauchy-Schwarz's and Young's inequalities in conjunction with Eq. 2.22, we have

$$
\begin{aligned}
&2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|(g_1(t,u),A_0u)|\\
\le\;&2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|A_0u||g_1(t,u)|\\
\le\;&\frac{\nu}{4}\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|A_0u|^2+c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|g_1(t,u)|^2\\
\le\;&\frac{\nu}{4}\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|A_0u|^2+c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|g_1(t,0)|^2\\
&+c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\|(u,\phi)\|_{\mathbb{V}}^2.
\end{aligned}
\tag{4.9}
$$

Thanks to the Agmon inequality, the Young inequality and the fact that $\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\|u\|^2\le 1$, we get

$$
\begin{aligned}
&2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|(B_0(u,u),A_0u)|\\
\le\;&c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|u|^{1/2}\|u\||A_0u|^{3/2}\\
\le\;&\tfrac{\nu}{4}\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|A_0u|^2+c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|u|^2\|u\|^2\|u\|^2\\
\le\;&\tfrac{\nu}{4}\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|A_0u|^2+c|(u,\phi)|_{\mathbb{H}}^2\|(u,\phi)\|_{\mathbb{V}}^2.
\end{aligned}
\tag{4.10}
$$

Note that

$$
\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\|g_2(t,u)\|_{L_2(\mathcal{U},V_1)}^2\le c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left[\|(u,\phi)\|_{\mathbb{V}}^2+\|g_2(t,0)\|_{L_2(\mathcal{U},V_1)}^2\right].
$$

Arguing similarly as in [6, Inequality (4.81)], we have

$$
\begin{aligned}
&2\mathcal{K}\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|(R_0(\epsilon A_1\phi,\phi),A_0u)|\\
\le\;&\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left[\tfrac{\nu}{4}|A_0u|^2+\tfrac{\epsilon}{4}|A_1^2\phi|^2\right]+c\|\phi\|^{10}.
\end{aligned}
\tag{4.11}
$$

Also, as in [6, Inequalities (4.82) and (4.83)], we have

$$
\begin{aligned}
&2\alpha\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left|\left(A_1f(\phi),A_1^2\phi\right)\right|\\
\le\;&2\alpha\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left|\left(f''(\phi)(A_1^{1/2}\phi)^2,A_1^2\phi\right)\right|+2\alpha\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left|\left(f'(\phi)A_1\phi,A_1^2\phi\right)\right|\\
\le\;&\tfrac{\epsilon}{4}\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)|A_1^2\phi|^2+c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left[\|\phi\|^6+\|\phi\|^{4\kappa+5}+\|\phi\|^{8\kappa+10}+|A_1\phi|^2\right].
\end{aligned}
\tag{4.12}
$$

We also note that

$$
\begin{aligned}
& 2\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left|\left(B_1(u,\phi),\, A_1^2\phi\right)\right| \\
\leq\ & c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\|u\|^{1/2}|A_0u|^{1/2}\|\phi\||A_1^2\phi| \\
\leq\ & \psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left[\tfrac{\nu}{4}|A_0u|^2+\tfrac{\epsilon}{2}|A_1^2\phi|^2\right]+c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\|u\|^2\|\phi\|^4 \\
\leq\ & \psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\left[\tfrac{\nu}{4}|A_0u|^2+\tfrac{\epsilon}{2}|A_1^2\phi|^2\right]+c\|\phi\|^4, \\
& 2\left|\psi''\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|\sum_{j=1}^{\infty}\left(g_2(t,u)e_j,\, A_0u\right)^2 \\
\leq\ & 2\left|\psi''\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|\|u\|^2\|g_2(t,u)\|_{L_2(\mathcal{U},V_1)}^2 \\
\leq\ & c\left|\psi''\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|\|u\|^2\|g_2(t,0)\|_{L_2(\mathcal{U},V_1)}^2+c\left|\psi''\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|\|u\|^4 \\
\leq\ & c\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\|g_2(t,0)\|_{L_2(\mathcal{U},V_1)}^2+c\left|\psi''\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|\|(u,\phi)\|_{\mathbb{V}}^4,
\end{aligned}
\tag{4.13}
$$

and by the Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
& 2\mathbb{E}\sup_{s\in[0,\tilde{\tau}_k]}\left|\int_0^s\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\sum_{j=1}^{\infty}\left(g_2(r,u)e_j,\, A_0u\right)d\beta_r^j\right| \\
\leq\ & 6\mathbb{E}\left[\int_0^{\tilde{\tau}_k}\left|\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|^2\sum_{j=1}^{\infty}\left(g_2(s,u)e_j,\, A_0u\right)^2ds\right]^{1/2} \\
\leq\ & 6\mathbb{E}\left[\int_0^{\tilde{\tau}_k}\left|\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|^2\|u\|^2\|g_2(s,u)\|_{L_2(\mathcal{U},V_1)}^2ds\right]^{1/2} \\
\leq\ & c\mathbb{E}\left[\int_0^{\tilde{\tau}_k}\left|\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|^2\|u\|^2\|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2ds\right]^{1/2} \\
& +c\mathbb{E}\left[\int_0^{\tilde{\tau}_k}\left|\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right|^2\|u\|^4ds\right]^{1/2} \\
=\ & c\mathbb{E}\left[\int_0^{\tilde{\tau}_k}\frac{1}{(1\ +\|(u,\phi)\|_{\mathbb{V}}^2)^2}\|u\|^2\|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2ds\right]^{1/2} \\
& +c\mathbb{E}\left[\int_0^{\tilde{\tau}_k}\frac{1}{(1+\|(u,\phi)\|_{\mathbb{V}}^2)^2}\|u\|^4ds\right]^{1/2}.
\end{aligned}
$$

Furthermoe, owing to the Young inequality in conjunction with the fact that $\dfrac{1}{(1+\|(u,\phi)\|_{\mathbb{V}}^2)^2}\leq 1$ and $\dfrac{\|u\|^4}{(1+\|(u,\phi)\|_{\mathbb{V}}^2)^2}\leq 1$, we obtain

$$
\begin{aligned}
\int_0^{\tilde{\tau}_k}\frac{\|u\|^2\|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2}{(1+\|(u,\phi)\|_{\mathbb{V}}^2)^2}ds &\leq c\int_0^{\tilde{\tau}_k}\frac{(1+\|u\|^4)\|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2}{(1+\|(u,\phi)\|_{\mathbb{V}}^2)^2}ds \\
&\leq c\int_0^{\tilde{\tau}_k}\|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2ds,
\end{aligned}
$$

and then

$$
\begin{aligned}
& c\mathbb{E}\left[\int_0^{\tilde{\tau}_k}\frac{\|u\|^2\|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2}{(1+\|(u,\phi)\|_{\mathbb{V}}^2)^2}ds\right]^{1/2}+c\mathbb{E}\left[\int_0^{\tilde{\tau}_k}\frac{\|u\|^4}{(1+\|(u,\phi)\|_{\mathbb{V}}^2)^2}ds\right]^{1/2} \\
\leq\ & c\mathbb{E}\left[\int_0^{T}\|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2ds\right]^{1/2}+cT^{1/2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&2\mathbb{E} \sup_{s\in[0,\tilde{\tau}_k]} \left| \int_0^s \psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right) \sum_{j=1}^{\infty} \left(g_2(r,u)e_j, A_0 u\right) d\beta_r^j \right| \\
&\leq c\mathbb{E}\left[\int_0^T \|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2 ds\right]^{1/2} + cT^{1/2},
\end{aligned}
\tag{4.14}
$$

where $\tilde{\tau}_k = \tau_k \wedge T = \min(\tau_k, T)$, and τ_k introduced in Theorem 4.1.

Now, we integrate with respect to time in Eq. 4.8 and then we take the supremum up to the stopping time $\tilde{\tau}_k = \tau_k \wedge T$ to the corresponding inequality. We recall that the stopping time τ_k has been introduced in Theorem 4.1. Denoting $\Omega_k = \{\omega \in \Omega : \tilde{\tau}_k = T\}$, we observe that $\Omega_k \uparrow \Omega$ as $k \to \infty$ by Theorem 4.1. By taking the expectation on Ω_k and, suppressing 1_{Ω_k} for simplicity of notation, using also the estimates (4.9)–(4.14), we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tilde{\tau}_k]}\left[\psi\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right] + \mathbb{E}\int_0^{\tilde{\tau}_k} \psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)[\nu|A_0u|^2 + \epsilon|A_1^2\phi|^2]ds \\
&\leq \mathbb{E}\left(\left[\psi\left(\|(u_0,\phi_0)\|_{\mathbb{V}}^2\right)\right]\right) + c\mathbb{E}\int_0^T |g_1(t,0)|^2 dt + c\mathbb{E}\int_0^T |(u,\phi)|_{\mathbb{H}}^2 \|(u,\phi)\|_{\mathbb{V}}^2 dt + cT \\
&\quad + c\mathbb{E}\int_0^T \|g_2(t,0)\|_{L_2(\mathcal{U},V_1)}^2 dt + + c\mathbb{E}\left[\int_0^T \|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2 ds\right]^{1/2} + cT^{1/2} \\
&\quad + c\mathbb{E}\int_0^T \left[\|\phi\|^4 + \|\phi\|^6 + \|\phi\|^{10} + \|\phi\|^{4\kappa+5} + \|\phi\|^{8\kappa+10}\right] dt,
\end{aligned}
\tag{4.15}
$$

where we have also use the fact $\psi'\left(\|(u,\phi)\|_{\mathbb{V}}^2\right) = \frac{1}{1+\|(u,\phi)\|_{\mathbb{V}}^2} \leq 1$.

Note that by Eq. 3.12, we have

$$
\begin{aligned}
&c\mathbb{E}\int_0^T |(u,\phi)|_{\mathbb{H}}^2 \|(u,\phi)\|_{\mathbb{V}}^2 dt + c\mathbb{E}\int_0^T \|\phi\|^4 dt \leq c\mathcal{Z}_4 + c\left(\mathcal{Z}_4\right)\left(1 + \mathcal{Z}_{4(\kappa+1)} + \mathcal{Z}_4\right)^{\frac{1}{2}}, \\
&c\mathbb{E}\int_0^T \|\phi\|^6 dt \leq c\mathcal{Z}_6 + c\left(\mathcal{Z}_6\right)^2\left(1 + \mathcal{Z}_{6(\kappa+1)} + \mathcal{Z}_6\right)^{\frac{1}{3}}, \\
&c\mathbb{E}\int_0^T \|\phi\|^{10} dt \leq c\mathcal{Z}_{10} + c\left(\mathcal{Z}_{10}\right)^4\left(1 + \mathcal{Z}_{10(\kappa+1)} + \mathcal{Z}_{10}\right)^{\frac{1}{5}}, \\
&c\mathbb{E}\int_0^T \|\phi\|^{4\kappa+5} dt \leq c\mathcal{Z}_{(4\kappa+5)} + c\left(\mathcal{Z}_{(4\kappa+5)}\right)^{\frac{4\kappa+3}{2}}\left(1 + \mathcal{Z}_{(4\kappa+5)(\kappa+1)} + \mathcal{Z}_{(4\kappa+5)}\right)^{\frac{2}{(4\kappa+5)}}, \\
&c\mathbb{E}\int_0^T \|\phi\|^{8\kappa+10} dt \leq c\mathcal{Z}_{(8\kappa+10)} + c\left(\mathcal{Z}_{(8\kappa+10)}\right)^{4\kappa+4}\left(1 + \mathcal{Z}_{(8\kappa+10)(\kappa+1)} + \mathcal{Z}_{(8\kappa+10)}\right)^{\frac{1}{4\kappa+5}}.
\end{aligned}
\tag{4.16}
$$

From the estimates (4.15)-(4.16) and writing out 1_{Ω_k} explicitly, we infer that

$$
\begin{aligned}
&\mathbb{E}\left(1_{\Omega_k}\sup_{[0,\tilde{\tau}_k]}\left[\psi\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right]\right) \\
&\leq \mathbb{E}\left(\left[\psi\left(\|(u_0,\phi_0)\|_{\mathbb{V}}^2\right)\right]\right) + c\mathbb{E}\int_0^T |g_1(t,0)|^2 dt + cT \\
&\quad + c\mathbb{E}\int_0^T \|g_2(t,0)\|_{L_2(\mathcal{U},V_1)}^2 dt + c\mathbb{E}\left[\int_0^T \|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2 ds\right]^{1/2} + cT^{1/2} \\
&\quad + c\mathcal{Z}_4 + c\left(\mathcal{Z}_4\right)\left(1 + \mathcal{Z}_{4(\kappa+1)} + \mathcal{Z}_4\right)^{\frac{1}{2}} + c\mathcal{Z}_6 + c\left(\mathcal{Z}_6\right)^2\left(1 + \mathcal{Z}_{6(\kappa+1)} + \mathcal{Z}_6\right)^{\frac{1}{3}} \\
&\quad + c\mathcal{Z}_{10} + c\left(\mathcal{Z}_{10}\right)^4\left(1 + \mathcal{Z}_{10(\kappa+1)} + \mathcal{Z}_{10}\right)^{\frac{1}{5}} \\
&\quad + c\mathcal{Z}_{(4\kappa+5)} + c\left(\mathcal{Z}_{(4\kappa+5)}\right)^{\frac{4\kappa+3}{2}}\left(1 + \mathcal{Z}_{(4\kappa+5)(\kappa+1)} + \mathcal{Z}_{(4\kappa+5)}\right)^{\frac{2}{(4\kappa+5)}} \\
&\quad + c\mathcal{Z}_{(8\kappa+10)} + c\left(\mathcal{Z}_{(8\kappa+10)}\right)^{4\kappa+4}\left(1 + \mathcal{Z}_{(8\kappa+10)(\kappa+1)} + \mathcal{Z}_{(8\kappa+10)}\right)^{\frac{1}{4\kappa+5}}.
\end{aligned}
\tag{4.17}
$$

Letting $k \to \infty$ in Eq. 4.17, using the monotone convergence theorem, we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{\in[0,T]} \left[\psi\left(\|(u,\phi)\|_{\mathbb{V}}^2\right)\right] \\
&\leq \mathbb{E}\left(\left[\psi\left(\|(u_0,\phi_0)\|_{\mathbb{V}}^2\right)\right]\right) + c\mathbb{E}\int_0^T |g_1(t,0)|^2 dt + cT \\
&\quad + c\mathbb{E}\int_0^T \|g_2(t,0)\|_{L_2(\mathcal{U},V_1)}^2 dt + c\mathbb{E}\left[\int_0^T \|g_2(s,0)\|_{L_2(\mathcal{U},V_1)}^2 ds\right]^{1/2} + cT^{1/2} \\
&\quad + c\mathcal{Z}_4 + c\left(\mathcal{Z}_4\right)\left(1+\mathcal{Z}_{4(\kappa+1)}+\mathcal{Z}_4\right)^{\frac{1}{2}} + c\mathcal{Z}_6 + c\left(\mathcal{Z}_6\right)^2\left(1+\mathcal{Z}_{6(\kappa+1)}+\mathcal{Z}_{\frac{6}{2}}\right)^{\frac{1}{3}} \\
&\quad + c\mathcal{Z}_{10} + c\left(\mathcal{Z}_{10}\right)^4\left(1+\mathcal{Z}_{10(\kappa+1)}+\mathcal{Z}_{10}\right)^{\frac{1}{5}} \\
&\quad + c\mathcal{Z}_{(4\kappa+5)} + c\left(\mathcal{Z}_{(4\kappa+5)}\right)^{\frac{4\kappa+3}{2}}\left(1+\mathcal{Z}_{(4\kappa+5)(\kappa+1)}+\mathcal{Z}_{(4\kappa+5)}\right)^{\frac{2}{(4\kappa+5)}} \\
&\quad + c\mathcal{Z}_{(8\kappa+10)} + c\left(\mathcal{Z}_{(8\kappa+10)}\right)^{4\kappa+4}\left(1+\mathcal{Z}_{(8\kappa+10)(\kappa+1)}+\mathcal{Z}_{(8\kappa+10)}\right)^{\frac{1}{4\kappa+5}} \\
&\equiv C(u_0,\phi_0,g_1,g_2,T).
\end{aligned} \tag{4.18}$$

which completes the proof of Theorem 3.2. □

We now go back to the proof of Theorem 3.1. First we establish the following result.

Lemma 4.2 *Let (u_n,ϕ_n) be the sequence of the Galerkin system (2.29) or (2.30). Then we have*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \log\left(1+\|(u_n(t),\phi_n(t))\|_{\mathbb{V}}^2\right)\right] \leq C_1(u_0,\phi_0,g_1,g_2,T), \\
&\mathbb{E}\left[\sup_{t\in[0,T]} \log\left(1+\|(u_n(t),\phi_n(t))-(u(t),\phi(t))\|_{\mathbb{V}}^2\right)\right] \leq C_2(u_0,\phi_0,g_1,g_2,T)
\end{aligned} \tag{4.19}$$

for all $n\in\mathbb{N}$.

Proof The proof of Eq. 4.19_1 follows the same steps as the proof of Theorem 3.2 and it is thus omitted.

Note that

$$\begin{aligned}
\log\left(1+\|(u_n,\phi_n)-(u,\phi)\|_{\mathbb{V}}^2\right) &\leq \log\left[\left(1+\|(u_n,\phi_n)\|_{\mathbb{V}}^2\right)^2\left(1+\|(u,\phi)\|_{\mathbb{V}}^2\right)^2\right] \\
&\leq 2\log\left[\left(1+\|(u_n,\phi_n)\|_{\mathbb{V}}^2\right)\right] + 2\log\left[\left(1+\|(u,\phi)\|_{\mathbb{V}}^2\right)\right].
\end{aligned} \tag{4.20}$$

Hence the inequality $(4.19)_2$ follows from Eqs. 4.20, $4.19_{\ 1}$ and 3.2. □

We can now give the proof of Theorem 3.1, which is the first stated main result of this paper.

Proof Let $\delta\in(0,1)$ be fix and let $K_n = \sup_{[0,T]} \log\left[(1+\|(u_n,\phi_n)-(u,\phi)\|_{\mathbb{V}}^2)\right]^{1-\delta}$. Note that for any $k>0$, we have

$$\begin{aligned}
\mathbb{P}(K_n\geq k) &= \mathbb{P}\left(\sup_{[0,T]} \log\left[(1+\|(u_n,\phi_n)-(u,\phi)\|_{\mathbb{V}}^2)\right] \geq \frac{k}{1-\delta}\right) \\
&\leq \mathbb{P}\left(\sup_{[0,T]} \|(u_n,\phi_n)-(u,\phi)\|_{\mathbb{V}}^2 \geq \frac{k}{1-\delta}\right)
\end{aligned} \tag{4.21}$$

since $\log(1+x) \leq x, x \geq 0$. Hence, thanks to Eq. 4.19 $_2$, 4.21 and 4.2, we infer that

$$\mathbb{P}\left(K_n \geq k\right) \rightarrow 0 \text{ as } n \rightarrow \infty \tag{4.22}$$

which proves the convergence in probability of the the sequence $\{K_n\}_n$.

Due to Eq. 4.19_2, we have

$$\sup_n \mathbb{E}\left[K_n^{\frac{1}{1-\delta}}\right] \leq C_2(u_0, \phi_0, g_1, g_2, T), \tag{4.23}$$

which proves that the sequence $\{K_n\}$ is uniformly integrable in $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. Hence, using de la Vallée-Poussin criterion for the uniform integrability (see e.g. [10]), we infer that $K_n \rightarrow 0$ in $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ as $n \rightarrow \infty$ and Theorem 3.1 is proven. □

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