Potential Analysis (2023) 58:203–223
https://doi.org/10.1007/s11118-021-09937-w

Comparison Geometry for Integral Radial Bakry-Émery Ricci Tensor Bounds

Jia-Yong Wu[1]

Received: 21 February 2021 / Accepted: 2 June 2021 / Published online: 18 June 2021

Abstract
In this paper we prove mean curvature comparisons and volume comparisons on a smooth metric measure space when the integral radial Bakry-Émery Ricci tensor and the potential function or its gradient are bounded. As applications, we prove diameter estimates and eigenvalue estimates on smooth metric measure spaces. These results not only give a supplement of the author's previous results under integral Bakry-Émery Ricci tensor bounds, but also are generalizations of the Wei-Wylie's pointwise results.

Keywords Bakry-Émery Ricci tensor · Smooth metric measure space · Integral curvature · Comparison theorem · Diameter estimate · Eigenvalue estimate

Mathematics Subject Classification (2010) Primary 53C20; Secondary 53C21 · 53C65

1 Introduction and Main Results

Classical comparison properties of the pointwise Ricci curvature condition, such as the mean curvature comparison and the volume comparison, are basic theories for Riemannian manifolds. See [25] for a survey and references therein. These comparison results were later generalized to the integral Ricci tensor condition, which are briefly described as follows. Given an n-dimensional complete Riemannian manifold (M, g), for each point $x \in M$, let $\lambda(x)$ be the smallest eigenvalue for the Ricci curvature $\mathrm{Ric} : T_x M \to T_x M$, and let

$$\mathrm{Ric}_-^H(x) := [(n-1)H - \lambda(x)]_+ = \max\{0, (n-1)H - \lambda(x)\},$$

the amount of the Ricci tensor below $(n-1)H$, where $H \in \mathbb{R}$. For any real number $p > 0$ and $R > 0$, we consider

$$\|\mathrm{Ric}_-^H\|_p(R) := \sup_{x\in M}\left(\int_{B(x,R)} (\mathrm{Ric}_-^H)^p \, dv\right)^{\frac{1}{p}},$$

✉ Jia-Yong Wu
wujiayong@shu.edu.cn

[1] Department of Mathematics, Shanghai University, Shanghai, 200444, China

which measures the amount of Ricci tensor lying below $(n-1)H$ in the L^p sense, where $B(x, R)$ is the geodesic ball with radius R and center x. It is easy to see that $\|\mathrm{Ric}_-^H\|_p(R) \equiv 0$ if and only if $\mathrm{Ric} \geq (n-1)H$. Under certain assumption of $\|\mathrm{Ric}_-^H\|_p(R)$, Petersen and Wei [15, 16] generalized classical comparison theorems to the integral case. For more related results, we refer the reader to [1, 2, 5–8, 13, 14, 17, 24] and references therein.

In another direction, Wei and Wylie [21] extended comparison results of Riemannian manifolds to smooth metric measure spaces. Recall that a complete smooth metric measure space (SMMS for short) is a triple $(M, g, e^{-f}dv)$, where (M, g) is an n-dimensional Riemannian manifold, dv is the volume element of the metric g, f is a smooth function on M and $e^{-f}dv$ is the weighted volume element. The Bakry-Émery Ricci tensor [3] and the f-Laplacian associated to $(M, g, e^{-f}dv)$ are respectively defined by

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\, f \quad \text{and} \quad \Delta_f := \Delta - \nabla f \cdot \nabla,$$

where Hess and Δ are the Hessian and Laplacian with respect to the metric g, respectively. The Bakry-Émery Ricci tensor and the f-Laplacian are related by the generalized Bochner formula

$$\Delta_f |\nabla u|^2 = 2|\mathrm{Hess}\, u|^2 + 2g(\nabla u, \nabla \Delta_f u) + 2\mathrm{Ric}_f(\nabla u, \nabla u)$$

for $u \in C^\infty(M)$. The Bakry-Émery Ricci tensor is also related to the gradient Ricci soliton defined by

$$\mathrm{Ric}_f = \lambda g$$

for some $\lambda \in \mathbb{R}$, which plays an important role in the singularities of the Ricci flow [9]. When Ric_f is bounded below and f or $|\nabla f|$ is bounded, Wei and Wylie [21] applied the generalized Bochner formula to prove various weighted comparisons and topological results on $(M, g, e^{-f}dv)$. Meanwhile, they expect that weighted comparisons can be extended to the case that Ric_f is bounded below in the integral sense.

Inspired by the above work, the author [23] generalized pointwise weighted comparison theorems [21] to the integral Bakry-Émery Ricci tensor setting. To be more precise, for each point $x \in (M, g, e^{-f}dv)$, we consider a weighted geometric quantity

$$\mathrm{Ric}_{f\,-}^H(x) := [(n-1)H - \lambda(x)]_+ = \max\{0, (n-1)H - \lambda(x)\},$$

where $H \in \mathbb{R}$ and $\lambda(x)$ is the smallest eigenvalue of $\mathrm{Ric}_f : T_xM \to T_xM$. When $\partial_r f \geq -a$ ($\partial_r := \nabla r$) for some constant $a \geq 0$, along a minimal geodesic segment r from x, we introduce a weighted L^p norm of $\mathrm{Ric}_{f\,-}^H$

$$\|\mathrm{Ric}_{f\,-}^H\|_{p,a}(R) := \sup_{x\in M}\left(\int_{B(x,R)} |\mathrm{Ric}_{f\,-}^H|^p \mathcal{A}_f e^{-at} dt d\theta_{n-1}\right)^{\frac{1}{p}},$$

where $\mathcal{A}_f(t, \theta)$ is the volume element of $e^{-f}dv_g = \mathcal{A}_f(t, \theta)dt \wedge d\theta_{n-1}$ in polar coordinate, and $d\theta_{n-1}$ is the volume element on unit sphere S^{n-1}. When $\partial_r f \geq -a$, we easily see that $\|\mathrm{Ric}_{f\,-}^H\|_{p,a}(R) \equiv 0$ if and only if $\mathrm{Ric}_f \geq (n-1)H$. In [23], the author proved many weighted comparison theorems on $(M, g, e^{-f}dv)$ when $\|\mathrm{Ric}_{f\,-}^H\|_{f,a}(R)$ is bounded and $\partial_r f \geq -a$. As applications, classical eigenvalue estimates, Sobolev constant estimates and Myers' type theorems, etc were generalized to the case of some assumptions of $\|\mathrm{Ric}_{f\,-}^H\|_{f,a}(R)$ and $\partial_r f$; see [10, 20, 23]. However, when f is bounded, there seem to be lack of effective comparison theorems under the integral Bakry-Émery Ricci tensor, though some progress has been made in [23].

In this paper we will prove some comparison results on $(M, g, e^{-f}dv)$ when the integral radial Bakry-Émery Ricci tensor is bounded and f or $\partial_r f$ is bounded. Our results are different from the case of [23] and seem to be new even in the manifold case. As applications, we prove some new Myers' type theorems and eigenvalue estimates.

To state our results, we fix some notations. On SMMS $(M, g, e^{-f}dv)$, for any point $x \in M$ and any $r(y) := d(y, x)$ a distance function from x to y, in geodesic polar coordinates (r, θ), we have another expression of $\mathrm{Ric}^H_{f\,-}$:

$$\rho(r, \theta) := [(n-1)H - \lambda(r, \theta)]_+ ,$$

where $H \in \mathbb{R}$ and $\lambda(r, \theta)$ be the smallest eigenvalue for Ric_f at the point (r, θ). Clearly,

$$(n-1)H - \mathrm{Ric}_f(\partial_r, \partial_r) \le \rho(r, \theta)$$

along that minimal geodesic segment from x; while $\rho(r, \theta) \equiv 0$ along the minimal geodesic segment r if and only if $\mathrm{Ric}_f(\partial_r, \partial_r) \ge (n-1)H$. Let m^n_H denote the mean curvature of the geodesic sphere in the model space (M^n_H, g_H), the n-dimensional simply connected space with constant sectional curvature H. For the weighted measure $e^{-f}dv$, we define the weighted mean curvature

$$m_f := m - \partial_r f,$$

which measures the relative rate of change of the weighted volume element of the geodesic sphere, where m is the mean curvature of the geodesic sphere in the outer normal direction.

Let us first state weighted mean curvature comparisons on $(M, g, e^{-f}dv)$ along the integral radial Bakry-Émery Ricci tensor.

Theorem 1.1 (Mean Curvature Comparison) *Let $(M, g, e^{-f}dv)$ be an n-dimensional smooth metric measure space with a base point $x \in M$. Fix $H \in \mathbb{R}$.*

(a) If $|f| \le k$ for some constant $k \ge 0$, along a minimal geodesic segment r from $x \in M$ (assume $r \le \frac{\pi}{4\sqrt{H}}$ when $H > 0$), then

$$m_f(r) \le m^{n+4k}_H(r) + \int_0^r \rho(t, \theta)dt$$

along that minimal geodesic segment from x. For the case $H > 0$ and $\frac{\pi}{4\sqrt{H}} \le r \le \frac{\pi}{2\sqrt{H}}$, then

$$m_f(r) \le \left(1 + \frac{4k}{n-1} \cdot \frac{1}{\sin(2\sqrt{H}r)}\right) m^n_H(r) + \int_0^r \rho(t, \theta)dt \tag{1}$$

along that minimal geodesic segment from x.

(b) If $\partial_r f \ge -a$ for some constant $a \ge 0$, along a minimal geodesic segment r from $x \in M$ (assume $r \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$), then

$$m_f(r) \le m^n_H(r) + a + \int_0^r \rho(t, \theta)dt$$

along that minimal geodesic segment from x. Equality holds if and only if the radial sectional curvatures are equal to H and $f(t) = f(x) - at$ for all $t < r$.

When $\rho = 0$, we have $\mathrm{Ric}_f(\partial_r, \partial_r) \ge (n-1)H$ and Theorem 1.1 recovers Wei-Wylie's results [21]. When $k = 0$, Theorem 1.1 reduces to the manifold cases, which seems to be firstly appeared in the literature. The estimate Eq. 1 will be used in the Myers' type diameter estimate.

As in the classical case, weighted mean curvature comparisons have many applications. First, we have weighted volume comparisons. On $(M, g, e^{-f}dv)$, the weighted volume of the ball $B(x,r)$ is defined by

$$V_f(x,r) := \int_0^r e^{-f}dv.$$

Let $V^n_H(R)$ be the volume of ball $B(O,R)$ in the model space (M^n_H, g_H), where $O \in M^n_H$. When $\partial_r f \geq -a$ for some constant $a \geq 0$ along all minimal geodesic segments from x, we introduce a new model volume according to constant a. That is, let $V^a_H(R)$ be the h-volume of ball $B(O,R)$ in the pointed smooth metric measure space

$$M^n_{H,a} = (M^n_H, g_H, e^{-h}dv_{g_H}, O),$$

where $O \in M^n_H$ and $h(x) = -a \cdot d(O,x)$. Indeed we have

$$V^a_H(R) := \int_0^R \int_{S^{n-1}} \mathcal{A}^a_H(t,\theta)\, d\theta_{n-1}dt = \int_0^R A^a_H(t)dt,$$

where $\mathcal{A}^a_H(t,\theta) = e^{at}\mathcal{A}_H(t,\theta)$, $A^a_H(t) = e^{at}A_H(t)$, $\mathcal{A}_H$ and A_H are the volume element and the volume of the geodesic sphere respectively in the model space (M^n_H, g_H).

Theorem 1.2 (Volume Comparison) *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space with a base point $x \in M$. Fix $H \in \mathbb{R}$. Assume that*

$$\int_0^\infty \rho(t,\theta)dt \leq l$$

along all minimal geodesic segments from $x \in M$, where $l \geq 0$ is a constant.

(a) If $|f| \leq k$ for some constant $k \geq 0$, then for $0 < r \leq R$ (assume $R \leq \frac{\pi}{4\sqrt{H}}$ when $H > 0$),

$$\frac{V_f(x,R)}{V^{n+4k}_H(R)} \leq \frac{V_f(x,r)}{V^{n+4k}_H(r)} \exp\left\{\int_0^R \left(e^{c(n,k,H)lt} - 1\right)\frac{A^{n+4k}_H(t)}{V^{n+4k}_H(t)}dt\right\},$$

where $c(n,k,H) := \frac{V(S^{n+4k-1})}{V(S^{n-1})}$ and $V(S^{n-1})$ is the area of the unit sphere $S^{n-1} \subset M^{n-1}_H$.

(b) If $\partial_r f \geq -a$ for some constant $a \geq 0$, along all minimal geodesic segments from $x \in M$, then for $0 < r \leq R$ (assume $R \leq \frac{\pi}{2\sqrt{H}}$ when $H > 0$),

$$\frac{V_f(x,R)}{V^a_H(R)} \leq \frac{V_f(x,r)}{V^a_H(r)} \exp\left\{\int_0^R \left(e^{lt} - 1\right)\frac{A^a_H(t)}{V^a_H(t)}dt\right\}.$$

Furthermore, when $r = 0$, we have

$$V_f(x,R) \leq V^a_H(R) \exp\left\{-f(x) + \int_0^R \left(e^{lt} - 1\right)\frac{A^a_H(t)}{V^a_H(t)}dt\right\}$$

for $R \geq 0$.

When $l = 0$, Theorem 1.2 returns to Wei-Wylie's results [21]. We remark that the term $\frac{V_f(x,r)}{V^{n+4k}_H(r)}$ $(k > 0)$ in Theorem 1.2 (a) blows up if $r \to 0$. If we let $r = 1$, then

$$V_f(x,R) \leq \frac{V_f(x,1)}{V^{n+4k}_H(1)} V^{n+4k}_H(R) \exp\left\{\int_0^R \left(e^{c(n,k,H)lt} - 1\right)\frac{A^{n+4k}_H(t)}{V^{n+4k}_H(t)}dt\right\} \tag{2}$$

for $R \geq 1$. This estimate will be improved when $H < 0$; see Theorem 3.3 in Section 3.

Next, we apply Theorem 1.1 to give Myers' type diameter estimates, which are regarded as generalizations of the Wei-Wylie's result [21].

Theorem 1.3 (Myers' Theorem) *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space. Fix $H \in \mathbb{R}^+$. Assume that*

$$\int_0^\infty \rho(t,\theta)dt \le l$$

along all minimal geodesic segments from every point $p \in M$, where $l \ge 0$ is a constant.

(a) If $|f| \le k$ for some constant $k \ge 0$, then M is compact and

$$\mathrm{diam}(M) \le \frac{\pi}{\sqrt{H}} + \frac{4k\sqrt{H} + 2l}{(n-1)H}.$$

(b) If $|\nabla f| \le a$ for some constant $a \ge 0$, then M is compact and

$$\mathrm{diam}(M) \le \frac{\pi}{\sqrt{H}} + \frac{2a + 2l}{(n-1)H}.$$

We point out that our integral assumption in Theorem 1.3 needs to hold for *every* point $p \in M$ and it seems to be a stronger condition. In Section 4, we can apply the index form argument to get another diameter estimate under a weaker assumption; see Theorem 4.1.

Finally, we apply volume comparisons to give a generalization of Cheng's eigenvalue estimates [4]. On an n-dimensional SMMS $(M, g, e^{-f}dv)$, we assume that $\partial_r f \ge -a$ for some constant $a \ge 0$, along all minimal geodesic segments from a point $x_0 \in M$. For any $H \in \mathbb{R}$ and $R > 0$ ($R \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$), we let $\lambda_1^D(B(x_0, R))$ be the first eigenvalue of the f-Laplacian with the Dirichlet condition in $B(x_0, R) \subseteq M$. We also let $\lambda_1^D(n, a, H, R)$ be the first eigenvalue of the h-Laplacian Δ_h, where $h(x) := -a \cdot d(\bar{x}_0, x)$, with the Dirichlet condition in a metric ball $B(\bar{x}_0, R) \subseteq M^n_{H,a}$. Then we have a weighted version of Petersen-Sprouse's result [14].

Theorem 1.4 (Cheng's Eigenvalue Estimate) *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space with $\partial_r f \ge -a$ for some constant $a \ge 0$, along all minimal geodesic segments from a point $x_0 \in M$. Given $H \in \mathbb{R}$, $R > 0$ (assume $R \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$), for every $\delta > 0$, there exists an $\epsilon = \epsilon(n, a, H, R)$ such that if*

$$\int_0^\infty \rho(t,\theta)dt \le \epsilon$$

along all minimal geodesic segments from the point $x_0 \in M$, then

$$\lambda_1^D(B(x_0, R)) \le (1+\delta)\ \lambda_1^D(n, a, H, R).$$

When $\rho \equiv 0$ and f is constant, Theorem 1.4 returns to Cheng's result [4]. In [23], the author proved another generalization of Cheng's eigenvalue estimates, but this result is different from that case. For the case $|f| \le k$, there seem to be essential obstacles to deriving Cheng's eigenvalue estimates because volume comparisons in this case depend on the volumes of *higher* dimensional geodesic balls.

The rest of this paper is organized as follows. In Section 2, we study mean curvature comparisons along the integral radial Bakry-Émery Ricci tensor. In particular we prove Theorem 1.1. In Section 3, we prove various volume comparisons, including Theorem 1.2 and the volume doubling. In Section 4, we apply Theorem 1.1 to prove Myers' diameter estimates (Theorem 1.3). We also apply the index form to give another diameter estimate

(Theorem 4.1). In Section 5, we apply the volume doubling to prove eigenvalue estimates (Theorem 1.4).

2 Mean Curvature Comparison

In this section, we will discuss mean curvature comparisons on $(M, g, e^{-f}dv)$ when the integral radial Bakry-Émery Ricci tensor and f or $\partial_r f$ are bounded. We shall prove Theorem 1.1. The proof mainly uses the arguments of Petersen and Wei [15], and Wei and Wylie [21]. First, we give a rough estimate on m_f which will be used in the proof of Myers' type diameter estimates.

Theorem 2.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional smooth metric measure space with a base point $x \in M$. Fix $H \in \mathbb{R}$. Then given any minimal geodesic segment from x and $r_0 > 0$,*

$$m_f(r) \le m_f(r_0) - (n-1)H(r - r_0) + \int_{r_0}^{r} \rho(t, \theta)dt$$

for $r \ge r_0$. Equality holds for some $r > r_0$ if and only if all the radial sectional curvatures are zero, $\operatorname{Hess} r \equiv 0$, *and $\partial_r \partial_r f = (n-1)H - \rho(r, \theta)$ along the geodesic from r_0 to r.*

Proof of Theorem 2.1 Let $u = r(y)$, where $r(y) = d(y, x)$ is the distance function. It is well-known that distance function r is almost smooth on M and also $|\nabla r| = 1$ holds where r is smooth. Applying u to the Bochner formula

$$\Delta |\nabla u|^2 = 2|\operatorname{Hess} u|^2 + 2g(\nabla u, \nabla \Delta u) + 2\operatorname{Ric}(\nabla u, \nabla u)$$

and using the fact $|\nabla r| = 1$, we get

$$0 = |\operatorname{Hess} r|^2 + \partial_r(\Delta r) + \operatorname{Ric}(\partial_r, \partial_r), \tag{3}$$

where $\partial_r = \nabla r$. Note that $\operatorname{Hess} r$ is the second fundamental from of the geodesic sphere and $\Delta r = m$, the mean curvature of the geodesic sphere. By the Schwarz inequality,

$$m' \le -\frac{m^2}{n-1} - \operatorname{Ric}(\partial_r, \partial_r). \tag{4}$$

Since $m_f := m - \partial_r f$, i.e. $m_f = \Delta_f r$, then

$$m'_f = m' - \partial_r \partial_r f,$$

and hence

$$m'_f \le -\frac{m^2}{n-1} - \operatorname{Ric}_f(\partial_r, \partial_r).$$

By the definition of $\rho(r, \theta)$, we get

$$\begin{aligned} m'_f &\le -\frac{m^2}{n-1} - (n-1)H - \rho(r, \theta) \\ &\le -(n-1)H + \rho(r, \theta). \end{aligned} \tag{5}$$

Integrating this inequality from r_0 to r gives the result.

To see the equality statement, suppose that

$$m'_f = -(n-1)H + \rho(r, \theta)$$

on an interval $[r_0, r]$, then from Eq. 5 we get $m = 0$ (i.e. $\Delta r = 0$). We also have

$$(n-1)H - \mathrm{Ric}_f(\partial_r, \partial_r) = \rho(r, \theta).$$

So,

$$m_f' = -\partial_r \partial_r f = -\mathrm{Ric}_f(\partial_r, \partial_r) = -(n-1)H + \rho(r, \theta).$$

This implies $\mathrm{Ric}(\partial_r, \partial_r) = 0$. Then from Eq. 3 we have $\mathrm{Hess}\, r = 0$, which implies the sectional curvatures must be zero. □

In the following we will prove Theorem 1.1.

Proof of Theorem 1.1 We start to prove part (a) of Theorem 1.1. From Eq. 4, we see that this inequality becomes equality if and only if the radial sectional curvatures are constant. So the mean curvature $m_H(r)$ of the n-dimensional model space satisfies

$$m_H' = -\frac{m_H^2}{n-1} - (n-1)H,$$

where

$$m_H(r) := (n-1)\frac{\mathrm{sn}_H'(r)}{\mathrm{sn}_H(r)},$$

and $\mathrm{sn}_H(r)$ is the unique function satisfying

$$\mathrm{sn}_H''(r) + H\mathrm{sn}_H(r) = 0$$

with $\mathrm{sn}_H(0) = 0$ and $\mathrm{sn}_H'(0) = 1$. So

$$\begin{aligned}(m - m_H)' &\leq -\frac{m^2 - m_H^2}{n-1} + (n-1)H - \mathrm{Ric}(\partial r, \partial r)\\ &\leq -\frac{m^2 - m_H^2}{n-1} + \partial_r \partial_r f + \rho(r, \theta),\end{aligned}$$

where we used the definition of ρ in the second inequality. Then we compute that

$$\begin{aligned}\left[\mathrm{sn}_H^2(m - m_H)\right]' &= \mathrm{sn}_H^2 \frac{2m_H}{n-1}(m - m_H) + \mathrm{sn}_H^2\left(-\frac{m^2 - m_H^2}{n-1} + \partial_r \partial_r f + \rho(r, \theta)\right)\\ &= -\mathrm{sn}_H^2(r)\frac{(m - m_H)^2}{n-1} + \mathrm{sn}_H^2(r)\partial_r \partial_r f + \mathrm{sn}_H^2(r)\rho(r, \theta)\\ &\leq \mathrm{sn}_H^2(r)\partial_r \partial_r f + \mathrm{sn}_H^2(r)\rho(r, \theta).\end{aligned}$$

Integrating the above inequality from 0 to r yields

$$\mathrm{sn}_H^2(r)m(r) \leq \mathrm{sn}_H^2(r)m_H(r) + \int_0^r \mathrm{sn}_H^2(t)\partial_t \partial_t f(t)dt + \int_0^r \mathrm{sn}_H^2(t)\rho(t, \theta)dt.$$

Integrating by parts on the above third term,

$$\mathrm{sn}_H^2(r)m_f(r) \leq \mathrm{sn}_H^2(r)m_H(r) - \int_0^r \partial_t f(t)(\mathrm{sn}_H^2)'(t)dt + \int_0^r \mathrm{sn}_H^2(t)\rho(t, \theta)dt, \quad (6)$$

where $m_f := m - \partial_r f$. Integrating by parts on the above third term again,

$$\mathrm{sn}_H^2(r)m_f(r) \leq \mathrm{sn}_H^2(r)m_H(r) - f(r)(\mathrm{sn}_H^2(r))' + \int_0^r f(t)(\mathrm{sn}_H^2)''(t)dt + \int_0^r \mathrm{sn}_H^2(t)\rho(t, \theta)dt. \quad (7)$$

We see that if $H\le 0$, then $(\mathrm{sn}_H^2)''(t)\ge 0$; if $H>0$ and $0<r\le\frac{\pi}{4\sqrt{H}}$, then $(\mathrm{sn}_H^2)''(t)\ge 0$. Hence when $|f|\le k$, in any case, we have

$$\mathrm{sn}_H^2(r)m_f(r)\le \mathrm{sn}_H^2(r)m_H(r)+2k(\mathrm{sn}_H^2(r))'+\int_0^r \mathrm{sn}_H^2(t)\rho(t,\theta)dt.$$

Noticing that

$$(\mathrm{sn}_H^2(r))'=2\mathrm{sn}_H(r)(\mathrm{sn}_H(r))'=\frac{2m_H(r)}{n-1}\mathrm{sn}_H^2(r)$$

and $\mathrm{sn}_H^2(t)$ is increasing, we finally get

$$m_f(r)\le m_H^{n+4k}(r)+\int_0^r\rho(t,\theta)dt$$

along that minimal geodesic segment from x. This proves the first inequality of theorem.

Next we prove the case $H>0$ and $\frac{\pi}{4\sqrt{H}}\le r\le\frac{\pi}{2\sqrt{H}}$. We start with Eq. 7 and give a delicate estimate. Since $m_H(r)\ge 0$ for $\frac{\pi}{4\sqrt{H}}\le r\le\frac{\pi}{2\sqrt{H}}$, we observe that

$$\begin{aligned}-f(r)(\mathrm{sn}_H^2(r))'&=-f(r)\frac{2m_H(r)}{n-1}\mathrm{sn}_H^2(r)\\&\le\frac{2k}{n-1}m_H(r)\mathrm{sn}_H^2(r).\end{aligned}$$

Also,

$$\begin{aligned}\int_0^r f(t)\cdot(\mathrm{sn}_H^2)''(t)dt&\le k\left(\int_0^{\frac{\pi}{4\sqrt{H}}}(\mathrm{sn}_H^2)''(t)dt-\int_{\frac{\pi}{4\sqrt{H}}}^r(\mathrm{sn}_H^2)''(t)dt\right)\\&=k\left(\frac{2}{\sqrt{H}}-\mathrm{sn}_H(2r)\right).\end{aligned}$$

Substituting the above two estimates into Eq. 7, we have

$$\begin{aligned}\mathrm{sn}_H^2(r)m_f(r)&\le\left(1+\frac{2k}{n-1}\right)m_H(r)\mathrm{sn}_H^2(r)+k\left(\frac{2}{\sqrt{H}}-\mathrm{sn}_H(2r)\right)\\&\quad+\int_0^r\mathrm{sn}_H^2(t)\rho(t,\theta)dt\\&=\left(1+\frac{4k}{n-1}\right)\mathrm{sn}_H^2(r)\frac{m_H(r)}{\sin(2\sqrt{H}r)}+\int_0^r\mathrm{sn}_H^2(t)\rho(t,\theta)dt\\&\le\left(1+\frac{4k}{n-1}\right)\mathrm{sn}_H^2(r)\frac{m_H(r)}{\sin(2\sqrt{H}r)}+\mathrm{sn}_H^2(r)\int_0^r\rho(t,\theta)dt.\end{aligned}$$

Hence,

$$m_f(r)\le\left(1+\frac{4k}{n-1}\cdot\frac{1}{\sin(2\sqrt{H}r)}\right)m_H(r)+\int_0^r\rho(t,\theta)dt$$

which completes the second inequality of theorem. Hence Theorem 1.1 (a) follows.

Under Theorem 1.1 (b) assumptions, we see that

$$(\mathrm{sn}_H^2)'(t)=2\mathrm{sn}_H(t)(\mathrm{sn}_H)'(t)\ge 0.$$

So if $\partial_r f\ge -a$, from Eq. 6, we have

$$\mathrm{sn}_H^2(r)m_f(r)\le\mathrm{sn}_H^2(r)m_H(r)+a\int_0^r(\mathrm{sn}_H^2)'(t)dt+\int_0^r\mathrm{sn}_H^2(t)\rho(t,\theta)dt$$

and the third inequality of theorem follows.

To see the equality statement, assume that $\partial_r f \geq -a$ and

$$m_f(r) = m_H^n(r) + a + \int_0^r \rho(t,\theta)dt$$

for some r. Substituting them into Eq. 6,

$$\begin{aligned} a\,\mathrm{sn}_H^2(r) + \mathrm{sn}_H^2(r^2)\int_0^r \rho(t,\theta)dt &\leq -\int_0^r \partial_t f(t)(\mathrm{sn}_H^2)'(t)dt + \int_0^r \mathrm{sn}_H^2(t)\rho(t,\theta)dt \\ &\leq a\int_0^r (\mathrm{sn}_H^2)'(t)dt + \int_0^r \mathrm{sn}_H^2(t)\rho(t,\theta)dt, \end{aligned}$$

where we used $\partial_r f \geq -a$. This implies $\rho(r,\theta) = 0$ along that minimal geodesic segment r from $x \in M$. In other words, $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$. Therefore the rigidity follows from the rigidity for the Wei-Wylie's mean curvature comparison; see Theorem 1.1 in [21]. □

3 Volume Comparison

In this section, we will apply mean curvature comparisons to prove volume comparisons on $(M, g, e^{-f}dv)$ when the integral radial Bakry-Émery Ricci tensor is bounded and f or $\partial_r f$ is bounded.

On an n-dimensional SMMS $(M^n, g, e^{-f}dv_g)$, let $\mathcal{A}_f(t,\theta) = e^{-f}\mathcal{A}(t,\theta)$ be the volume element of the weighted volume form $e^{-f}dv_g = \mathcal{A}_f(t,\theta)dt \wedge d\theta_{n-1}$ in polar coordinate (r,θ), where $\mathcal{A}(t,\theta)$ is the standard volume element of the metric g. Let

$$A_f(x,r) = \int_{S^{n-1}} \mathcal{A}_f(r,\theta)d\theta_{n-1},$$

be the weighted volume of the geodesic sphere $S(x,r) = \{y \in M | d(x,y) = r\}$, and let $A_H(r)$ be the volume of the geodesic sphere $S(x,r)$ in the model space (M_H^n, g_H), the n-dimensional simply connected space with constant sectional curvature H. Moreover, the weighted volume of the ball $B(x,r) = \{y \in M | d(x,y) \leq r\}$ is defined by

$$V_f(x,r) = \int_0^r A_f(x,t)dt.$$

When $\partial_r f \geq -a$ for some constant $a \geq 0$, along all minimal geodesic segments from $x \in M$, we modify the usual model space (M_H^n, g_H) to the weighted model space $M_{H,a}^n = (M_H^n, g_H, e^{-h}dv_{g_H}, O)$, where $O \in M_H^n$, and $h(x) = -a \cdot d(x,O)$. Let $\mathcal{A}_H^a$ be the h-volume element in $M_{H,a}^n$. That is,

$$\mathcal{A}_H^a(r) = e^{ar}\mathcal{A}_H(r),$$

where $\mathcal{A}_H$ is the Riemannian volume element in (M_H^n, g_H). The corresponding h-volume of the geodesic sphere in the weighted model space $M_{H,a}^n$ is defined by

$$A_H^a(r) = \int_{S^{n-1}} \mathcal{A}_H^a(r,\theta)d\theta_{n-1}.$$

The h-volume of the ball $B(O,r) \subset M_H^n$ is defined by

$$V_H^a(r) = \int_0^r A_H^a(t)dt.$$

In order to prove Theorem 1.2, we first apply Theorem 1.1 to prove area comparisons of the geodesic spheres.

Theorem 3.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional smooth metric measure space with base point $x \in M$. Fix $H \in \mathbb{R}$. Assume that*

$$\int_0^\infty \rho(t,\theta)dt \leq l$$

along all minimal geodesic segments from $x \in M$, where $l \geq 0$ is a constant.

(a) If $|f| \leq k$ for some constant $k \geq 0$, then for $0 < r \leq R$ (assume $R \leq \frac{\pi}{4\sqrt{H}}$ when $H > 0$),

$$\frac{A_f(x,R)}{A_H^{n+4k}(R)} \leq e^{c(n,k,H)Rl}\frac{A_f(x,r)}{A_H^{n+4k}(r)} \tag{8}$$

where $c(n,k,h) := \frac{V(S^{n+4k-1})}{V(S^{n-1})}$ and $V(S^{n-1})$ is the area of the unit sphere $S^{n-1} \subset M_H^{n-1}$.

(b) If $\partial_r f \geq -a$ for some constant $a \geq 0$, along all minimal geodesic segments from $x \in M$, then for $0 < r \leq R$ (assume $R \leq \frac{\pi}{2\sqrt{H}}$ when $H > 0$),

$$\frac{A_f(x,R)}{A_H^a(R)} \leq e^{Rl}\frac{A_f(x,r)}{A_H^a(r)}. \tag{9}$$

Proof of Theorem 3.1 Applying

$$\mathcal{A}_f' = m_f\mathcal{A}_f \quad \text{and} \quad (\mathcal{A}_H^{n+4k})' = m_H^{n+4k}\mathcal{A}_H,$$

we compute that

$$\frac{d}{dt}\left(\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H^{n+4k}(t)}\right) = (m_f - m_H^{n+4k})\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H^{n+4k}(t)}.$$

Then by Theorem 1.1 (a), we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{A_f(x,t)}{A_H^{n+4k}(t)}\right) &= \frac{1}{V(S^{n-1})}\int_{S^{n-1}}\frac{d}{dt}\left(\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H^{n+4k}(t)}\right)d\theta_{n-1}\\
&= \frac{V(S^{n+4k-1})}{V(S^{n-1})}\frac{1}{A_H^{n+4k}(t)}\int_{S^{n-1}}\left(\int_0^t \rho(\tau,\theta)d\tau\right)\mathcal{A}_f(t,\theta)d\theta_{n-1}\\
&\leq \frac{c(n,k,H)l}{A_H^{n+4k}(t)}\int_{S^{n-1}}\mathcal{A}_f(t,\theta)d\theta_{n-1}\\
&= c(n,k,H)l\frac{A_f(x,t)}{A_H^{n+4k}(t)},
\end{aligned}$$

where $c(n,k,H) := \frac{V(S^{n+4k-1})}{V(S^{n-1})}$ and $V(S^{n-1})$ is the area of the unit sphere $S^{n-1} \subset M_H^{n-1}$. Here we used the relation

$$A_H^{n+4k}(t) = \int_{S^{n+4k-1}}\mathcal{A}_H^{n+4k}(t)d\theta = V(S^{n+4k-1})\mathcal{A}_H^{n+4k}(t)$$

in the above second equality. Separating variables and integrating from r to R, we immediately get Eq. 8.

Next we shall prove Eq. 9. We apply

$$\mathcal{A}_f' = m_f\mathcal{A}_f \quad \text{and} \quad \mathcal{A}_H^{a\,\prime} = (m_H + a)\mathcal{A}_H^a$$

to compute that

$$\frac{d}{dt}\left(\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H^a(t)}\right) = (m_f - m_H - a)\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}_H^a(t)}.$$

Using this, by Theorem 1.1 (b) and our theorem assumption, we estimate that

$$\begin{aligned}\frac{d}{dt}\left(\frac{A_f(x,t)}{A^a_H(t)}\right) &= \frac{1}{V(S^{n-1})}\int_{S^{n-1}}\frac{d}{dt}\left(\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}^a_H(t)}\right)d\theta_{n-1}\\ &= \frac{1}{V(S^{n-1})}\int_{S^{n-1}}(m_f - m_H - a)\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}^a_H(t)}d\theta_{n-1}\\ &\le \frac{1}{V(S^{n-1})}\int_{S^{n-1}}\left(\int_0^\infty \rho(\tau,\theta)d\tau\right)\frac{\mathcal{A}_f(t,\theta)}{\mathcal{A}^a_H(t)}d\theta_{n-1}\\ &\le l\,\frac{A_f(x,t)}{A^a_H(t)}.\end{aligned}$$

Separating variables and integrating from r to R, we get Eq. 9. □

Similar to the argument of Petersen and Wei [15], we will apply Theorem 3.1 to complete the proof of Theorem 1.2.

Proof of Theorem 1.2 We first prove part (a). Recall that

$$\frac{V_f(x,r)}{V^{n+4k}_H(r)} = \frac{\int_0^r A_f(x,t)dt}{\int_0^r A^{n+4k}_H(t)dt}.$$

So we have

$$\frac{d}{dr}\left(\frac{V_f(x,r)}{V^{n+4k}_H(r)}\right) = \frac{A_f(x,r)\int_0^r A^{n+4k}_H(t)dt - A^{n+4k}_H(r)\int_0^r A_f(x,t)dt}{(V^{n+4k}_H(r))^2}. \tag{10}$$

Notice that, by Theorem 3.1 (a), for $t \le r$,

$$A_f(x,r)A^{n+4k}_H(t) - A^{n+4k}_H(r)A_f(x,t) \le (e^{c(n,k,H)lr}-1)A^{n+4k}_H(r)A_f(x,t).$$

Substituting this into Eq. 10 yields

$$\frac{d}{dr}\left(\frac{V_f(x,r)}{V^{n+4k}_H(r)}\right) \le \left(e^{c(n,k,H)lr}-1\right)\frac{A^{n+4k}_H(r)}{V^{n+4k}_H(r)}\left(\frac{V_f(x,r)}{V^{n+4k}_H(r)}\right).$$

Separating variables and integrating from r to R ($r \le R$), we get

$$\begin{aligned}\frac{V_f(x,R)}{V^{n+4k}_H(R)} &\le \frac{V_f(x,r)}{V^{n+4k}_H(r)}\exp\left\{\int_r^R\left(e^{c(n,k,H)lt}-1\right)\frac{A^{n+4k}_H(t)}{V^{n+4k}_H(t)}dt\right\}\\ &\le \frac{V_f(x,r)}{V^{n+4k}_H(r)}\exp\left\{\int_0^R\left(e^{c(n,k,H)lt}-1\right)\frac{A^{n+4k}_H(t)}{V^{n+4k}_H(t)}dt\right\}\end{aligned}$$

and Theorem 1.2 (a) follows.

Next we shall prove Theorem 1.2 (b). The proof is very similar to the arguments of part (a) in Theorem 1.2. For the completeness we provide a detailed proof. It is known that

$$\frac{V_f(x,r)}{V^a_H(r)} = \frac{\int_0^r A_f(x,t)dt}{\int_0^r A^a_H(t)dt}.$$

So we compute

$$\frac{d}{dr}\left(\frac{V_f(x,r)}{V^a_H(r)}\right) = \frac{A_f(x,r)\int_0^r A^a_H(t)dt - A^a_H(r)\int_0^r A_f(x,t)dt}{(V^a_H(r))^2}. \tag{11}$$

By Eq. 9, we see that

$$A_f(x,r)A_H^a(t) - A_H^a(r)A_f(x,t) \le (e^{lr}-1)A_H^a(r)A_f(x,t)$$

for $t \le r$. Substituting this into Eq. 11 yields

$$\frac{d}{dr}\left(\frac{V_f(x,r)}{V_H^a(r)}\right) \le \left(e^{lr}-1\right)\frac{A_H^a(r)}{V_H^a(r)}\left(\frac{V_f(x,r)}{V_H^a(r)}\right).$$

Separating variables and integrating from r to R ($r \le R$), we have

$$\begin{aligned}\frac{V_f(x,R)}{V_H^a(R)} &\le \frac{V_f(x,r)}{V_H^a(r)}\exp\left\{\int_r^R \left(e^{lt}-1\right)\frac{A_H^a(t)}{V_H^a(t)}dt\right\}\\ &\le \frac{V_f(x,r)}{V_H^a(r)}\exp\left\{\int_0^R \left(e^{lt}-1\right)\frac{A_H^a(t)}{V_H^a(t)}dt\right\}\end{aligned}$$

and Theorem 1.2 (b) follows. □

The weighted volume comparisons immediately yield volume doubling properties of smooth metric measure spaces.

Corollary 3.2 (Volume Doubling) *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space.*

(a) Assume that $|f| \le k$ for some constant $k \ge 0$. Given $H \in \mathbb{R}$, $\alpha > 1$ and $R > 0$(assume $R \le \frac{\pi}{4\sqrt{H}}$ when $H > 0$), there is an $\epsilon = \epsilon(n,k,H,R,\alpha)$ such that if

$$\int_0^\infty \rho(t,\theta)dt \le \epsilon,$$

along all minimal geodesic segments from $x \in M$, then for all $0 < r_1 < r_2 \le R$,

$$\frac{V_f(x,r_2)}{V_f(x,r_1)} \le \alpha\frac{V_H^{n+4k}(r_2)}{V_H^{n+4k}(r_1)}.$$

(b) Assume that $\partial_r f \ge -a$ for some constant $a \ge 0$, along all minimal geodesic segments from $x \in M$. Given $H \in \mathbb{R}$, $\alpha > 1$ and $R > 0$ (assume $R \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$), there is an $\epsilon = \epsilon(n,a,H,R,\alpha)$ such that if

$$\int_0^\infty \rho(t,\theta)dt \le \epsilon,$$

along all minimal geodesic segments from $x \in M$, then for all $0 < r_1 < r_2 \le R$,

$$\frac{V_f(x,r_2)}{V_f(x,r_1)} \le \alpha\frac{V_H^a(r_2)}{V_H^a(r_1)}.$$

Proof of Corollary 3.2 We only prove part (a); the proof of part (b) is similar. Assume that $\int_0^\infty \rho(t,\theta)dt \le l$ along all minimal geodesic segments from $x \in M$, where $l \ge 0$ is a constant. Since $|f| \le k$, by Theorem 1.2 (a), for $0 < r_1 < r_2 \le R$,

$$\begin{aligned}\frac{V_f(x,r_2)}{V_H^{n+4k}(r_2)} &\le \frac{V_f(x,r_1)}{V_H^{n+4k}(r_1)}\exp\left\{\int_{r_1}^{r_2}\left(e^{c(n,k,H)lt}-1\right)\frac{A_H^{n+4k}(t)}{V_H^{n+4k}(t)}dt\right\}\\ &\le \frac{V_f(x,r_1)}{V_H^{n+4k}(r_1)}\exp\left\{\int_0^R\left(e^{c(n,k,H)lt}-1\right)\frac{A_H^{n+4k}(t)}{V_H^{n+4k}(t)}dt\right\},\end{aligned} \tag{12}$$

where $c(n,k,H) := \frac{V(S^{n+4k-1})}{V(S^{n-1})}$. Notice that the right hand side of integral quantity is finite (depending on R) because that

$$\lim_{t\to 0}(e^{c(n,k,H)lt}-1)\frac{A_H^{n+4k}(t)}{V_H^{n+4k}(t)}=0.$$

Set

$$F(\sigma) := \int_0^R \left(e^{c(n,k,H)\sigma t}-1\right)\frac{A_H^{n+4k}(t)}{V_H^{n+4k}(t)}dt.$$

We see that $F(0) = 0$ and $e^{F(0)} = 1$. Moreover, the function $F(\sigma)$ is continuous with respect to the parameter σ. Therefore, for any $\alpha > 1$, there exists a number $\epsilon = \epsilon(n,k,H,R,\alpha)$ (as long as ϵ is small enough) such that if $\int_0^\infty \rho(t,\theta)dt \le \epsilon$, then

$$e^{F(\epsilon)} \le \alpha.$$

Hence the conclusion follows. □

In the end of this section, we will give an absolute volume comparison when $H < 0$ by modifying the argument of Jaramillo [12], which is an improvement of Eq. 2. When $\rho \equiv 0$, this result returns to Jaramillo's result [12].

Theorem 3.3 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space with a base point $x \in M$. Fix $H < 0$. Assume that*

$$\int_0^\infty \rho(t,\theta)dt \le l$$

along all minimal geodesic segments from $x \in M$, where $l \ge 0$ is a constant. If $|f| \le k$ for some constant $k \ge 0$, then

$$V_f(x,R) \le e^{3k}\int_0^R \mathcal{A}_H(t)e^{\cosh(2\sqrt{-H}t)+lt}dt$$

for all $R \ge 0$.

Proof of Theorem 3.3 Recall that in the course of proving Theorem 1.1 (a), by Eq. 7 and the increase of $\mathrm{sn}_H^2(r)$, we indeed prove that

$$m_f(r) \le m_H(r) - f(r)\frac{(\mathrm{sn}_H^2(r))\prime}{\mathrm{sn}_H^2(r)} + \int_0^r f(t)\frac{(\mathrm{sn}_H^2)\prime\prime(t)}{\mathrm{sn}_H^2(r)}dt + \int_0^r \rho(t,\theta)dt$$

along any a minimal geodesic segment from x, where $\mathrm{sn}_H(r) = \frac{1}{\sqrt{-H}}\sinh\sqrt{H}r$, since $H < 0$. Integrating the above inequality from r_1 to r_2 ($r_2 \ge r_1$) gives

$$\begin{aligned}\int_{r_1}^{r_2} m_f(r)dr \le& \int_{r_1}^{r_2} m_H(r)dr - \int_{r_1}^{r_2} f(r)\frac{(\mathrm{sn}_H^2(r))\prime}{\mathrm{sn}_H^2(r)}dr + \int_{r_1}^{r_2}\frac{1}{\mathrm{sn}_H^2(r)}\left[\int_0^r f(t)(\mathrm{sn}_H^2)\prime\prime(t)dt\right]dr\\ &+\int_{r_1}^{r_2}\left(\int_0^r \rho(t,\theta)dt\right)dr.\end{aligned}$$

Notice that

$$
\begin{aligned}
&-\int_{r_1}^{r_2} f(r)\frac{(\mathrm{sn}_H^2(r))'}{\mathrm{sn}_H^2(r)}dr+\int_{r_1}^{r_2}\frac{1}{\mathrm{sn}_H^2(r)}\left[\int_0^r f(t)(\mathrm{sn}_H^2)''(t)dt\right]dr\\
&\quad=-2\sqrt{-H}\int_{r_1}^{r_2} f(r)\coth\sqrt{-H}r dr-2H\int_{r_1}^{r_2}\mathrm{csch}^2\sqrt{-H}r\left[\int_0^r f(t)\cosh 2\sqrt{-H}t dt\right]dr\\
&\quad=-2\sqrt{-H}\int_{r_1}^{r_2} f(r)\coth\sqrt{-H}r dr-2H\left[-\frac{\coth\sqrt{-H}r}{\sqrt{-H}}\int_0^r f(t)\cosh 2\sqrt{-H}t dt\right]_{r_1}^{r_2}\\
&\qquad-4H\int_{r_1}^{r_2}\frac{\coth\sqrt{-H}r}{\sqrt{-H}}f(r)\sinh^2\sqrt{-H}r dr-2H\int_{r_1}^{r_2}\frac{\coth\sqrt{-H}r}{\sqrt{-H}}f(r)dr.
\end{aligned}
$$

Using the assumption $|f|\le k$, we further have

$$
\begin{aligned}
&-\int_{r_1}^{r_2} f(r)\frac{(\mathrm{sn}_H^2(r))'}{\mathrm{sn}_H^2(r)}dr+\int_{r_1}^{r_2}\frac{1}{\mathrm{sn}_H^2(r)}\left[\int_0^r f(t)(\mathrm{sn}_H^2)''(t)dt\right]dr\\
&\quad\le k\coth\sqrt{-H}r_2\sinh(2\sqrt{-H}r_2)+k\coth\sqrt{-H}r_1\sinh(2\sqrt{-H}r_1)\\
&\qquad+2k\left(\sinh^2\sqrt{-H}r_2-\sinh^2\sqrt{-H}r_1\right)\\
&\quad=2k\left[\cosh(2\sqrt{-H}r_2)+1\right].
\end{aligned}
$$

Therefore, for $r_1\le r_2$, we have

$$
\int_{r_1}^{r_2} m_f(r)dr\le\int_{r_1}^{r_2} m_H(r)dr+2k\left[\cosh(2\sqrt{-H}r_2)+1\right]+l(r_2-r_1),
$$

where we used $|f|\le k$ and $\int_0^\infty\rho(t,\theta)dt\le l$. This implies

$$
\ln\left(\frac{\mathcal{A}_f(r_2,\theta)}{\mathcal{A}_f(r_1,\theta)}\right)\le\ln\left(\frac{\mathcal{A}_H(r_2)}{\mathcal{A}_H(r_1)}\right)+2k\left[\cosh(2\sqrt{-H}r_2)+1\right]+l(r_2-r_1)
$$

for $r_1\le r_2$, and hence

$$
\mathcal{A}_f(r_2,\theta)\mathcal{A}_H(r_1)\le\mathcal{A}_f(r_1,\theta)\mathcal{A}_H(r_2)e^{2k\left[\cosh(2\sqrt{-H}r_2)+1\right]+lr_2}.
$$

for all $r_1\le r_2$. Integrating both sides of the inequality over S^{n-1} with respect to θ gives

$$
\mathcal{A}_H(r_1)\int_{S^{n-1}}\mathcal{A}_f(r_2,\theta)d\theta\le\mathcal{A}_H(r_2)e^{2k\left[\cosh(2\sqrt{-H}r_2)+1\right]+lr_2}\int_{S^{n-1}}\mathcal{A}_f(r_1,\theta)d\theta
$$

for $r_1\le r_2$. Then integrating both sides of the inequality with respect to r_1 from 0 to R_1,

$$
V_H(r_1)\int_{S^{n-1}}\mathcal{A}_f(r_2,\theta)d\theta\le V_f(x,R_1)\mathcal{A}_H(r_2)e^{2k\left[\cosh(2\sqrt{-H}r_2)+1\right]+lr_2}
$$

for $R_1\le r_2$. Finally integrating both sides of the inequality with respect to r_2 from 0 to R_2,

$$
V_H(R_1)V_f(x,R_2)\le V_f(x,R_1)\int_0^{R_2}\mathcal{A}_H(r_2)e^{2k\left[\cosh(2\sqrt{-H}r_2)+1\right]+lr_2}dr_2
$$

for $R_1\le R_2$. Namely,

$$
\frac{V_H(R_1)}{V_f(x,R_1)}\le\frac{\int_0^{R_2}\mathcal{A}_H(r_2)e^{2k\left[\cosh(2\sqrt{-H}r_2)+1\right]+lr_2}dr_2}{V_f(x,R_2)}
$$

for $R_1 \leq R_2$. Letting $R_1 \to 0$, the left hand side tends to $e^{f(x)}$ and hence

$$V_f(x, R_2) \leq e^{f(x)} \int_0^{R_2} \mathcal{A}_H(r_2) e^{2k\left[\cosh(2\sqrt{-H}r_2)+1\right]+lr_2} dr_2$$
$$\leq e^{3k} \int_0^{R_2} \mathcal{A}_H(r_2) e^{\cosh(2\sqrt{-H}r_2)+lr_2} dr_2$$

for all $R_2 \geq 0$. This finishes the proof. □

4 Myers' Theorem

In this section, we will discuss some Myers' type diameter estimates on $(M, g, e^{-f} dv)$ when the integral radial Bakry-Émery Ricci tensor and f or $|\nabla f|$ are bounded. First, we will apply mean curvature comparisons of Section 2 to prove Theorem 1.3. The proof uses the excess function which is similar to the Wei-Wylie's argument [21]; see also [22].

Proof of Theorem 1.3 We first prove part (a). Choose two any points p_1 and p_2 in (M, g, f) such that $d(p_1, p_2) \geq \frac{\pi}{\sqrt{H}}$ and set

$$B := d(p_1, p_2) - \frac{\pi}{\sqrt{H}}.$$

Let

$$r_1(x) = d(p_1, x) \quad \text{and} \quad r_2(x) = d(p_2, x),$$

and let $e(x)$ be the excess function for the points p_1 and p_2, that is,

$$e(x) := d(p_1, x) + d(p_2, x) - d(p_1, p_2).$$

The excess function measures how much the triangle inequality fails to be an equality. By the triangle inequality, we obviously have $e(x) \geq 0$ and $e(\gamma(t)) = 0$, where γ is a minimal geodesic from p_1 to p_2. Hence $\Delta_f(e(\gamma(t))) \geq 0$ in the barrier sense. Let

$$y_1 = \gamma\left(\frac{\pi}{2\sqrt{H}}\right) \quad \text{and} \quad y_2 = \gamma\left(\frac{\pi}{2\sqrt{H}} + B\right).$$

Then we see that $r_i(y_i) = \frac{\pi}{2\sqrt{H}}$, $i = 1, 2$. Furthermore, by the estimate Eq. 1 of Theorem 1.1 and our assumption, we have

$$\begin{aligned}\Delta_f(r_i(y_i)) &\leq 2k\sqrt{H} + \int_0^\infty \rho(t, \theta) dt \\ &\leq 2k\sqrt{H} + l.\end{aligned} \tag{13}$$

Noticing that $r_1(y_2) > \frac{\pi}{2\sqrt{H}}$, we can not give an upper estimate for $\Delta_f(r_1(y_2))$ by directly using Theorem 1.1. But we can apply Theorem 2.1 and Eq. 13 to get that

$$\begin{aligned}\Delta_f(r_1(y_2)) &\leq 2k\sqrt{H} - B(n-1)H + \int_0^\infty \rho(t, \theta) dt \\ &\leq 2k\sqrt{H} - B(n-1)H + l.\end{aligned} \tag{14}$$

Combining Eq. 13 and Eq. 14, we get that

$$\begin{aligned}0 \leq \Delta_f(e(y_2)) &= \Delta_f(r_1(y_2)) + \Delta_f(r_2(y_2)) \\ &\leq 4k\sqrt{H} - B(n-1)H + 2l,\end{aligned}$$

which implies

$$B \leq \frac{4k\sqrt{H}+2l}{(n-1)H}$$

and hence

$$d(p_1, p_2) \leq \frac{\pi}{\sqrt{H}} + \frac{4k\sqrt{H}+2l}{(n-1)H}.$$

Since p_1 and p_2 are arbitrary two points, this completes the proof of part (a).

The proof of part (b) is almost the same as the part (a) and the main difference is that we apply Theorem 1.1 (b) instead of the estimate Eq. 1. So we omit it here. □

In the end of this section, we will apply the index form technique to get another Myers' type diameter estimate. In this case, the integral assumption is weaker than that of Theorem 1.3 (a). The proof is inspired by the argument of Limoncu [11]; see also [18].

Theorem 4.1 *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space. Fix a point $p \in M$ and $H \in \mathbb{R}^+$. Assume that*

$$\int_0^\infty \rho(t,\theta)dt \leq l$$

along all minimal geodesic segments from the point p, where $l \geq 0$ is a constant. If $|f| \leq k$ for some constant $k \geq 0$, then M is compact and

$$\mathrm{diam}(M) \leq \frac{2\pi}{\sqrt{H}}\sqrt{1+\frac{8k}{(n-1)\pi}+\frac{l^2}{(n-1)^2H\pi^2}}+\frac{2l}{(n-1)H}.$$

We would like to point out that Tadano [19] also proved a Myers' type diameter estimate for the integral radial Bakry-Émery Ricci tensor. But his curvature condition is different from our case.

Before proving the theorem, let us recall some notations. Let X, Y, Z be three smooth vector fields on Riemannian manifold (M, g). For any smooth function $f \in C^\infty(M)$, the gradient vector field and Hessian of f are defined by

$$g(\nabla f, X) = df(X) \quad \text{and} \quad \mathrm{Hess}\, f(X,Y) = g(\nabla_X \nabla f, Y),$$

respectively. The Riemannian curvature tensor and the Ricci curvature are defined by

$$\mathrm{Rm}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \quad \text{and} \quad \mathrm{Ric}_g(X,Y) = \sum_{i=1}^n g(\mathrm{Rm}(e_i, X)Y, e_i),$$

respectively, where $\{e_i\}_{i=1}^n$ denotes an orthonormal frame of (M, g).

Proof of Theorem 4.1 On $(M, g, e^{-f}dv)$, for the fixed point $p \in M$, let any point $q \in M$ and let σ be a minimizing unit speed geodesic segment from p to q of length L. Consider a parallel orthonormal frame $\{e_1 = \dot{\sigma}, e_2, ..., e_n\}$ along σ and a smooth function $\phi \in C^\infty([0, L])$ such that $\phi(0) = \phi(L)$=0, and we have

$$I(\phi e_i, \phi e_i) = \int_0^L \left[g(\dot{\phi} e_i, \dot{\phi} e_i) - g(\mathrm{Rm}(\phi e_i, \dot{\sigma})\dot{\sigma}, \phi e_i)\right] dt,$$

where $I(\cdot,\cdot)$ deontes the index form of the geodesic segment σ. Summing i from 1 to n in the above equality and using $g(\mathrm{Rm}(\dot\sigma,\dot\sigma)\dot\sigma,\dot\sigma)=0$, we get

$$\sum_{i=2}^{n} I(\phi e_i,\phi e_i)=\int_0^L\Big[(n-1)\dot\phi^2-\phi^2\,\mathrm{Ric}_g(\dot\sigma,\dot\sigma)\Big]dt.$$

According to the definition of ρ, we have

$$\begin{aligned}\sum_{i=2}^{n} I(\phi e_i,\phi e_i)&\leqslant\int_0^L\Big[(n-1)(\dot\phi^2-H\phi^2)+\phi^2\,\mathrm{Hess}\,f(\dot\sigma,\dot\sigma)\Big]dt+\int_0^L\phi^2\rho(t,\theta)dt\\&=\int_0^L\Big[(n-1)(\dot\phi^2-H\phi^2)+\phi^2 g(\nabla_{\dot\sigma}\nabla f,\dot\sigma)\Big]dt+\int_0^L\phi^2\rho(t,\theta)dt\\&=\int_0^L\Big[(n-1)(\dot\phi^2-H\phi^2)+\phi^2\dot\sigma(g(\nabla f,\dot\sigma))\Big]dt+\int_0^L\phi^2\rho(t,\theta)dt,\end{aligned}\tag{15}$$

where we used the parallelism of the Riemannian metric g and $\nabla_{\dot\sigma}\dot\sigma=0$ in the last equality. Along the geodesic segment $\sigma(t)$, we get that

$$\begin{aligned}\phi^2\dot\sigma\,(g(\nabla f,\dot\sigma))&=\phi^2\frac{d}{dt}\,(g(\nabla f,\dot\sigma))\\&=\frac{d}{dt}\Big(\phi^2 g(\nabla f,\dot\sigma)\Big)-2\phi\dot\phi g(\nabla f,\dot\sigma)\\&=\frac{d}{dt}\Big(\phi^2 g(\nabla f,\dot\sigma)\Big)+2f\frac{d}{dt}\left(\phi\dot\phi\right)-2\frac{d}{dt}\left(f\phi\dot\phi\right),\end{aligned}$$

where we used $g(\nabla f,\dot\sigma)=\frac{df}{dt}(\sigma(t))$ in the last equality. Then integrating the both sides of the above equality, we get

$$\begin{aligned}\int_0^L\phi^2\dot\sigma(g(\nabla f,\dot\sigma))dt&=\phi^2 g(\nabla f,\dot\sigma)\Big|_0^L+\int_0^L 2f\frac{d}{dt}\left(\phi\dot\phi\right)dt-2f\phi\dot\phi\Big|_0^L\\&=2\int_0^L f\frac{d}{dt}\left(\phi\dot\phi\right)dt,\end{aligned}$$

where we used $\phi(0)=\phi(L)=0$ in the last equality. Since $|f|\leqslant k$ by the theorem assumption, then

$$\int_0^L\phi^2\dot\sigma(g(\nabla f,\dot\sigma))dt\le 2k\int_0^L\left|\frac{d}{dt}(\phi\dot\phi)\right|dt.$$

Substituting this into Eq. 15, we get that

$$\sum_{i=2}^{n} I(\phi e_i,\phi e_i)\le(n-1)\int_0^L(\dot\phi^2-H\phi^2)dt+2k\int_0^L\left|\frac{d}{dt}(\phi\dot\phi)\right|dt+\int_0^L\phi^2\rho(t,\theta)dt.$$

If we take $\phi(t)=\sin(\frac{\pi t}{L})$, then

$$\dot\phi(t)=\frac{\pi}{L}\cos\left(\frac{\pi t}{L}\right)\quad\text{and}\quad\phi\dot\phi=\frac{\pi}{2L}\sin\left(\frac{2\pi t}{L}\right).$$

We also know

$$\int_0^\infty\sin^2\left(\frac{\pi t}{L}\right)\rho(t,\theta)dt\le\int_0^\infty\rho(t,\theta)dt\le l$$

from our assumption. We collect these results together and the above estimate becomes

$$\sum_{i=2}^{n} I(\phi e_i, \phi e_i) \leq (n-1)\int_0^L \left[\frac{\pi^2}{L^2}\cos^2\left(\frac{\pi t}{L}\right) - H\sin^2\left(\frac{\pi t}{L}\right)\right] dt + 2k\left(\frac{\pi}{L}\right)^2 \int_0^L \left|\cos\frac{2\pi t}{L}\right| dt + l.$$

We simplify it and have that

$$\sum_{i=2}^{n} I(\phi e_i, \phi e_i) \leq -\frac{1}{2L}\left[(n-1)HL^2 - (n-1)\pi^2 - 8\pi k\right] + l.$$

Since σ is a minimizing geodesic, then

$$\sum_{i=2}^{n} I(\phi e_i, \phi e_i) \geq 0$$

and we must have

$$-\frac{1}{2L}\left[(n-1)HL^2 - (n-1)\pi^2 - 8\pi k\right] + l \geq 0.$$

This gives

$$L \leq \frac{\pi}{\sqrt{H}}\sqrt{1 + \frac{8k}{(n-1)\pi} + \frac{l^2}{(n-1)^2 H\pi^2}} + \frac{l}{(n-1)H}.$$

Therefore for any two points $q_1, q_2 \in M$, we have

$$d(q_1, q_2) \leq d(p, q_1) + d(p, q_2) \leq 2L$$

and the result follows. □

Remark 4.2 The index form argument also gives a Myers' type diameter estimate when the integral radial Bakry-Émery Ricci tensor bounds and $\partial_r f$ is bounded below along geodesics. To save the length of the paper, we omit them here.

5 Eigenvalue Estimate

In this section we will apply the volume doubling of Section 3 (Corollary 3.2 (b)) to prove Theorem 1.4 by following the argument of [14] and [23].

Proof of Theorem 1.4 Recall that $B(\bar{x}_0, R)$, where $R \leq \frac{\pi}{2\sqrt{H}}$ when $H > 0$ is a metric ball in the weighted model space $M^n_{H,a}$. Let $\lambda_1^D(n, a, H, R)$ be the first eigenvalue of the h-Laplacian Δ_h with the Dirichlet condition in $M^n_{H,a}$, where $h(x) = -a \cdot d(\bar{x}_0, x)$. Let $u(x) = \phi(r)$ be the corresponding eigenfunction of $\lambda_1^D(n, a, H, R)$ such that

$$\phi'' + (m_H + a)\phi' + \lambda_1^D(n, a, H, R)\phi = 0$$

with $\phi(0) = 1$ and $\phi(R) = 0$. Since $\phi' < 0$ on $[0, R]$, we see that $0 \le \phi \le 1$. Now we consider the Rayleigh quotient of $u(x) = \phi(d(x_0, x))$. We compute that

$$\begin{aligned}\int_{B(x_0,R)} |\nabla u|^2 e^{-f} dv &= \int_{S^{n-1}} \int_0^R (\phi')^2 \mathcal{A}_f(t,\theta)\, dt d\theta_{n-1} \\ &= \int_{S^{n-1}} \left(\phi\phi' \mathcal{A}_f \Big|_0^R - \int_0^R \phi(\phi'\mathcal{A}_f)'\, dt \right) d\theta_{n-1} \\ &= -\int_{S^{n-1}} \int_0^R \phi(\phi'' + m_f\phi')\mathcal{A}_f\, dt d\theta_{n-1} \\ &= -\int_{S^{n-1}} \int_0^R \phi(\phi'' + (m_H^n + a)\phi')\mathcal{A}_f\, dt d\theta_{n-1} \\ &\quad -\int_{S^{n-1}} \int_0^R (m_f - m_H^n - a)\phi\phi'\mathcal{A}_f\, dt d\theta_{n-1}.\end{aligned}$$

Noticing that

$$\phi'' + (m_H + a)\phi' = -\lambda_1^D(n, a, H, R)\phi$$

so

$$\begin{aligned}\int_{B(x_0,R)} |\nabla u|^2 e^{-f} dv \le \lambda_1^D(n, a, H, R) &\int_{S^{n-1}} \int_0^R \phi^2 \mathcal{A}_f\, dt d\theta_{n-1} \\ &+ \int_{S^{n-1}} \int_0^R (m_f - m_H - a)_+ |\phi'| \mathcal{A}_f\, dt d\theta_{n-1}.\end{aligned}$$

Hence the Rayleigh quotient satisfies

$$Q := \frac{\int_{B(x_0,R)} |\nabla u|^2 e^{-f} dv}{\int_{B(x_0,R)} u^2 e^{-f} dv} \le \lambda_1^D(n, a, H, R) + \frac{\int_{S^{n-1}} \int_0^R (m_f - m_H^n - a)_+ |\phi'| \mathcal{A}_f\, dt d\theta_{n-1}}{\int_{S^{n-1}} \int_0^R \phi^2 \mathcal{A}_f\, dt d\theta_{n-1}}. \tag{16}$$

Next we will estimate the last term of the above inequality by choosing a proper function ϕ. Now we choose the first value $r = r(n, a, H, R)$ such that $\phi(r) = 1/2$. Then the last error term can be estimated as follows:

$$\begin{aligned}&\frac{\int_{S^{n-1}} \int_0^R (m_f - m_H^n - a)_+ |\phi'| \mathcal{A}_f}{\int_{S^{n-1}} \int_0^R \phi^2 \mathcal{A}_f} \\ &\le \frac{\left(\int_{S^{n-1}} \int_0^R (m_f - m_H^n - a)_+^2 \mathcal{A}_f\right)^{\frac{1}{2}} \left(\int_{S^{n-1}} \int_0^R |\phi'|^2 \mathcal{A}_f\right)^{\frac{1}{2}}}{\frac{1}{2} V_f^{\frac{1}{2}}(x_0, r) \left(\int_{S^{n-1}} \int_0^R \phi^2 \mathcal{A}_f\right)^{\frac{1}{2}}} \\ &= 2\left(\frac{\int_{S^{n-1}} \int_0^R (m_f - m_H^n - a)_+^2 \mathcal{A}_f}{V_f(x_0, r)}\right)^{\frac{1}{2}} \sqrt{Q},\end{aligned}$$

where we used the Cauchy-Schwarz inequality and

$$\int_{S^{n-1}} \int_0^R \phi^2 \mathcal{A}_f \ge \frac{1}{4} V_f(x_0, r)$$

in the above second inequality. On the other hand, if $\int_0^\infty \rho(t,\theta)dt \le \epsilon(n, a, H, R)$ is very small along all minimal geodesic segments from $x_0 \in M$, by Corollary 3.2 (b), we have the volume doubling

$$\frac{V_f(x_0,R)}{V_f(x_0,r)}\le 4\frac{V_H^a(R)}{V_H^a(r)}.$$

Substituting this into the above error estimate,

$$\frac{\int_{S^{n-1}}\int_0^R(m_f-m_H^n-a)_+|\phi'|\mathcal{A}_f}{\int_{S^{n-1}}\int_0^R\phi^2\mathcal{A}_f}\le 4\left(\frac{V_H^a(R)}{V_H^a(r)}\right)^{\frac{1}{2}}\left(\frac{\int_{S^{n-1}}\int_0^R(m_f-m_H^n-a)_+^2\mathcal{A}_f}{V_f(x_0,R)}\right)^{\frac{1}{2}}\sqrt{Q}.$$

Since $\int_0^\infty\rho(t,\theta)dt\le\epsilon(n,H,a,R)$ by the assumption of theorem, we observe that

$$\int_{S^{n-1}}\int_0^R(m_f-m_H^n-a)_+^2\mathcal{A}_f\le\epsilon^2V_f(x_0,R).$$

Hence we finally get

$$\frac{\int_{S^{n-1}}\int_0^R(m_f-m_H^{n+4k})_+|\phi'|\mathcal{A}_f}{\int_{S^{n-1}}\int_0^R\phi^2\mathcal{A}_f}\le C(n,a,H,R)\epsilon\sqrt{Q}$$

for some constant $C(n,a,H,R)$ depending on n, a, H and R. Substituting this estimate into Eq. 16, we have

$$Q\le\lambda_1^D(n,a,H,R)+C(n,a,H,R)\epsilon\sqrt{Q},$$

which implies the desired result. □

Acknowledgements The author would like to thank Homare Tadano for providing the manuscript [19] and pointing out a mini omission in the proof of Theorem 4.1. He also thanks the referee for a very careful reading of the paper and helpful suggestions. This work was partially supported by the Natural Science Foundation of Shanghai (17ZR1412800).

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Aubry, E.: Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature. Ann. Sci. Ecole Norm. Sup. **40**, 675–695 (2007)
2. Aubry, E.: Bounds on the volume entropy and simplicial volume in Ricci curvature l^p-bounded from below. Int. Math. Res. Not. IMRN **10**, 1933–1946 (2009)
3. Bakry, D., Émery, M.: Diffusion Hypercontractivitives, in Séminaire De ProbabilitéS XIX, 1983/1984. In: Lecture Notes in Math., vol. 1123, pp. 177–206. Springer, Berlin (1985)
4. Cheng, S.-Y.: Eigenvalue comparison theorems and its geometric applications. Math. Z. **143**, 289–297 (1975)
5. Dai, X.-Z., Petersen, P., Wei, G.-F.: Integral pinching theorems. Manu. Math. **101**, 143–152 (2000)
6. Dai, X.-Z., Wei, G.-F.: A heat kernel lower bound for integral Ricci curvature. Michigan Math. Jour. **52**, 61–69 (2004)
7. Dai, X.-Z., Wei, G.-F., Zhang, Z.-L.: Local sobolev constant estimate for integral Ricci curvature bounds. Adv. Math. **325**, 1–33 (2018)
8. Gallot, S.: Isoperimetric inequalities based on integral norms of Ricci curvature. Astérisque, (157-158), 191–216, 1988. Colloque Paul lévy sur les Processus Stochastiques (Palaiseau (1987)
9. Hamilton, R.: The formation of singularities in the Ricci flow. Surv Different Geom **2**, 7–136 (1995). International Press
10. Li, F.-J., Wu, J.-Y., Zheng, Y.: Myers' type theorem for integral bakry-Émery Ricci tensor bounds. Results Math. **76**(1), 32 (2021)

11. Limoncu, M.: The Bakry-Emery Ricci tensor and its applications to some compactness theorems. Math. Z. **271**, 715–722 (2012)
12. Jaramillo, M.: Fundamental groups of spaces with Bakry-Emery ricci tensor bounded below. J. Geom. Anal. **25**, 1828–1858 (2015)
13. Olivé, X.R., Seto, S., Wei, G.-F., Zhang, Q.-S.: Zhong-yang type eigenvalue estimate with integral curvature condition. Math. Z. **296**, 595–613 (2020)
14. Petersen, P., Sprouse, C.: Integral curvature bounds, distance estimates and applications. J. Differ. Geom. **50**, 269–298 (1998)
15. Petersen, P., Wei, G.-F.: Relative volume comparison with integral curvature bounds. GAFA **7**, 1031–1045 (1997)
16. Petersen, P., Wei, G.-F.: Analysis and geometry on manifolds with integral Ricci curvature bounds. II Trans. AMS. **353**, 457–478 (2000)
17. Seto, S., Wei, G.-F.: First eigenvalue of the p-Laplacian under integral curvature condition. Nonlinear Anal. **163**, 60–70 (2017)
18. Tadano, H.: Remark on a diameter bound for complete Riemannian manifolds with positive bakry-Émery Ricci curvature. Differ. Geom. Appl. **44**, 136–143 (2016)
19. Tadano, H.: m-Bakry-Émery Ricci curvatures, Riccati inequalities, and bounded diameters preprint (2021)
20. Wang, L.-L., Wei, G.-F.: Local Sobolev constant estimate for integral bakry-Émery Ricci curvature. Pac. J. Math. **300**, 233–256 (2019)
21. Wei, G.-F., Wylie, W.: Comparison geometry for the bakry-Émery Ricci tensor. J. Differ. Geom. **83**, 377–405 (2009)
22. Wu, J.-Y.: Myers' type theorem with the bakry-Émery Ricci tensor. Ann. Glob. Anal. Geom. **54**, 541–549 (2018)
23. Wu, J.-Y.: Comparison geometry for integral bakry-Émery Ricci tensor bounds. J. Geom. Anal. **29**, 828–867 (2019)
24. Zhang, Q.-S., Zhu, M.: Li-yau gradient bounds on compact manifolds under nearly optimal curvature conditions. J. Funct. Anal. **275**, 478–515 (2018)
25. Zhu, S.-H.: The Comparison Geometry of Ricci Curvature, Comparison Geometry (Berkeley, CA, 1993-94), Volume 30 of Math. Sci. Res. Inst. Publ, pp. 221–262. Cambridge Univ Press, Cambridge (1997)