



Compactness Characterizations of Commutators on Ball Banach Function Spaces

Jin Tao¹ · Dachun Yang¹ · Wen Yuan¹ · Yangyang Zhang¹

Received: 19 January 2021 / Accepted: 8 September 2021 / Published online: 18 October 2021
© The Author(s), under exclusive licence to Springer Nature B.V. 2021

Abstract

Let X be a ball Banach function space on \mathbb{R}^n . Let Ω be a Lipschitz function on the unit sphere of \mathbb{R}^n , which is homogeneous of degree zero and has mean value zero, and let T_Ω be the convolutional singular integral operator with kernel $\Omega(\cdot)/|\cdot|^n$. In this article, under the assumption that the Hardy–Littlewood maximal operator \mathcal{M} is bounded on both X and its associated space, the authors prove that the commutator $[b, T_\Omega]$ is compact on X if and only if $b \in \text{CMO}(\mathbb{R}^n)$. To achieve this, the authors mainly employ three key tools: some elaborate estimates, given in this article, on the norm of X about the commutators and the characteristic functions of some measurable subsets, which are implied by the assumed boundedness of \mathcal{M} on X and its associated space as well as the geometry of \mathbb{R}^n ; the complete John–Nirenberg inequality in X obtained by Y. Sawano et al.; the generalized Fréchet–Kolmogorov theorem on X also established in this article. All these results have a wide range of applications. Particularly, even when $X := L^{p(\cdot)}(\mathbb{R}^n)$ (the variable Lebesgue space), $X := L^p(\mathbb{R}^n)$ (the mixed-norm Lebesgue space), $X := L^\Phi(\mathbb{R}^n)$ (the Orlicz space), and $X := (E_\Phi^q)_t(\mathbb{R}^n)$ (the Orlicz-slice space or the generalized amalgam space), all these results are new.

Keywords Ball Banach function space · Commutator · Convolutional singular integral operator · BMO · CMO · Extrapolation · Fréchet–Kolmogorov theorem

Mathematics Subject Classification (2010) Primary 47B47; Secondary 42B20 · 42B25 · 42B30 · 42B35 · 46E30

This project is partially supported by the National Natural Science Foundation of China (Grant Nos. 11971058, 12071197, 12122102 and 11871100) and the National Key Research and Development Program of China (Grant No. 2020YFA0712900).

✉ Dachun Yang
dcyang@bnu.edu.cn

Extended author information available on the last page of the article.

1 Introduction

Let Ω be a Lipschitz function on the unit sphere of \mathbb{R}^n , which is homogeneous of degree zero and has mean value zero, namely,

$$|\Omega(x) - \Omega(y)| \leq |x - y| \text{ for any } x, y \in \mathbb{S}^{n-1}, \tag{1.1}$$

$$\Omega(\mu x) := \Omega(x) \text{ for any } \mu \in (0, \infty) \text{ and } x \in \mathbb{S}^{n-1}, \tag{1.2}$$

and

$$\int_{\mathbb{S}^{n-1}} \Omega(x) d\sigma(x) = 0, \tag{1.3}$$

here and thereafter, $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ denotes the unit sphere in \mathbb{R}^n and $d\sigma$ the area measure on \mathbb{S}^{n-1} . To study the factorization theorem of the Hardy space, Coifman et al. [24] initiated the study of the commutator $[b, T_\Omega](f) := bT_\Omega(f) - T_\Omega(bf)$, where $b \in \text{BMO}(\mathbb{R}^n)$ and T_Ω denotes the Calderón–Zygmund operator defined by setting, for any suitable function f and any $x \in \mathbb{R}^n$,

$$\begin{aligned} T_\Omega(f)(x) &:= \text{p. v.} \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^n} f(y) dy \\ &:= \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon < |x - y| < 1/\varepsilon} \frac{\Omega(x - y)}{|x - y|^n} f(y) dy, \end{aligned} \tag{1.4}$$

here and thereafter, $\varepsilon \rightarrow 0^+$ means $\varepsilon \in (0, \infty)$ and $\varepsilon \rightarrow 0$. The commutator of this type plays key roles in harmonic analysis (see, for instance, [4, 5, 15, 17, 34, 63, 69]), partial differential equations (see, for instance, [16, 18, 75]), and quasiregular mappings (see, for instance, [47]).

The first significant result in this direction was made by Coifman et al. [24], which characterizes the boundedness of such type commutators on the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$, via the well-known space $\text{BMO}(\mathbb{R}^n)$. Recall that the *space* $\text{BMO}(\mathbb{R}^n)$, introduced by John and Nirenberg [51], is defined to be the set of all locally integrable functions f on \mathbb{R}^n such that

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} := \sup_{\text{ball } B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, and $f_B := \frac{1}{|B|} \int_B f(x) dx$ for any given ball $B \subset \mathbb{R}^n$. Precisely, Coifman et al. [24] proved that, if a function $b \in \text{BMO}(\mathbb{R}^n)$, then the commutator $[b, T_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$ for any given $p \in (1, \infty)$, and also that, if $[b, R_j]$ is bounded on $L^p(\mathbb{R}^n)$ for any Riesz transform $R_j, j \in \{1, \dots, n\}$, then $b \in \text{BMO}(\mathbb{R}^n)$. Moreover, Uchiyama [76] proved that $[b, T_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$ for any given $p \in (1, \infty)$ if and only if $b \in \text{BMO}(\mathbb{R}^n)$. Later, such boundedness characterizations were also established on various function spaces: for instance, Di Fazio and Ragusa [30] on Morrey spaces, Lu et al. [58] on weighted Lebesgue spaces, and Karlovich and Lerner [52] on variable Lebesgue spaces.

As for the compactness characterization of commutators, in [76] Uchiyama first proved that $[b, T_\Omega]$ is compact on $L^p(\mathbb{R}^n)$ for any given $p \in (1, \infty)$ if and only if $b \in \text{CMO}(\mathbb{R}^n)$, where $\text{CMO}(\mathbb{R}^n)$ denotes the closure of infinitely differentiable functions with compact support in $\text{BMO}(\mathbb{R}^n)$. This characterization of compactness was also extended to Morrey spaces in [19], and to weighted Lebesgue spaces in [23, 37]. However, to the best of our knowledge, for other known function spaces, such as mixed-norm Lebesgue spaces, variable Lebesgue spaces, Orlicz spaces, and Orlicz-slice spaces (see, respectively, Sections 4.2, 4.3,

4.5, and 4.6 below for their histories and definitions), the equivalent characterization of the compactness of commutators corresponding to these aforementioned spaces are still unknown so far. Therefore, it is natural to ask whether or not there exists a unified theory on the equivalent characterization for the boundedness and the compactness of commutators on all aforementioned function spaces. In this article, we give an *affirmative* answer to this question on so-called ball Banach function spaces.

Recall that the ball (quasi-)Banach function space was introduced by Sawano et al. [71] (see also Definition 2.1 below), which contains all aforementioned function spaces as special cases. For more studies of ball Banach function spaces, we refer the reader to [14, 43, 68, 77–80, 83, 84]. Very recently, Chaffee and Cruz-Uribe [13], and Guo et al. [36] studied the necessity of the boundedness of commutators on ball Banach function spaces. However, the sufficiency of the boundedness of commutators on the ball Banach function space X and the equivalent characterization of their compactness on X are still unknown.

In what follows, we always let $(X, \|\cdot\|_X)$ be a ball Banach function space satisfying the following additional assumption which, when it is used, is explicitly indicated in the context.

Assumption 1.1 *The Hardy–Littlewood maximal operator \mathcal{M} [see Eq. 2.5 below for its definition] is bounded on X and X' ; here and thereafter, X' denotes the associate space of X (see Definition 2.3 below for its definition).*

Motivated by the aforementioned results, in this article, we establish the following equivalent characterizations of the boundedness and the compactness of commutators on X .

Theorem 1.2 *Let X be a ball Banach function space satisfying Assumption 1.1, Ω a homogeneous function of degree zero satisfying Eqs. 1.1, 1.2, and 1.3, T_Ω as in Eq. 1.4, and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then*

- (i) $[b, T_\Omega]$ is bounded on X if and only if $b \in \text{BMO}(\mathbb{R}^n)$;
- (ii) $[b, T_\Omega]$ is compact on X if and only if $b \in \text{CMO}(\mathbb{R}^n)$.

Indeed, we prove Theorem 1.2 under much weaker assumptions on Ω ; see Theorems 2.17 and 2.22 [and also Remark 2.23(ii)] below for the boundedness, as well as Theorems 3.1 and 3.2 (and also Remark 3.3) below for the compactness. To obtain these results, we need to overcome the essential difficulty caused by the lack of the explicit expression of the norm of X , via mainly employing three key tools: some elaborate lower and upper estimates, obtained in Propositions 3.14 and 3.16 below, on the norm of X about the commutators and the characteristic functions of some measurable subsets, which are implied by the assumed boundedness of \mathcal{M} on X and its associated space as well as the geometry of \mathbb{R}^n ; the complete John–Nirenberg inequality in X obtained by Sawano et al. in [48]; the generalized Fréchet–Kolmogorov theorem on X established in Theorem 3.6 below. All these results have a wide range of applications, which not only recover several well-known results but also yield some new ones. Particularly, even when $X := L^{p(\cdot)}(\mathbb{R}^n)$ (the variable Lebesgue space), $X := L^{\vec{p}}(\mathbb{R}^n)$ (the mixed-norm Lebesgue space), $X := L^\Phi(\mathbb{R}^n)$ (the Orlicz space), and $X := (E^q_\Phi)_t(\mathbb{R}^n)$ (the Orlicz-slice space or the generalized amalgam space), all these results are new. It should be mentioned that, applying the necessity of boundedness, obtained in Theorem 2.22 below, into six concrete examples of ball Banach function spaces in Section 4, we obtain even better results than [13] and [36] for the necessity of the boundedness of commutators. In addition, the equivalent characterization of the compactness, obtained in Theorems 3.1 and 3.2 below, coincides with Guo et al. [37, Theorems 1.4 and 1.5] about the convolutional singular integral operator on the weighted Lebesgue space.

To be precise, the remainder of this article is organized as follows.

In Section 2, we first show that $[b, T_\Omega]$ is bounded on X for any given $b \in \text{BMO}(\mathbb{R}^n)$ in Theorem 2.17 via the extrapolation theorem. It should be pointed out that ball Banach function spaces are embedded into weighted Lebesgue spaces (see Lemma 2.12 below), which guarantees that the Calderón–Zygmund commutator under consideration is well defined on ball Banach function spaces (see Proposition 2.14 below). Observe that the extrapolation theorem plays an essential role in establishing the boundedness of operators on ball Banach function spaces, which is a bridge connecting the ball Banach function space and the weighted Lebesgue space. Moreover, combining the technique of the local mean oscillation as in [37, 56] and a fine inequality on the norm of X (see Lemma 2.21 below), we also show that, if $[b, T_\Omega]$ is bounded on X , then $b \in \text{BMO}(\mathbb{R}^n)$ in Theorem 2.22 below. As a consequence, Theorem 1.2(i) is a direct corollary of Theorems 2.17 and 2.22.

Section 3 is devoted to Theorem 1.2(ii) which can be easily deduced from two more general results: Theorem 3.1 below (the sufficiency) and Theorem 3.2 below (the necessity). To prove these, we need to overcome some essential difficulties by borrowing some basic ideas from the proof of the recent result on the weighted Lebesgue space given by Guo et al. [37]; see also Uchiyama [76] for the corresponding one on the Lebesgue space, and Chen et al. [19] for the corresponding one on the Morrey space. However, their calculations are no longer completely feasible for the ball Banach function space X because they used the following three crucial properties of the norm under consideration, which are not available for $\|\cdot\|_X$: the Lebesgue dominated convergence theorem, the translation invariance, and the explicit expression of the norm. In the proof of Theorem 3.1, using a skillful decomposition and the smooth truncated technique given by Clop and Cruz [23], we successfully avoid the translation invariance as in Uchiyama [76] or Chen et al. [19]. Moreover, we establish a new criterion on the compactness of a set in the ball Banach function space X (see Theorem 3.6 below), which generalizes the Fréchet–Kolmogorov theorem in [12, 19, 23, 38] to the present setting. Indeed, via establishing a new Minkowski-type inequality for the ball quasi-Banach function space X (see Lemma 3.4 below), we drop the assumption that X has an absolutely continuous norm in [38, Theorem 3.1]. In the proof of Theorem 3.2, since we do not have the aforementioned three key properties on the norm $\|\cdot\|_X$, nearly all the corresponding calculations used in [19, 37, 76] are unworkable in the present setting. To overcome these difficulties, we need to improve the method used in [37]. Indeed, we first establish the lower estimate of commutators in Proposition 3.14 via the aforementioned technique of the local mean oscillation; we then apply an equivalent characterization of $\text{BMO}(\mathbb{R}^n)$ via the ball Banach function space obtained in [48] (see also Lemma 3.15 below) to establish the upper estimate of commutators (see Proposition 3.16 below); from Propositions 3.14 and 3.16, we finally deduce the desired necessity of the equivalent characterization on the compactness of commutators.

In Section 4, we apply all these results obtained in Sections 2 and 3, respectively, to $X := M_r^p(\mathbb{R}^n)$ (the Morrey space) or to $X := L_\omega^p(\mathbb{R}^n)$ (the weighted Lebesgue space), and we find that, even for these well-known function spaces, some of our results also improve the known results (see Remark 4.4 below for more details). Moreover, to the best of our knowledge, when we apply all these results obtained in Sections 2 and 3, respectively, to $X := L^{p(\cdot)}(\mathbb{R}^n)$ (the variable Lebesgue space), $X := L^{\vec{p}}(\mathbb{R}^n)$ (the mixed-norm Lebesgue space), $X := L^\Phi(\mathbb{R}^n)$ (the Orlicz space), or $X := (E_\Phi^r)_t(\mathbb{R}^n)$ (the Orlicz-slice space or the generalized amalgam space), all these results are totally new.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_+^n := (\mathbb{Z}_+)^n$. We always denote by C a positive constant which is independent of the

main parameters, but it may vary from line to line. We also use $C_{(\alpha,\beta,\dots)}$ to denote a positive constant depending on the indicated parameters α, β, \dots . The symbol $f \lesssim g$ means that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If $f \leq Cg$ and $g = h$ or $g \leq h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \leq h$. The symbol $\lfloor s \rfloor$ for any $s \in \mathbb{R}$ denotes the largest integer not greater than s . We use $\mathbf{0}$ to denote the origin of \mathbb{R}^n and let $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$. If E is a subset of \mathbb{R}^n , we denote by $\mathbf{1}_E$ its characteristic function and by E^c the set $\mathbb{R}^n \setminus E$. Furthermore, for any $\alpha \in (0, \infty)$ and any ball $B := B(x_B, r_B)$ in \mathbb{R}^n , with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, we let $\alpha B := B(x_B, \alpha r_B)$. Finally, for any $q \in [1, \infty]$, we denote by q' its conjugate exponent, namely, $1/q + 1/q' = 1$.

2 Boundedness Characterization of Commutators on Ball Banach Function Spaces

In this section, we first present some known facts on the ball quasi-Banach function space X in Section 2.1, and then establish the characterization of the boundedness on X of commutators in Section 2.2.

2.1 Ball Quasi-Banach Function Spaces

We now recall some preliminaries on ball quasi-Banach function spaces introduced in [71]. Denote by the symbol $\mathcal{M}(\mathbb{R}^n)$ the set of all measurable functions on \mathbb{R}^n . For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and

$$\mathbb{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty)\}. \tag{2.1}$$

Definition 2.1 A quasi-Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is called a *ball quasi-Banach function space* if it satisfies

- (i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;
- (ii) $|g| \leq |f|$ almost everywhere implies that $\|g\|_X \leq \|f\|_X$;
- (iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$;
- (iv) $B \in \mathbb{B}$ implies that $\mathbf{1}_B \in X$, where \mathbb{B} is as in Eq. 2.1.

Moreover, a ball quasi-Banach function space X is called a *ball Banach function space* if the norm of X satisfies the triangle inequality: for any $f, g \in X$,

$$\|f + g\|_X \leq \|f\|_X + \|g\|_X, \tag{2.2}$$

and, for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$, depending on B , such that, for any $f \in X$,

$$\int_B |f(x)| dx \leq C_{(B)} \|f\|_X. \tag{2.3}$$

Remark 2.2 (i) Observe that, in Definition 2.1, if we replace any ball B by any bounded measurable set E , we obtain its another equivalent formulation.

- (ii) Recall that a quasi-Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is called a *Banach function space* if it is a ball Banach function space and it satisfies Definition 2.1(iv) with ball replaced by any measurable set of *finite measure* (see, for instance, [10, Chapter 1, Definitions 1.1 and 1.3]). It is easy to see that every Banach function space is a ball Banach function space, and the converse is not necessary to be true. As was mentioned in [71, p. 9] and

[78, Section 5], the family of ball Banach function spaces includes Morrey spaces, mixed-norm Lebesgue spaces, variable Lebesgue spaces, weighted Lebesgue spaces, and Orlicz-slice spaces, all of which are not necessary to be Banach function spaces.

The following notion of the associate space of a ball Banach function space can be found, for instance, in [10, Chapter 1, Definitions 2.1 and 2.3].

Definition 2.3 For any ball Banach function space X , the *associate space* (also called the *Köthe dual*) X' is defined by setting

$$X' := \{f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{X'} < \infty\}, \tag{2.4}$$

where, for any $f \in \mathcal{M}(\mathbb{R}^n)$,

$$\|f\|_{X'} := \sup_{\{g \in X : \|g\|_X = 1\}} \|fg\|_{L^1(\mathbb{R}^n)}$$

and $\|\cdot\|_{X'}$ is called the *associate norm* of $\|\cdot\|_X$.

Remark 2.4 By [71, Proposition 2.3], we know that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

The following lemma is just [85, Lemma 2.6].

Lemma 2.5 *Let X be a ball quasi-Banach function space satisfying the triangle inequality as in Eq. 2.2. Then X coincides with its second associate space X'' . In other words, a function f belongs to X if and only if it belongs to X'' and, in that case,*

$$\|f\|_X = \|f\|_{X''}.$$

The following Hölder inequality is a direct corollary of both Definition 2.1(i) and Eq. 2.4 (see [10, Theorem 2.4]).

Lemma 2.6 *Let X be a ball quasi-Banach function space satisfying the triangle inequality as in Eq. 2.2, and X' its associate space. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq \|f\|_X \|g\|_{X'}.$$

Also, recall the following notion of the convexity of ball quasi-Banach spaces, which is a part of [71, Definition 2.6].

Definition 2.7 Let X be a ball quasi-Banach function space and $p \in (0, \infty)$. The *p -convexification* X^p of X is defined by setting $X^p := \{f \in \mathcal{M}(\mathbb{R}^n) : |f|^p \in X\}$ equipped with the quasi-norm $\|f\|_{X^p} := \| |f|^p \|_X^{1/p}$.

In what follows, we denote by the symbol $L^1_{loc}(\mathbb{R}^n)$ the set of all locally integrable functions on \mathbb{R}^n . The *Hardy–Littlewood maximal operator* \mathcal{M} is defined by setting, for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \tag{2.5}$$

where the supremum is taken over all balls $B \in \mathbb{B}$ containing x .

For any $\theta \in (0, \infty)$, the *powered Hardy–Littlewood maximal operator* $\mathcal{M}^{(\theta)}$ is defined by setting, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^{(\theta)}(f)(x) := \{\mathcal{M}(|f|^\theta)(x)\}^{1/\theta}. \tag{2.6}$$

The following lemma is a part of [85, Remark 2.19(i)].

Lemma 2.8 *Let $\theta \in (0, \infty)$ and X be a ball quasi-Banach function space. Assume that there exists a positive constant C such that, for any $f \in \mathcal{M}(\mathbb{R}^n)$,*

$$\|\mathcal{M}^{(\theta)}(f)\|_X \leq C \|f\|_X.$$

Then there exists a positive constant \tilde{C} such that, for any ball $B \in \mathbb{B}$ and $\beta \in [1, \infty)$,

$$\|\mathbf{1}_{\beta B}\|_X \leq \tilde{C} \beta^{n/\theta} \|\mathbf{1}_B\|_X, \tag{2.7}$$

where the positive constant C is independent of $B \in \mathbb{B}$ and β .

Remark 2.9 From [71, Lemma 2.15(ii)], we deduce that, if \mathcal{M} is bounded on X , then there exists an $\eta \in (1, \infty)$ such that $\mathcal{M}^{(\eta)}$ is bounded on X , where $\mathcal{M}^{(\eta)}$ is as in Eq. 2.6 with θ replaced by η .

2.2 Sufficiency and Necessity of the Boundedness of Commutators

In this subsection, we obtain the sufficiency and the necessity of the boundedness of commutators, respectively, in Theorems 2.17 and 2.22 below.

First, recall the following notion of Muckenhoupt weights $A_p(\mathbb{R}^n)$ (see, for instance, [33]).

Definition 2.10 An $A_p(\mathbb{R}^n)$ -weight ω , with $p \in [1, \infty)$, is a locally integrable and nonnegative function on \mathbb{R}^n satisfying that, when $p \in (1, \infty)$,

$$[\omega]_{A_p(\mathbb{R}^n)} := \sup_{B \in \mathbb{B}} \left[\frac{1}{|B|} \int_B \omega(x) dx \right] \left\{ \frac{1}{|B|} \int_B [\omega(x)]^{\frac{1}{1-p}} dx \right\}^{p-1} < \infty$$

and, when $p = 1$,

$$[\omega]_{A_1(\mathbb{R}^n)} := \sup_{B \in \mathbb{B}} \frac{1}{|B|} \int_B \omega(x) dx \left[\|\omega^{-1}\|_{L^\infty(B)} \right] < \infty,$$

where \mathbb{B} is as in Eq. 2.1. Define $A_\infty(\mathbb{R}^n) := \bigcup_{p \in [1, \infty)} A_p(\mathbb{R}^n)$.

Definition 2.11 Let $p \in (0, \infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$. The *weighted Lebesgue space* $L^p_\omega(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on \mathbb{R}^n such that

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right]^{\frac{1}{p}} < \infty.$$

By the proof of [25, Theorem 10.1], we have the following technical lemma, which plays a vital role in the proof of Proposition 2.14 below.

Lemma 2.12 *Let X be a ball Banach function space satisfying Assumption 1.1, and $p_0 \in [1, \infty)$. Then $X \subset \bigcup_{\omega \in A_{p_0}(\mathbb{R}^n)} L^{p_0}_\omega(\mathbb{R}^n)$.*

Similarly to [25, Theorem 10.1], we have the following conclusion, whose proof is a slight modification of the corresponding one of [25, Theorem 10.1]; we omit the details.

Lemma 2.13 *Let X be a ball Banach function space satisfying Assumption 1.1, and $p_0 \in [1, \infty)$. Let \mathcal{F} be the set of all pairs of nonnegative measurable functions (F, G) such that, for any given $\omega \in A_{p_0}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} [F(x)]^{p_0} \omega(x) dx \leq C_{(p_0, [\omega]_{A_{p_0}(\mathbb{R}^n)})} \int_{\mathbb{R}^n} [G(x)]^{p_0} \omega(x) dx,$$

where the positive constant $C_{(p_0, [\omega]_{A_{p_0}(\mathbb{R}^n)})}$ is independent of (F, G) , but depending on p_0 and $[\omega]_{A_{p_0}(\mathbb{R}^n)}$. Then there exists a positive constant C_0 such that, for any $(F, G) \in \mathcal{F}$ with $\|F\|_X < \infty$,

$$\|F\|_X \leq C_0 \|G\|_X.$$

To study the boundedness of commutators in this article, we modify Lemma 2.13 as follows.

Proposition 2.14 *Let X be a ball Banach function space satisfying Assumption 1.1, and $s \in [1, \infty)$. Let \mathcal{T} be an operator satisfying, for any given $\omega \in A_s(\mathbb{R}^n)$ and any $f \in L^s_\omega(\mathbb{R}^n)$,*

$$\|\mathcal{T}(f)\|_{L^s_\omega(\mathbb{R}^n)} \leq C_{(s, [\omega]_{A_s(\mathbb{R}^n)})} \|f\|_{L^s_\omega(\mathbb{R}^n)},$$

where $C_{(s, [\omega]_{A_s(\mathbb{R}^n)})}$ is a positive constant independent of f , but depending on s and $[\omega]_{A_s(\mathbb{R}^n)}$. Then there exists a positive constant C such that, for any $f \in X$,

$$\|\mathcal{T}(f)\|_X \leq C \|f\|_X. \tag{2.8}$$

Proof Let X be a ball quasi-Banach function space satisfying Assumption 1.1, and $s \in [1, \infty)$. To show Eq. 2.8, let

$$\mathcal{F} := \left\{ (\max\{|\mathcal{T}(f)|, N\} \mathbf{1}_{B(\mathbf{0}, N)}, |f|) : f \in \bigcup_{\omega \in A_s(\mathbb{R}^n)} L^s_\omega(\mathbb{R}^n), N \in \mathbb{N} \right\}.$$

Then, by the assumption on \mathcal{T} , we obtain, for any given $\omega \in A_s(\mathbb{R}^n)$ and any $f \in \bigcup_{\omega \in A_s(\mathbb{R}^n)} L^s_\omega(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\mathcal{T}(f)(x)|^s \omega(x) dx \leq \left[C_{(s, [\omega]_{A_s(\mathbb{R}^n)})} \right]^s \int_{\mathbb{R}^n} |f(x)|^s \omega(x) dx,$$

which, together with Lemma 2.13, further implies that, for any $f \in \bigcup_{\omega \in A_s(\mathbb{R}^n)} L^s_\omega(\mathbb{R}^n)$ and $N \in \mathbb{N}$,

$$\|\max\{|\mathcal{T}(f)|, N\} \mathbf{1}_{B(\mathbf{0}, N)}\|_X \lesssim \|f\|_X.$$

From this and Definition 2.1(iii), it follows that, for any $f \in \bigcup_{\omega \in A_s(\mathbb{R}^n)} L^s_\omega(\mathbb{R}^n)$,

$$\|\mathcal{T}(f)\|_X \lesssim \|f\|_X. \tag{2.9}$$

By Lemma 2.12, we know that $X \subset \bigcup_{\omega \in A_s(\mathbb{R}^n)} L^s_\omega(\mathbb{R}^n)$, which, combined with Eq. 2.9, implies the desired boundedness and hence completes the proof of Proposition 2.14. \square

In order to introduce singular integral operators with homogeneous kernel, we now state the following notion of the L^∞ -Dini condition.

Definition 2.15 A function $\Omega \in L^\infty(\mathbb{S}^{n-1})$ is said to satisfy the L^∞ -Dini condition if

$$\int_0^1 \frac{\omega_\infty(\tau)}{\tau} d\tau < \infty, \tag{2.10}$$

where, for any $\tau \in (0, 1)$,

$$\omega_\infty(\tau) := \sup_{\{x, y \in \mathbb{S}^{n-1} : |x-y| < \tau\}} |\Omega(x) - \Omega(y)|.$$

Recall that the symbol $a \rightarrow 0^+$ means that $a \in (0, \infty)$ and $a \rightarrow 0$. Through this article, assuming that Ω satisfies Eqs. 1.2 and 1.3, and the L^∞ -Dini condition, a linear operator T_Ω is called a *singular integral operator with homogeneous kernel* Ω (see, for instance, [58, p. 53, Corollary 2.1.1]) if, for any $f \in L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$, and for any $x \in \mathbb{R}^n$, Eq. 1.4 holds true. Let X be a ball quasi-Banach function space satisfying Assumption 1.1. By Lemma 2.12 and [31, Corollary 7.13], we know that, for any $f \in X$, $T_\Omega(f)(x)$ exists for almost every $x \in \mathbb{R}^n$.

For any given $b \in L^1_{loc}(\mathbb{R}^n)$, the commutator $[b, T_\Omega]$ is defined by setting, for any bounded function f with compact support, and for any $x \in \mathbb{R}^n$,

$$[b, T_\Omega](f)(x) := b(x)T_\Omega(f)(x) - T_\Omega(bf)(x). \tag{2.11}$$

To prove Theorem 2.17, we need the following weighted $L^p(\mathbb{R}^n)$ boundedness of the commutator $[b, T_\Omega]$, which is a part of [58, Theorem 2.4.4].

Lemma 2.16 Let $p \in (1, \infty)$, $\omega \in A_p(\mathbb{R}^n)$, and $q \in (1, \infty]$ satisfy $q' \leq p$ with $1/q + 1/q' = 1$. Assume that $b \in \text{BMO}(\mathbb{R}^n)$, $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfies Eqs. 1.2 and 1.3, and T_Ω is a singular integral operator with homogeneous kernel Ω . Then there exists a positive constant $C_{(p, \Omega, [\omega]_{A_p(\mathbb{R}^n)})}$, depending on p , Ω , and $[\omega]_{A_p(\mathbb{R}^n)}$, such that, for any $f \in L^p_\omega(\mathbb{R}^n)$,

$$\begin{aligned} & \int_{\mathbb{R}^n} |[b, T_\Omega](f)(x)|^p \omega(x) dx \\ & \leq C_{(p, \Omega, [\omega]_{A_p(\mathbb{R}^n)})} \|b\|_{\text{BMO}(\mathbb{R}^n)}^p \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx. \end{aligned}$$

Then we immediately have the following sufficiency of the boundedness of commutators on ball Banach function spaces.

Theorem 2.17 Let X be a ball Banach function space satisfying Assumption 1.1. Let $q \in (1, \infty]$. Assume that $b \in \text{BMO}(\mathbb{R}^n)$, $\Omega \in L^q(\mathbb{S}^{n-1})$ satisfies Eqs. 1.2 and 1.3, and T_Ω is a singular integral operator with homogeneous kernel Ω . Then there exists a positive constant C such that, for any $f \in X$,

$$\|[b, T_\Omega](f)\|_X \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_X.$$

Proof Using Lemma 2.16 and Proposition 2.14, we immediately complete the proof of Theorem 2.17. □

Remark 2.18 Let $X := L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$. Assume that $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfies Eqs. 1.1, 1.2, and 1.3. Then, in this case, Theorem 2.17 coincides with the classical conclusion in [24, Theorem 1]. Compared with the assumptions on Ω in [24, Theorem 1], the assumptions on Ω in Theorem 2.17 are much weaker.

Now, we show the necessity of the boundedness of commutators. To this end, we need three key lemmas, namely, Lemmas 2.19, 2.20, and 2.21, respectively.

First, recall that, for any given measurable function f , the *non-increasing rearrangement* of f is defined by setting, for any $t \in (0, \infty)$,

$$f^*(t) := \inf\{\alpha \in (0, \infty) : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < t\};$$

for any given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and any given ball $B \subset \mathbb{R}^n$, the *local mean oscillation* of f on B is defined by setting, for any $\lambda \in (0, 1)$,

$$\omega_\lambda(f; B) := \inf_{c \in \mathbb{C}} \{[(f - c)\mathbf{1}_B]^*(\lambda|B|)\}. \tag{2.12}$$

The following characterization of $\text{BMO}(\mathbb{R}^n)$ is a part of [37, Lemma 2.5]; see also [56, Lemma 2.1].

Lemma 2.19 *Let $\lambda \in (0, 1/2]$. Then there exists a positive constant C such that, for any $f \in \text{BMO}(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{\text{BMO}(\mathbb{R}^n)} \leq \sup_{\text{ball } B \subset \mathbb{R}^n} \omega_\lambda(f; B) \leq C\|f\|_{\text{BMO}(\mathbb{R}^n)}.$$

Moreover, the following geometrical lemma is just [37, Proposition 4.1] with cubes replaced by balls.

Lemma 2.20 *Let $\lambda \in (0, 1)$ and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy that there exists an open set $\Lambda \subset \mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Λ . Then there exist an $\varepsilon_0 \in (0, \infty)$ and a $k_0 \in (10\sqrt{n}, \infty)$, depending only on Ω and n , such that, for any given ball $B(x_0, r_0) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r_0 \in (0, \infty)$, there exist an $x_1 \in \mathbb{R}^n$ and measurable sets $E \subset B(x_0, r_0)$ with $|E| = \frac{1}{2}|B(x_0, r_0)|$, $F \subset B(x_1, r_0)$ with $|x_1 - x_0| = 2k_0r_0$ and $|F| = \frac{1}{2}|B(x_1, r_0)|$, and $G \subset E \times F$ with $|G| \geq \frac{\lambda}{8}|B(x_0, r_0)|^2$ having the following properties:*

- (i) *for any $x \in E$ and $y \in F$, $\omega_\lambda(b; B) \leq |b(x) - b(y)|$;*
- (ii) *$\Omega(\frac{x-y}{|x-y|})$ and $b(x) - b(y)$ do not change sign on $E \times F$;*
- (iii) *for any $(x, y) \in G$, $|\Omega(\frac{x-y}{|x-y|})| \geq \varepsilon_0$.*

In addition, the following lemma shows that, for any ball B , the converse of Lemma 2.6 also holds true with f and g replaced by $\mathbf{1}_B$, which is a part of [49, Lemma 2.2 and Remark 2.3].

Lemma 2.21 *Let X be a ball Banach function space such that \mathcal{M} is bounded on X . Then there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$,*

$$\frac{1}{|B|} \|\mathbf{1}_B\|_X \|\mathbf{1}_B\|_{X'} \leq C.$$

In what follows, for any operator \mathcal{L} mapping X into itself, we use $\|\mathcal{L}\|_{X \rightarrow X}$ to denote its *operator norm*. Also, it is natural to assume that, for any given $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, and any bounded measurable set $F \subset \mathbb{R}^n$, $[b, T_\Omega](\mathbf{1}_F)$ has the following integral representation: for any $x \in \mathbb{R}^n \setminus \bar{F}$,

$$[b, T_\Omega](\mathbf{1}_F)(x) = \int_F [b(x) - b(y)] \frac{\Omega(x - y)}{|x - y|^n} dy. \tag{2.13}$$

Theorem 2.22 *Let X be a ball Banach function space and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$. Assume that \mathcal{M} is bounded on X . Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy that there exists an open set $\Lambda \subset \mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Λ . If $[b, T_\Omega]$ is bounded on X and satisfies Eq. 2.13, then $b \in \text{BMO}(\mathbb{R}^n)$ and there exists a positive constant C , independent of b , such that*

$$\|b\|_{\text{BMO}(\mathbb{R}^n)} \leq C \|[b, T_\Omega]\|_{X \rightarrow X}.$$

Proof Let $\lambda \in (0, 1/2]$. To prove this theorem, by Lemma 2.19, it suffices to show that there exists a positive constant C , independent of b , such that, for any ball $B \subset \mathbb{R}^n$,

$$\omega_\lambda(b; B) \leq C \|[b, T_\Omega]\|_{X \rightarrow X}. \tag{2.14}$$

Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $B := B(x_0, r_0)$ with $x_0 \in \mathbb{R}^n$ and $r_0 \in (0, \infty)$. Let ε_0, k_0, G, E , and F be as in Lemma 2.20. Then, by (i) and (iii) of Lemma 2.20, we conclude that

$$\omega_\lambda(b; B)|G| \leq \frac{1}{\varepsilon_0} \int_G |b(x) - b(y)| \left| \Omega \left(\frac{x - y}{|x - y|} \right) \right| dx dy.$$

From this, the fact that $|x - y| \leq 2(k_0 + 1)r_0$ for any $(x, y) \in G$, Lemma 2.20(ii), $|G| \geq \frac{\lambda}{8}|B|^2$, Eq. 2.13, and the observation $\bar{E} \cap \bar{F} = \emptyset$, we deduce that

$$\begin{aligned} \omega_\lambda(b; B) &\leq \frac{[2(k_0 + 1)r_0]^n}{\varepsilon_0|G|} \int_G |b(x) - b(y)| \frac{|\Omega(\frac{x-y}{|x-y|})|}{|x-y|^n} dx dy \\ &\leq \frac{8[2(k_0 + 1)r_0]^n}{\varepsilon_0\lambda|B|^2} \int_E \left| \int_F [b(x) - b(y)] \frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^n} dy \right| dx \\ &\leq \frac{8[2(k_0 + 1)r_0]^n}{\varepsilon_0\lambda|B|^2} \int_E |[b, T_\Omega](\mathbf{1}_F)(x)| dx, \end{aligned}$$

which, combined with Lemmas 2.6 and 2.21, Eq. 2.7, and the fact that $[b, T_\Omega]$ is bounded on X , further implies that

$$\begin{aligned} \omega_\lambda(b; B) &\leq \frac{8[2(k_0 + 1)r_0]^n}{\varepsilon_0\lambda|B|^2} \|[b, T_\Omega](\mathbf{1}_F)\|_X \|\mathbf{1}_B\|_{X'} \\ &\lesssim \frac{1}{|B|} \|[b, T_\Omega]\|_{X \rightarrow X} \|\mathbf{1}_{k_0 B}\|_X \|\mathbf{1}_B\|_{X'} \\ &\lesssim \|[b, T_\Omega]\|_{X \rightarrow X} \frac{1}{|B|} \|\mathbf{1}_B\|_X \|\mathbf{1}_B\|_{X'} \\ &\lesssim \|[b, T_\Omega]\|_{X \rightarrow X}, \end{aligned}$$

where the implicit positive constants depend only on $\lambda, k_0, \varepsilon_0$, and n . This finishes the proof of Eq. 2.14 and hence of Theorem 2.22. □

Remark 2.23 (i) The necessity of the boundedness of commutators was also obtained by Guo et al. [36, Theorem 2.1], under the assumption that there exists an open set $\Lambda \subset \mathbb{S}^{n-1}$ such that, for any $x \in \mathbb{S}^{n-1}$, $c \leq \Omega(x) \leq C$ with constants c and C satisfying $0 < c < C$ or $C < c < 0$, while, in Theorem 2.22, Ω is assumed to never vanish and never change sign on Λ . Thus, Theorem 2.22 generalizes the corresponding conclusion of [36, Theorem 2.1].

(ii) It is easy to see that Theorem 1.2(i) is a direct corollary of Theorems 2.17 and 2.22.

3 Compactness Characterization of Commutators on Ball Banach Function Spaces

In this section, applying Theorems 2.17 and 2.22, we further investigate the compactness of the commutator on ball Banach function spaces.

In what follows, the space $\text{CMO}(\mathbb{R}^n)$ is defined to be the closure in $\text{BMO}(\mathbb{R}^n)$ of $C_c^\infty(\mathbb{R}^n)$ [the set of all infinitely differentiable functions on \mathbb{R}^n with compact support]. Recall that the Hardy–Littlewood operator \mathcal{M} is defined in Eq. 2.5, and the commutator $[b, T_\Omega]$ in Eq. 2.11.

Theorem 3.1 *Let X be a ball Banach function space satisfying Assumption 1.1, $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy Eqs. 1.2, 1.3, and 2.10, and T_Ω be a singular integral operator with homogeneous kernel Ω . If $b \in \text{CMO}(\mathbb{R}^n)$, then the commutator $[b, T_\Omega]$ is compact on X .*

Theorem 3.2 *Let X be a ball Banach function space satisfying Assumption 1.1, and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy that there exists an open set $\Lambda \subset \mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Λ . If $[b, T_\Omega]$ is compact on X and satisfies Eq. 2.13, then $b \in \text{CMO}(\mathbb{R}^n)$.*

Remark 3.3 It is easy to see that the assumptions on Ω in Theorems 3.1 and 3.2 are much weaker than the Lipschitz condition which was also used in Uchiyama [76, Theorems 1 and 2], and hence Theorem 1.2(ii) is a direct corollary of Theorems 3.1 and 3.2.

The proofs of Theorems 3.1 and 3.2 are given, respectively, in Sections 3.1 and 3.2 below.

3.1 Proof of Theorem 3.1

To show Theorem 3.1, we need several key lemmas. The first one is the following Minkowski-type inequality for ball Banach function spaces.

Lemma 3.4 *Let X be a ball Banach function space, E a measurable subset of \mathbb{R}^n , and F a measurable function on $\mathbb{R}^n \times E$. Then*

$$\left\| \int_E |F(\cdot, y)| dy \right\|_X \leq |E| \sup_{y \in E} \|F(\cdot, y)\|_X.$$

Proof By Lemma 2.5 and the fact that X is a ball Banach function space, we have

$$\begin{aligned} \left\| \int_E |F(\cdot, y)| dy \right\|_X &= \left\| \int_E |F(\cdot, y)| dy \right\|_{X''} \\ &= \sup_{\{g \in X': \|g\|_{X'}=1\}} \left| \int_{\mathbb{R}^n} \int_E |F(x, y)| dy g(x) dx \right|. \end{aligned} \tag{3.1}$$

From the Tonelli theorem and Lemma 2.6, it follows that, for any $g \in X'$ such that $\|g\|_{X'} = 1$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_E F(x, y) dy g(x) dx \right| \\ & \leq \int_{\mathbb{R}^n} \int_E |F(x, y)| |g(x)| dy dx = \int_E \int_{\mathbb{R}^n} |F(x, y)| |g(x)| dx dy \\ & \leq \int_E \sup_{y \in E} \int_{\mathbb{R}^n} |F(x, y)| |g(x)| dx dy \lesssim \int_E \sup_{y \in E} \|F(\cdot, y)\|_X \|g\|_{X'} dy \\ & \sim |E| \sup_{y \in E} \|F(\cdot, y)\|_X, \end{aligned}$$

which, together with Eq. 3.1, then implies the desired inequality. This finishes the proof of Lemma 3.4. □

Definition 3.5 Let $\epsilon \in (0, \infty)$, \mathcal{F} be a subset of the ball Banach function space X , and $\mathcal{G} \subset \mathcal{F}$. Then \mathcal{G} is called an ϵ -net of \mathcal{F} if, for any $f \in \mathcal{F}$, there exists a $g \in \mathcal{G}$ such that $\|f - g\|_X < \epsilon$. Moreover, if \mathcal{G} is an ϵ -net of \mathcal{F} and the cardinality of \mathcal{G} is finite, then \mathcal{G} is called a *finite ϵ -net* of \mathcal{F} . Furthermore, \mathcal{F} is said to be *totally bounded* if, for any $\epsilon \in (0, \infty)$, there exists a finite ϵ -net. In addition, \mathcal{F} is said to be *relatively compact* if the closure in X of \mathcal{F} is compact.

From the Hausdorff theorem (see, for instance, [81, p. 13, Theorem]), it follows that a subset \mathcal{F} of a ball Banach function space X is relatively compact if and only if \mathcal{F} is totally bounded due to the completeness of X .

Next, we give a sufficient condition for subsets of ball Banach function spaces to be totally bounded, which is a generalization in X of the well-known Fréchet–Kolmogorov theorem in $L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$. In what follows, we use $C_c(\mathbb{R}^n)$ to denote the set of all continuous functions with compact support.

Theorem 3.6 *Let X be a ball Banach function space. Then a subset \mathcal{F} of X is totally bounded if the set \mathcal{F} satisfies the following three conditions:*

- (i) \mathcal{F} is bounded, namely,

$$\sup_{f \in \mathcal{F}} \|f\|_X < \infty;$$
- (ii) \mathcal{F} uniformly vanishes at infinity, namely, for any given $\epsilon \in (0, \infty)$, there exists a positive constant M such that, for any $f \in \mathcal{F}$,

$$\|f \mathbf{1}_{\{x \in \mathbb{R}^n: |x| > M\}}\|_X < \epsilon;$$
- (iii) \mathcal{F} is uniformly equicontinuous, namely, for any given $\epsilon \in (0, \infty)$, there exists a positive constant ρ such that, for any $f \in \mathcal{F}$ and $\xi \in \mathbb{R}^n$ with $|\xi| \in [0, \rho)$,

$$\|f(\cdot + \xi) - f(\cdot)\|_X < \epsilon.$$

Conversely, assume that X satisfies the following additional assumptions that $C_c(\mathbb{R}^n)$ is dense in X and, for any $f \in X$ and $y \in \mathbb{R}^n$,

$$\|f\|_X = \|f(\cdot + y)\|_X. \tag{3.2}$$

If a subset \mathcal{F} of X is totally bounded, then \mathcal{F} satisfies (i) through (iii).

Proof We first show the first part of this theorem. To achieve this, let $\mathcal{F} \subset X$ satisfy (i), (ii), and (iii). We now prove that \mathcal{F} is totally bounded. To this end, by the fact that X is a Banach space, it suffices to find a finite ϵ -net of \mathcal{F} for any given $\epsilon \in (0, \infty)$.

For any $i \in \mathbb{Z}$, let $\mathcal{R}_i := [2^{-i}, 2^i]^n$. From (ii), we deduce that there exists an $M \in \mathbb{N}$ such that, for any $f \in \mathcal{F}$,

$$\|f - f\mathbf{1}_{\mathcal{R}_M}\|_X < \epsilon/3.$$

Therefore, to find a finite ϵ -net of \mathcal{F} , it suffices to find a finite $(2\epsilon/3)$ -net of $\{f\mathbf{1}_{\mathcal{R}_M}\}_{f \in \mathcal{F}}$. To achieve this, we use the following finite dimensional method similar to that used in [23, 38].

First, by (iii), we conclude that there exists an $i_\epsilon \in \mathbb{Z}$ such that, for any $f \in \mathcal{F}$ and $\xi \in \mathcal{R}_{i_\epsilon}$,

$$\|f(\cdot + \xi) - f(\cdot)\|_X < 2^{-n}\epsilon/3. \tag{3.3}$$

Observe that, for any $x \in \mathcal{R}_M$, there exists a unique dyadic cube $Q_x := \prod_{j=1}^n [m_j 2^{i_\epsilon}, (m_j + 1)2^{i_\epsilon}]$ which has the side length 2^{i_ϵ} and contains x for some integers $\{m_j\}_{j=1}^n$. Let $\mathcal{Q} := \{Q_x : x \in \mathcal{R}_M\}$. Moreover, for any $f \in \mathcal{F}$ and $x \in \mathbb{R}^n$, let

$$\Phi(f\mathbf{1}_{\mathcal{R}_M})(x) := \begin{cases} f_{Q_x} = \frac{1}{|Q_x|} \int_{Q_x} f(y) dy & \text{if } x \in \mathcal{R}_M, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \mathcal{R}_M. \end{cases}$$

By Eq. 2.3, we know that Φ is well defined, namely, for any given $f \in \mathcal{F}$ and any $x \in \mathcal{R}_M$, $\Phi(f\mathbf{1}_{\mathcal{R}_M})(x) < \infty$.

Now, we estimate $\|f\mathbf{1}_{\mathcal{R}_M} - \Phi(f\mathbf{1}_{\mathcal{R}_M})\|_X$ for any given $f \in \mathcal{F}$. To this end, notice that, for any $x \in \mathcal{R}_M$,

$$\begin{aligned} & |[f(x) - f_{Q_x}]\mathbf{1}_{Q_x}(x)| \\ &= \left| \frac{1}{|Q_x|} \int_{Q_x} [f(x) - f(y)] dy \mathbf{1}_{Q_x}(x) \right| \\ &= \frac{1}{|Q_x|} \int_{Q_x} |f(x) - f(y)| dy \mathbf{1}_{Q_x}(x) \\ &\leq \frac{1}{|Q_x|} \int_{\mathcal{R}_{i_\epsilon}} |f(x) - f(x + \xi)| d\xi \mathbf{1}_{Q_x}(x). \end{aligned}$$

From this, we deduce that, for any $x \in \mathcal{R}_M$,

$$\left| \sum_{Q \in \mathcal{Q}} [f(x) - f_Q]\mathbf{1}_Q(x) \right| \leq 2^{-ni_\epsilon} \int_{\mathcal{R}_{i_\epsilon}} |f(x) - f(x + \xi)| d\xi \mathbf{1}_{\mathcal{R}_M}(x)$$

and hence, by Lemma 3.4 and Eq. 3.3, we have

$$\begin{aligned} & \|f\mathbf{1}_{\mathcal{R}_M} - \Phi(f\mathbf{1}_{\mathcal{R}_M})\|_X \\ &= \left\| \sum_{Q \in \mathcal{Q}} [f(\cdot) - f_Q]\mathbf{1}_Q(\cdot) \right\|_X \\ &\leq 2^{-ni_\epsilon} \left\| \int_{\mathcal{R}_{i_\epsilon}} |f(\cdot) - f(\cdot + \xi)| d\xi \mathbf{1}_{\mathcal{R}_M}(\cdot) \right\|_X \\ &\leq 2^{-ni_\epsilon} |\mathcal{R}_{i_\epsilon}| \sup_{\xi \in \mathcal{R}_{i_\epsilon}} \|f(\cdot) - f(\cdot + \xi)\|_X \\ &= 2^n \sup_{\xi \in \mathcal{R}_{i_\epsilon}} \|f(\cdot) - f(\cdot + \xi)\|_X < \epsilon/3. \end{aligned} \tag{3.4}$$

This is the desired estimate.

In addition, it is easy to see that $\{\Phi(f\mathbf{1}_{\mathcal{R}_M})\}_{f \in \mathcal{F}}$ is a bounded subset of a finite dimensional Banach space, which implies that $\{\Phi(f\mathbf{1}_{\mathcal{R}_M})\}_{f \in \mathcal{F}}$ has a finite $(\epsilon/3)$ -net. This, together with Eq. 3.4, shows that there exists a finite $(2\epsilon/3)$ -net of $\{f\mathbf{1}_{\mathcal{R}_M}\}_{f \in \mathcal{F}}$, which completes the proof of the first part of this theorem.

Now, we show the second part of this theorem. Let \mathcal{F} be a totally bounded set of X . Then, by the definition of totally bounded sets, we easily know that (i) holds true.

Next, for any given $\epsilon \in (0, \infty)$, let $\{U_1, \dots, U_m\}$ be a finite $(\epsilon/3)$ -net of \mathcal{F} , and choose $g_j \in U_j$ for any $j \in \{1, \dots, m\}$. By the additional assumption that $C_c(\mathbb{R}^n)$ is dense in X , we may assume that g_j also belongs to $C_c(\mathbb{R}^n)$ for any $j \in \{1, \dots, m\}$, which implies that there exists a positive constant M such that, for any $j \in \{1, \dots, m\}$,

$$g_j \mathbf{1}_{\{x \in \mathbb{R}^n: |x| > M\}} = 0.$$

Thus, for any given $j \in \{1, \dots, m\}$, if $f \in U_j$, then $\|f - g_j\|_X < \epsilon/3$ and hence

$$\begin{aligned} & \|f \mathbf{1}_{\{x \in \mathbb{R}^n: |x| > M\}}\|_X \\ & \leq \|(f - g_j) \mathbf{1}_{\{x \in \mathbb{R}^n: |x| > M\}}\|_X + \|g_j \mathbf{1}_{\{x \in \mathbb{R}^n: |x| > M\}}\|_X \\ & \leq \|f - g_j\|_X + \|g_j \mathbf{1}_{\{x \in \mathbb{R}^n: |x| > M\}}\|_X < \epsilon/3 < \epsilon. \end{aligned}$$

This shows that (ii) holds true.

Finally, for any given $j \in \{1, \dots, m\}$, by $g_j \in C_c(\mathbb{R}^n)$, we conclude that there exists a positive constant ρ such that, for any $\xi \in \mathbb{R}^n$ with $|\xi| < \rho$,

$$\|g_j(\cdot + \xi) - g_j(\cdot)\|_X < \epsilon/3. \tag{3.5}$$

Moreover, for any given $f \in \mathcal{F}$, there exists a $g_j \in U_j \cap C_c(\mathbb{R}^n)$ with some $j \in \{1, \dots, m\}$ such that $\|f - g_j\|_X < \epsilon/3$, which, combined with Eqs. 3.2 and 3.5, further implies that, for any $\xi \in \mathbb{R}^n$ with $|\xi| < \rho$,

$$\begin{aligned} & \|f(\cdot + \xi) - f(\cdot)\|_X \\ & \leq \|f(\cdot + \xi) - g_j(\cdot + \xi)\|_X + \|g_j(\cdot + \xi) - g_j(\cdot)\|_X + \|g_j - f\|_X \\ & < \epsilon. \end{aligned}$$

This shows that (iii) holds true, which completes the proof of the second part of this theorem and hence of Theorem 3.6. □

Remark 3.7 (i) In the second part of Theorem 3.6, the additional assumption Eq. 3.2 is reasonable because, even when X is the weighted Lebesgue space, if $f \in X$, then $f(\cdot + \xi)$ may not be in X even when $|\xi|$ is small.

(ii) If X has an absolutely continuous quasi-norm, then $C_c(\mathbb{R}^n)$ is dense in X [see Proposition 3.8 below]. Recall that a ball quasi-Banach function space X is said to have an *absolutely continuous quasi-norm* if, for any $f \in X$ and any sequence of measurable sets, $\{E_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$, satisfying that $\mathbf{1}_{E_j} \rightarrow 0$ almost everywhere as $j \rightarrow \infty$, $\|f \mathbf{1}_{E_j}\|_X \rightarrow 0$ as $j \rightarrow \infty$.

Proposition 3.8 *Let X be a ball quasi-Banach function space having an absolutely continuous quasi-norm. Then $C_c(\mathbb{R}^n)$ is dense in X .*

Proof Without loss of generality, we may let $f \in X$ be a non-negative measurable function on \mathbb{R}^n . Then there exists an increasing sequence of non-negative simple functions, $\{f_j\}_{j \in \mathbb{N}}$, which converges pointwise to f as $j \rightarrow \infty$. From this and [10, p. 16, Proposition 3.6], it

follows that, for any given $\varepsilon \in (0, \infty)$, there exists a simple function $g := \sum_{k=1}^N \lambda_k \mathbf{1}_{E_k} \in \{f_j\}_{j \in \mathbb{N}}$ such that

$$\|f - g\|_X < \varepsilon,$$

where, for any $k \in \{1, \dots, N\}$, E_k is a measurable set and λ_k is a positive constant. Furthermore, by the inner regularity of the Lebesgue measure and [10, p. 16, Proposition 3.6], we know that there exists a simple function $h := \sum_{k=1}^N \lambda_k \mathbf{1}_{F_k}$ such that

$$\|g - h\|_X < \varepsilon,$$

where, for any $k \in \{1, \dots, N\}$, $F_k \subset E_k$ is a compact set, which, together with the outer regularity of the Lebesgue measure and [10, p. 16, Proposition 3.6], further implies that there exists a simple function $u := \sum_{k=1}^N \lambda_k \mathbf{1}_{U_k}$ such that

$$\|u - h\|_X < \varepsilon,$$

where, for any $k \in \{1, \dots, N\}$, $F_k \subset U_k$ is a bounded open set. Then, using the Urysohn lemma, we obtain $f_0 \in C_c(\mathbb{R}^n)$ satisfying that $0 \leq f_0 - h \leq u - h$ and hence

$$\|f_0 - h\|_X \leq \|u - h\|_X < \varepsilon.$$

By the above estimates, we conclude that $\|f - f_0\|_X < \varepsilon$. This finishes the proof of Proposition 3.8. \square

Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy Eqs. 1.2 and 1.3, and the L^∞ -Dini condition, and T_Ω be a singular integral operator with homogeneous kernel Ω . To show Theorem 3.1, we first establish the boundedness of the maximal operator T_Ω^* of a family of truncated transforms $\{T_{\Omega,\eta}\}_{\eta \in (0,\infty)}$ defined as follows. For any given $\eta \in (0, \infty)$ and for any $f \in X$ and $x \in \mathbb{R}^n$, let

$$T_{\Omega,\eta}f(x) := \int_{\{y \in \mathbb{R}^n : |x-y| > \eta\}} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

The maximal operator T_Ω^* is defined by setting, for any $f \in X$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} T_\Omega^*f(x) &:= \sup_{\eta \in (0,\infty)} |T_{\Omega,\eta}f(x)| \\ &= \sup_{\eta \in (0,\infty)} \left| \int_{\{y \in \mathbb{R}^n : |x-y| > \eta\}} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|. \end{aligned} \tag{3.6}$$

We point out that Proposition 3.10 below ensures that, for any $f \in X$, T_Ω^*f in Eq. 3.6 is well defined.

Recall that the following weighted $L^p(\mathbb{R}^n)$ boundedness of the maximal operator T_Ω^* is a part of [58, Theorem 2.1.8].

Lemma 3.9 *Let $p \in (1, \infty)$, $\omega \in A_p(\mathbb{R}^n)$, and $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy Eqs. 1.2 and 1.3, and the L^∞ -Dini condition. Assume that T_Ω^* is a singular integral operator with homogeneous kernel Ω . Then there exists a positive constant $C_{(p,\Omega, [\omega]_{A_p(\mathbb{R}^n)})}$ such that, for any $f \in L^p_\omega(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} [T_\Omega^*(f)(x)]^p \omega(x) dx \leq C_{(p,\Omega, [\omega]_{A_p(\mathbb{R}^n)})} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.$$

As an immediate consequence of Lemma 3.9, we have the following conclusion.

Proposition 3.10 *Let X be a ball Banach function space satisfying Assumption 1.1, $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy Eqs. 1.2, 1.3, and 2.10, and T_Ω^* be the maximal operator as in Eq. 3.6. Then there exists a positive constant C such that, for any $f \in X$,*

$$\|T_\Omega^*(f)\|_X \leq C \|f\|_X, \tag{3.7}$$

and, for any $f \in X$ and almost every $x \in \mathbb{R}^n$,

$$T_\Omega(f)(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon < |x-y| < 1/\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy. \tag{3.8}$$

Proof Using Lemma 3.9 and Proposition 2.14, we immediately obtain Eq. 3.7. Moreover, from Lemma 2.12, we deduce that $X \subset \bigcup_{\omega \in A_s(\mathbb{R}^n)} L_\omega^s(\mathbb{R}^n)$ for any $s \in [1, \infty)$. By this, Lemma 3.9 and [31, Theorem 2.2], we know that, for any $f \in X$ and almost every $x \in \mathbb{R}^n$,

$$T_\Omega(f)(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon < |x-y| < 1/\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy$$

and hence Eq. 3.8 holds true. This finishes the proof of Proposition 3.10. □

Next, we recall the following smooth truncated technique as in [23] (see also [55]). Let $\varphi \in C^\infty([0, \infty))$ satisfy

$$0 \leq \varphi \leq 1 \quad \text{and} \quad \varphi(x) = \begin{cases} 1, & x \in [0, 1/2], \\ 0, & x \in [1, \infty]. \end{cases}$$

Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy Eqs.1.2 and 1.3, and the L^∞ -Dini condition. For any $\kappa \in (0, \infty)$ and any $x, y \in \mathbb{R}^n$, define $K_\kappa(x, y) := \frac{\Omega(x-y)}{|x-y|^n} [1 - \varphi(\frac{|x-y|}{\kappa})]$. Let X be a ball Banach function space satisfying Assumption 1.1. By Lemma 2.12 and [33, Lemma 7.4.5], we know that, for any $f \in X, \kappa \in (0, \infty)$, and $x \in \mathbb{R}^n$,

$$T_\Omega^{(\kappa)} f(x) := \int_{\mathbb{R}^n} K_\kappa(x, y) f(y) dy < \infty.$$

Remark 3.11 Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy Eqs. 1.2, 1.3, and 2.10. Then, for any given $\kappa \in (0, \infty)$, K_κ satisfies the following smoothness condition: there exists a positive constant C such that, for any $x, y, \xi \in \mathbb{R}^n$ with $|\xi| \leq |x-y|/2$,

$$|K_\kappa(x, y) - K_\kappa(x + \xi, y)| \leq C \left[\frac{1}{|x-y|^n} \omega_\infty \left(\frac{4|\xi|}{|x-y|} \right) + \frac{|\xi|}{|x-y|^{n+1}} \right]. \tag{3.9}$$

Indeed, by Eq. 1.2, we conclude that, for any $x, y, \xi \in \mathbb{R}^n$ with $|\xi| \leq |x-y|/2$,

$$\begin{aligned} & |\Omega(x-y) - \Omega(x+\xi-y)| \\ &= \left| \Omega \left(\frac{x-y}{|x-y|} \right) - \Omega \left(\frac{x+\xi-y}{|x+\xi-y|} \right) \right| \leq \omega_\infty \left(\frac{4|\xi|}{|x-y|} \right) \end{aligned}$$

From this, $\Omega \in L^\infty(\mathbb{R}^n)$, the mean value theorem, and the definition of φ , it follows that, for any $x, y, \xi \in \mathbb{R}^n$ with $|\xi| \leq |x - y|/2$,

$$\begin{aligned} & |K_\kappa(x, y) - K_\kappa(x + \xi, y)| \\ & \leq \left| \frac{\Omega(x - y)}{|x - y|^n} - \frac{\Omega(x + \xi - y)}{|x + \xi - y|^n} \right| \left| 1 - \varphi\left(\frac{|x - y|}{\kappa}\right) \right| \\ & \quad + \left| \frac{\Omega(x + \xi - y)}{|x + \xi - y|^n} \right| \left| \varphi\left(\frac{|x + \xi - y|}{\kappa}\right) - \varphi\left(\frac{|x - y|}{\kappa}\right) \right| \\ & \lesssim \left| \frac{\Omega(x - y) - \Omega(x + \xi - y)}{|x - y|^n} \right| + \left| \frac{1}{|x + \xi - y|^n} - \frac{1}{|x - y|^n} \right| \\ & \quad + \frac{\|\varphi'\|_{L^\infty(\mathbb{R}_+)}}{|x + \xi - y|^n} \left| \frac{|x + \xi - y|}{\kappa} - \frac{|x - y|}{\kappa} \right| \mathbf{1}_{\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{1}{3}\kappa \leq |x - y| \leq 2\kappa\}}(x, y) \\ & \lesssim \frac{1}{|x - y|^n} \omega_\infty\left(\frac{4|\xi|}{|x - y|}\right) + \frac{|\xi|}{|x - y|^{n+1}} \\ & \quad + \frac{1}{\kappa} \frac{|\xi|}{|x - y|^n} \mathbf{1}_{\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{1}{3}\kappa \leq |x - y| \leq 2\kappa\}}(x, y) \\ & \lesssim \frac{1}{|x - y|^n} \omega_\infty\left(\frac{4|\xi|}{|x - y|}\right) + \frac{|\xi|}{|x - y|^{n+1}}, \end{aligned}$$

where the implicit positive constants are independent of κ, x, y , and ξ .

Lemma 3.12 *Let $b \in C_c^\infty(\mathbb{R}^n)$, X be a ball Banach function space, and $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy Eqs. 1.2, 1.3, and 2.10. Then there exists a positive constant C such that, for any $\kappa \in (0, \infty)$, $f \in X$, and $x \in \mathbb{R}^n$,*

$$\left| [b, T_\Omega](f)(x) - [b, T_\Omega^{(\kappa)}](f)(x) \right| \leq C\kappa \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \mathcal{M}f(x).$$

Moreover, if \mathcal{M} is bounded on X , then

$$\lim_{\kappa \rightarrow 0^+} \left\| [b, T_\Omega] - [b, T_\Omega^{(\kappa)}] \right\|_{X \rightarrow X} = 0.$$

Proof Let $f \in X$. For any $x \in \mathbb{R}^n$, by the mean value theorem and Eq. 3.8, we have

$$\begin{aligned} & \left| [b, T_\Omega](f)(x) - [b, T_\Omega^{(\kappa)}](f)(x) \right| \\ & = \left| \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon < |x - y| < 1/\varepsilon} [b(x) - b(y)] \frac{\Omega(x - y)}{|x - y|^n} \varphi\left(\frac{|x - y|}{\kappa}\right) f(y) dy \right| \\ & \leq \int_{\{y \in \mathbb{R}^n : |x - y| \leq \kappa\}} |b(x) - b(y)| \frac{|\Omega(x - y)|}{|x - y|^n} |f(y)| dy \\ & \leq \|\Omega\|_{L^\infty(\mathbb{R}^n)} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \sum_{j=0}^\infty \int_{\{y \in \mathbb{R}^n : \frac{\kappa}{2^{j+1}} < |x - y| \leq \frac{\kappa}{2^j}\}} |x - y| \frac{|f(y)|}{|x - y|^n} dy \end{aligned}$$

$$\begin{aligned} &\leq \|\Omega\|_{L^\infty(\mathbb{R}^n)} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \sum_{j=0}^\infty \frac{\kappa}{2^j} \left(\frac{\kappa}{2^{j+1}}\right)^{-n} \int_{B(x, \frac{\kappa}{2^j})} |f(y)| dy \\ &\leq \kappa 2^n \|\Omega\|_{L^\infty(\mathbb{R}^n)} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \sum_{j=0}^\infty 2^{-j} \mathcal{M}f(x) \\ &\lesssim \kappa \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \mathcal{M}f(x). \end{aligned}$$

Moreover, if \mathcal{M} is bounded on X , then

$$\left\| [b, T_\Omega](f) - [b, T_\Omega^{(\kappa)}](f) \right\|_X \lesssim \kappa \|\mathcal{M}f\|_X \lesssim \kappa \|f\|_X,$$

which implies that $\lim_{\kappa \rightarrow 0^+} \|[b, T_\Omega] - [b, T_\Omega^{(\kappa)}]\|_{X \rightarrow X} = 0$ and hence completes the proof of Lemma 3.12. \square

Proof of Theorem 3.1 Let $b \in \text{CMO}(\mathbb{R}^n)$. By the definition of $\text{CMO}(\mathbb{R}^n)$, we know that, for any given $\varepsilon \in (0, \infty)$, there exists a $b^{(\varepsilon)} \in C_c^\infty(\mathbb{R}^n)$ such that $\|b - b^{(\varepsilon)}\|_{\text{BMO}(\mathbb{R}^n)} < \varepsilon$. Then, by the boundedness of $[b - b^{(\varepsilon)}, T_\Omega]$ on X (see Theorem 2.17), we obtain, for any given $\varepsilon \in (0, \infty)$ and for any $f \in X$,

$$\begin{aligned} &\left\| [b, T_\Omega](f) - [b^{(\varepsilon)}, T_\Omega](f) \right\|_X \\ &= \left\| [b - b^{(\varepsilon)}, T_\Omega](f) \right\|_X \lesssim \left\| b - b^{(\varepsilon)} \right\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_X \lesssim \varepsilon \|f\|_X. \end{aligned}$$

From this, Lemma 3.12, and the fact that the limit of compact operators is also a compact operator, it follows that, to prove Theorem 3.1, it suffices to show that, for any $b \in C_c^\infty(\mathbb{R}^n)$ and any $\kappa \in (0, \infty)$ small enough, $[b, T_\Omega^{(\kappa)}]$ is a compact operator on X . To this end, by the definition of compact operators, it suffices to prove that, for any bounded subset $\mathcal{F} \subset X$,

$$[b, T_\Omega^{(\kappa)}]\mathcal{F} := \left\{ [b, T_\Omega^{(\kappa)}](f) : f \in \mathcal{F} \right\}$$

is relatively compact. To achieve this, from Theorem 3.6, we deduce that it suffices to show that $[b, T_\Omega^{(\kappa)}]\mathcal{F}$ satisfies the conditions (i) through (iii) of Theorem 3.6 for any given $b \in C_c^\infty(\mathbb{R}^n)$ and $\kappa \in (0, \infty)$ small enough.

By Theorem 2.17 and Lemma 3.12, we conclude that $[b, T_\Omega^{(\kappa)}]$ is bounded on X for any given $\kappa \in (0, \infty)$, which implies that $[b, T_\Omega^{(\kappa)}]\mathcal{F}$ satisfies the condition (i) of Theorem 3.6.

Next, since $b \in C_c^\infty(\mathbb{R}^n)$, it follows that there exists a positive constant R_0 such that $\text{supp}(b) \subset B(\mathbf{0}, R_0)$. Let $M \in (2R_0, \infty)$. Then, for any $y \in B(\mathbf{0}, R_0)$ and $x \in \mathbb{R}^n$ with $|x| \in (M, \infty)$, we have $|x - y| \sim |x|$. Moreover, by this, $\Omega \in L^\infty(\mathbb{S}^{n-1})$, and Lemma 2.6, we conclude that, for any $f \in \mathcal{F}$ and $x \in \mathbb{R}^n$ with $|x| \in (M, \infty)$,

$$\begin{aligned} &\left| [b, T_\Omega^{(\kappa)}](f)(x) \right| \\ &\leq \int_{\mathbb{R}^n} |b(x) - b(y)| \frac{|\Omega(x - y)|}{|x - y|^n} |f(y)| dy \lesssim \int_{B(\mathbf{0}, R_0)} \frac{|f(y)|}{|x|^n} dy \\ &\lesssim \frac{1}{|x|^n} \|f\|_X \|\mathbf{1}_{B(\mathbf{0}, R_0)}\|_{X'} \lesssim \frac{1}{|x|^n}. \end{aligned}$$

From this and Lemma 2.8 with θ replaced by $\eta \in (1, \infty)$ in Remark 2.9, we deduce that

$$\begin{aligned} & \left\| [b, T_\Omega^{(\kappa)}](f)\mathbf{1}_{\{x \in \mathbb{R}^n: |x| > M\}} \right\|_X \\ & \lesssim \sum_{j=0}^\infty \left\| \frac{1}{|\cdot|^n} \mathbf{1}_{\{x \in \mathbb{R}^n: 2^j M < |x| \leq 2^{j+1} M\}} \right\|_X \lesssim \sum_{j=0}^\infty \frac{\|\mathbf{1}_{\{x \in \mathbb{R}^n: |x| \leq 2^{j+1} M\}}\|_X}{(2^j M)^n} \\ & \lesssim \sum_{j=0}^\infty \frac{(2^{j+1} M)^{n/\eta}}{(2^j M)^n} \lesssim \sum_{j=0}^\infty \frac{1}{(2^j M)^{n(1-1/\eta)}} \lesssim \frac{1}{M^{n(1-1/\eta)}}. \end{aligned}$$

Therefore, the condition (ii) of Theorem 3.6 holds true for $[b, T_\Omega^{(\kappa)}]\mathcal{F}$.

It remains to prove that $[b, T_\Omega^{(\kappa)}]\mathcal{F}$ also satisfies the condition (iii) of Theorem 3.6. For any $f \in \mathcal{F}$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} & [b, T_\Omega^{(\kappa)}](f)(x) - [b, T_\Omega^{(\kappa)}](f)(x + \xi) \\ & = \int_{\mathbb{R}^n} [b(x) - b(y)]K_\kappa(x, y)f(y) dy \\ & \quad - \int_{\mathbb{R}^n} [b(x + \xi) - b(y)]K_\kappa(x + \xi, y)f(y) dy \\ & = [b(x) - b(x + \xi)] \int_{\mathbb{R}^n} K_\kappa(x, y)f(y) dy \\ & \quad + \int_{\mathbb{R}^n} [b(x + \xi) - b(y)][K_\kappa(x, y) - K_\kappa(x + \xi, y)]f(y) dy \\ & =: L_1(x) + L_2(x). \end{aligned} \tag{3.10}$$

We first estimate $L_1(x)$. Observe that, by the mean value theorem and the definition of K_κ ,

$$\begin{aligned} |L_1(x)| & \leq \xi \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \left| \int_{\{y \in \mathbb{R}^n: |x-y| \geq \frac{\kappa}{2}\}} \left[K_\kappa(x, y) - \frac{\Omega(x-y)}{|x-y|^n} \right] f(y) dy \right. \\ & \quad \left. + \int_{\{y \in \mathbb{R}^n: |x-y| \geq \frac{\kappa}{2}\}} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right| \\ & \lesssim \xi \left[\int_{\{y \in \mathbb{R}^n: \kappa \geq |x-y| \geq \frac{\kappa}{2}\}} \left| \frac{\Omega(x-y)}{|x-y|^n} \right| |f(y)| dy + T_\Omega^* f(x) \right] \\ & \lesssim \xi [\mathcal{M}f(x) + T_\Omega^* f(x)], \end{aligned}$$

where the implicit positive constants are independent of f , ξ , and x . From this, the boundedness of \mathcal{M} on X , and Proposition 3.10, we deduce that

$$\|L_1\|_X \lesssim |\xi| \|f\|_X. \tag{3.11}$$

Now, we estimate $L_2(x)$. Observe that, for any $x, y, \xi \in \mathbb{R}^n$ with $|x - y| < \kappa/4$ and $|\xi| < \kappa/8$, we have $|x - y|/\kappa < 1/2$ and $|x + \xi - y|/\kappa < 1/2$, which implies that $\varphi(|x - y|/\kappa) = 0 = \varphi(|x + \xi - y|/\kappa)$ and hence

$$K_\kappa(x, y) = 0 = K_\kappa(x + \xi, y). \tag{3.12}$$

Besides, for any $x, y, \xi \in \mathbb{R}^n$ with $|x - y| \geq \kappa/4$ and $|\xi| < \kappa/8$, we have $|\xi| \leq |x - y|/2$. From this, Eqs. 3.9 and 3.12, we deduce that, for any given $\xi \in \mathbb{R}^n$ with $|\xi| < \kappa/8$,

$$\begin{aligned} |L_2(x)| &\lesssim |\xi| \int_{\{y \in \mathbb{R}^n: |x-y| \geq \frac{\kappa}{4}\}} \frac{|f(y)|}{|x-y|^{n+1}} dy \\ &\quad + \int_{\{y \in \mathbb{R}^n: |x-y| \geq \frac{\kappa}{4}\}} \frac{|f(y)|}{|x-y|^n} \omega_\infty\left(\frac{4|\xi|}{|x-y|}\right) dy \\ &\lesssim |\xi| \sum_{k=0}^\infty (2^k \kappa)^{-(n+1)} \int_{\{y \in \mathbb{R}^n: 2^k \frac{\kappa}{4} \leq |x-y| < 2^{k+1} \frac{\kappa}{4}\}} |f(y)| dy \\ &\quad + \sum_{k=0}^\infty (2^k \kappa)^{-n} \omega_\infty\left(\frac{|\xi|}{2^{k-4}\kappa}\right) \int_{\{y \in \mathbb{R}^n: 2^k \frac{\kappa}{4} \leq |x-y| < 2^{k+1} \frac{\kappa}{4}\}} |f(y)| dy \\ &\lesssim \left[|\xi| + \sum_{k=0}^\infty \omega_\infty\left(\frac{|\xi|}{2^{k-4}\kappa}\right) \int_{2^{-(k+1)}}^{2^{-k}} \frac{d\tau}{\tau} \right] \mathcal{M}f(x) \\ &\lesssim \left[|\xi| + \int_0^1 \omega_\infty\left(\frac{32|\xi|}{\kappa} \tau\right) \frac{d\tau}{\tau} \right] \mathcal{M}f(x) \\ &\sim \left[|\xi| + \int_0^{\frac{32|\xi|}{\kappa}} \omega_\infty(\tau) \frac{d\tau}{\tau} \right] \mathcal{M}f(x) \end{aligned}$$

and hence

$$\|L_2\|_X \lesssim \left[|\xi| + \int_0^{\frac{32|\xi|}{\kappa}} \omega_\infty(\tau) \frac{d\tau}{\tau} \right] \|f\|_X. \tag{3.13}$$

Combining Eqs. 3.10, 3.11, and 3.13, and the L^∞ -Dini condition, we have

$$\lim_{|\xi| \rightarrow 0^+} \left\| [b, T_\Omega^{(\kappa)}](f)(\cdot + \xi) - [b, T_\Omega^{(\kappa)}](f)(\cdot) \right\|_X = 0,$$

which implies the condition (iii) of Theorem 3.6. Thus, $[b, T_\Omega^{(\kappa)}]$ is a compact operator for any given $b \in C_c^\infty(\mathbb{R}^n)$ and $\kappa \in (0, \infty)$. This finishes the proof of Theorem 3.1. \square

3.2 Proof of Theorem 3.2

We begin with recalling the following equivalent characterization of $\text{CMO}(\mathbb{R}^n)$ in terms of the local mean oscillation, which is just [37, Theorem 3.3].

Lemma 3.13 *Let $f \in \text{BMO}(\mathbb{R}^n)$ and $\lambda \in (0, 1/2)$. Then $f \in \text{CMO}(\mathbb{R}^n)$ if and only if f satisfies the following three conditions:*

- (i) $\lim_{a \rightarrow 0^+} \sup_{|B|=a} \omega_\lambda(f; B) = 0$;
- (ii) $\lim_{a \rightarrow \infty} \sup_{|B|=a} \omega_\lambda(f; B) = 0$;
- (iii) $\lim_{d \rightarrow \infty} \sup_{B \cap B(\mathbf{0}, d) = \emptyset} \omega_\lambda(f; B) = 0$,

where the local mean oscillation $\omega_\lambda(f; B)$ is as in Eq. 2.12.

To prove Theorem 3.2, we establish the lower and the upper estimates of commutators on X , respectively, in Propositions 3.14 and 3.16 below.

Proposition 3.14 *Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\lambda \in (0, 1)$, and X be a ball Banach function space. Assume that \mathcal{M} is bounded on X and $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfies that there exists an open set $\Lambda \subset \mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Λ . Let $B := B(x_0, r_0)$, k_0, ε_0, E , and F be as in Lemma 2.20, and $[b, T_\Omega]$ satisfy Eq. 2.13. Then there exists a positive constant $C_{(\lambda, k_0, \varepsilon_0, n)}$, depending only on $\lambda, k_0, \varepsilon_0$, and n , such that, for any measurable set $Q \subset \mathbb{R}^n$ with $|Q| \leq \frac{\lambda}{8}|B(x_0, r_0)|$,*

$$\omega_\lambda(b; B)\|\mathbf{1}_F\|_X \leq C_{(\lambda, k_0, \varepsilon_0, n)} \|[b, T_\Omega](\mathbf{1}_F)\mathbf{1}_{E \setminus Q}\|_X.$$

Proof Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\lambda \in (0, 1)$, and $B := B(x_0, r_0)$ with $x_0 \in \mathbb{R}^n$ and $r_0 \in (0, \infty)$; let ε_0, k_0, G, E , and F be as in Lemma 2.20; and let Q be a measurable set in \mathbb{R}^n with $|Q| \leq \frac{\lambda}{8}|B(x_0, r_0)|$. Then, by (i) and (iii) of Lemma 2.20, we conclude that

$$\begin{aligned} &\omega_\lambda(b; B(x_0, r_0)) \|[E \setminus Q] \times F] \cap G| \\ &\leq \frac{1}{\varepsilon_0} \int_{[(E \setminus Q) \times F] \cap G} |b(x) - b(y)| \left| \Omega \left(\frac{x - y}{|x - y|} \right) \right| dx dy. \end{aligned}$$

From this, the fact that $|x - y| \leq 2(k_0 + 1)r_0$ for any $(x, y) \in G$, Lemma 2.20(ii), the observations

$$\begin{aligned} |[E \setminus Q] \times F] \cap G| &\geq |G| - |Q||F| \\ &\geq \frac{\lambda}{8}|B(x_0, r_0)|^2 - \frac{\lambda}{8}|B(x_0, r_0)| \frac{|B(x_0, r_0)|}{2} \\ &= \frac{\lambda}{16}|B(x_0, r_0)|^2 \end{aligned}$$

as well as $\overline{E} \cap \overline{F} = \emptyset$, and Eq. 2.13, we deduce that

$$\begin{aligned} &\omega_\lambda(b; B(x_0, r_0)) \\ &\leq \frac{[2(k_0 + 1)r_0]^n}{\varepsilon_0|[E \setminus Q] \times F] \cap G|} \int_{[(E \setminus Q) \times F] \cap G} \frac{|b(x) - b(y)|}{|x - y|^n} \left| \Omega \left(\frac{x - y}{|x - y|} \right) \right| dx dy \\ &\leq \frac{16[2(k_0 + 1)r_0]^n}{\lambda \varepsilon_0 |B(x_0, r_0)|^2} \int_{E \setminus Q} \left| \int_F \frac{b(x) - b(y)}{|x - y|^n} \Omega \left(\frac{x - y}{|x - y|} \right) dy \right| dx \\ &\lesssim \frac{1}{|B(x_0, r_0)|} \int_{E \setminus Q} |[b, T_\Omega](\mathbf{1}_F)| dx, \end{aligned}$$

which, combined with Lemma 2.6, $F \subset 4k_0B(x_0, r_0)$, $E \subset B(x_0, r)$, and Lemma 2.21, further implies that

$$\begin{aligned} &\omega_\lambda(b; B(x_0, r_0))\|\mathbf{1}_F\|_X \\ &\lesssim \frac{\|\mathbf{1}_F\|_X}{|B(x_0, r_0)|} \|[b, T_\Omega](\mathbf{1}_F)\mathbf{1}_{E \setminus Q}\|_X \|\mathbf{1}_{E \setminus Q}\|_{X'} \\ &\lesssim \frac{\|\mathbf{1}_{4k_0B(x_0, r_0)}\|_X \|\mathbf{1}_{B(x_0, r_0)}\|_{X'}}{|B(x_0, r_0)|} \|[b, T_\Omega](\mathbf{1}_F)\mathbf{1}_{E \setminus Q}\|_X \\ &\lesssim \|[b, T_\Omega](\mathbf{1}_F)\mathbf{1}_{E \setminus Q}\|_X. \end{aligned}$$

This finishes the proof of Proposition 3.14. □

To establish the upper estimate of commutators, we need the following equivalent BMO-norm characterization on ball Banach function spaces, namely, Lemma 3.15 below, which is just [48, Theorems 1.2] and an essential tool needed in this article.

Lemma 3.15 *Let X be a ball Banach function space such that M is bounded on X' and, for any $b \in L^1_{\text{loc}}(\mathbb{R}^n)$,*

$$\|b\|_{\text{BMO}_X} := \sup_B \frac{1}{\|\mathbf{1}_B\|_X} \| |b - b_B| \mathbf{1}_B \|_X,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. Then there exists a positive constant C such that, for any $b \in \text{BMO}(\mathbb{R}^n)$,

$$C^{-1} \|b\|_{\text{BMO}(\mathbb{R}^n)} \leq \|b\|_{\text{BMO}_X} \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}.$$

Next, we give the upper estimate of commutators on X as follows.

Proposition 3.16 *Let $b \in \text{BMO}(\mathbb{R}^n)$ and X be a ball Banach function space. Assume that \mathcal{M} is bounded on X and X' , and $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfies that there exists an open set $\Lambda \subset \mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Λ . Let $B(x_0, r_0)$ and $F \subset B(x_1, r_0)$ be as in Lemma 2.20, and $[b, T_\Omega]$ satisfy Eq. 2.13. Then there exist positive constants C, d_0 , and δ such that, for any $d \in (0, \infty)$ with $d \geq d_0$,*

$$\| [b, T_\Omega](\mathbf{1}_F \mathbf{1}_{B(x_0, 2^{d+1}r_0) \setminus B(x_0, 2^d r_0)}) \|_X \leq C 2^{-\delta d n} d \|b\|_{\text{BMO}(\mathbb{R}^n)} \|\mathbf{1}_F\|_X,$$

where the positive constants C and δ are independent of d, b as well as $B(x_0, r_0)$, and d_0 is a large constant depending only on k_0 in Lemma 2.20.

Proof Without loss of generality, we may assume that $\|b\|_{\text{BMO}(\mathbb{R}^n)} = 1$. Let $b \in \text{BMO}(\mathbb{R}^n)$ and $B := B(x_0, r_0)$ with $x_0 \in \mathbb{R}^n$ and $r_0 \in (0, \infty)$; let ε_0, k_0, G, E , and $F \subset B(x_1, r_0)$ be as in Lemma 2.20; and let d_0 be a positive constant such that $2^{d_0} \in (4k_0, \infty)$. Then, for any given positive constant $d \geq d_0$ and for any $x \in B(x_0, 2^{d+1}r_0) \setminus B(x_0, 2^d r_0)$ and $y \in B(x_1, r_0)$, we have $|x - y| \sim 2^d r_0$. By this, Eq. 2.13, and Lemma 2.6, we conclude that, for any $x \in B(x_0, 2^{d+1}r_0) \setminus B(x_0, 2^d r_0)$,

$$\begin{aligned} & |[b, T_\Omega](\mathbf{1}_F)(x)| \\ &= \left| \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^n} \mathbf{1}_F(y) dy \right| \\ &\leq \int_{\mathbb{R}^n} |b(x) - b_{B(x_1, r_0)} + b_{B(x_1, r_0)} - b(y)| \frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^n} \mathbf{1}_F(y) dy \\ &\leq |b(x) - b_{B(x_1, r_0)}| \int_{\mathbb{R}^n} \frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^n} \mathbf{1}_F(y) dy \\ &\quad + \int_{\mathbb{R}^n} |b_{B(x_1, r_0)} - b(y)| \frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^n} \mathbf{1}_F(y) dy \\ &\lesssim \frac{\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}}{2^{dn} r_0^n} |b(x) - b_{B(x_1, r_0)}| \|\mathbf{1}_F\|_X \|\mathbf{1}_F\|_{X'} \\ &\quad + \frac{\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}}{2^{dn} r_0^n} \|\mathbf{1}_F\|_X \| |b - b_{B(x_1, r_0)}| \mathbf{1}_F \|_{X'} \\ &\sim H_1(x) + H_2(x), \end{aligned} \tag{3.14}$$

where

$$H_1(x) := \frac{\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}}{2^{dn} r_0^n} |b(x) - b_{B(x_1, r_0)}| \|\mathbf{1}_F\|_X \|\mathbf{1}_F\|_{X'}$$

and

$$H_2(x) := \frac{\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}}{2^{dn} r_0^n} \|\mathbf{1}_F\|_X \| |b - b_{B(x_1, r_0)}| \mathbf{1}_F \|_{X'}.$$

Observe that $B(x_1, 2^d r_0) \ni x_0$. Thus, for any $y \in B(x_0, 2^{d+1} r_0)$, we have

$$|y - x_1| \leq |y - x_0| + |x_0 - x_1| \leq 2^{d+1} r_0 + 2^d r_0 = 3 \cdot 2^d r_0,$$

which implies that

$$B(x_0, 2^{d+1} r_0) \subset B(x_1, 2^{d+2} r_0). \tag{3.15}$$

Moreover, by $\|b\|_{\text{BMO}(\mathbb{R}^n)} = 1$, it is easy to see that

$$|b_{B(x_1, r_0)} - b_{2^{d+2} B(x_1, r_0)}| \leq (d + 2)2^n.$$

From this, Eqs. 3.14 and 3.15, and Lemmas 3.15, we deduce that

$$\begin{aligned} & \| \mathbf{H}_1 \mathbf{1}_{B(x_0, 2^{d+1} r_0) \setminus B(x_0, 2^d r_0)} \|_X \\ & \lesssim 2^{-dn} r_0^{-n} \| \mathbf{1}_F \|_X \| \mathbf{1}_F \|_{X'} \| |b - b_{B(x_1, r_0)}| \mathbf{1}_{B(x_0, 2^{d+1} r_0) \setminus B(x_0, 2^d r_0)} \|_X \\ & \lesssim 2^{-dn} r_0^{-n} \| \mathbf{1}_F \|_X \| \mathbf{1}_F \|_{X'} \| |b - b_{B(x_1, r_0)}| \mathbf{1}_{B(x_1, 2^{d+2} r_0)} \|_X \\ & \lesssim 2^{-dn} r_0^{-n} \| \mathbf{1}_F \|_X \| \mathbf{1}_F \|_{X'} \\ & \quad \times \left[\| |b - b_{B(x_1, 2^{d+2} r_0)}| \mathbf{1}_{B(x_1, 2^{d+2} r_0)} \|_X + d \| \mathbf{1}_{B(x_1, 2^{d+2} r_0)} \|_X \right] \\ & \lesssim 2^{-dn} r_0^{-n} d \| \mathbf{1}_F \|_X \| \mathbf{1}_F \|_{X'} \| \mathbf{1}_{B(x_1, 2^{d+2} r_0)} \|_X, \end{aligned}$$

which, combined with Lemma 2.8 with θ replaced by $\eta \in (1, \infty)$ in Remark 2.9, the conclusion $F \subset B(x_1, r_0)$ of Lemma 2.20, and Lemma 2.21 with B replaced by $B(x_1, r_0)$, further implies that

$$\begin{aligned} & \| \mathbf{H}_1 \mathbf{1}_{B(x_0, 2^{d+1} r_0) \setminus B(x_0, 2^d r_0)} \|_X \\ & \lesssim 2^{-(1-1/\eta)dn} d \| \mathbf{1}_F \|_X \| \mathbf{1}_{B(x_1, r_0)} \|_{X'} \| \mathbf{1}_{B(x_1, r_0)} \|_X r_0^{-n} \\ & \lesssim 2^{-(1-1/\eta)dn} d \| \mathbf{1}_F \|_X. \end{aligned} \tag{3.16}$$

Similarly, by Eq. 3.14, the fact $F \subset B(x_1, r_0)$ again, and Lemmas 2.5, 3.15, 2.8, and 2.21, we conclude that

$$\begin{aligned} & \| \mathbf{H}_2 \mathbf{1}_{B(x_0, 2^{d+1} r_0) \setminus B(x_0, 2^d r_0)} \|_X \\ & \lesssim 2^{-dn} r_0^{-n} \| \mathbf{1}_F \|_X \| |b - b_{B(x_1, r_0)}| \mathbf{1}_F \|_{X'} \| \mathbf{1}_{B(x_0, 2^{d+1} r_0) \setminus B(x_0, 2^d r_0)} \|_X \\ & \lesssim 2^{-dn} r_0^{-n} \| \mathbf{1}_F \|_X \| |b - b_{B(x_1, r_0)}| \mathbf{1}_{B(x_1, r_0)} \|_{X'} \| \mathbf{1}_{B(x_1, 2^d r_0)} \|_X \\ & \lesssim 2^{-(1-1/\eta)dn} d \| \mathbf{1}_F \|_X \| \mathbf{1}_{B(x_1, r_0)} \|_{X'} \| \mathbf{1}_{B(x_1, r_0)} \|_X r_0^{-n} \\ & \lesssim 2^{-(1-1/\eta)dn} d \| \mathbf{1}_F \|_X. \end{aligned} \tag{3.17}$$

Combining Eqs. 3.14, 3.16, and 3.17, and letting $\delta := 1 - 1/\eta$, we then complete the proof of Proposition 3.16. □

Proof of Theorem 3.2 By Theorem 2.22, we conclude that $b \in \text{BMO}(\mathbb{R}^n)$ and then, without loss of generality, we may assume that $\|b\|_{\text{BMO}(\mathbb{R}^n)} = 1$. To show $b \in \text{CMO}(\mathbb{R}^n)$, we use a contradiction argument via Lemma 3.13. Now, observe that, if $b \notin \text{CMO}(\mathbb{R}^n)$, then b does not satisfy at least one of (i), (ii), and (iii) of Lemma 3.13. To finish the proof of this theorem, we only need to show that, if b does not satisfy one of (i), (ii), and (iii) of Lemma 3.13, then $[b, T_\Omega]$ is not compact on X . We prove this by three cases on b as follows.

Case i) Suppose that b does not satisfy Lemma 3.13(i). In this case, there exist a constant $\delta_0 \in (0, 1)$ and a sequence $\{B_j\}_{j \in \mathbb{N}}$ of balls, with $|B_j| \rightarrow 0$ as $j \rightarrow \infty$, such that, for any $j \in \mathbb{N}$,

$$\omega_\lambda(b; B_j) \geq \delta_0, \tag{3.18}$$

where $\lambda \in (0, 1/2)$ and $\omega_\lambda(b; B_j)$ is as in Eq. 2.12 with f and B replaced, respectively, by b and B_j . For any given ball $B := B(x_0, r_0)$, let E and F be the sets associated with B in Lemma 2.20,

$$f := \|\mathbf{1}_F\|_X^{-1} \mathbf{1}_F,$$

and $2C_0 := C_{(\lambda, k_0, \varepsilon_0, n)}$ as in Proposition 3.14. Then, by Proposition 3.14, we conclude that, for any measurable set $Q \subset \mathbb{R}^n$ with $|Q| \leq \frac{\lambda}{8}|B|$,

$$\|[b, T_\Omega](f)\mathbf{1}_{E \setminus Q}\|_X \geq 2C_0\omega_\lambda(b; B). \tag{3.19}$$

For such chosen C_0 and δ_0 , by Proposition 3.16, we know that there exists a positive constant d_0 such that

$$\begin{aligned} & \left\| [b, T_\Omega](f)\mathbf{1}_{\mathbb{R}^n \setminus B(x_0, 2^{d_0}r_0)} \right\|_X \\ & \leq \sum_{k=0}^\infty \left\| [b, T_\Omega](f)\mathbf{1}_{B(x_0, 2^{d_0+k+1}r_0) \setminus B(x_0, 2^{d_0+k}r_0)} \right\|_X \leq C_0\delta_0. \end{aligned} \tag{3.20}$$

Take a subsequence of balls $\{B_j\}_{j \in \mathbb{N}}$, still denoted by $\{B_j\}_{j \in \mathbb{N}}$, such that, for any $j \in \mathbb{N}$,

$$\frac{|B_{j+1}|}{|B_j|} \leq \min \left\{ \frac{\lambda^2}{64}, 2^{-2d_0n} \right\}.$$

Let $\tilde{B}_j := (|B_{j-1}|/|B_j|)^{1/2n} B_j$ for any $j \in \mathbb{N}$ and $j \geq 2$. Then it is easy to see that, for any $j \in \mathbb{N}$ and $j \geq 2$,

$$\left(\frac{|B_{j-1}|}{|B_j|} \right)^{\frac{1}{2n}} \geq 2^{d_0} \quad \text{and} \quad |\tilde{B}_j| \leq \frac{\lambda}{8}|B_{j-1}|.$$

From this and the monotonicity of $\{B_j\}_{j \in \mathbb{N}}$, we deduce that, for any integers k and j with $k > j \geq 2$,

$$2^{d_0} B_k \subset \tilde{B}_k \quad \text{and} \quad |\tilde{B}_k| \leq \frac{\lambda}{8}|B_{k-1}| \leq \frac{\lambda}{8}|B_j|. \tag{3.21}$$

Now, for any $j \in \mathbb{N}$, let E_j and F_j be the sets associated with B_j as in Lemma 2.20 with B replaced by B_j , and

$$f_j := \|\mathbf{1}_{F_j}\|_X^{-1} \mathbf{1}_{F_j}.$$

Then, for any integers k and j with $k > j \geq 2$, by Eqs. 3.19, 3.18, 3.21, and 3.20, we conclude that

$$\|[b, T_\Omega](f_j)\mathbf{1}_{E_j \setminus \tilde{B}_k}\|_X \geq 2C_0\omega_\lambda(b; B) \geq 2C_0\delta_0$$

and

$$\|[b, T_\Omega](f_k)\mathbf{1}_{E_j \setminus \tilde{B}_k}\|_X \leq \|[b, T_\Omega](f_k)\mathbf{1}_{\mathbb{R}^n \setminus 2^{d_0}B_k}\|_X \leq C_0\delta_0,$$

which further implies that

$$\begin{aligned} & \|[b, T_\Omega](f_j) - [b, T_\Omega](f_k)\|_X \\ & \geq \|[b, T_\Omega](f_j) - [b, T_\Omega](f_k)\mathbf{1}_{E_j \setminus \tilde{B}_k}\|_X \\ & \geq \|[b, T_\Omega](f_j)\mathbf{1}_{E_j \setminus \tilde{B}_k}\|_X - \|[b, T_\Omega](f_k)\mathbf{1}_{E_j \setminus \tilde{B}_k}\|_X \geq C_0\delta_0. \end{aligned}$$

Therefore, $\{[b, T_\Omega]f_j\}_{j \in \mathbb{N}}$ is not relatively compact in X , which leads to a contradiction with the compactness of $[b, T_\Omega]$ on X . This shows that b satisfies Lemma 3.13(i), which is the desired conclusion.

Case ii) Suppose that b does not satisfy Lemma 3.13(ii). In this case, similarly to above Case i), there exist a $\delta_0 \in (0, 1)$ and a sequence $\{B_j\}_{j \in \mathbb{N}}$ of balls such that, for any $j \in \mathbb{N}$,

$$\omega_\lambda(b; B_j) \geq \delta_0 \quad \text{and} \quad \frac{|B_j|}{|B_{j+1}|} \leq \min \left\{ \frac{\lambda^2}{64}, 2^{-2d_0 n} \right\},$$

where C_0 and d_0 are as in Case i) such that Eqs. 3.19 and 3.20 hold true. For any $j \in \mathbb{N}$, let E_j, F_j , and f_j be as in Case i), and $\tilde{B}_j := (|B_j|/|B_{j-1}|)^{1/2n} B_{j-1}$ for any $j \geq 2$. Then it is easy to see that, for any integers k and j with $2 \leq k \leq j$,

$$2^{d_0} B_{k-1} \subset \tilde{B}_k \quad \text{and} \quad |\tilde{B}_k| \leq \frac{\lambda}{8} |B_j|.$$

Using a method similar to that used in Case i), we conclude that

$$\|[b, T_\Omega](f_j) - [b, T_\Omega](f_k)\|_X \geq C_0 \delta_0,$$

and hence $\{[b, T_\Omega]f_j\}_{j \in \mathbb{N}}$ is not relatively compact in X , which is a contradiction. This shows that b satisfies Lemma 3.13(ii), which is also the desired conclusion.

Case iii) Suppose that b does not satisfy Lemma 3.13(iii). In this case, there exist a $\delta_0 \in (0, 1)$ and a sequence $\{B_j\}_{j \in \mathbb{N}}$ of balls such that, for any $j \in \mathbb{N}$,

$$\omega_\lambda(b; B_j) \geq \delta_0. \tag{3.22}$$

From this and Cases i) and ii), we deduce that there exist a constant $d_1 \in [d_0, \infty)$ with d_0 as in Lemma 2.20, and a subsequence of balls $\{B_j\}_{j \in \mathbb{N}}$, still denoted by $\{B_j\}_{j \in \mathbb{N}}$, such that

$$|B_j| \sim 1, \quad \forall j \in \mathbb{N}$$

and

$$2^{d_1} B_i \cap 2^{d_1} B_j = \emptyset, \quad \forall i \neq j.$$

For any $j \in \mathbb{N}$, let E_j, F_j, f_j , and C_0 be as in Case i). Notice that, for any positive integers k and j ,

$$\left(2^{d_0} B_k \cap E_j\right) \subset \left(2^{d_1} B_k \cap 2^{d_1} B_j\right) = \emptyset.$$

By this, Proposition 3.14 with $Q := \emptyset$, and Eq. 3.22, we conclude that, for any positive integers k and j ,

$$\begin{aligned} \|[b, T_\Omega](f_j) \mathbf{1}_{E_j \setminus 2^{d_0} B_k}\|_X &= \|[b, T_\Omega](f_j) \mathbf{1}_{E_j}\|_X \\ &\geq 2C_0 \omega_\lambda(b; B) \geq 2C_0 \delta_0. \end{aligned} \tag{3.23}$$

Moreover, from Proposition 3.16, we deduce that, for any positive integers k and j ,

$$\|[b, T_\Omega](f_k) \mathbf{1}_{E_j \setminus 2^{d_0} B_k}\|_X \leq \|[b, T_\Omega](f_k) \mathbf{1}_{\mathbb{R}^n \setminus 2^{d_0} B_k}\|_X \leq C_0 \delta_0. \tag{3.24}$$

Combining Eqs. 3.23 and 3.24, we obtain

$$\begin{aligned} &\|[b, T_\Omega](f_j) - [b, T_\Omega](f_k)\|_X \\ &\geq \|[b, T_\Omega](f_j) - [b, T_\Omega](f_k)\|_{E_j \setminus 2^{d_0} B_k} \\ &\geq \|[b, T_\Omega](f_j) \mathbf{1}_{E_j \setminus 2^{d_0} B_k}\|_X - \|[b, T_\Omega](f_k) \mathbf{1}_{E_j \setminus 2^{d_0} B_k}\|_X \geq C_0 \delta_0 \end{aligned}$$

and hence $\{[b, T_\Omega]f_j\}_{j \in \mathbb{N}}$ is not relatively compact in X , which is a contradiction. This shows that b satisfies Lemma 3.13(iii), which completes the proof of Theorem 3.2. \square

4 Applications

In this section, we apply Theorems 2.17, 2.22, 3.1, and 3.2, respectively, to six concrete examples of ball Banach function spaces, namely, Morrey spaces (see Section 4.1 below), mixed-norm Lebesgue spaces (see Section 4.2 below), variable Lebesgue spaces (see Section 4.3 below), weighted Lebesgue spaces (see Section 4.4 below), Orlicz spaces (see Section 4.5 below), and Orlicz-slice spaces (see Section 4.6 below). Observe that, among these six examples, only variable Lebesgue spaces and Orlicz spaces are Banach function spaces as in Remark 2.2(ii), while the other four examples are ball Banach function spaces, which are not necessary to be Banach function spaces.

4.1 Morrey Spaces

Recall that, due to the applications in elliptic partial differential equations, the Morrey space $M_r^p(\mathbb{R}^n)$ with $0 < r \leq p < \infty$ was introduced by Morrey [60] in 1938. In recent decades, there exists an increasing interest in applications of Morrey spaces to various areas of analysis such as partial differential equations, potential theory, and harmonic analysis; see, for instance, [1, 2, 21, 50, 72–74, 82].

Definition 4.1 Let $0 < r \leq p < \infty$. The *Morrey space* $M_r^p(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on \mathbb{R}^n such that

$$\|f\|_{M_r^p(\mathbb{R}^n)} := \sup_{B \in \mathbb{B}} |B|^{1/p-1/r} \|f\|_{L^r(B)} < \infty,$$

where \mathbb{B} is as in Eq. 2.1 (the set of all balls of \mathbb{R}^n).

Remark 4.2 Observe that, as was pointed out in [71, p. 86], $M_r^p(\mathbb{R}^n)$ may not be a Banach function space, but it is a ball Banach function space as in Definition 2.1.

Let $1 < r \leq p < \infty$. From [21, Theorem 1], it follows that the Hardy–Littlewood maximal operator \mathcal{M} is bounded on $M_r^p(\mathbb{R}^n)$. Recall that the associate space of the Morrey space is the block space (see, for instance, [70, Theorem 4.1]) and \mathcal{M} is bounded on block spaces (see, for instance, [20, Theorem 3.1] and [39, Lemma 5.7]). Using these and Definition 2.7, we can easily show that \mathcal{M} is bounded on X' , where $X := M_r^p(\mathbb{R}^n)$. Thus, all the assumptions of the main theorems in Sections 2 and 3 are satisfied. Using Theorems 2.17, 2.22, 3.1, and 3.2, we obtain the following characterization of the boundedness and the compactness of commutators on Morrey spaces, respectively, via $BMO(\mathbb{R}^n)$ and $CMO(\mathbb{R}^n)$.

Theorem 4.3 Let $1 < r \leq p < \infty$. Then Theorems 2.17, 2.22, 3.1, and 3.2 hold true with X replaced by $M_r^p(\mathbb{R}^n)$.

Remark 4.4 (i) The boundedness of commutators on Morrey spaces was first obtained by Di Fazio and Ragusa [30, Theorem 1]. Indeed, Di Fazio and Ragusa [30] proved Theorem 4.3 under the assumption that $\Omega \in \text{Lip}(\mathbb{S}^{n-1})$ satisfies Eqs. 1.2 and 1.3, which is a special case of Theorem 4.3.

(ii) Let $1 < r \leq p < \infty$. Theorem 3.1 with X replaced by $M_r^p(\mathbb{R}^n)$ was obtained by Chen et al. [19, Theorem 1.1]. On the other hand, Chen et al. [19, Theorem 1.2] showed the necessity under the assumption that $\Omega \in \text{Lip}(\mathbb{S}^{n-1})$ satisfies Eqs. 1.2 and 1.3, which is stronger than Theorem 4.3.

4.2 Mixed-norm Lebesgue Spaces

The mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ was studied by Benedek and Panzone [9] in 1961, which can be traced back to Hörmander [42]. Later on, in 1970, Lizorkin [57] further developed both the theory of multipliers of Fourier integrals and estimates of convolutions in the mixed-norm Lebesgue spaces. Particularly, in order to meet the requirements arising in the study of the boundedness of operators, partial differential equations, and some other analysis subjects, the real-variable theory of mixed-norm function spaces, including mixed-norm Morrey spaces, mixed-norm Hardy spaces, mixed-norm Besov spaces, and mixed-norm Triebel–Lizorkin spaces, has rapidly been developed in recent years (see, for instance, [22, 32, 44–46, 64, 65]).

Definition 4.5 Let $\vec{p} := (p_1, \dots, p_n) \in (0, \infty]^n$. The mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on \mathbb{R}^n such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} \cdots dx_n \right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \dots, n\}$.

In this subsection, for any $\vec{p} := (p_1, \dots, p_n) \in (0, \infty)^n$, we always let $p_- := \min\{p_1, \dots, p_n\}$ and $p_+ := \max\{p_1, \dots, p_n\}$.

Let $\vec{p} \in (1, \infty)^n$. Then \mathcal{M} is bounded on $L^{\vec{p}}(\mathbb{R}^n)$ (see, for instance, [44, Lemma 3.5]). Applying this and the dual theorem (see, for instance, [9, Theorem 1.a]), we can easily show that \mathcal{M} is bounded on X' , where $X := L^{\vec{p}}(\mathbb{R}^n)$. Thus, all the assumptions of the main theorems in Sections 2 and 3 are satisfied. Using Theorems 2.17, 2.22, 3.1, and 3.2, we obtain the following characterization of the boundedness and the compactness of commutators on mixed-norm Lebesgue spaces.

Theorem 4.6 Let $\vec{p} := (p_1, \dots, p_n) \in (1, \infty)^n$. Then Theorems 2.17, 2.22, 3.1, and 3.2 hold true with X replaced by $L^{\vec{p}}(\mathbb{R}^n)$.

Remark 4.7 To the best of our knowledge, Theorem 4.6 is totally new.

4.3 Variable Lebesgue Spaces

Let $p(\cdot) : \mathbb{R}^n \rightarrow [0, \infty)$ be a measurable function. Then the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on \mathbb{R}^n such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left[\frac{|f(x)|}{\lambda} \right]^{p(x)} dx \leq 1 \right\} < \infty.$$

We refer the reader to [26, 28, 54, 61, 62] for more details on variable Lebesgue spaces, and [27] for the study on variable Hardy spaces.

For any measurable function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$, in this subsection, we let

$$\tilde{p}_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad \tilde{p}_+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

If $1 < \tilde{p}_- \leq \tilde{p}_+ < \infty$, then, similarly to the proof of [29, Theorem 3.2.13], we know that $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space and hence a ball Banach function space.

A measurable function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ is said to be *globally log-Hölder continuous* if there exist a $p_\infty \in \mathbb{R}$ and a positive constant C such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \leq C \frac{1}{\log(e + 1/|x - y|)}$$

and

$$|p(x) - p_\infty| \leq C \frac{1}{\log(e + |x|)}.$$

Let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ be a globally log-Hölder continuous function satisfying $1 < \tilde{p}_- \leq \tilde{p}_+ < \infty$. The boundedness of the Hardy–Littlewood maximal operator on variable Lebesgue spaces was obtained in [29, Theorem 4.3.8]; see also [26, Theorem 3.16]. Furthermore, from this and the dual theorem (see, for instance, [26, Theorem 2.80]), we deduce that M is bounded on X' , where $X := L^{p(\cdot)}(\mathbb{R}^n)$. Thus, all the assumptions of the main theorems in Sections 2 and 3 are satisfied. Using Theorems 2.17, 2.22, 3.1, and 3.2, we obtain the following characterization of the boundedness and the compactness of commutators on variable Lebesgue spaces.

Theorem 4.8 *Let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ be a globally log-Hölder continuous function satisfying $1 < \tilde{p}_- \leq \tilde{p}_+ < \infty$. Then Theorems 2.17, 2.22, 3.1, and 3.2 hold true with X replaced by $L^{p(\cdot)}(\mathbb{R}^n)$.*

Remark 4.9 The boundedness characterization of commutators on variable Lebesgue spaces was first studied by Karlovich and Lerner [52, Theorem 1.1]; meanwhile, they pointed out in [52, Remark 4.3] that the corresponding conclusion also holds true in Banach function spaces. Moreover, Guo et al. [36, Theorem 2.1] proved a generalization for the necessity part in ball Banach function spaces, based on a weaker assumption than [52, Theorem 1.1(b)]. Furthermore, Theorem 2.22 generalizes the corresponding conclusion of [36, Theorem 2.1]; see Remark 2.23(i). As for the compactness characterization, to the best of our knowledge, the corresponding conclusions of Theorem 4.8 are totally new.

4.4 Weighted Lebesgue Spaces

It is worth pointing out that a weighted Lebesgue space with an $A_\infty(\mathbb{R}^n)$ -weight may not be a Banach function space; see [71, Section 7.1]. From [3, Theorem 3.1(b)], it follows that, for any $p \in (1, \infty)$,

$$\mathcal{M} \text{ is bounded on } L_\omega^p(\mathbb{R}^n) \text{ if and only if } \omega \in A_p(\mathbb{R}^n). \tag{4.1}$$

Therefore, $L_\omega^p(\mathbb{R}^n)$ satisfies Assumption 2.8 for any given $p \in (1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Moreover, from [29, Theorem 2.7.4], we deduce that, when $p \in (1, \infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$,

$$[L_\omega^p(\mathbb{R}^n)]' = L_{\omega^{1-p'}}^{p'}(\mathbb{R}^n),$$

where $[L_\omega^p(\mathbb{R}^n)]'$ denotes the associated space of $L_\omega^p(\mathbb{R}^n)$ as in Eq. 2.4 with $X := L_\omega^p(\mathbb{R}^n)$. By this, Eq. 4.1, and the observation that

$$\omega \in A_p(\mathbb{R}^n) \text{ if and only if } \omega^{1-p'} \in A_{p'}(\mathbb{R}^n),$$

we conclude that \mathcal{M} is bounded on $[L_\omega^p(\mathbb{R}^n)]'$. Thus, all the assumptions of the main theorems in Sections 2 and 3 are satisfied. Using Theorems 2.17, 2.22, 3.1, and 3.2, we immediately obtain the following characterization of the boundedness and the compactness of commutators on weighted Lebesgue spaces.

Theorem 4.10 *Let $p \in (1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Then Theorems 2.17, 2.22, 3.1, and 3.2 hold true with X replaced by $L^p_\omega(\mathbb{R}^n)$.*

Remark 4.11 The boundedness characterization of Theorem 4.10 was obtained in [58, p. 129, Theorem 2.4.3], [35, Theorem 1.3], and [56, Theorem 1.1] under weaker assumptions on both the kernel of the operator under consideration and the weight under consideration. The compactness characterization of Theorem 4.10 coincides with that of [37, Theorems 1.4 and 1.5].

4.5 Orlicz Spaces

Birnbaum and Orlicz [11] (see also Orlicz [66]) introduced the Orlicz space which is another generalization of $L^p(\mathbb{R}^n)$. Since then, Orlicz spaces have been well developed and widely used in harmonic analysis, partial differential equations, potential theory, probability, and some other fields of analysis; see, for instance, [6, 59, 67] and their references.

First, we recall the notions of both Orlicz functions and Orlicz spaces.

Definition 4.12 A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is called an *Orlicz function* if it is non-decreasing and satisfies $\Phi(0) = 0$, $\Phi(t) > 0$ whenever $t \in (0, \infty)$, and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$.

An Orlicz function Φ as in Definition 4.12 is said to be of *lower* (resp., *upper*) *type* p with $p \in \mathbb{R}$ if there exists a positive constant $C_{(p)}$, depending on p , such that, for any $t \in [0, \infty)$ and $s \in (0, 1)$ [resp., $s \in [1, \infty)$],

$$\Phi(st) \leq C_{(p)}s^p\Phi(t).$$

A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is said to be of *positive lower* (resp., *upper*) *type* if it is of lower (resp., upper) type p for some $p \in (0, \infty)$.

Definition 4.13 Let Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+ . The *Orlicz space* $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on \mathbb{R}^n such that

$$\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\} < \infty.$$

It is well known that, if $p_\Phi^-, p_\Phi^+ \in (1, \infty)$, then the dual space of $L^\Phi(\mathbb{R}^n)$ is $L^\Psi(\mathbb{R}^n)$, where Ψ denotes the *complementary function* defined by setting

$$\Psi(t) := \sup \{xt - \Phi(x) : x \in [0, \infty)\}$$

for any $t \in [0, \infty)$ (see [83, Definition 2.14]) of Φ . Moreover, $L^\Phi(\mathbb{R}^n)$ is a Banach function space and hence a ball Banach function space. Furthermore, \mathcal{M} is bounded on $L^\Phi(\mathbb{R}^n)$ and $L^\Psi(\mathbb{R}^n)$. These basic properties can be found in, for instance, [71, Subsection 7.6]. Thus, all the assumptions of the main theorems in Sections 2 and 3 are satisfied. Using Theorems 2.17, 2.22, 3.1, and 3.2, we immediately obtain the following characterization of the boundedness and the compactness of commutators on Orlicz spaces.

Theorem 4.14 *Let $p_\Phi^-, p_\Phi^+ \in (1, \infty)$ and Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+ . Then Theorems 2.17, 2.22, 3.1, and 3.2 hold true with X replaced by $L^\Phi(\mathbb{R}^n)$.*

Remark 4.15 To the best of our knowledge, Theorem 4.14 is totally new.

4.6 Orlicz-slice Spaces

Now, we recall the notion of Orlicz-slice spaces.

Definition 4.16 Let $t, r \in (0, \infty)$ and Φ be an Orlicz function with positive lower type p_{Φ}^{-} and positive upper type p_{Φ}^{+} . The Orlicz-slice space $(E_{\Phi}^r)_t(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on \mathbb{R}^n such that

$$\|f\|_{(E_{\Phi}^r)_t(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left[\frac{\|f\mathbf{1}_{B(x,t)}\|_{L^{\Phi}(\mathbb{R}^n)}}{\|\mathbf{1}_{B(x,t)}\|_{L^{\Phi}(\mathbb{R}^n)}} \right]^r dx \right\}^{\frac{1}{r}} < \infty.$$

Remark 4.17 By [83, Lemma 2.28], we know that the Orlicz-slice space $(E_{\Phi}^r)_t(\mathbb{R}^n)$ is a ball Banach function space, but it may not be a Banach function space (see, for instance, [85, Remark 7.41(i)]).

The Orlicz-slice space was introduced by Zhang et al. [83], which is a generalization of slice spaces proposed by Auscher and Mourgoglou [7, 8] and Wiener amalgam spaces in [40, 41, 53]. Let $t \in (0, \infty)$, $r \in (1, \infty)$, and Φ be an Orlicz function with positive lower type $p_{\Phi}^{-} \in (1, \infty)$ and positive upper type $p_{\Phi}^{+} \in (1, \infty)$. Then \mathcal{M} is bounded on $(E_{\Phi}^r)_t(\mathbb{R}^n)$ with the implicit positive constant independent of t (see [83, Proposition 2.22]). Besides, from [83, Theorem 2.26], it follows that

$$[(E_{\Phi}^r)_t(\mathbb{R}^n)]' = (E_{\Psi}^r)_t(\mathbb{R}^n),$$

where Ψ is the complementary function of Φ . By this, [83, Proposition 2.22 and Lemma 4.4], we conclude that \mathcal{M} is bounded on $[(E_{\Phi}^r)_t(\mathbb{R}^n)]'$. Thus, all the assumptions of the main theorems in Sections 2 and 3 are satisfied. Using Theorems 2.17, 2.22, 3.1, and 3.2, we immediately obtain the following characterization of the boundedness and the compactness of commutators on Orlicz-slice spaces.

Theorem 4.18 Let $t \in (0, \infty)$, $r, p_{\Phi}^{-}, p_{\Phi}^{+} \in (1, \infty)$, and Φ be an Orlicz function with positive lower type p_{Φ}^{-} and positive upper type p_{Φ}^{+} . Then Theorems 2.17, 2.22, 3.1, and 3.2 hold true with X replaced by $(E_{\Phi}^r)_t(\mathbb{R}^n)$.

Remark 4.19 To the best of our knowledge, Theorem 4.18 is totally new and, even for slice spaces in [7, 8], namely, $(E_{\Phi}^r)_t(\mathbb{R}^n)$ with $\Phi(u) := u^p$ for any $u \in [0, \infty)$ and $p \in (1, \infty)$, it is also new.

Acknowledgements The authors would like to thank Professors Huoxiong Wu and Qingying Xue for some useful discussions on the subject of this article. The authors would also like to thank both two referees and the associate editor for their carefully reading and several motivating remarks which indeed improve the quality of this article.

References

1. Adams, D.R.: Morrey Spaces. Birkhäuser/Springer, Cham (2015)
2. Adams, D.R., Xiao, J.: Nonlinear potential analysis on Morrey spaces and their capacities. Ind. Univ. Math. J. **53**, 1629–1663 (2004)

3. Andersen, K.F., John, R.T.: Weighted inequalities for vector-valued maximal functions and singular integrals. *Studia Math.* **69**, 19–31 (1980)
4. Arai, R., Nakai, E.: Commutators of Calderón–Zygmund and generalized fractional integral operators on generalized Morrey spaces. *Rev. Mat. Complut.* **31**, 287–331 (2018)
5. Arai, R., Nakai, E.: An extension of the characterization of CMO and its application to compact commutators on Morrey spaces. *J. Math. Soc. Japan* **72**, 507–539 (2020)
6. Astala, K., Iwaniec, T., Koskela, P., Martin, G.: Mappings of BMO-bounded distortion. *Math. Ann.* **317**, 703–726 (2000)
7. Auscher, P., Mourougolou, M.: Representation and uniqueness for boundary value elliptic problems via first order systems. *Rev. Mat. Iberoam.* **35**, 241–315 (2019)
8. Auscher, P., Prisuelos-Arribas, C.: Tent space boundedness via extrapolation. *Math. Z.* **286**, 1575–1604 (2017)
9. Benedek, A., Panzone, R.: The space L^p , with mixed norm. *Duke Math. J.* **28**, 301–324 (1961)
10. Bennett, C., Sharpley, R.: *Interpolation of Operators*. Pure Appl. Math., vol. 129. Academic Press, Boston, MA (1988)
11. Birnbaum, Z., Orlicz, W.: Über die verallgemeinerung des begriffes der zueinander konjugierten potenzten. *Studia Math.* **3**, 1–67 (1931)
12. Bokayev, N.A., Burenkov, V.I., Matin, D.T.: On precompactness of a set in general local and global Morrey-type spaces. *Eurasian. Math. J.* **8**, 109–115 (2017)
13. Chaffee, L., Cruz-Uribe, D.: Necessary conditions for the boundedness of linear and bilinear commutators on Banach function spaces. *Math. Inequal. Appl.* **21**, 1–16 (2018)
14. Chang, D.-C., Wang, S., Yang, D., Zhang, Y.: Littlewood–Paley characterizations of Hardy-type spaces associated with ball quasi-Banach function spaces. *Complex Anal. Oper. Theory* **14**, Paper No. 40, 1–33 (2020)
15. Chen, J., Hu, G.: Compact commutators of rough singular integral operators. *Canad. Math. Bull.* **58**, 19–29 (2015)
16. Chen, Y., Deng, Q., Ding, Y.: Commutators with fractional differentiation for second-order elliptic operators on \mathbb{R}^n . *Commun. Contemp. Math.* **22**(1950010), 1–29 (2020)
17. Chen, Y., Ding, Y.: L^p bounds for the commutators of singular integrals and maximal singular integrals with rough kernels. *Trans. Amer. Math. Soc.* **367**, 1585–1608 (2015)
18. Chen, Y., Ding, Y., Hong, G.: Commutators with fractional differentiation and new characterizations of BMO-Sobolev spaces. *Anal. PDE* **9**, 1497–1522 (2016)
19. Chen, Y., Ding, Y., Wang, X.: Compactness of commutators for singular integrals on Morrey spaces. *Canad. J. Math.* **64**, 257–281 (2012)
20. Cheung, K., Ho, K.-P.: Boundedness of Hardy–Littlewood maximal operator on block spaces with variable exponent. *Czechoslovak Math. J.* **64**(139), 159–171 (2014)
21. Chiarenza, F., Frasca, M.: Morrey spaces and Hardy–Littlewood maximal function. *Rend. Mat. Appl.* (7) **7**, 273–279 (1987)
22. Cleanthous, G., Georgiadis, A.G., Nielsen, M.: Discrete decomposition of homogeneous mixed-norm Besov spaces. In: *Functional Analysis, Harmonic Analysis, and Image Processing: a Collection of Papers in Honor of Björn Jawerth*, vol. 693, pp. 167–184. *Contemp. Math. Amer. Math. Soc.*, Providence (2017)
23. Clop, A., Cruz, V.: Weighted estimates for Beltrami equations. *Ann. Acad. Sci. Fenn. Math.* **38**, 91–113 (2013)
24. Coifman, R.R., Rochberg, R., Weiss, G.: Factorization theorems for Hardy spaces in several variables. *Ann. of Math. (2)* **103**, 611–635 (1976)
25. Cruz-Uribe, D.V.: Extrapolation and factorization. [arXiv:1706.02620](https://arxiv.org/abs/1706.02620)
26. Cruz-Uribe, D.V., Fiorenza, A.: *Variable Lebesgue Space. Foundations and Harmonic Analysis. Applied and Numerical Harmonic Analysis*. Birkhäuser/Springer, Heidelberg (2013)
27. Cruz-Uribe, D.V., Wang, L.A.D.: Variable Hardy spaces. *Ind. Univ. Math. J.* **63**, 447–493 (2014)
28. Diening, L., Hästö, P., Roudenko, S.: Function spaces of variable smoothness and integrability. *J. Funct. Anal.* **256**, 1731–1768 (2009)
29. Diening, L., Harjulehto, P., Hästö, P., Ružička, M.: *Lebesgue and Sobolev Spaces with Variable Exponents*. *Lecture Notes in Mathematics*, vol. 2017. Springer, Heidelberg (2011)
30. Di Fazio, G., Ragusa, M.A.: Commutators and Morrey spaces. *Boll. Un. Mat. Ital. A (7)* **5**, 323–332 (1991)
31. Duoandikoetxea, J.: *Fourier Analysis*. *Graduate Studies in Mathematics*, vol. 29. American Mathematical Society, Providence (2001)
32. Georgiadis, A.G., Johnsen, J., Nielsen, M.: Wavelet transforms for homogeneous mixed-norm Triebel–Lizorkin spaces. *Monatsh Math.* **183**, 587–624 (2017)

33. Grafakos, L.: Classical Fourier Analysis. Third edition. Graduate Texts in Mathematics, vol. 249. Springer, New York (2014)
34. Guliyev, V., Omarova, M., Sawano, Y.: Boundedness of intrinsic square functions and their commutators on generalized weighted Orlicz–Morrey spaces. *Banach J. Math. Anal.* **9**, 44–62 (2015)
35. Guo, X., Hu, G.: On the commutators of singular integral operators with rough convolution kernels. *Canad. J. Math.* **68**, 816–840 (2016)
36. Guo, W., Lian, J., Wu, H.: The unified theory for the necessity of bounded commutators and applications. *J. Geom. Anal.* **30**, 3995–4035 (2020)
37. Guo, W., Wu, H., Yang, D.: A revised on the compactness of commutators. *Canad. J. Math.* <https://doi.org/10.4153/S0008414X20000644> (2020)
38. Guo, W., Zhao, G.: On relatively compact sets in quasi-Banach function spaces. *Proc. Amer. Math. Soc.* **148**, 3359–3373 (2020)
39. Ho, K.-P.: Atomic decomposition of Hardy–Morrey spaces with variable exponents. *Ann. Acad. Sci. Fenn. Math.* **40**, 31–62 (2015)
40. Ho, K.-P.: Dilation operators and integral operators on amalgam space (L_p, l_q) . *Ric. Mat.* **68**, 661–677 (2019)
41. Holland, F.: Harmonic analysis on amalgams of L^p and l^q . *J. London Math. Soc. (2)* **10**, 295–305 (1975)
42. Hörmander, L.: Estimates for translation invariant operators in L^p spaces. *Acta Math.* **104**, 93–140 (1960)
43. Huang, L., Chang, D.-C., Yang, D.: Fourier transform of Hardy spaces associated with ball quasi-Banach function spaces. *Appl Anal.* <https://doi.org/10.1142/S0219530521500135> (2021)
44. Huang, L., Liu, J., Yang, D., Yuan, W.: Atomic and Littlewood–Paley characterizations of anisotropic mixed-norm Hardy spaces and their applications. *J. Geom. Anal.* **29**, 1991–2067 (2019)
45. Huang, L., Liu, J., Yang, D., Yuan, W.: Dual spaces of anisotropic mixed-norm Hardy spaces. *Proc. Amer. Math. Soc.* **147**, 1201–1215 (2019)
46. Huang, L., Yang, D.: On function spaces with mixed norms — a survey. *J. Math. Study* **54**, 262–336 (2021)
47. Iwaniec, T.: L^p -theory of quasiregular mappings. In: *Quasiconformal Space Mappings. Lecture Notes in Math.* vol. 1508, pp. 39–64. Springer, Berlin (1992)
48. Izuki, M., Noi, T., Sawano, Y.: The John–Nirenberg inequality in ball Banach function spaces and application to characterization of BMO. *J. Inequal. Appl. Paper No. 268*, pp. 11 (2019)
49. Izuki, M., Sawano, Y.: Characterization of BMO via ball Banach function spaces. *Vestn. St.-Peterbg. Univ. Mat. Mekh. Astron.* **4(62)**, 78–86 (2017)
50. Jia, H., Wang, H.: Decomposition of Hardy–Morrey spaces. *J. Math. Anal. Appl.* **354**, 99–110 (2009)
51. John, F., Nirenberg, L.: On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* **14**, 415–426 (1961)
52. Karlovich, A., Lerner, A.: Commutators of singular integrals on generalized L^p spaces with variable exponent. *Publ. Mat.* **49**, 111–125 (2005)
53. Kikuchi, N., Nakai, E., Tomita, N., Yabuta, K., Yoneda, T.: Calderón–Zygmund operators on amalgam spaces and in the discrete case. *J. Math. Anal. Appl.* **335**, 198–212 (2007)
54. Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak. Math. J.* **41(116)**, 592–618 (1991)
55. Krantz, S.G., Li, S.Y.: Boundedness and compactness of integral operators on spaces of homogeneous type and applications. II. *J. Math. Anal. Appl.* **258**, 642–657 (2001)
56. Lerner, A.K., Ombrosi, S., Rivera-Ríos, I.P.: Commutators of singular integrals revisited. *Bull. Lond. Math. Soc.* **51**, 107–119 (2019)
57. Lizorkin, P.I.: Multipliers of Fourier integrals and estimates of convolutions in spaces with mixed norm. applications. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **34**, 218–247 (1970)
58. Lu, S., Ding, Y., Yan, D.: *Singular Integrals and Related Topics*. World Scientific Publishing Co. Pte. Ltd., Hackensack (2007)
59. Martínez, S., Wolanski, N.: A minimum problem with free boundary in Orlicz spaces. *Adv. Math.* **218**, 1914–1971 (2008)
60. Morrey, C.B.: On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.* **43**, 126–166 (1938)
61. Nakano, H.: *Modulated Semi-Ordered Linear Spaces*. Maruzen, Tokyo (1950)
62. Nakano, H.: *Topology of Linear Topological Spaces*. Maruzen, Tokyo (1951)
63. Nakamura, S., Sawano, Y.: The singular integral operator and its commutator on weighted Morrey spaces. *Collect. Math.* **68**, 145–174 (2017)
64. Nogayama, T.: Mixed Morrey spaces. *Positivity* **23**, 961–1000 (2019)
65. Nogayama, T., Ono, T., Salim, D., Sawano, Y.: Atomic decomposition for mixed Morrey spaces. *J. Geom. Anal.* **31**, 9338–9365 (2021)

66. Orlicz, W.: Über eine gewisse Klasse von räumen vom typus B. *Bull. Inst. Acad. Pol. Ser. A* **8**, 207–220 (1932)
67. Rao, M.M., Ren, Z.D.: *Theory of Orlicz Spaces. Monographs and Textbooks in Pure and Applied Mathematics*, vol. 146. Marcel Dekker, Inc., New York (1991)
68. Sawano, Y.: *Theory of Besov Spaces. Developments in Mathematics*, vol. 56. Springer, Singapore (2018)
69. Sawano, Y., Shirai, S.: Compact commutators on Morrey spaces with non-doubling measures. *Georgian Math. J.* **15**, 353–376 (2008)
70. Sawano, Y., Tanaka, H.: The Fatou property of block spaces. *J. Math. Sci. Univ. Tokyo* **22**, 663–683 (2015)
71. Sawano, Y., Ho, K.-P., Yang, D., Yang, S.: Hardy spaces for ball quasi-Banach function spaces. *Dissertationes. Math. (Rozprawy Mat.)* **525**, 1–102 (2017)
72. Sawano, Y., Di Fazio, G., Hakim, D.: *Morrey Spaces. Introduction and Applications to Integral Operators and PDE's*, vol. I. Chapman and Hall/CRC, New York (2020)
73. Sawano, Y., Di Fazio, G., Hakim, D.: *Morrey Spaces: Introduction and Applications to Integral Operators and PDE's*, vol. II. Chapman and Hall/CRC, New York (2020)
74. Tao, J., Yang, D.: Boundedness and compactness characterizations of Cauchy integral commutators on Morrey spaces. *Math. Methods Appl. Sci.* **42**, 1631–1651 (2019)
75. Tao, J., Yang, D., Yang, D.: Beurling–Ahlfors commutators on weighted Morrey spaces and applications to Beltrami equations. *Potential Anal.* **53**, 1467–1491 (2020)
76. Uchiyama, A.: On the compactness of operators of Hankel type. *Tôhoku Math. J. (2)* **30**, 163–171 (1978)
77. Wang, F., Yang, D., Yang, S.: Applications of Hardy spaces associated with ball quasi-Banach function spaces. *Results Math.* **75**, Paper No. 26, 1–58 (2020)
78. Wang, S., Yang, D., Yuan, W., Zhang, Y.: Weak Hardy-type spaces associated with ball quasi-Banach function spaces II: Littlewood–Paley characterizations and real interpolation. *J. Geom. Anal.* **31**, 631–696 (2021)
79. Yan, X., Yang, D., Yuan, W.: Intrinsic square function characterizations of several Hardy-type spaces — a survey. *Anal. Theory Appl.* (to appear)
80. Yan, X., Yang, D., Yuan, W.: Intrinsic square function characterizations of Hardy spaces associated with ball quasi-Banach function spaces. *Front. Math. China* **15**, 769–806 (2020)
81. Yosida, K.: *Functional Analysis. Classics in Mathematics*. Springer, Berlin (1995)
82. Yuan, W., Sickel, W., Yang, D.: *Morrey and Campanato Meet Besov, Lizorkin and Triebel. Lecture Notes in Mathematics*, vol. 2005. Springer, Berlin (2010)
83. Zhang, Y., Yang, D., Yuan, W., Wang, S.: Real-variable characterizations of Orlicz-slice Hardy spaces. *Anal. Appl. (Singap.)* **17**, 597–664 (2019)
84. Zhang, Y., Huang, L., Yang, D., Yuan, W.: New ball Campanato-type function spaces and their applications. *J. Geom. Anal.* (to appear)
85. Zhang, Y., Wang, S., Yang, D., Yuan, W.: Weak Hardy-type spaces associated with ball quasi-Banach function spaces I: Decompositions with applications to boundedness of Calderón–Zygmund operators. *Sci. China Math.* **64**, 2007–2064 (2021)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Affiliations

Jin Tao¹ · Dachun Yang¹  · Wen Yuan¹ · Yangyang Zhang¹

Jin Tao
jintao@mail.bnu.edu.cn

Wen Yuan
wenyuan@bnu.edu.cn

Yangyang Zhang
yangzhang@mail.bnu.edu.cn

¹ Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China