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$C^{1,\alpha}$ Regularity for Degenerate Fully Nonlinear Elliptic Equations with Neumann Boundary Conditions

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Abstract
In this paper, we establish $C^{1,\alpha}$ regularity up to the boundary for a class of degenerate fully nonlinear elliptic equations with Neumann boundary conditions. Our main result Theorem 2.1 constitutes the boundary analogue of the interior $C^{1,\alpha}$ regularity result established in Imbert and Silvestre (Adv. Math. **233**: 196–206, 2013) for equations with similar structural assumptions. The proof of our main result is achieved via compactness arguments combined with new boundary Hölder estimates for equations which are uniformly elliptic when the gradient is either small or large.

Keywords Pucci's extremal operator · Degenerate elliptic · Viscosity solutions · Regularity

Mathematics Subject Classification (2010) Primary 35J60 · 35D40

1 Introduction

In this paper, we are concerned with the regularity up to the boundary for solutions to fully nonlinear equations of the type

$$|Du|^{\beta} F\left(D^2u, x\right) = f, \tag{1.1}$$

with Neumann boundary conditions, where $\beta \geq 0$, F is uniformly elliptic and $F(0, x) = 0$. Equation 1.1 constitutes a subfamily of a class of nonlinear elliptic equations studied in a series of papers by Birindelli and Demengel starting with [11]. We note that such equations are not uniformly elliptic, they are either degenerate or singular depending on whether $\beta > 0$ or $\beta < 0$. In the singular case (i.e. when $\beta < 0$), the authors in [11] proved many important

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results like comparison principles and Liouville type properties. See also [12] for regularity results in this case.

In the degenerate case (i.e. when $\beta > 0$), the first breakthrough was made by Imbert and Silvestre in [22] where the authors proved the interior $C^{1,\alpha}$ regularity for solutions to such equations as in Eq. 1.1. A fairly simple example as in [22] shows that solutions to such equations cannot be more regular than $C^{1,\alpha}$ even when $F(D^2u) = \Delta u$. Subsequently, optimal $C^{1,\alpha}$ regularity results in case of concave F have been obtained in the recent interesting work [5]. We note that the proof of the $C^{1,\alpha}$ result in [22] is based on successful adaptation of compactness arguments inspired by the ideas as in the fundamental work of Caffarelli in [16] (see also [15]). We also refer the reader to the paper [13] for $C^{1,\alpha}$ results in case Dirichlet boundary conditions. Our main result Theorem 2.1 below thus complements the regularity results previously obtained in [22] and [13].

Now, in order to put things in the right perspective, we note that getting a $C^{1,\alpha}$ regularity result in general amounts to show that the graph of the solution u can be touched by an affine function so that the error is of order $r^{1+\alpha}$ in a ball of radius r for every r small enough. The proof of this is based on iterative argument where one ensures improvement of flatness at every successive scale. At each step, via rescaling, it reduces to show that if $< p, x > +u$ solves Eq. 1.1 in B_1, then the oscillation of u is strictly smaller in a smaller ball up to a linear function. This is accomplished via compactness arguments which crucially relies on apriori estimates. Now for a u which solves Eq. 1.1, we have that $u- < p, x >$ is a solution to

$$|Dv + p|^{\beta} F(D^2v, x) = f. \tag{1.2}$$

Therefore, in order to make such a compactness argument work for $\beta > 0$, it is important to get equicontinuous estimates for equations of the type (1.2) independent of $|p|$. This is precisely done in [22] using Hölder estimates for small slopes (i.e. when $|p|$ is small) established by the same authors in their previous work [23] (see also [25]) combined with a new Lipschitz estimate for large slopes which they obtain by adapting the Ishii-Lion's approach as in [20] to their setting.

In this paper, we follow a strategy similar to that in [22] with appropriate adaptations. For small slopes, we establish analogous boundary Hölder estimates as in [23] for Neumann conditions by the method of sliding cusps introduced in the same paper [23]. However for large slopes, we could not find a suitable adaptation of the Ishii-Lion's approach in our setting for getting equicontinuous estimates. We note that although such an approach has been implemented for global oblique derivative problems by Barles in [7], nevertheless a suitable localization of such an approach in case of non-homogeneous boundary conditions is not clear to us. Therefore, in order to overcome such an obstruction, we employ the method of Savin as in [30] based on sliding paraboloids in order to obtain equicontinuity estimates for large slopes. More precisely, we adapt a certain quantitative version of Savin's method due to Colombo and Figalli in [18]. We also note that such oscillation estimates are in fact established for more general fully nonlinear operators (with structural assumptions as in SC1)-SC3) in Section 4) and we believe that this aspect could possibly be of independent interest and may find other applications. Finally for a historical account, we note that the method of sliding paraboloids seems to have originated first in a slightly different context in the work of Cabre in [14].

As the reader will observe, the implementation of either of these approaches for Neumann boundary conditions is somewhat delicate. For instance in the case of small slopes, because of certain technical obstructions, our proof of the L^{ϵ} estimate as in Theorem 3.6 is based on the Calderon-Zygmund decomposition instead of the growing ink spot lemma as used in [23]. Moreover for large slopes, unlike that in [22], since our oscillation estimate as stated in Theorem 4.9 below only holds at large enough scales, therefore the compactness arguments in our setting required some appropriate modifications.

The paper is organized as follows. In Section 2, we introduce basic notations and then state our main result. In Section 3, we establish uniform boundary Hölder estimates for small slopes by the method of sliding cusps. In Section 4, we obtain analogous equicontinuous estimates for large slopes via sliding paraboloids. In Section 5, we finally prove our main result Theorem 2.1 using the compactness method which crucially relies on the regularity estimates proved in Sections 3 and 4. Finally we refer the reader to [28] for Lipschitz regularity results for equations of the type (1.1) in the singular case with homogeneous Neumann conditions.

In closing, we would like to mention that it remains to be seen whether similar regularity results can be obtained for more general oblique derivative conditions. This is an interesting aspect to which we would like to come back in a future study. Finally we would like the reader to note that Neumann regularity results are also useful in the context of Signorini type obstacle problems. See for instance [2, 6, 17, 27] and [31] to name a few. We would also like to refer to the recent interesting paper [29] where subsequent to our work, the optimal $C^{1,\alpha}$ regularity result for the Neumann boundary value problem (2.4) has been obtained in the case when F is concave. It is to be noted that the compactness arguments in [29] crucially uses the estimates in the present work.

2 Notations and the Statement of the Main Result

For a given $r > 0$ and $x \in \mathbb{R}^n$, we denote by $B_r(x)$ the ball of radius r centered at $x = (x', x_n)$ and the set $B_r(x) \cap \{y : y_n > 0\}$ by $B_r^+(x)$. When $x = 0$, we will occasionally denote such sets by B_r and B_r^+ respectively. Also the set $\{x_n = 0\} \cap B_r$ will be denoted by B_r^0. Likewise $Q_r(x)$ will denote a cube of length r centered at x. In particular, if $x = 0$, we will use the simpler notation Q_r for such a set. Q_r^0 will refer to the set $Q_r \cap \{y_n = 0\}$. For $x_0 \in \{y_n = 0\}$, we also define the upper half cube of side length r as follows:

$$Q_r^+(x_0) = \left\{ x \in \mathbb{R}^n \mid |x' - x_0'|_\infty < \frac{r}{2} \text{ and } 0 < x_n < r \right\},$$

Finally $S(n)$ will denote the set of all $n \times n$ real symmetric matrices. The norm of a matrix $M \in S(n)$ is defined as

$$||M|| \overset{\text{def}}{=} \sum_{i=1}^{n} |\lambda_i| \tag{2.1}$$

where λ_i's are the eigenvalues of M counted with multiplicity. up to

Now we list our basic structural assumptions. We will assume that F as in Eq. 1.1 is uniformly elliptic with ellipticity bounds λ and Λ, i.e.

$$\lambda||N^+|| - \Lambda||N^-|| + F(M, x) \leq F(M + N, x) \leq F(M, x) + \Lambda||N^+|| - \lambda||N^-||, \tag{2.2}$$

where N^+ and N^- denote the positive and negative parts of a symmetric matrix N respectively. Moreover, we will also assume that

$$|F(M, x) - F(M, y)| \leq \omega(|x - y|)\|M\|, \tag{2.3}$$

for some modulus of continuity ω. We now state our main result.

2.1 Statement of the Main Result

Theorem 2.1 *Let u be a viscosity solution to the following Neumann problem*

$$\begin{cases} |Du|^{\beta} F\left(D^2u, x\right) = f, \text{ in } \Omega \cap B_1(0), 0 \in \partial\Omega, \beta \geq 0, \\ u_{\nu} = g, \text{ on } \partial\Omega \cap B_1(0), \end{cases} \tag{2.4}$$

where F satisfies the structural assumptions in Eqs. 2.2 *and* 2.3*, Ω is a C^2 domain, $f \in C(\overline{\Omega})$ and $g \in C^{\alpha_0}(\partial\Omega)$ for some $\alpha_0 > 0$. Then we have that $u \in C^{1,\alpha}(\overline{\Omega \cap B_{1/2}(0)})$ for some $\alpha > 0$ depending on $n, \lambda, \Lambda, \omega, \beta, \alpha_0$ and the C^2 character of Ω. Here ν denotes the outward unit normal to Ω.*

Remark 2.2 For the precise notion of viscosity solutions to fully nonlinear Neumann problems, we refer the reader to [24]. The Neumann condition in Eq. 2.4 similar to that in [26] is to be interpreted in the following sense: If a smooth function ϕ locally touches u from above (or below) at $x_0 \in \partial\Omega$(i.e. in $\overline{\Omega} \cap B_r(x_0)$ for some $r > 0$), then $\phi_{\nu}(x_0) \leq$ (or $\geq$) $g(x_0)$.

From Theorem 2.1, the following corollary can be deduced.

Corollary 2.3 *Let u be a viscosity solution to the following Robin boundary problem*

$$\begin{cases} |Du|^{\beta} F\left(D^2u, x\right) = f, \text{ in } \Omega \cap B_1(0), 0 \in \partial\Omega, \beta \geq 0, \\ u_{\nu} + h(x)u = g, \text{ on } \partial\Omega \cap B_1(0), \end{cases} \tag{2.5}$$

where F satisfies the assumptions in Eqs. 2.2 *and* 2.3*, $\Omega \in C^2$, $f \in C(\overline{\Omega})$ and $h, g \in C^{\alpha_0}(\partial\Omega)$ for some $\alpha_0 > 0$. Then $u \in C^{1,\alpha}(\overline{B_{1/2} \cap \Omega})$ for some $\alpha > 0$ depending on $n, \lambda, \Lambda, \omega, \beta, \alpha_0$ and the C^2 character of Ω.*

3 Hölder Estimates up to the Boundary for Equations which are Uniformly Elliptic when the Gradient is Large

In this section we establish uniform non-perturbative Hölder estimates for equations of the type (1.2) for small $|p|'s$ (say when $|p| \leq a_0$ for some $a_0 > 0$). We first note that this in turn is equivalent to getting similar estimates for small $|p|$ (say $|p| \leq a_0$) up to the boundary for equations of the type

$$F\left(D^2u, x\right) = \frac{f}{|Du + p|^{\beta}}$$

which lends itself a uniformly elliptic structure when say $|Du|$ satisfies $|Du| > 2a_0 + 1$ in the viscosity sense. Therefore, this reduces to getting uniform Hölder estimates for equations which are uniformly elliptic when the gradient is large. We thus introduce the relevant framework similar to that in [23].

For a given $\gamma > 0$ and $0 < \lambda < \Lambda$, let $\mathcal{P}^{\pm}_{\lambda,\Lambda,\gamma}$ be defined by

$$\mathcal{P}^{+}_{\lambda,\Lambda,\gamma}\left(D^2v, Dv\right) = \begin{cases} \Lambda \mathrm{tr} D^2v^+ - \lambda \mathrm{tr} D^2v^- + \Lambda|Dv|, & \text{if } |Dv| \geq \gamma \\ +\infty, & \text{otherwise} \end{cases} \tag{3.1}$$

and

$$\mathcal{P}^{-}_{\lambda,\Lambda,\gamma}\left(D^2v, Dv\right) = \begin{cases} \lambda \mathrm{tr} D^2v^+ - \Lambda \mathrm{tr} D^2v^- - \Lambda|Dv|, & \text{if } |Dv| \geq \gamma \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.2}$$

When the context is clear, we will frequently denote $\mathcal{P}^{\pm}_{\lambda,\Lambda,\gamma}$ simply by $\mathcal{P}^{\pm}$. We first recall the interior C^{α} estimate as established in Theorem 1.1 in [23].

Theorem 3.1 *For any continuous function $u : \overline{B_1} \longrightarrow \mathbb{R}$, satisfying in the viscosity sense,*

$$\begin{cases} \mathcal{P}^{-}(D^2u, Du) \leq C_0 \text{ in } B_1, \\ \mathcal{P}^{+}(D^2u, Du) \geq -C_0 \text{ in } B_1, \\ \|u\|_{L^{\infty}(B_1)} \leq C_0, \end{cases} \tag{3.3}$$

we have that $u \in C^{\alpha}\left(B_{\frac{1}{2}}(0)\right)$ for some α depending on λ, Λ and the dimension n. Furthermore, the following estimate holds,

$$\|u\|_{C^{\alpha}\left(B_{\frac{1}{2}}(0)\right)} \leq C(n, \lambda, \Lambda, \gamma, C_0). \tag{3.4}$$

Remark 3.2 It is clear from the definition of $\mathcal{P}^{\pm}(M, p)$ that if u satisfies $\mathcal{P}^{+}\left(D^2u, Du\right) \geq L$ $\left(\text{resp., } \mathcal{P}^{-}\left(D^2u, Du\right) \leq L\right)$ in Ω, then the rescaled function $v(x) = Mu(x_0 + rx)$ satisfies

$$\mathcal{P}^{+}_{r,M}\left(D^2v, Dv\right) \geq Mr^2L \; \left(\text{resp. } \mathcal{P}^{-}_{r,M}\left(D^2v, Dv\right) \leq MLr^2\right) \text{ in } \frac{1}{r}\Omega - x_0,$$

where

$$\mathcal{P}^{+}_{r,M}\left(D^2v, Dv\right) = \begin{cases} \Lambda \mathrm{tr} D^2v^{+} - \lambda \mathrm{tr} D^2v^{-} + r\Lambda|Dv|, & \text{if } |Dv| \geq rM\gamma \\ +\infty, & \text{otherwise.} \end{cases}$$

Similarly, $\mathcal{P}^{-}_{r,M}\left(D^2v, Dv\right)$ is also defined.

We now proceed with our proof of analogous boundary estimates. In Sections 3 and 4, we only restrict to the case when $\partial\Omega = \{x_n = 0\}$. In Section 5, we then show how to reduce to flat boundary conditions. The following result is the measure to uniform estimate at the boundary, which is analogue to Lemma 3.1 in [23].

Theorem 3.3 *There exist two small constants $\epsilon_0 > 0$ and $\delta > 0$, and a large constant $K > 0$, such that if $\gamma \leq \epsilon_0$, then for any lower semicontinuous function $u : \overline{Q_1^{+}} \longrightarrow \mathbb{R}$ satisfying*

$$\begin{cases} u \geq 0 \text{in } Q_1^{+}, \\ \mathcal{P}^{-}\left(D^2u, Du\right) \leq 1 \text{ in } Q_1^{+}, \\ u_{x_n} \leq 0 \text{ on } Q_1^{0}, \\ |\{u > K\} \cap Q_1^{+}| \geq (1-\delta)|Q_1^{+}|, \end{cases} \tag{3.5}$$

we have that $u > 1$ in $Q^{+}_{\frac{1}{16\sqrt{n}}}$.

Proof The proof is divided into three steps.

Step 1: Similar to that in [23], we first assume that u is a classical solution of Eq. 3.5, i.e. let $u \in C^2(Q_1^{+}) \cap C\left(\overline{Q_1^{+}}\right)$ and satisfies the Neumann condition in the viscosity sense. Suppose on the contrary that for all ϵ_0, δ and K such that for which Eq. 3.5

holds, there exists $x_0 \in Q^+_{\frac{1}{16\sqrt{n}}}$ such that $u(x_0) \leq 1$. Let us consider the following set $G = \{u > K\} \cap Q^+_{\frac{1}{16\sqrt{n}}}$. Given $x \in G$, let $y \in \overline{Q^+_1}$ be a point such that

$$u(y) + 10|y - x|^{1/2} = \min_{\overline{Q^+_1}} \left\{u(z) + 10|z - x|^{1/2}\right\} \tag{3.6}$$

i.e. we slide the cusp with vertex at x till touches the graph of u for the first time. Now on one hand, since $u \geq 0$ in $\overline{Q^+_1}$ and $x \in G \subset B^+_{1/8}$, therefore we have

$$\begin{aligned} u(\xi) + 10|\xi - x|^{\frac{1}{2}} &\geq 10|\xi - x|^{\frac{1}{2}} && (3.7)\\ &> 5\sqrt{\frac{7}{2}}, && (3.8)\end{aligned}$$

for any $\xi \in \partial Q^+_1 \cap \{x_n > 0\}$. On the other hand,

$$\begin{aligned} u(x_0) + 10|x_0 - x|^{\frac{1}{2}} &\leq 1 + 10 \times \left(\frac{1}{4}\right)^{\frac{1}{2}} && (3.9)\\ &= 6 < 5\sqrt{\frac{7}{2}}. && (3.10)\end{aligned}$$

This shows that $y \notin \partial Q^+_1 \cap \{x_n > 0\}$. We now show that $y \notin Q^0_1$. If that is not the case, then since $u_{x_n} \leq 0$ in the viscosity sense, therefore necessarily we must have

$$\limsup_{h \to 0} \left[\frac{\phi\left(y', he_n\right) - \phi\left(y', 0\right)}{h}\right] \leq 0 \tag{3.11}$$

where $\phi(\cdot) = -10|\cdot - x|^{\frac{1}{2}}$. However a direct calculation shows that the quantity in Eq. 3.11 equals

$$-5\frac{(y-x)}{|y-x|^{\frac{3}{2}}} \cdot e_n = \frac{5x_n}{|y-x|^{\frac{3}{2}}} > 0 \quad (\text{since} \;\; y_n = 0),$$

which is a contradiction to Eq. 3.11. Therefore, the minimum will never be achieved on the boundary and thus $y \in Q^+_1$. At this point, the rest of the proof is similar to that in the in interior case (see Proposition 3.3 in [23]) but we nevertheless provide the details for the sake of completeness.

Let $K = 1 + 5\sqrt{\frac{7}{2}}$. In this way, we can ensure that $u(y) < K$. In particular $x \neq y$ and therefore $|z - x|^{\frac{1}{2}}$ is differentiable at $z = y$. Note that for one value of x, there can be more than one y where the minimum is achieved. However, the value of y determines x completely since we must have

$$Du(y) = 5(x - y)|y - x|^{-\frac{3}{2}}.$$

Let us now set $\psi(\xi) = -10|\xi|^{\frac{1}{2}}$. Then from the extrema conditions, we have

$$\begin{aligned} Du(y) &= D\psi(y - x), && (3.12)\\ D^2u(y) &\geq D^2\psi(y - x). && (3.13)\end{aligned}$$

The relations (3.12) and (3.13), together with $\mathcal{P}^-\left(D^2u, Du\right) \leq 1$, imply that

$$|D^2u(y)| \leq C\left(1 + |D^2\psi(y - x)| + |D\psi(y - x)|\right), \tag{3.14}$$

as long as $\epsilon_0 \le \min_{B_{\sqrt{n}}} |D\psi|$. Note that over here, C only depends on the ellipticity constants and the dimension. Since for each value of y, there is only one value of x, so we can define a map $\tau(y) := x$. Let U be the domain of τ. It is clear that $U \subset \{z : u(z) < K\}$ and $\tau(U) = G$.

By putting $x = \tau(y)$ in Eq. 3.12 and employing the chain rule, we get

$$D^2u(y) = D^2\psi(y - \tau(y))(I - D\tau(y)).$$

Solving for $D\tau$ and using the estimate (3.14), we get

$$|D\tau(y)| \le 1 + C\frac{1 + |D^2\psi(y-x)| + |D\psi(y-x)|}{|D^2\psi(y-x)|} \le \tilde{C}. \tag{3.15}$$

The reader should note over here in Eq. 3.15, we crucially used the fact that all the eigenvalues of $D^2\psi$ are comparable. Now, since

$$\frac{\left|Q^+_{\frac{1}{16\sqrt{n}}}\right|}{|Q_1^+|} \ge c(n),$$

therefore in view of the last condition in Eq. 3.5 and the fact that $U \subset \{z \mid u(z) < K\}$, we obtain

$$(1 - c\delta)|Q^+_{\frac{1}{16\sqrt{n}}}| \le |G| = \int_U |\text{Det}\, \tau(y)| dy \le C|U| \le C\delta|Q_1^+|.$$

This is a contradiction if δ is small enough. This completes the proof of *Step 1*.

Step 2: Assuming that the Theorem 3.3 holds for semiconcave supersolutions, we now show that this in turn implies that the conclusion remains true for lower semi-continuous supersolution u.

Let u be a merely lower semi continuous supersolution defined in $\overline{Q_1^+}$. Let $v := \min\{u, 2K\}$, where K is as in *Step 1*. Note that v is still a supersolution because it is the minimum of two supersolutions. Indeed, suppose that $v - \phi$ has minimum at x_0. There are two possibilities:

$$\begin{cases} 1) \quad x_0 \in Q_1^+ \text{ or} \\ 2) \quad x_0 \in Q_1^0. \end{cases} \tag{3.16}$$

We first note that there two possible subcases under the Case 1).

(1a) If $v(x_0) = u(x_0)$, then we have

$$u(x_0) - \phi(x_0) = v(x_0) - \phi(x_0) \le v(x) - \phi(x) \le u(x) - \phi(x).$$

In this case, the desired differential inequality is seen to be valid for ϕ because u satisfies such an inequality in the viscosity sense.

(1b) Suppose instead that $v(x_0) = 2K$, then we have

$$2K - \phi(x_0) = v(x_0) - \phi(x_0) \le v(x) - \phi(x) \le 2K - \phi(x)$$

and conclusion in this case follows from the extrema conditions for ϕ. Similarly the Neumann condition when Case 2) holds is seen to be satisfied.

As in [23], for a given $\delta > 0$, we now consider the inf-convolution of v defined as follows:

$$v_\epsilon(x) = \inf_{y \in \overline{Q^+_{1-\tilde{\delta}}}} \left(v(y) + \frac{1}{2\epsilon}|y - x|^2 \right),$$

where $\tilde{\delta} = \delta/2$. For any $x \in Q_1^+$, using the fact that $v_{x_n} \leq 0$, it follows in a standard way that the infimum above will be achieved at any point $y_0 \in \overline{Q_{1-\tilde{\delta}}^+} \setminus Q_1^0$. See for instance the proof of Lemma 5.2 in [26].

We now make the following claim.

Claim For any $\epsilon > 0$ satisfying $2\sqrt{2K\epsilon} < \delta/4$, v_ϵ is supersolution to the following problem

$$\begin{cases} \mathcal{P}^-(D^2 v_\epsilon, Dv_\epsilon) \leq 1 \text{ in } Q_{1-\delta}^+, \\ (v_\epsilon)_{x_n} \leq 0 \text{ on } Q_{1-\delta}^0. \end{cases} \tag{3.17}$$

The proof of this claim follows exactly the same way as that of Lemma 5.3 in [26] and so we skip the details. Then by noting that v_ϵ is semiconcave and satisfies (3.17), we can now apply the conclusion of *Step 1* to v_ϵ and then by a limiting argument as in the proof of Proposition 3.4 in [23], we thus conclude that the assertion in *Step 2* holds.

Step 3: Finally the fact that the conclusion of Theorem 3.3 holds when u is a semiconcave viscosity supersolution of Eq. 3.5 follows by repeating the interior arguments as in the proof of Proposition 3.5 in [23]. Note that the Neumann condition $u_{x_n} \leq 0$ ensures that as in *Step 1* that the minimum in Eq. 3.6 is attained on the set $Q_1^+ \setminus \{x_n = 0\}$. This finishes the proof.

□

3.1 Barrier Function and Doubling Type Lemma

As mentioned in the introduction, since our proof of the L^ϵ estimate relies on Calderon-Zygmund decomposition instead of the growing Ink-spot lemma as employed in [23] because of certain technical obstructions, therefore we need a somewhat adjusted doubling type lemma as stated in Theorem 3.4 below.

Similar to [23], we consider the function

$$V(x) = |x|^{-\sigma} + \epsilon_{n,\sigma} x_n = h(x) + \epsilon_{n,\sigma} x_n,$$

where $\epsilon_{n,\sigma} > 0$ is a positive constant depending on σ and n and will be subsequently chosen. We let $r = |x|$. As the reader will see, unlike the interior case as in [23], this additional term $\epsilon_{n,\sigma} x_n$ accounts for the adjustment required due to the presence of the Neumann condition. Using $D^2 V = D^2 h$ and also the fact that h is radial, we can assert that the eigenvalues of $D^2 h(x)$, for $x \neq 0$, are $-\sigma r^{-\sigma-2}$ with multiplicity $n-1$ and $\sigma(\sigma+1) r^{-\sigma-2}$ with multiplicity 1. Therefore, for $x \neq 0$, we have

$$\mathcal{P}^-\Big(D^2 V(x), DV(x)\Big) = \lambda\sigma(\sigma+1) r^{-\sigma-2} - \Lambda(n-1)\sigma r^{-\sigma-2} - \Lambda|\epsilon_{n,\sigma} e_n - \sigma r^{-\sigma-2} x|, \tag{3.18}$$

as long as $|DV(x)| \geq \gamma$. A standard calculation shows

$$\begin{aligned} \mathcal{P}^-\Big(D^2 V(x), DV(x)\Big) &= \lambda\sigma(\sigma+1) r^{-\sigma-2} - \Lambda(n-1)\sigma r^{-\sigma-2} - \Lambda|\epsilon_\sigma e_n - \sigma r^{-\sigma-2} x| \\ &= \sigma r^{-\sigma-2}\Big(\lambda(\sigma+1) - \Lambda(n-1) - \Lambda|\Big(\frac{\epsilon_{n,\sigma}}{\sigma}\Big) r^{\sigma+2} e_n - x|\Big) \\ &\geq \sigma r^{-\sigma-2}\Big(\lambda(\sigma+1) - \Lambda(n-1) - \frac{\Lambda\epsilon_{n,\sigma} r^{\sigma+2}}{\sigma} - \Lambda r\Big). \end{aligned} \tag{3.19}$$

The next lemma corresponds to the spread of the positivity set needed to apply the Calderon-Zygmund type lemma in the upper half space.

Theorem 3.4 *There exists an $\epsilon_0 > 0$ depending on the ellipticity and dimension such that if $\gamma \le \epsilon_0$, $u : Q_{8n}^+ \longrightarrow \mathbb{R}$, satisfies*

$$\begin{cases} u \ge 0 \ in Q_{8n}^+, \\ \mathcal{P}_{\lambda,\Lambda,\gamma}^-\left(D^2u, Du\right) \le 1 \ in \ Q_{8n}^+, \\ u_{x_n} \le 0 \ on \ Q_{8n}^0, \end{cases} \tag{3.20}$$

and $u > K$ on $Q^+_{\frac{1}{16\sqrt{n}}}$ for a sufficiently large K (depending on $\Lambda, \lambda, n, \gamma$), then $u > 1$ in Q_3^+.

Proof We first observe that

$$B^+_{\frac{1}{32\sqrt{n}}} \subset Q^+_{\frac{1}{16\sqrt{n}}} \quad \text{and} \quad Q_3^+ \subset B^+_{3\sqrt{n}} \subset B^+_{4n} \subset Q^+_{8n}.$$

Then we consider the following barrier function:

$$\mathscr{B}(x) = \frac{K}{2[32\sqrt{n}]^\sigma}\left[|x|^{-\sigma} - (4n)^{-\sigma} + \epsilon_{n,\sigma}\left(x_n - 8n\sqrt{n}\right)\right], \tag{3.21}$$

with $\epsilon_{n,\sigma} = (128n)^{-8(\sigma+2)}$.

For any value of $K \ge 1$, we note that $\mathscr{B}$ has the following properties:

(1) $\mathscr{B}(x) \le 0$ for any $|x| \ge 4n$.
(2) $\mathscr{B}(x) \le \frac{K}{2} < K$ for any $x \in \partial B_{\frac{1}{32\sqrt{n}}}$. In particular for any $x \in \partial B_{\frac{1}{32\sqrt{n}}} \cap \{x_n > 0\}$, $\mathscr{B}(x) < K$.
(3) For any $x \ne 0$ such that $x_n = 0$, we have that $\frac{\partial \mathscr{B}}{\partial x_n}(x) = \frac{\epsilon_{n,\sigma} K}{2[32\sqrt{n}]^\sigma}$. In particular,

$$\frac{\partial \mathscr{B}}{\partial x_n}(x) > 0 \tag{3.22}$$

for $x \ne 0$.

We now choose σ sufficiently large such that the following holds:

$$\lambda(\sigma+1) - \Lambda(n-1) - \frac{\Lambda\left(8n\sqrt{n}\right)^{\sigma+2}}{\sigma(128n)^{8(\sigma+2)}} - \Lambda\left(8n\sqrt{n}\right) \ge 2. \tag{3.23}$$

Having chosen σ, it is always possible to choose $K \ge 1$ (sufficiently large), such that following inequalities hold:

(1) $|D\mathscr{B}(x)| \ge \gamma$ in Q_{8n}^+,
(2) $|\mathscr{B}(x)| > 1$ in $B^+_{3\sqrt{n}}$,
(3) $\mathcal{P}^-\left(D^2\mathscr{B}, D\mathscr{B}\right) \ge 2$ in Q_{8n}^+.

Now, we claim that $u \ge \mathscr{B}$ in $\left(B_{4n} \setminus B_{\frac{1}{32\sqrt{n}}}\right) \cap \{x_n \ge 0\}$. If not, then there exists an $z_0 \in \left(B_{4n} \setminus B_{\frac{1}{32\sqrt{n}}}\right) \cap \{x_n \ge 0\}$ which corresponds to a negative minimum of $u - \mathscr{B}$ in that same set. Then there are two possibilities:

(1) $(z_0)_n = 0$. In this case, we must have $\frac{\partial \mathscr{B}}{\partial x_n} \le 0$ due to Eq. 3.20 which in view of Eq. 3.22 above is not possible.

(2) z_0 is an interior point. In this case, we again have a contradiction to Eq. 3.20 since $\mathcal{P}^-\left(D^2\mathscr{B}, D\mathscr{B}\right) \geq 2$ in Q_{8n}^+.

This proves the claim. Therefore for $\epsilon = \min_{B^+_{\frac{1}{32\sqrt{n}}}} (u/K-1)$, we obtain $u \geq (1+\epsilon)K > 1$ in $B^+_{3\sqrt{n}}$. □

As a consequence, we have the following corollary which is the key ingredient in our proof of L^ϵ estimate.

Corollary 3.5 *There exist small constants $\epsilon_0 > 0$ and $\delta > 0$ and a large constant $K > 0$, such that if $\gamma \leq \epsilon_0$, then for any lower semicontinuous function $u : \overline{Q_{8n}^+} \longrightarrow \mathbb{R}$, satisfying*

$$\begin{cases} u \geq 0 \;\; in \;\; Q_{8n}^+, \\ \mathcal{P}^-\left(D^2u, Du\right) \leq 1 \; in \; Q_{8n}^+, \\ u_{x_n} \leq 0 \; on \; Q_{8n}^0, \\ |\{u > K\} \cap Q_1^+| > (1-\delta)|Q_1^+|, \end{cases} \tag{3.24}$$

we have $u > 1$ in Q_3^+.

Proof Let K_1 and K_2 be the (renamed) constants from Theorems 3.3 and 3.4 respectively. We claim that K can be taken to be K_1K_2. With such a choice of K, we note that the function $v = u/K_2$ satisfies the assumption of Theorem 3.3. From there we conclude that $v > 1$ in $Q^+_{\frac{1}{16\sqrt{n}}}$, i.e, $u > K_2$ in $Q^+_{\frac{1}{16\sqrt{n}}}$. Now we can apply the doubling result Theorem 3.4 to finally obtain that $u > 1$ in Q_3^+. □

We now state and prove the boundary version of the L^ϵ estimate.

Theorem 3.6 *There exists a small enough $\epsilon, \epsilon_0 > 0$ such that if $\gamma \leq \epsilon_0$, then for any u satisfying*

$$\begin{cases} u \geq 0 \;\; in \;\; Q_{8n}^+, \\ \mathcal{P}^-\left(D^2u, Du\right) \leq 1 \; in \; Q_{8n}^+, \\ u_{x_n} \leq 0 \; on \; Q_{8n}^0, \\ \inf_{Q_3^+} u \leq 1, \end{cases} \tag{3.25}$$

we have

$$|\{u > t\} \cap Q_1^+| \leq \tilde{C} t^{-\epsilon}, \quad t > 0. \tag{3.26}$$

Proof In order to prove (3.26), note that it suffices to show that for $\delta > 0$ as in Corollary 3.5,

$$|\left\{u > (C_0K)^m\right\} \cap Q_1^+| \leq (1-\delta/2)^m |Q_1^+| \tag{3.27}$$

for K as in Corollary 3.5 and C_0 sufficiently large which will be chosen below. For $m = 1$, since $\inf_{Q_3^+} u \leq 1$ so by Corollary 3.5 we find

$$|\{u > K\} \cap Q_1^+| \leq (1-\delta)|Q_1^+|.$$

Now assume that the result is true for $m-1$, that is,

$$|\left\{u > (C_0K)^{m-1}\right\} \cap Q_1^+| \leq \left(1 - \frac{\delta}{2}\right)^{m-1} |Q_1^+|. \tag{3.28}$$

Let us set

$$A_m = \left\{u > (C_0K)^m\right\} \cap Q_1^+ \quad \text{and} \quad A_{m-1} = \left\{u > (C_0K)^{m-1}\right\} \cap Q_1^+.$$

We claim that

$$|A_m| \leq (1-\delta/2)|A_{m-1}|. \tag{3.29}$$

If not, then by the Calderon-Zygmund lemma applied to cubes in the upper half space, we have that there exists a dyadic cube Q such that

$$|A_m \cap Q| > (1-\delta/2)|Q| \tag{3.30}$$

and $2Q = \tilde{Q} \not\subset A_{m-1}$, i.e. there is a point $x_1 \in \tilde{Q}$ such that $u(x_1) \leq (C_0K)^{m-1}$. Let us consider the following cases:

Case 1 Suppose $Q = Q_{\frac{1}{2^i}}(x_0)$ such that $|(x_0', (x_0)_n) - (x_0', 0)| \geq \frac{4n}{2^i}$. In this case, it is easy to observe that $Q_{\frac{8n}{2^i}}(x_0) \subset Q_{8n}^+$. Therefore, the rescaled function $\tilde{u} : Q_{8n} \longrightarrow \mathbb{R}$, defined by $\tilde{u}(y) = \frac{1}{(C_0K)^{m-1}} u\left(x_0 + \frac{e_n}{2^{i+1}} + \frac{y}{2^i}\right)$ satisfies the following differential inequality

$$\begin{cases} \tilde{u} \geq 0 \ \text{ in } \ Q_{8n}, \\ \mathcal{P}^-\left(D^2\tilde{u}, D\tilde{u}\right) \leq 1 \text{ in } Q_{8n}, \\ \tilde{u}(y_1) \leq 1 \ \text{ for some } \ y_1 \in Q_3, \end{cases} \tag{3.31}$$

for a smaller γ in view of the discussion in Remark 3.2. Therefore, we can employ the interior version of Corollary 3.5 to conclude that

$$|\{\tilde{u} > K\} \cap Q_1| \leq (1-\delta/2)|Q_1|, \tag{3.32}$$

which in particular implies

$$|A_m \cap Q| = |\left\{u > (C_0K)^m\right\} \cap Q| \leq (1-\delta/2)|Q|. \tag{3.33}$$

This contradicts (3.30).

Case 2 Now suppose instead that either $Q = Q_{\frac{1}{2^i}}(x_0)$ or $Q = Q^+_{\frac{1}{2^i}}(x_0)$ with

$$|(x_0', (x_0)_n) - (x_0', 0)| \leq \frac{4n}{2^i}.$$

In this case, due to the nature of the Calderon-Zygmund decomposition for cubes in the upper half space, there are two possibilities

$$[i] \ (x_0)_n = 0 \ \text{ or}$$
$$[ii] \ (x_0)_n \geq \frac{1}{2^i}.$$

In Case 2 [i], we again consider the rescaled function $\tilde{u} : Q_{8n}^+ \longrightarrow \mathbb{R}$ defined by

$$\tilde{u}(y) = \frac{1}{(C_0K)^{m-1}} u\left(x_0 + \frac{y}{2^i}\right), \tag{3.34}$$

which satisfies the following differential inequality

$$\begin{cases} \tilde{u} \geq 0 \ \text{ in } \ Q_{8n}^+, \\ \mathcal{P}^-\left(D^2\tilde{u}, D\tilde{u}\right) \leq 1 \text{ in } Q_{8n}^+, \\ \tilde{u}_{x_n} \leq 0 \text{ on } Q_{8n}^0, \\ \text{and } \ \tilde{u}(z_1) \leq 1 \ \text{ for some } z_1 \in Q_3^+ \end{cases} \tag{3.35}$$

Therefore by Corollary 3.5 we note that the following holds,

$$|\{\tilde{u} > K\} \cap Q_1^+| \leq (1 - \delta/2)|Q_1^+|. \tag{3.36}$$

This implies that

$$|\left\{u > (C_0 K)^m\right\} \cap Q| \leq (1 - \delta/2)|Q|, \tag{3.37}$$

which then contradicts (3.30) as before.

Instead if Case 2 [(ii)] happens, i.e. say $(x_0)_n \geq \frac{1}{2^i}$. Now since we also have that $(x_0)_n \leq 4n/2^i$, therefore, given δ_0 such that $0 < \delta_0 < 1$, there exists a cube $Q^{\delta_0} \subset Q_1^+$ of size comparable to Q which contains Q such that $\text{dist}(Q^{\delta_0}, \{x_n = 0\}) = \delta_0/2^i$. We now make the following claim.

Claim If C_0 is large enough, then the function

$$v(y) = \frac{u(y)}{(C_0 K)^{m-1}} > K \text{ in } Q^{\delta_0}.$$

Proof of the claim: Suppose on the contrary that there exists a point $y_0 \in Q^{\delta_0}$ such that

$$v(y_0) \leq K.$$

Then the function defined by $w(z) = \frac{v(z)}{K}$, satisfies $w(y_0) \leq 1$. So by the interior L^ϵ estimate we have

$$|\{w > t\} \cap Q^{\delta_0}| \leq C(\epsilon, \delta_0) t^{-\epsilon} |Q^{\delta_0}|.$$

Note that such an estimate is a consequence of the interior L^ϵ estimate in [23] followed by a standard covering argument. We also note that the constant $C = C(\epsilon, \delta_0)$ can be chosen to be independent of i in view of scale invariance of the estimates (note that the size of both Q^{δ_0} as well as Q are comparable to $\frac{1}{2^i}$) , see for instance Remark 3.2. Therefore, in particular,

$$|\{w > C_0\} \cap Q^{\delta_0}| \leq C(\epsilon, \delta_0) C_0^{-\epsilon} |Q^{\delta_0}|. \tag{3.38}$$

Now we note that since

$$\{w > C_0\} = \{v > C_0 K\} = \left\{u > (C_0 K)^m\right\},$$

therefore this implies that the following holds,

$$|\left\{u > (C_0 K)^m\right\} \cap Q^{\delta_0}| \leq C(\epsilon, \delta_0) C_0^{-\epsilon} |Q^{\delta_0}|.$$

Then using (3.30), we have

$$|\{w > C_0\} \cap Q| = |\left\{u > (C_0 K)^m\right\} \cap Q| > (1 - \delta/2)|Q|. \tag{3.39}$$

Now, we choose the smallest cube $\hat{Q}^+$ with base at $\{x_n = 0\}$ which contains Q^{δ_0} and we also set $\tilde{C}(\delta_0) = \frac{|Q^{\delta_0}|}{|\hat{Q}^+|}$. Note that we have that $\tilde{C}(\delta_0) \to 1$ as $\delta_0 \to 0$. Thus we can choose δ_0 sufficiently small such that $\tilde{C}(\delta_0) > (1 - \delta)$, where δ is from Corollary 3.5. We then let $C(\delta_0) = |Q|/|Q^{\delta_0}|$. It is easy to see that $C(\delta_0)$ is bounded from below uniformly as $\delta_0 \to 0$. Therefore we have from Eq. 2

$$|\{w > C_0\} \cap Q^{\delta_0}| > (1 - \delta/2)|Q| = (1 - \delta/2) C(\delta_0) |Q^{\delta_0}|. \tag{3.40}$$

At this point, if we choose C_0 sufficiently large such that $2C(\epsilon, \delta_0) C_0^{-\epsilon} < (1 - \delta/2) C(\delta_0)$, then from Eq. 3.38 we obtain

$$|\{w > C_0\} \cap Q^{\delta_0}| < (1 - \delta/2) C(\delta_0) |Q^{\delta_0}|$$

which contradicts (3.40). This proves the claim.

Consequently, we have

$$\begin{aligned} |\{v > K\} \cap \hat{Q}^+| &\geq |\{v > K\} \cap Q^{\delta_0}| \quad \left(\text{since } \; Q^{\delta_0} \subset \hat{Q}^+\right) \\ &= |Q^{\delta_0}| \quad \left(\text{since } \; v > K \; \text{ in } \; Q^{\delta_0}\right) \\ &= \tilde{C}(\delta_0)|\hat{Q}^+| \\ &> (1-\delta)|\hat{Q}^+|. \end{aligned} \tag{3.41}$$

Therefore by invoking Corollary 3.5, we conclude that $v > 1$ in $3\hat{Q}^+$ and hence $v > 1$ in $\tilde{Q}$ since $\tilde{Q} \subset 3\hat{Q}^+$. Now given that $A_{m-1} = \{u > (C_0 K)^{m-1}\} \cap Q_1^+ = \{v > 1\} \cap Q_1^+$, therefore this contradicts the fact that $\tilde{Q} \not\subset A_{m-1}$. The conclusion of the Theorem thus follows. □

We also need the following uniform estimate as in Theorem 3.8 below which is a consequence of a scaled version of the above L^ϵ estimate. Such an estimate plays a crucial role in the proof of Hölder regularity of the solutions up to the boundary similar to that in the interior case as in [23]. Before stating such a result, we make the following important remark.

Remark 3.7 Given $\epsilon_0 > 0$ as in Theorem 3.6, we will choose C_1 large enough in the hypothesis of Theorem 3.8 below such that $\epsilon_0 > \frac{2}{C_1 \Lambda}$ where Λ is the ellipticity upper bound.

Theorem 3.8 *There exist small constants $\tilde{\epsilon}_0, c_0 > 0$ and $\alpha, r_0 \in (0, 1)$ such that if $\gamma \leq \tilde{\epsilon}_0$, then for any lower semicontinuous function $u : B^+_{(4n)r} :\longrightarrow \mathbb{R}$ satisfying the following differential inequalities for $r \leq r_0$,*

$$\begin{cases} u \geq 0 \; \textit{in} \; B^+_{(4n)r}, \\ \mathcal{P}^-\left(D^2 u, Du\right) \leq \epsilon_1/2 \; \textit{in} \; B^+_{(4n)r}, \\ u_{x_n} \leq g \; \textit{on} \; B^0_{(4n)r}, \\ \|g\|_{L^\infty(B^0_{4n})} \leq \frac{\epsilon_1}{C_1 \Lambda} \\ \textit{and} \; |\{u > r^\alpha\} \cap Q_r^+| \geq \frac{1}{2}|Q_r^+|, \end{cases} \tag{3.42}$$

we have,

$$u > c_0 r^\alpha \tag{3.43}$$

in Q^+_{3r}. In particular, $u > c_0 r^\alpha$ in B_r^+.

Proof Let $\tau > 1$ be such that

$$C\tau^{-\epsilon} < \frac{|Q_1^+|}{2} \tag{3.44}$$

where C and $\epsilon > 0$ are the constants from the L^ϵ estimate as in Theorem 3.6 above. Now, consider the following function

$$\tilde{u} \; : \; B^+_{4n} \longrightarrow \mathbb{R},$$

defined by

$$\tilde{u}(x) = \tau r^{-\alpha} u(rx) + \frac{\tau \epsilon_1}{\Lambda C_1} r^{1-\alpha}(4n - x_n). \tag{3.45}$$

where ϵ_1 will be chosen later. Then we have that $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u} \geq 0 \text{ in } B^+_{(4n)}, \\ \mathcal{P}^-_{\lambda,\Lambda,\tilde{\gamma}}\left(D^2\tilde{u}, D\tilde{u}\right) \leq \left[\frac{\epsilon_1\tau}{C_1} + \frac{\epsilon_1\tau}{2}\right] r^{2-\alpha} \text{ in } B^+_{(4n)}, \\ \tilde{u}_{x_n} \leq 0 \text{ on } B^0_{4n}, \end{cases} \tag{3.46}$$

with $\tilde{\gamma} = \left(\gamma\tau + \frac{2\epsilon_1\tau}{\Lambda C_1}\right) r^{1-\alpha}$. Furthermore, we have

$$|\{\tilde{u} > \tau\} \cap Q^+_1| \geq \frac{1}{2}|Q^+_1| \geq C\tau^{-\epsilon}. \tag{3.47}$$

Now let us choose $\epsilon_1 = \tau^{-1}$. Then we have that $\tilde{\gamma} = \left(\gamma\tau + \frac{2}{\Lambda C_1}\right) r^{1-\alpha}$. We now fix $\alpha \in (0, 1/2)$. Then by choosing r_0 small enough we can ensure that

$$\mathcal{P}^-_{\lambda,\Lambda,\tilde{\gamma}}\left(D^2\tilde{u}, D\tilde{u}\right) \leq 1 \tag{3.48}$$

Moreover with ϵ_0 as in Theorem 3.6, we note that in view of our choice of C_1 in Remark 3.7, if we have

$$\gamma \leq \tilde{\epsilon_0} \overset{\text{def}}{=} \left(\epsilon_0\epsilon_1 - \frac{2\epsilon_1}{\Lambda C_1}\right),$$

then we can ensure that $\tilde{\gamma} \leq \epsilon_0$.

In such a case, necessarily we must have

$$\tilde{u} > 1 \text{ in } Q^+_3, \tag{3.49}$$

otherwise by applying the L^ϵ estimate in Theorem 3.6, we will obtain a contradiction to Eq. 3.47. We thus obtain from Eq. 3.49 that

$$u > \epsilon_1 r^\alpha - C_2\epsilon_1 r \tag{3.50}$$

in Q^+_{3r}. The desired estimate (3.43) now follows from Eq. 3.50 in a standard way provided r_0 is adjusted further depending also on C_2. □

With Theorem 3.8 in hand, we can now repeat the arguments in [23] to conclude the Hölder decay of u at a boundary point. The Hölder regularity up to the boundary consequently follows by a standard real analysis argument by combining the boundary estimate with the interior estimate in [23]. We close this section by stating such a result.

Theorem 3.9 *For any continuous function* $u : \overline{B^+_1} \longrightarrow \mathbb{R}$, *such that*

$$\begin{cases} \mathcal{P}^-\left(D^2u, Du\right) \leq C_0 \;\; in \; B^+_1, \\ \mathcal{P}^+\left(D^2u, Du\right) \geq -C_0 \;\; in \; B^+_1, \\ u_{x_n} = g \;\; on \;\; B^0_1, \\ \|g\|_{L^\infty(B^0_1)} \leq C_0, \end{cases}$$

we have $u \in C^\alpha\left(\overline{B^+_{\frac{1}{2}}}\right)$ *for some* $\alpha > 0$ *depending on* λ, Λ *and the dimension.*

4 Equicontinuous Estimates up to the Boundary for Equations which are Uniformly Elliptic when the Gradient is Small

In this section we obtain equicontinuous estimates for equations of the type (1.2) for large slopes, i.e. when $|p|$ is large. As we have already mentioned in the introduction, since an appropriate generalization of the doubling variable argument of Ishii and Lions to our Neumann problem is not clear to us, therefore we instead adapt the method of Savin as in [30] based on sliding paraboloids.

Now in order to see that the method of sliding paraboloids can be applied in this situation (which is tailor-made for equations which are uniformly elliptic when the gradient is small), we note that (1.2) can be rewritten as

$$\left|\frac{Du}{|p|}+\frac{p}{|p|}\right|^{\beta} F(D^2u)=\frac{f}{|p|^{\beta}}.$$

Therefore, for large enough $|p|$, getting equicontinuity estimates for Eq. 1.2 reduces to getting such estimates for equations of the following type

$$\begin{cases} |e+\sigma Du|^{\beta}F(D^2u,x) = f \text{ in } B_1^+, \\ u_{x_n} = g \text{ on } B_1^0, \end{cases} \tag{4.1}$$

where $|e|=1$, $0<\sigma\le 1$ and $F: S(n)\times\mathbb{R}^n \longrightarrow \mathbb{R}$, is a uniformly elliptic operator, i.e.

$$\lambda\|Y\| \le F(X+Y,x)-F(X,x)\le \Lambda\|Y\|, \tag{4.2}$$

for all $X, Y \in S(n)$ with $Y\ge 0$. Note that the equation in Eq. 4.1 has a uniformly elliptic structure when $|Du|$ is small. More precisely when $|Du|\le \frac{1}{2\sigma}$, it follows by triangle inequality using $|e|=1$ that

$$\frac{1}{2}\le |e+\sigma Du|\le \frac{3}{2}$$

and thus from Eq. 4.2 it follows that the operator

$$\tilde F(M,q,x)=|e+q\sigma|^{\beta}F(M,x)$$

satisfies

$$\left(\frac{1}{2}\right)^{\beta}\lambda\|Y\|\le \tilde F(X+Y,q,x)-\tilde F(X,q,x)\le \left(\frac{3}{2}\right)^{\beta}\Lambda\|Y\| \tag{4.3}$$

for all $Y\ge 0$ and $|q|\le\frac{1}{2\sigma}$. Since $0<\sigma\le 1$, we thus see that $\tilde F$ is uniformly elliptic at least for $(q,M)\in B_{1/2}(0)\times S(n)$ which is independent of $\sigma\in(0,1)$.

In our discussion, we will however be considering slightly more general degenerate elliptic operators as in [30]. More precisely, we consider fully nonlinear operators of the type $\tilde F: S(n)\times\mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}$, which satisfies the following structural conditions

SC1) $\tilde F$ is degenerate elliptic, that is,

$$\tilde F(X+Y,q,x)\ge \tilde F(X,q,x) \text{ for all } X,Y\in S(n),\ Y\ge 0 \text{ and } (q,x)\in\mathbb{R}^n\times\mathbb{R}^n.$$

SC2) $\tilde F(0,q,x)=0$ for all $(q,x)\in\mathbb{R}^n\times\mathbb{R}^n$.

SC3) $\tilde F$ is uniformly elliptic in a small neighbourhood of 0, that is, there is a $\delta>0$ such that

$$\lambda\|Y\|\le \tilde F(X+Y,q,x)-\tilde F(X,q,x)\le\Lambda\|Y\|,$$

for some $0<\lambda<\Lambda$, $q\in B_\delta$, $X,Y\in S(n)$ and $Y\ge 0$ and $x\in\mathbb{R}^n$.

Note that it is clear that the operator $\tilde F(X,q,x)=|e+\sigma q|^{\beta}F(X,x)$ satisfies the structural conditions SC1), SC2) and SC3) with ellipticity bounds $\left(\frac{1}{2}\right)^{\beta}\lambda$ and $\left(\frac{3}{2}\right)^{\beta}\Lambda$ for $\delta=\frac{1}{2\sigma}$.

Let us now consider the following problem:

$$\begin{cases} \tilde F(D^2u,Du,x) \le f \text{ in } B_1^+, \\ u_{x_n} \le 0 \text{ on } B_1^0, \end{cases} \tag{4.4}$$

where $\tilde{F}$ satisfies SC1)-SC3). The following lemma is a boundary version of Lemma 2.3 in [18] which in turn is inspired by the ideas in the proof of Lemma 2.1 in [30].

Lemma 4.1 *Let u be a viscosity solution to* Eq. 4.4. *Fix $a \in (0, \delta/2)$, let $B \subset B_1^+$ be a compact set, and define $A \subset \overline{B_1^+}$ to be the set of contact points of paraboloid with vertices in B and opening $-a$, namely the set of points $x \in \overline{B_1^+}$ such that there exists $y \in B$ which satisfies*

$$\inf_{\xi \in B_1^+} \left\{ \frac{a}{2}|y-\xi|^2 + u(\xi) \right\} = \frac{a}{2}|y-x|^2 + u(x). \tag{4.5}$$

Then there exists universal constant $c_1 > 0$ such that

$$c_1|B| \le |A| + \int_A \frac{|f(x)|^n}{a^n} dx \tag{4.6}$$

Proof Since B is compact subset of B_1^+, therefore for any $y \in B$, $y_n > 0$. Therefore the contact point $x \notin B_1^0$. For if $x \in A \cap B_1^0$, then the paraboloid

$$P^y(\xi) = u(x) + \frac{a}{2}|x-y|^2 - \frac{a}{2}|\xi - y|^2$$

touches u at $x \in B_1^0$ from below and also $(P^y)_{\xi_n}(x) = ay_n > 0$, which contradicts the Neumann condition in the viscosity formulation as in Eq. 4.4 above. At this point, we can essentially repeat the arguments as in Lemma 2.3 in [18]. Note that although Lemma 2.3 in [18] deals with C^2 solutions, but nevertheless the proof can be generalized to semiconcave solutions using Alexandrov's theorem and then to arbitrary viscosity solutions using inf convolution. See for instance the proof of Lemma 2.1 in [30]. □

We now let A_a be the set of all $x \in B_1^+$ such that $u(x) \le a$ and the function u can be touched from below at x with a paraboloid of opening $-a$ with vertex in $\overline{B_1^+}$, namely there exists $y \in \overline{B_1^+}$ such that

$$\inf_{z \in B_1^+} \left[u(z) + \frac{a}{2}|y-z|^2 \right] = u(x) + \frac{a}{2}|y-x|^2 \tag{4.7}$$

The next result is the boundary version of the Lemma 2.4 in [18] . See also the corresponding Lemma 2.2 in [30].

Lemma 4.2 *Let u be as in* Eq. 4.4. *Also let $a > 0$ and $x_0 \in B_1^0$ such that $B_{4r}^+(x_0) \subset B_1^+$. Then there exist universal constants C_b and c_b and μ_b, such that if $a \le \frac{\delta}{C_b}$, $\|f\|_{L^\infty(B_1^+)} \le \mu_b a$ and*

$$B_r^+(x_0) \cap A_a \neq \emptyset, \tag{4.8}$$

then

$$c_b \left| B_r^+(x_0) \right| \le \left| B_{\frac{r}{16}}^+(x_0) \cap A_{aC_b} \right|. \tag{4.9}$$

Proof By Eq. 4.8, there exists $x_1 \in B_r^+(x_0) \cap A_a$. So by the definition of A_a, there exists $y_1 \in B_1^+$ such that the paraboloid

$$Q_{y_1}(\xi) = u(x_1) + \frac{a}{2}|x_1 - y_1|^2 - \frac{a}{2}|\xi - y_1|^2, \tag{4.10}$$

satisfies

$$\begin{cases} Q_{y_1}(\xi) & \leq u(\xi) \ \ \forall\, \xi \in B_1^+, \\ Q_{y_1}(x_1) & = u(x_1). \end{cases} \tag{4.11}$$

We now make the following claim.

Claim There exists $z \in B^+_{\frac{r}{32}}(x_0)$ such that

$$u(z) \leq Q_{y_1}(z) + C_1 a r^2. \tag{4.12}$$

In order to prove the claim, let us consider the function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$, defined by

$$\phi(x) = \begin{cases} \frac{1}{\alpha}(32^\alpha - 1), & \text{if } |x| < \frac{1}{32} \\ \frac{1}{\alpha}(|x|^{-\alpha} - 1), & \text{if } \frac{1}{32} \leq |x| \leq 1 \\ 0, & \text{if } |x| > 1 \end{cases} \tag{4.13}$$

where α is to be chosen later. In terms of ϕ, we then define $\psi : B_r^+(x_0) \longrightarrow \mathbb{R}$ in the following way,

$$\psi(x) = Q_{y_1}(x) + ar^2\phi\left(\frac{x - x_0}{r}\right) - \epsilon a r^2 (r - x_n), \tag{4.14}$$

where ϵ is a sufficiently small number which will be chosen below. We note that for x satisfying $\frac{r}{32} < |x - x_0| < r$, the function ψ is smooth. Moreover, for any x in the above set we have

$$D\psi(x) = -a(x - y_1) + ar D\phi\left(\frac{x - x_0}{r}\right) + \epsilon a r^2 e_n. \tag{4.15}$$

Thus it follows that

$$\begin{aligned} |D\psi(x)| &\leq 4a + a\frac{r^{1+\alpha}}{|x - x_0|^\alpha} \\ &\leq a\left(4 + 32^{1+\alpha}\right) < \delta, \end{aligned} \tag{4.16}$$

provided $C_b \geq \left(4 + 32^{1+\alpha}\right)$ and consequently F is uniformly elliptic in the above region. In view of $SC3$) we have

$$\begin{aligned} \tilde{F}\left(D^2\psi(x), D\psi(x), x\right) - f(x) &\geq \lambda \left\|\left(D^2\psi(x)\right)^+\right\| - \Lambda \left\|\left(D^2\psi(x)\right)^-\right\| - \|f\|_{L^\infty(B_1^+)} \\ &\geq a\left[\left(\lambda(1+\alpha) - \sqrt{n-1}\Lambda\right)\frac{r^{\alpha+2}}{|x - x_0|^\alpha} - \lambda - \sqrt{n-1}\Lambda - \mu_b\right]. \end{aligned} \tag{4.17}$$

Consequently, if we choose α sufficiently large, then we obtain

$$\tilde{F}\left(D^2\psi(x), D\psi(x), x\right) - f(x) > 0 \ \text{ in } \ B_r^+(x_0) \cap \{x_n > 0\} \cap \left\{\frac{r}{32} < |x - x_0| < r\right\}. \tag{4.18}$$

Also for $\bar{x} \in B_1^0$, we observe that

$$\partial_{x_n}\psi(\bar{x}) = a(y_1)_n + a\epsilon r^2 > 0. \tag{4.19}$$

We denote by z the point where $\min_{x \in \overline{B_r^+(x_0)}}(u - \psi)$ is achieved. We now choose $\epsilon > 0$ sufficiently small such that

$$-ar^2\phi\left(\frac{x_1 - x_0}{r}\right) + ar^2(r - (x_1)_n)\epsilon < 0. \tag{4.20}$$

Note that although the choice of ϵ depends on x_1 but as we will see, it doesn't affect the final conclusion. Equation 4.20 implies

$$u(x_1) - \psi(x_1) = Q_{y_1}(x_1) - \psi(x_1) = -ar^2\phi\left(\frac{x_1 - x_0}{r}\right) + ar^2(r - (x_1)_n)\epsilon < 0. \quad (4.21)$$

Moreover on $\partial B_r(x_0) \cap \{x_n > 0\}$, we have

$$\begin{aligned} u(x) &\geq Q_{y_1}(x) \\ &\geq Q_{y_1}(x) - \epsilon ar^2(r - x_n) \qquad \left(\text{since } (ar^2(r - x_n) \geq 0)\right) \\ &= \psi(x). \end{aligned}$$

Now we note that since $u_{x_n} \leq 0$ on B_1^0 (in the viscosity sense), so in view of Eq. 4.19, we can deduce that $u - \psi$ cannot attain minimum on $\left\{\frac{r}{32} < |x - x_0| < r\right\} \cap \left\{x_n = 0\right\} \cup \partial B_r(x_0) \cap \{x_n > 0\}$. Therefore there exists $z \in B^+_{\frac{r}{32}}(x_0)$ such that

$$\begin{aligned} u(z) &< \psi(z), \qquad \text{(thanks to (4.21))} \\ &\leq Q_{y_1}(z) + ar^2\phi\left(\frac{z - x_0}{r}\right) - \epsilon ar^2(r - z_n) \\ &\leq Q_{y_1}(z) + ar^2\phi\left(\frac{z - x_0}{r}\right) \quad \left(\text{since } \left(ar^2(r - z_n) \geq 0\right)\right) \\ &\leq Q_{y_1}(z) + C_1 ar^2. \end{aligned}$$

For a given $L > 0$ and $y \in B_{\frac{r}{128}}(z) \cap \{y_n > z_n\}$, we consider the paraboloid

$$P_y(x) = Q_{y_1}(x) - L\frac{a}{2}|x - y|^2. \quad (4.22)$$

It is easy to check that for each y, P_y is a paraboloid with opening $-(L+1)a$ and vertex $\frac{y_1 + Ly}{1+L}$. We slide it from below till it touches the graph of u for the first time. We claim that the contact point $\bar{x} \in B^+_{\frac{r}{32}}(z)$ provided L is large enough. In order to prove such a claim, we make the following observations.

(i) Suppose $\bar{x} \in B_1^0$, then

$$\partial_{x_n}(P_y)(\bar{x}) = a(y_1)_n + Lay_n > La(z_n) > 0. \quad (4.23)$$

Now since $\partial_{x_n} u \leq 0$ on B_1^0 (in the viscosity sense), therefore P_y cannot touch u from below at points in B_1^0.

(ii) Suppose instead $\bar{x}$ satisfies $|\bar{x} - z| \geq \frac{r}{32}$, then using $u \geq Q_{y_1}$ on B_1^+, we find that the following holds,

$$u(\bar{x}) - Q_{y_1}(\bar{x}) + \frac{La}{2}|\bar{x} - y|^2 \geq \frac{La}{2}\left(\frac{r}{32}\right)^2. \quad (4.24)$$

On the other hand since

$$\begin{aligned} &\min_{\overline{B_1^+}}\left\{u(x) - Q_{y_1}(x) + \frac{La}{2}|x - y|^2\right\} \\ &\leq u(z) - Q_{y_1}(z) + \frac{La}{2}|y - z|^2 \qquad (4.25) \\ &\leq C_1 ar^2 + \frac{La}{2}\left(\frac{r}{128}\right)^2, \end{aligned}$$

thus by choosing L large enough and by taking into account (4.24) and (4.25), we find that the contact point $\bar{x} \in B^+_{\frac{r}{32}}(z) \subset B^+_{\frac{r}{16}}(x_0)$.

We now show that at the contact point $\bar{x}$, we have $u(\bar{x}) \leq La$ provided L is further adjusted. Indeed, since

$$Q_{y_1}(x_1) = u(x_1) \leq a$$

and also

$$\begin{aligned} Q_{y_1}(\bar{x}) &= u(x_1) + \frac{a}{2}|x_1 - y_1|^2 - \frac{a}{2}|\bar{x} - y_1|^2 \\ &\leq a + 4a = 5a, \end{aligned}$$

hence from Eq. 4.25 (since $\bar{x}$ is the point where the minimum in Eq. 4.25 is achieved), we find

$$\begin{aligned} u(\bar{x}) &\leq Q_{y_1}(\bar{x}) - L\frac{a}{2}|\bar{x} - y|^2 + C_1 a r^2 + \frac{La}{2}\left(\frac{r}{128}\right)^2 \\ &\leq 5a + Car^2 + \frac{La}{2}\left(\frac{r}{128}\right)^2 \leq La, \end{aligned}$$

provided L is sufficiently large. Now as y varies in $B_{\frac{r}{128}}(z) \cap \{y_n \geq z_n\}$, the set of vertices of the paraboloids as in Eq. 4.22 falls in the region

$$\tilde{R} \stackrel{\text{def}}{=} \left[B\left(\frac{y_1 + Lz}{1+L}, \frac{Lr}{128(1+L)}\right) \bigcap \left\{\xi_n \geq \frac{(y_1)_n + Lz_n}{1+L}\right\}\right], \tag{4.26}$$

therefore by applying Lemma 4.1, we get

$$\begin{aligned} c_0|\tilde{R}| &\leq |B^+_{\frac{r}{16}(x_0)} \cap A_{a(L+1)}| + \left(\frac{\|f\|^n_{L^\infty(B_1^+)}}{a^n}\right)|B^+_{\frac{r}{16}(x_0)}| \\ &\leq |B^+_{\frac{r}{16}(x_0)} \cap A_{a(L+1)}| + \left(\frac{\mu_b}{16}\right)^n |B^+_r(x_0)|. \end{aligned} \tag{4.27}$$

Then we observe that

$$|\tilde{R}| = C|B^+_r(x_0)| \tag{4.28}$$

for some constant C independent of r. From (4.27) and (4.28) we finally obtain

$$\left[c_0 C - \left(\frac{\mu_b}{16}\right)^n\right]|B^+_r(x_0)| \leq |B^+_{\frac{r}{16}(x_0)} \cap A_{aL}|.$$

and thus the conclusion of the lemma follows. □

We note that the interior analogue of the lemma above is crucially needed to apply the measure decay estimate in [18] and [30] which is the key ingredient needed to obtain quantitative oscillation decay estimates. In our situation, in order to combine the boundary and interior estimate, we also need the following additional lemma.

Lemma 4.3 *Let* $a > 0$, *and suppose that* $B^+_{4r}(x_0) \subset B^+_1$ *and* $(x_0)_n \geq \frac{r}{16}$. *Suppose that* u *is a viscosity solution of* Eq. 4.4. *Then there exists universal constants* C_{ib}, c_{ib} *and* $\mu_{ib} > 0$ *such that if*

$$\begin{cases} \|f\|_{L^\infty(B_1^+)} \leq a\mu_{ib}, \\ a \leq \frac{\delta}{C_{ib}} \text{ and} \\ B^+_r(x_0) \cap A_a \neq \emptyset, \end{cases}$$

then

$$\left|B^+_{\frac{r}{16}}(x_0) \cap A_{aC_{ib}}| \geq c_{ib}|B^+_r(x_0)\right|. \tag{4.29}$$

Proof The proof of this Lemma is similar to that of Lemma 4.2. We nevertheless give a sketch of it for the sake of completeness.

By our assumption, there exists $x_1 \in \left(B_r(x_0) \cap \mathbb{R}^n_+\right) \cap A_a$. So from the definition of A_a, for some $y_1 \in B_1^+$, we have that the paraboloid

$$Q_{y_1}(\xi) = u(x_1) + \frac{a}{2}|y_1 - x_1|^2 - \frac{a}{2}|y_1 - \xi|^2, \tag{4.30}$$

satisfies

$$\begin{cases} u(\xi) & \geq Q_{y_1}(\xi) \;\; \forall \;\; \xi \in B_1^+, \\ u(x_1) & = Q_{y_1}(x_1). \end{cases}$$

We now claim that there exists $z \in B_{\frac{r}{16}}(x_0) \subset B_1^+$ (since $(x_0)_n \geq \frac{r}{16}$) such that

$$u(z) \leq Q_{y_1} + C_2 a r^2. \tag{4.31}$$

for some universal C_2. In order to prove the claim, we consider the following function $\Psi : \overline{B_r(x_0)} \longrightarrow \mathbb{R}$, defined by

$$\Psi(x) = Q_{y_1}(x) + ar^2\phi\left(\frac{x - x_0}{r}\right), \tag{4.32}$$

with ϕ as in Eq. 4.13. Again we can choose α large enough so that the following differential inequality is ensured

$$\begin{cases} F(D^2\Psi, D\Psi, x) > f(x) \;\text{ in }\; \left\{x \mid \frac{r}{32} < |x - x_0| < r\right\} \cap \{x_n > 0\}, \\ \qquad\qquad \Psi_{x_n} > 0 \;\text{ on }\; \left\{x \mid \frac{r}{32} < |x - x_0| < r\right\} \cap \{x_n = 0\}. \end{cases} \tag{4.33}$$

We only check the second condition since the first one is as in the previous lemma. Suppose that $\frac{r}{32} < |\bar{x} - x_0| < r$ and also that $\bar{x}_n = 0$. Then we have that

$$\partial_{x_n}\Psi(\bar{x}) = a(y_1)_n + \left(\frac{r}{|\bar{x} - x_0|}\right)^{\alpha+2}\frac{(x_0)_n}{r} > 0. \tag{4.34}$$

At this point, by arguing as in the proof of the previous lemma, we conclude that the point of minimum in

$$\min_{\overline{B_r(x_0)} \cap \{x \,\mid\, x_n \geq 0\}} \{u - \Psi\} \tag{4.35}$$

is realized in $\overline{B_{r/32}(x_0)}$. The rest of the arguments can then be repeated and the conclusion of the lemma follows similarly. □

Finally, we state the interior version of the above measure estimate. (see Lemma 2.4 in [18]).

Lemma 4.4 *Let u be a solution to the second order differential inequality in* Eq. 4.4. *Let $a > 0$, and $B_{4r}(x_0) \subset B_1^+$. Then there exist universal constants C_i and c_i and μ_i, such that if $a \leq \frac{\delta}{C_i}$, $\|f\|_{L^\infty(B_1^+)} \leq \mu_i a$ and*

$$B_r(x_0) \cap A_a \neq \emptyset, \tag{4.36}$$

then

$$c_i|B_r(x_0)| \leq |B_{\frac{r}{16}}(x_0) \cap A_{C_i a}|. \tag{4.37}$$

4.1 Boundary Version of Measure Decay

We now prove a boundary version of the covering lemma that corresponds to lemma 2.3 in [30]. Similar to the interior case, such a covering lemma is one of the crucial ingredients in our proof of the oscillation decay estimate as in Theorem 4.9 below.

Lemma 4.5 *Let D_0, D_1 be two closed sets satisfying*

$$\emptyset \neq D_0 \subset D_1 \subset \overline{B^+_{r_0}}.$$

and $\sigma_1, \sigma_2, \sigma_3 \in (0,1)$ be such that for $r_0 \leq \frac{1}{14}$, the following hypotheses are satisfied,

$$H(I) \begin{cases} \text{Whenever } x \in B^0_{r_0} \text{ and for some } r > 0, \text{ one has} \\ (i)\ \ B^+_{4r}(x) \subset B^+_1, \\ (ii)\ \ B^+_{\frac{r}{16}}(x) \subset B^+_{r_0}, \\ (iii)\ \ \overline{B^+_r(x)} \cap D_0 \neq \emptyset, \\ \text{then,} \\ \left|B^+_{\frac{r}{16}}(x) \cap D_1\right| \geq \sigma_1 \left|B^+_r(x)\right|. \end{cases}$$

$$H(II) \begin{cases} \text{Whenever } x \in B^+_{r_0} \text{ and for some } r > 0, \text{ one has} \\ (i)\ \ x_n \geq \dfrac{r}{16}, \\ (ii)\ \ B^+_{4r}(x) \subset B^+_1, \\ (iii)\ \ B_{\frac{r}{16}}(x) \subset B^+_{r_0}, \\ (iv)\ \ \overline{B^+_r(x)} \cap D_0 \neq \emptyset, \\ \text{then,} \\ |(B_{\frac{r}{16}}(x) \cap D_1| = \left|B^+_{\frac{r}{16}}(x) \cap D_1\right| \geq \sigma_2 \left|B^+_r(x)\right|. \end{cases}$$

$$H(III) \begin{cases} \text{Whenever } x \in B^+_{r_0} \text{ and for some } r > 0, \text{ one has} \\ (i)\ \ B_{4r}(x) \subset B^+_1, \\ (ii)\ \ B_{\frac{r}{16}}(x) \subset B^+_{r_0}, \\ (iii)\ \ \overline{B_r(x)} \cap D_0 \neq \emptyset, \\ \text{then,} \\ \left|(B_{\frac{r}{16}}(x) \cap D_1\right| \geq \sigma_3 |B_r(x)|. \end{cases}$$

In that case, we have that the following estimate holds,

$$\left|B^+_{r_0} \setminus D_1\right| \leq (1-\sigma) \left|B^+_{r_0} \setminus D_0\right|, \tag{4.38}$$

for some $\sigma \in (0,1)$.

Proof Given $x_0 \in B^+_{r_0} \setminus D_0$, set $\bar{r} = \text{dist}\{x_0, D_0\} \leq 2r_0$. Let us also define $r = \frac{8}{7}\bar{r}$. We will first show that for some $\sigma > 0$, the following estimate holds,

$$\left|B_{\frac{r}{4}}(x_0) \cap B^+_{r_0} \cap D_1\right| \geq \sigma \left|B_r(x_0) \cap B^+_{r_0}(x_0)\right|. \tag{4.39}$$

The proof of Eq. 4.39 is based on a case by case argument depending on the distance of x_0 from $\{x_n = 0\}$. Note that there are 4 possibilities.

Case (i) $x_0 \in B^0_{r_0}$.
Case (ii) $0 < (x_0)_n < \frac{r}{16} = \frac{\bar{r}}{14}$.
Case (iii) $\frac{r}{16} \le (x_0)_n < r_0 - \frac{r}{16}$.
Case (iv) $r_0 - \frac{r}{16} \le (x_0)_n \le r_0$.

Case-(i) In this case let us define

$$x_1 = x_0 - \frac{r}{16}\frac{x_0}{|x_0|} \in B^0_{r_0}.$$

when $|x_0| > 0$. Otherwise, we take $x_1 = x_0$. Then it is easy to observe that the following hold:

(a) $B^+_{\frac{r}{16}}(x_1) \subset B^+_{r_0}$,
(b) $B^+_{\frac{r}{16}}(x_1) \subset B^+_{\frac{r}{8}}(x_0)$.
(c) $B^+_r(x_1) \cap D_0 \neq \emptyset$.

(a) and (b) are easy consequences of triangle inequality. (c) can be seen as follows. Since $\bar{r} = \text{dist}\{x_0, D_0\}$, therefore there exists $z_0 \in D_0$ such that $|x_0 - z_0| = \bar{r}$. Thus

$$\begin{aligned} |z_0 - x_1| &\le |z_0 - x_0| + |x_0 - x_1| \\ &< \bar{r} + \frac{\bar{r}}{14} = \frac{15\bar{r}}{14} < \frac{16\bar{r}}{14} = r. \end{aligned}$$

This implies that $z_0 \in B^+_r(x_1)$ and hence $z_0 \in B^+_r(x_1) \cap D_0$. Then we observe that the following holds,

(d) $B^+_{4r}(x_1) \subset B^+_1$.

In fact, since $(x_1)_n = 0$, $|x_1| \le r_0$, and $r \le 3r_0$, therefore if $x \in B^+_{4r}(x_1)$, then

$$|x| \le |x - x_1| + |x_1| < 4r + r_0 \le 13r_0 < 1.$$

Therefore in this situation we see that the conditions in H(I) are satisfied and consequently we have

$$\left|B^+_{\frac{r}{16}}(x_1) \cap D_1\right| \ge \sigma_1 \left|B^+_r(x_1)\right|. \tag{4.40}$$

Thus from Eq. 4.40, we find

$$\begin{aligned} \sigma_1|B_r(x_0) \cap B^+_{r_0}| &\le \sigma_1|B^+_r(x_0)| \\ &= \sigma_1|B^+_r(x_1)| \text{ (since the measure is translation invariant)} \\ &\le |B^+_{\frac{r}{16}}(x_1) \cap D_1| \quad \text{(by Eq. 4.40)} \\ &\le |B^+_{\frac{r}{8}}(x_0) \cap B^+_{r_0} \cap D_1| \text{ (by observation } (a), (b)) \\ &\le |B_{\frac{r}{8}}(x_0) \cap B^+_{r_0} \cap D_1| \\ &\le |B_{\frac{r}{4}}(x_0) \cap B^+_{r_0} \cap D_1|. \end{aligned} \tag{4.41}$$

Equation 4.39 thus follows in this case. We now consider Case (ii).

In this case we have $0 < (x_0)_n < \frac{\bar{r}}{14} = \frac{r}{16}$. Let us consider the following shifted point corresponding to x_0.

$$x_1 = \begin{cases} P(x_0) - \frac{\bar{r}}{14}\frac{P(x_0)}{|P(x_0)|}, & \text{if } P(x_0) \neq 0 \\ 0, & \text{if } \ P(x_0) = 0 \end{cases} \tag{4.42}$$

where $P(x_0)$ is the projection of x_0 on $\{x \in \mathbb{R}^n \ | \ x_n = 0\}$. We first note that $(x_1)_n = 0$. Moreover we easily observe that the following hold,

(a') $B^+_{\frac{r}{16}}(x_1) \subset B^+_{r_0}$.
(b') $\overline{B^+_r(x_1) \cap D_0} \neq \emptyset$.
(c') $B^+_{\frac{r}{16}}(x_1) \subset B^+_{\frac{r}{4}}(x_0) \subset B_{\frac{r}{4}}(x_0)$.
(d') $B^+_{4r}(x_1) \subset B^+_{14r_0} \subset B^+_1$ since $r_0 \leq \frac{1}{14}$.

(a'), (c') and (d') follow easily from triangle inequality. (b') can be seen as follows. As in Case i), let $z_0 \in D_0$ be such that $|x_0 - z_0| = \bar{r}$. Then

$$|x_1 - z_0| \leq |x_1 - P(x_0)| + |P(x_0) - x_0| + |x_0 - z_0| \quad < \frac{\bar{r}}{14} + \frac{\bar{r}}{14} + \bar{r} = \frac{8\bar{r}}{7} = r,$$

(b') thus follows.

In view of the observations (a'),(b') and (d') and (HI), we get

$$\left|B^+_{\frac{r}{16}}(x_1) \cap D_1\right| \geq \sigma_1 \left|B^+_r(x_1)\right|. \tag{4.43}$$

We then note that

(a'') $\left|B^+_r(x_1)\right| = \left|B^+_r(P(x_0))\right|$ (because $(x_1)_n = (P(x_0))_n = 0$).
(b'') $|B^+_r(P(x_0))| = |B^+_r(x_0) \cap \{x \mid x_n \geq (x_0)_n\}|$ (because the measure is translation invariant).
(c'') $|B^+_r(x_0) \cap \{x \mid x_n \geq (x_0)_n\}| = \frac{1}{2}|B_r(x_0)|$.

Thus

$$\begin{aligned} \left|B_{\frac{r}{4}}(x_0) \cap B^+_{r_0} \cap D_1\right| &\geq \left|B^+_{\frac{r}{16}}(x_1) \cap D_1\right| \ \big(\text{by } (c')\big) \\ &\geq \sigma_1 \left|B^+_r(x_1)\right| \quad (\text{by Eq. 4.43}) \\ &= \sigma_1 \left|B^+_r(P(x_0))\right| \quad (\text{by } (a'')) \\ &= \frac{\sigma_1}{2}|B_r(x_0)| \quad (\text{by } (b'') \text{ and } (c'')) \\ &\geq \frac{\sigma_1}{2}|B_r(x_0) \cap B^+_{r_0}(0)|. \end{aligned} \tag{4.44}$$

Equation 4.39 thus follows in this case as well.

We now look at Case (iii). In this case similar to that of Case (ii), we consider the following shifted point corresponding to x_0,

$$x_1 = \begin{cases} x_0 - \frac{\bar{r}}{14}\frac{P(x_0)}{|P(x_0)|}, & \text{if } P(x_0) \neq 0 \\ x_0, & \text{if } \ P(x_0) = 0. \end{cases}$$

We then make the following observations.

(e') From the choice of x_1 and the fact $\frac{r}{16} < (x_0)_n = (x_1)_n < r_0 - \frac{r}{16}$, we find that

$$B_{\frac{r}{16}}(x_1) \subset B^+_{r_0} \ \text{ and } \ B_{\frac{r}{16}}(x_1) \subset B_{\frac{r}{4}}(x_0).$$

(f') By arguing as in the previous case, we also have

$$\overline{B_r^+(x_1)} \cap D_0 \neq \emptyset.$$

(h') Likewise we have $B_{4r}^+(x_1) \subset B_{14r_0}^+ \subset B_1^+(0)$.

So in view of above observations (e'), (f') and (h'), we find that the conditions in H(II) are satisfied and consequently we have

$$|B_{\frac{r}{16}}(x_1) \cap D_1| \geq \sigma_2 |B_r^+(x_1)|. \tag{4.45}$$

Now in order to get appropriate measure estimate in terms of ball centered at x_0 instead of x_1, let us also observe that

(d") Since $(x_0)_n = (x_1)_n$, hence

$$|B_r^+(x_1)| = |B_r^+(x_0)|.$$

Therefore, we have

$$\begin{aligned} \left|\left(B_{\frac{r}{4}}(x_0) \cap B_{r_0}^+\right) \cap D_1\right| &\geq \left|B_{\frac{r}{16}}(x_1) \cap D_1\right| \quad \big(\text{by } (e')\big) \\ &\geq \sigma_2 \left|B_r^+(x_1)\right| \quad \big(\text{by Eq. 4.45}\big) \\ &= \sigma_2 \left|B_r^+(x_0)\right| \quad (\text{by } (d")) \\ &\geq \sigma_2 |B_r(x_0) \cap B_{r_0}^+| \end{aligned} \tag{4.46}$$

We finally note that Case (iv) corresponds to the interior case and therefore by repeating the arguments as in [18] (given that H(III) holds) we will have

$$\left|B_{\frac{r}{4}}(x_0) \cap B_{r_0}^+ \cap D_1\right| \geq \sigma_3 \left|B_r(x_0) \cap B_{r_0}^+(x_0)\right|. \tag{4.47}$$

Thus in view of Eqs .4.41, 4.44, 4.46 and 4.47, it is clear that the estimate in Eq. 4.39 follows by letting $\sigma = \min\{\sigma_1/2,\ \sigma_2, \sigma_3\}$.

Now, for every $x \in B_{r_0}^+ \setminus D_0$, we consider the ball centered at x of radius $r := \text{dist}\{x,\ D_0\}$. Then by applying Vitali covering's Lemma to this family, we can extract a sub-family $\{B_{r_j}(x_j)\}$ such that the balls $\left\{B_{\frac{r_j}{3}}(x_j)\right\}$ are disjoint. In particular, $\left\{B_{\frac{r_j}{4}}(x_j)\right\}'$ s are disjoint. Hence,

$$\begin{aligned} \left|B_{r_0}^+ \setminus D_0\right| &\leq \sum_j \left|\left(B_{r_j}(x_j) \cap B_{r_0}^+\right) \setminus D_0\right| \\ &\leq \sigma^{-1} \sum_j \left|\left(B_{\frac{r_j}{4}}(x_j) \cap B_{r_0}^+\right) \cap (D_1 \setminus D_0)\right| \\ &\leq \sigma^{-1} \left|B_{r_0}^+ \cap (D_1 \setminus D_0)\right|. \end{aligned} \tag{4.48}$$

From Eq. 4.48 it follows that,

$$\begin{aligned} \left|B_{r_0}^+ \setminus D_1\right| &= \left|B_{r_0}^+ \setminus D_0\right| - \left|B_{r_0}^+ \cap (D_1 \setminus D_0)\right| \\ &\leq (1-\sigma) \left|B_{r_0}^+ \setminus D_0\right|. \end{aligned} \tag{4.49}$$

This finishes the proof. □

Now, we are ready to prove the main oscillation decay result in this section. Before stating such a result, we make the following remarks.

Remark 4.6 From now on, we let $C = \max\{C_i, C_{ib}, C_b\}$, $c = \min\{c_i, c_{ib}, c_b\}$ and $\mu = \min\{\mu_i, \mu_{ib}, \mu_b\}$, where triplet (C_b, c_b, μ_b), $(C_{ib}, c_{ib}, \mu_{ib})$ and (C_i, c_i, μ_i) are respectively from the Lemmas 4.2, 4.3 and 4.4. It is clear from the proofs that if we replace such triplets in the hypothesis of the respective Lemmas by (C, c, μ) then we get that the concluding inequality holds in all lemmas with A_{Ca} instead of A_{C_ba}, $A_{C_{ib}a}$ and A_{C_ia}.

Remark 4.7 We would also like to remark that from here onwards, we would deal with the following non-homogeneous Neumann boundary value problem,

$$\begin{cases} \tilde{F}(D^2u, Du, x) = f \text{ in } B_1^+, \\ u_{x_n} = g \text{ on } B_1^0. \end{cases} \tag{4.50}$$

Theorem 4.8 *Let $u \in C(B_1^+ \cup B_1^0)$ be a viscosity solution (4.50) where $\tilde{F}$ satisfies the structure conditions SC1)-SC3) and $f \in C(\overline{B_1^+})$. Let λ, Λ and δ be as in SC1)- SC3). Then there exist universal constants $\nu, \epsilon, \rho, \theta \in (0, 1)$ such that if for some δ' satisfying $\delta' \leq \theta\delta$ the following hold,*

$$\begin{cases} \|f\|_{L^\infty(B_1^+)} \leq \epsilon\delta', \\ \|g\|_{L^\infty(B_1^0)} \leq \epsilon\delta', \\ osc_{B_1^+} u \leq \delta' \end{cases} \tag{4.51}$$

then

$$osc_{B_\rho^+} u \leq (1 - \nu)\delta'. \tag{4.52}$$

Proof We closely follow the ideas as in the proof of Proposition 2.2 in [18] with suitable modifications in our situation. Let c_1 be the constant from Lemma 4.1, when the fully nonlinear operator $\tilde{F}$ under consideration is uniformly elliptic with ellipticity constants λ, Λ in the region $p \in B_{\frac{\delta}{2}}$ instead of B_δ. Also we fix r_0 sufficiently small so that Lemma 4.5 holds and then let $r_1 = \frac{r_0}{16}$. Let $\nu < \frac{1}{6}$ and $\mathfrak{N}$ be universal constants to be chosen later such that additionally the following is satisfied,

$$\mathfrak{N}\nu << 1. \tag{4.53}$$

Let us set

$$a = \mathfrak{N}\nu\delta' \text{ and } m = \inf_{B_1^+} u. \tag{4.54}$$

Suppose that there exists $x_0 \in B^+_{\frac{r_0}{2}}$ such that

Assertion A:

$$u(x_0) + \|g\|_{L^\infty(B_1^0)} - m < \frac{3}{2}\nu\delta' \tag{4.55}$$

as well as

$$\sup_{B_{r_1}^+} u - \|g\|_{L^\infty(B_1^0)} - m > \frac{\delta'}{2}. \tag{4.56}$$

We now make the following claim.

Claim The *Assertion A* is false, i.e. both the inequalities (4.55), (4.56) cannot hold at the same time.

Subsequently we show that this leads to the validity of the oscillation decay as asserted in Eq. 4.52 above.

In order to prove the claim we assume on the contrary that both the inequalities are correct and then derive a contradiction.

Let us consider the following function

$$w = u - \|g\|_{L^\infty(B_1^0)} x_n. \tag{4.57}$$

Then we note that w satisfies the following differential inequality in the viscosity sense

$$\begin{cases} F_1(D^2 w, Dw, x) \le f \text{ in } B_1^+, \\ w_{x_n} \le 0 \text{ on } B_1^0, \end{cases} \tag{4.58}$$

where $F_1(M, p, x) = \tilde{F}(M, p + \|g\|_{L^\infty(B_1^0)} e_n, x)$ and $e_n = (0, 0, ..., 1)$. We have assumed that $\|g\|_{L^\infty(B_1^0)} \le \epsilon\delta'$ so that if we choose $\epsilon < \frac{\nu}{2} \le \frac{1}{2}$, then we have that

$$\|g\|_{L^\infty(B_1^0)} \le \frac{\delta'}{2} \le \frac{\theta\delta}{2} \le \frac{\delta}{2} \ \big(\text{since } \ \theta \in (0, 1)\big).$$

Consequently, F_1 is uniformly elliptic with the same ellipticity constant provided $p \in B_{\frac{\delta}{2}}$.

Let us then consider the non-negative function

$$v = u - m + (1 - x_n)\|g\|_{L^\infty(B_1^0)}. \tag{4.59}$$

It is easy to observe that v satisfies (4.58) in the viscosity sense because it differs from w by a constant. We now let $\tilde{A}_a$ to be the set of points in B_1^+, where v is bounded above by a and can be touched by a paraboloid of opening $-a$ with vertex in B_1^+.

Step 1: We first show that given any $\eta > 0$ sufficiently small depending on r_1, the following estimate holds

$$\left|B_{r_0}^+ \cap \tilde{A}_a\right| > \frac{c_0}{2} \left|B_{r_1}^+ \cap \{y \mid y_n > \eta\}\right|, \tag{4.60}$$

with c_0 being independent of η.

In order to prove the claim, for every $y \in B_{r_1}^+ \cap \{y \mid y_n > \eta\}$ let us consider the following paraboloid

$$P_y(x) = \frac{a}{2}\left[(r_0 - r_1)^2 - |x - y|^2\right].$$

Since given x for which $|x| \ge r_0$, we have that $|x - y| \ge |x| - |y| \ge r_0 - r_1$, therefore $P_y(x) \le 0 \le v$ for all $x \in \{z : 1 > |z| \ge r_0\} \cap \{z_n > 0\}$.

On the other hand, for all $x \in B_{\frac{r_0}{2}}^+$, we find that $|x - y| \le |x| + |y| \le \frac{r_0}{2} + r_1$. Thus

$$\begin{cases} P_y(x) \ge \dfrac{a}{2}\left[(r_0 - r_1)^2 - \left(\dfrac{r_0}{2} + r_1\right)^2\right] \\ \quad = \dfrac{\mathfrak{N}\nu\delta' r_0^2}{2}\left[\left(\dfrac{15}{16}\right)^2 - \left(\dfrac{9}{16}\right)^2\right] \\ \quad > \dfrac{3\nu\delta'}{2} \\ \quad \ge u(x_0) - m + (1 - (x_0)_n)\|g\|_{L^\infty(B_1^0)} = v(x_0) \text{ (by Eq. 4.55)}, \end{cases} \tag{4.61}$$

where in the second line above, we have chosen $\mathfrak{N}$ sufficiently large so that the third step in Eq. 4.61 above follows. Since (4.61) holds for $x \in B_{\frac{r_0}{2}}^+$, therefore, in particular, $P_y(x_0) > \frac{3\nu\delta'}{2}$.

Note also that $P_y(x) \le a$ for all $x, y \in B_1^+$. Let us now slide the paraboloids P_y from below till it touches the function v for the first time. Let $\tilde{A}$ denotes the set of contact points as y varies in $B_{r_1}^+ \cap \{y_n > \eta\}$. Since the function v satisfies (4.58), therefore P_y will not touch the function at any $\tilde{x} \in B_1^0$. Otherwise by our choice of y, we would get

$$\begin{cases} 0 \ge \partial_{x_n}(P_y)(\tilde{x}) \text{ (because } v \text{ satisfies Eq. 4.58)} \\ = a(y - \tilde{x})_n \\ = a y_n \ge a\eta > 0 \ \big(\text{by the choice of } y\big), \end{cases} \tag{4.62}$$

which is a contradiction. Therefore, in view of the above observations, we can infer that all contact points $\{\tilde{x}\}'s$ lie inside $B_{r_0}^+$. Moreover thanks to Eq. 4.61, the following holds:

$$\begin{cases} 0 > v(x_0) - \frac{3}{2}\nu\delta' \\ \ge v(x_0) - P_y(x_0) \quad \text{(by Eq. 4.61)} \\ \ge \min_{z \in B_1^+} \{v(z) - P_y(z)\} \\ = v(\tilde{x}) - P_y(\tilde{x}) \text{ (for a contact point } \tilde{x}) \\ \ge v(\tilde{x}) - a \ \big(\text{since } (P_y(x) \le a)\big). \end{cases} \tag{4.63}$$

This implies that $\tilde{A} \subset \tilde{A}_a \cap B_{r_0}^+$. Thus by applying Lemma 4.1 with $B = \overline{B_{r_1}^+} \cap \{z_n \ge \eta\}$, we obtain

$$\begin{cases} \left|B_{r_0}^+ \cap \tilde{A}_a\right| \ge |\tilde{A}| \\ \ge c_1 \left|B_{r_1}^+ \cap \{y_n > \eta\}\right| - \frac{\|f\|^n_{L^\infty(B_1^+)}}{a^n}|\tilde{A}| \\ \ge c_1 \left|B_{r_1}^+ \cap \{y_n > \eta\}\right| - \frac{\epsilon^n}{\mathfrak{N}^n \nu^n}|\tilde{A}| \text{ (using Eqs 4.51 and 4.54)} \\ \ge c_1 \left|B_{r_1}^+ \cap \{y_n > \eta\}\right| - \frac{\epsilon^n}{\mathfrak{N}^n \nu^n}|B_{r_0}^+ \cap \tilde{A}_a|. \end{cases} \tag{4.64}$$

Now, by choosing $\epsilon > 0$ sufficiently small such that

$$\frac{\epsilon^n}{\mathfrak{N}^n \nu^n} < \frac{1}{2}, \tag{4.65}$$

we obtain (4.60) with $c_0 = \frac{c_1}{2}$. This finishes the proof of Step 1.

Step 2: We now show that there exists $\tilde{\sigma} \in (0, 1)$ and $\tilde{C} > 0$ such that the following estimate holds

$$\left|B_{r_0}^+ \setminus \tilde{A}_{a\tilde{C}^{k_0}}\right| \le (1 - \tilde{\sigma})^{k_0} \left|B_{r_0}^+\right|, \tag{4.66}$$

provided $\tilde{C}^{k_0+1} a \le \frac{\delta}{2}$. From Eq. 4.60, we find that

$$B_{r_0}^+ \cap \tilde{A}_a \ne \emptyset. \tag{4.67}$$

It is also clear that since the sets $\tilde{A}_{a\tilde{C}^k}$ are increasing with respect to k, therefore,

$$B_{r_0}^+ \cap \tilde{A}_{a\tilde{C}^k} \ne \emptyset \text{ for all } k \in \mathbb{N}, \tag{4.68}$$

where $\tilde{C}$ is the constant as in Remark 4.6 corresponding to $\delta/2$ instead of δ. Note that the hypothesis of the Lemmas 4.2, 4.3 and 4.4 are satisfied with $\tilde{C}^k a$ instead of a as long as $a\tilde{C}^{k+1} \le \frac{\delta}{2}$.

Thus that for every $k \in \mathbb{N}$, satisfying $a\tilde{C}^{k+1} \leq \delta/2$ we can apply Lemma 4.5 to the closed sets

$$D_0 = \overline{B_{r_0}^+} \cap \tilde{A}_{a\tilde{C}^k} \text{ and } D_1 = \overline{B_{r_0}^+} \cap \tilde{A}_{a\tilde{C}^{k+1}}, \tag{4.69}$$

to assert that

$$\left|B_{r_0}^+ \setminus \tilde{A}_{a\tilde{C}^{k+1}}\right| \leq (1-\tilde{\sigma}) \left|B_{r_0}^+ \setminus \tilde{A}_{a\tilde{C}^k}\right|. \tag{4.70}$$

Proceeding inductively, we obtain

$$\left|B_{r_0}^+ \setminus \tilde{A}_{a\tilde{C}^k}\right| \leq (1-\tilde{\sigma})^k \left|B_{r_0}^+\right|, \tag{4.71}$$

which completes the proof of Step 2.

Step 3: We now define the following set

$$E = \left\{x \in B_{r_0}^+ \mid u(x) - m + (x_n - 1)\|g\|_{L^\infty(B_1^0)} > \frac{\delta'}{4}\right\}. \tag{4.72}$$

Then we claim that the following estimate holds for any $\eta > 0$ sufficiently small,

$$|E| \geq \frac{c_1}{2}\left|B_{r_1}^+ \cap \{y_n > \eta\}\right|, \tag{4.73}$$

where c_1 is the constant from Lemma 4.1, when the operator under consideration is uniformly elliptic for $|p| < \delta/2$.

In order to prove (4.73), for each $y \in B_{r_1}^+ \cap \{y_n > \eta\}$, we consider the following paraboloid

$$S_y(x) = \frac{\delta'}{(r_0 - r_1)^2}|x-y|^2 + \frac{\delta'}{4}. \tag{4.74}$$

By using the fact that $r_1 = r_0/16$, it is easy to observe that for all $x, y \in B_{r_1}^+$, we have

$$S_y(x) \leq \frac{\delta'}{2}. \tag{4.75}$$

Now using (4.56), we find

$$\sup_{B_{r_1}^+} S_y(x) \leq \frac{\delta'}{2} < \sup_{B_{r_1}^+} u - \|g\|_{L^\infty(B_1^0)} - m \leq \sup_{B_{r_1}^+}\left(u + x_n\|g\|_{L^\infty(B_1^0)}\right) - \|g\|_{L^\infty(B_1^0)} - m. \tag{4.76}$$

On the other hand for $x \in \{x \mid |x| \geq r_0\} \cap \{x_n > 0\}$ since $S_y(x) > \delta'$, therefore by Eq. 4.51, we have

$$\begin{cases} S_y(x) > \delta' \geq u(x) - m \quad \text{(by 4.51 and from the definition of } m \text{ as in 4.54)} \\ \qquad \geq u(x) - m + (x_n - 1)\|g\|_{L^\infty(B_1^0)} \quad \left(\text{since } (x_n - 1)\|g\|_{L^\infty(B_1^0)} \leq 0\right). \end{cases} \tag{4.77}$$

Also for any $\bar{x} \in B_1^0$ and $y \in B_{r_1}^+ \cap \{y_n > \eta\}$, we observe that

$$\partial_{x_n}(S_y)(\bar{x}) = \frac{-2\delta' y_n}{(r_0 - r_1)^2} < 0. \tag{4.78}$$

We now let

$$\tilde{v} = u + (x_n - 1)\|g\|_{L^\infty(B_1^0)} - m. \tag{4.79}$$

Then we observe that $\tilde{v}$ satisfies the following differential inequalities in the viscosity sense

$$\begin{cases} F_2\left(D^2\tilde{v}, D\tilde{v}, x\right) \geq f \text{ in } B_1^+, \\ \tilde{v}_{x_n} \geq 0 \text{ on } B_1^0, \end{cases} \tag{4.80}$$

where $F_2(X, p, x) = \tilde{F}(X, p - \|g\|_{L^\infty(B_1^0)} e_n, x)$, which is again uniformly elliptic as long as $p \in B_{\frac{\delta}{2}}$.

Now we slide the paraboloids S_y from above until it touches the graph of $\tilde{v}$. In view of Eqs. 4.75, 4.76, 4.77 and 4.80, all contact points lie inside $B_{r_0}^+$. We denote by K the set of all contact points as y varies inside $B_{r_1}^+ \cap \{y_n > \eta\}$. We now apply Lemma 4.1 from "above" to $\tilde{v}$, i.e. more precisely, we apply that lemma to the function $-\tilde{v}$ which is touched from below by $-S_y(x)$. Note that in this case we have that $a = \frac{2\delta'}{(r_0 - r_1)^2} \leq \frac{2\theta\delta}{(r_0 - r_1)^2}$ since $\delta' \leq \theta\delta$. Therefore, if θ is chosen sufficiently small then we can ensure that $0 < a < \frac{\delta}{4}$. We then observe that $-\tilde{v}$ satisfies the following inequalities

$$\begin{cases} G(D^2(-\tilde{v}), D(-\tilde{v}), x) \leq -f \text{ in } B_1^+, \\ (-\tilde{v})_{x_n} \leq 0 \text{ on } B_1^0, \end{cases} \tag{4.81}$$

in the viscosity sense, where $G(X, p, x) = -F_2(-X, -p) = -\tilde{F}(-X, -p - \|g\|_{L^\infty(B_1^0)} e_n, x)$, which is again uniformly elliptic for $p \in B_{\frac{\delta}{2}}$. Therefore by applying Lemma 4.1, we get

$$\begin{cases} |K| \geq c_1 |B_{r_1}^+ \cap \{y_n > \eta\}| - \dfrac{\|f\|^n_{L^\infty(B_1^+)}}{a^n} |K| \\ \quad \geq c_1 |B_{r_1}^+ \cap \{y_n > \eta\}| - |K| \dfrac{\epsilon^n}{\mathfrak{N}^n \nu^n}. \end{cases} \tag{4.82}$$

At this point by using (4.65) we obtain the following estimate

$$|K| \geq \frac{c_1}{2} |B_{r_1}^+ \cap \{y_n > \eta\}|. \tag{4.83}$$

Now we note that because of Eq. 4.76, at any contact point $x \in K$, we have $\tilde{v} \geq \frac{\delta'}{4}$ and therefore $K \subset E$. Consequently, we can assert that Eq. 4.73 holds. This completes the proof of Step 3.

Step 4: (*Conclusion*.)

Let $k_0 \in \mathbb{N}$ be the largest integer such that $\tilde{C}^{k_0+1} a \leq \frac{\delta'}{4}$. Now since $\delta' \leq \delta$, so by using the estimate (4.66) in Step 2 we have

$$|B_{r_0}^+ \setminus \tilde{A}_{a\tilde{C}^{k_0}}| \leq (1 - \tilde{\sigma})^{k_0} |B_{r_0}^+|. \tag{4.84}$$

Now for $x \in B_1^+$, we make the crucial observation that the following inclusion holds:

$$\begin{cases} E = \left\{ x \in B_{r_0}^+ \ |\tilde{v}(x) > \dfrac{\delta'}{4} \right\} \\ \subset \left\{ x \in B_{r_0}^+ \ | \ v(x) > \dfrac{\delta'}{4} \right\} (\text{since } v \geq \tilde{v}) \\ \subset \left\{ x \in B_{r_0}^+ \ | \ v(x) > a\tilde{C}^{k_0} \right\} \left(\text{since } a\tilde{C}^{k_0} < \dfrac{\delta'}{4} \right) \\ \subset B_{r_0}^+ \setminus \tilde{A}_{a\tilde{C}^{k_0}} \left(\text{by definition of } \tilde{A}_{a\tilde{C}^{k_0}} \right). \end{cases} \tag{4.85}$$

Using Eqs. 4.73, 4.84 and 4.85, we have

$$\frac{c_1}{2} |B_{r_1}^+ \cap \{y_n > \eta\}| \leq |E| \leq (1 - \tilde{\sigma})^{k_0} |B_{r_0}^+|. \tag{4.86}$$

Now letting $\eta \to 0$, we obtain

$$\frac{c_1}{2}|B^+_{r_1}| \leq |E| \leq (1-\tilde{\sigma})^{k_0}|B^+_{r_0}|. \tag{4.87}$$

Now note that using $a = \mathfrak{N}\nu\delta'$, we have that

$$k_0 \sim |\log_{\tilde{C}}(\mathfrak{N}\nu)|. \tag{4.88}$$

At this point we first let $\mathfrak{N}$ large enough so that all previous arguments apply. Subsequently if ν is chosen small enough, then thanks to Eq. 4.88, we have that k_0 becomes too large so that Eq. 4.87 is violated (note that $r_1 = \frac{r_0}{16}$). This leads to a contradiction.

Note that we can accordingly choose ϵ sufficiently small such that Eq. 4.65 holds as well.

Therefore, we finally obtain that for appropriately chosen $\mathfrak{N}, \nu, \epsilon$ as above, either Eqs. 4.55 or 4.56 fails. Suppose first that Eq. 4.55 fails. Then since $\|g\|_{L^\infty(B^0_1)} \leq \delta'\epsilon < \frac{\delta'\nu}{2}$ (by our choice of ϵ), therefore we have;

$$u(x) - m \geq \nu\delta' \quad \text{for all} \quad x \in B^+_{r_1},$$

where we also use the fact that $r_1 < r_0/2$. Consequently, Eq. 4.52 follows with $\rho = r_1$. Now, suppose instead that Eq. 4.56 fails. Then in this case we have that

$$\sup_{B^+_{r_1}} u - \|g\|_{L^\infty(B^0_1)} - m \leq \frac{\delta'}{2},$$

that is,

$$\sup_{B^+_{r_1}} u - m \leq \frac{2\delta'}{3},$$

since $\|g\|_{L^\infty(B^0_1)} < \frac{\nu\delta'}{2}$ and $\nu < 1/3$. Thus, Eq. 4.52 again follows in view of the fact that $\frac{2}{3} < (1-\nu)$. This finishes the proof of the theorem. □

As a consequence of Theorem 4.8, we also have the following rescaled boundary oscillation estimate whose proof is identical to that of Theorem 2.1 in [18].

Theorem 4.9 *With $\tilde{F}, u, f, g$ as in Theorem 4.8, we have that there exists universal $\nu, \kappa, \epsilon, \rho \in (0,1)$ such that if $\delta' > 0$ and $k \in \mathbb{N}$ satisfy*

$$\begin{cases} osc_{B^+_1} u \leq \delta' \leq \rho^k \kappa \delta, \\ ||f||_{L^\infty(B^+_1)} \leq \epsilon\delta', \\ ||g||_{L^\infty(B^0_1)} \leq \epsilon\delta', \end{cases} \tag{4.89}$$

then

$$osc_{B^+_{\rho^s}} u \leq (1-\nu)^s\delta', \ for\ s = 0, ..., k+1. \tag{4.90}$$

5 Improvement of Flatness and the Proof of our Main Result

We now establish our main result Theorem 2.1 using the non perturbative Hölder estimates proved in Sections 3 and 4. We first show how to reduce the considerations to flat boundary conditions.

5.1 Reduction to Flat Boundary Conditions

Since $\Omega \in C^2$, we can flatten the boundary using coordinates which employs the distance function to the boundary $\partial\Omega$. See for instance Lemma 14.16 in [19] or the Appendix in [9]. We crucially note that such coordinates preserve the Neumann boundary conditions unlike standard flattening which changes Neumann conditions to oblique derivative conditions in general. Consequently, without loss of generality, we may consider the following flat boundary value problem

$$\begin{cases} \langle A(x)Du, Du\rangle^{\beta/2} F\left(D^2u, Du, x\right) = f \text{ in } B_1^+, \\ u_{x_n} = g \text{ on } B_1^0, \end{cases} \tag{5.1}$$

where A is a uniformly elliptic positive definite matrix with Lipschitz coefficients. Moreover such a transformation ensures that the resulting F is uniformly elliptic in D^2u and Lipschitz in Du. Without loss of generality, we will also assume that $\beta > 0$ since the case $\beta = 0$ is classical.

5.2 Improvement of Flatness

We first state and prove a compactness result for a perturbed variant of Eq. 5.1. This can be regarded as the boundary analogue of Lemma 4.2 in [18].

Lemma 5.1 *Let u be such that $|u| \leq 1$ and is a viscosity solution to the following Neumann problem,*

$$\begin{cases} \langle A(x)(Du+p), (Du+p)\rangle^{\beta/2} F\left(D^2u, Du, x\right) = f \textit{ in } B_1^+, \\ u_{x_n} = g \textit{ on } B_1^0, \end{cases} \tag{5.2}$$

where $p \in \mathbb{R}^n$, A is Lipschitz and uniformly elliptic and F is uniformly elliptic in M with ellipticity bounds λ and Λ, Lipschitz in the gradient variable q and continuous in x with a modulus of continuity ω. Also suppose $|F(0,0,0)| \leq 1$. Furthermore, assume that $f \in C(\overline{B_1^+})$, $||f||_{L^\infty(B_1^+)} \leq 1$ and $g \in C^{\alpha_0}\left(B_1^0\right)$ with $||g||_{C^{\alpha_0}} \leq 1$. Then given $\epsilon' > 0$, there exists $L = L(\epsilon') > 0$, such that if $|p| > L$, $|D_qF| \leq \frac{1}{L}$, then there exists $v \in C^{1,\alpha'}\left(\overline{B_{1/2}^+}\right)$ for some α' universal (with a universal $C^{1,\alpha'}$ estimate) such that

$$||u - v||_{L^\infty\left(B_{1/2}^+\right)} \leq \epsilon'. \tag{5.3}$$

Proof We first note that the Eq. 5.29 can be rewritten as

$$\begin{cases} \langle A(x)\left(\frac{Du}{|p|} + e\right), \frac{Du}{|p|} + e\rangle^{\beta/2} F\left(D^2u, Du, x\right) = \frac{f}{|p|^\beta} \text{ in } B_1^+, \\ u_{x_n} = g \text{ on } B_1^0, \end{cases} \tag{5.4}$$

where $e = \frac{p}{|p|}$. Therefore, we see that u satisfies a uniformly elliptic PDE when $|Du| \leq \frac{|p|}{2}$. Suppose on the contrary, the assertion is not true. Then there exist an $\epsilon_0 > 0$ and a sequence of $u_k's$, $p_k's$, $F_k's$, $f_k's$, $g_k's$ with $|u_k| \leq 1$, $|f_k| \leq 1$, $||g_k||_{C^{\alpha_0}} \leq 1$, $|D_qF_k| \leq \frac{1}{k}$, $|p_k| > k$, such that $F_k's$ have the same ellipticity bounds λ, Λ, are equicontinuous in x with modulus ω and u_k solves the following problem:

$$\begin{cases} \langle A(x)\left(\frac{Du_k}{|p_k|}+e_k\right), \frac{Du_k}{|p_k|}+e_k\rangle^{\beta/2} F_k\left(D^2u_k, Du_k, x\right) = \frac{f_k}{|p_k|^\beta} \text{ in } B_1^+,\ e_k = \frac{p_k}{|p_k|} \\ (u_k)_{x_n} = g_k \text{ on } B_1^0 \end{cases} \tag{5.5}$$

and such that u_k's are not ϵ_0 close to any $v \in C^{1,\alpha'}(\overline{B_{1/2}^+})$. We now rewrite the first equation in Eq. 5.5 as follows:

$$\langle A(x)\left(\frac{Du_k}{|p_k|}+e_k\right), \frac{Du_k}{|p_k|}+e_k\rangle^{\beta/2}\left(F_k\left(D^2u_k, Du_k, x\right) - F_k(0, Du_k, x)\right) = \frac{f_k}{|p_k|^\beta} \tag{5.6}$$
$$-\langle A(x)\left(\frac{Du_k}{|p_k|}+e_k\right), \frac{Du_k}{|p_k|}+e_k\rangle^{\beta/2} F_k(0, Du_k, x) \text{ in } B_1^+.$$

Now, notice that the operators in Eq. 5.6 above satisfy the structural assumptions SC1)-SC3) as in Section 4 and are uniformly elliptic for $|Du_k| \leq \frac{|p_k|}{2}$. Before proceeding further, we make the following important discursive remark.

Remark 5.2 Over here, the reader should note that the reason as to why we subtract off $F_k(0, Du_k, x)$ is to ensure that SC2) holds. Note that even if we start with F satisfying SC2), after flattening such a condition is not necessarily preserved.

Now similar to the proof of Lemma 4.2 in [18], we look at the following rescaled functions

$$w_k(x) = \theta_k(u_k(x) - u(0)), \tag{5.7}$$

where

$$\theta_k = \max\left\{\frac{1}{|p_k|}, ||D_q F_k||\right\}. \tag{5.8}$$

Then, it follows that w_k solves:

$$\langle A(x)\left(\frac{Dw_k}{\theta_k|p_k|}+e_k\right), \frac{Dw_k}{\theta_k|p_k|}+e_k\rangle^{\beta/2}\theta_k(F_k\left(D^2w_k/\theta_k, Dw_k/\theta_k, x\right) - F_k(0, Dw_k/\theta_k, x))$$
$$= \theta_k\frac{f_k}{|p_k|^\beta} - \theta_k\langle A(x)\left(\frac{Dw_k}{\theta_k|p_k|}+e_k\right), \frac{Dw_k}{\theta_k|p_k|}+e_k\rangle^{\beta/2} F_k(0, Dw_k/\theta_k, x) \text{ in } B_1^+. \tag{5.9}$$

Moreover, w_k satisfies in the viscosity sense the Neumann condition $(w_k)_{x_n} = \theta_k g_k$. Also from Eq. 5.9 it follows that w_k solves a degenerate elliptic problem which is uniformly elliptic independent of k when $|Dw_k| \leq 1/2 = \delta$. Now let $\rho, \kappa, \epsilon, \nu$ be as in Theorem 4.9 corresponding to $\delta = \frac{1}{2}$. In the region of uniform ellipticity it is easily seen that the scalar term

$$\tilde{f}_k = \theta_k\frac{f_k}{|p_k|^\beta}$$
$$-\theta_k\langle A(x)\left(\frac{Dw_k}{\theta_k|p_k|}+e_k\right), \frac{Dw_k}{\theta_k|p_k|}+e_k\rangle^{\beta/2} F_k(0, Dw_k/\theta_k, x) \tag{5.10}$$

satisfies $|\tilde{f}_k| \leq C_0\theta_k$. This follows from the expression of θ_k as in Eq. 5.8. Likewise, we have that $|\theta_k g_k| \leq \theta_k$. We now let $\delta' = \frac{C_0\theta_k}{\epsilon}$. For a given k, let m_k be the largest integer such that

$$\frac{C_0\theta_k}{\epsilon} \leq \rho^{m_k}\kappa\delta.$$

Note that $m_k \to \infty$ as $k \to \infty$. Then it follows from the estimate in Theorem 4.9 that

$$\operatorname{osc}_{B^+_{\rho^s}} w_k \leq C(1-\nu)^s \theta_k, \ \ s = 1, ..., m_k.$$

Scaling back to u_k we obtain

$$|u_k(x) - u_k(0)| \leq C|x|^\alpha \text{ as long as } |x| \geq \rho^{m_k},$$

where $\alpha = -\log_\rho(1-\nu)$. Likewise one has a similar Hölder estimate at every boundary point in $B^0_{3/4}$. The interior version of such estimates follows from [18]. This is enough to show that $\{u_k\}$;s are equicontinuous up to $\{x_n = 0\}$ and consequently Arzela-Ascoli can be applied. Therefore, there exists a subsequence which we still denote by $\{u_k\}$ which converges in $\overline{B^+_{3/4}}$ to some v_0. By passing to another subsequence, we can also assume that $F_k \to F_0$ which has the same ellipticity bounds and is independent of q (since $D_q F_k \to 0$), $e_k \to e_0$ with $|e_0| = 1$ and also $g_k \to g_0$ in C^{α_0}. In a standard way, one can show that since $\frac{f_k}{|p_k|^\beta} \to 0$, therefore v_0 is a viscosity solution to

$$\begin{cases} \langle A(x)e_0, e_0\rangle^{\beta/2} F_0\left(D^2 v_0, x\right) = 0 \ \text{ in } B^+_{3/4}, \\ (v_0)_{x_n} = g_0 \text{ on } B^0_{3/4}. \end{cases} \tag{5.11}$$

For relevant stability results, we refer to Proposition 2.1 in [24]. Now since $\langle A(x)e_0, e_0\rangle^{\beta/2} > 0$, therefore, we can conclude that v_0 is a solution to

$$\begin{cases} F_0\left(D^2 v_0, x\right) = 0 \text{ in } B^+_{3/4}, \\ (v_0)_{x_n} = g_0 \text{ on } B^0_{3/4}. \end{cases} \tag{5.12}$$

Now, from the regularity results in [26], it follows that $v_0 \in C^{1,\alpha'}\left(\overline{B^+_{1/2}}\right)$ for some $\alpha' > 0$ with universal bounds which immediately leads to a contradiction for large enough $k's$. □

Before we state and prove the improvement of flatness result for the perturbed equations as in Lemma 5.1, we first introduce a few universal parameters. Let

$$F(M, q, x) : S(n) \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},$$

such that F is uniformly elliptic in M with ellipticity constants λ, Λ, Lipschitz in q with Lipschitz bound say 1 and continuous in x with some modulus of continuity ω. Also assume that $|F(0, 0, 0)| \leq 1$. Let $\alpha', C > 0$ be universal constants such that the following estimate holds

$$||w||_{C^{1,\alpha'}\left(\overline{B^+_{1/2}}\right)} \leq C, \tag{5.13}$$

for any w which is a viscosity solution to the following problem:

$$\begin{cases} F\left(D^2 w, Dw, x\right) = 0 \text{ in } \ B^+_{3/4}, \\ |w| \leq 1, \\ w_{x_n} = g \text{ on } \ B^0_{3/4} \quad \text{and} \quad ||g||_{C^{\alpha_0}} \leq 1 \text{ for some fixed } \alpha_0 > 0. \end{cases} \tag{5.14}$$

The existence of such α', C follows from the regularity results in [26]. We also note that from Eq. 5.13, the following estimate can be deduced,

$$|w(x) - \tilde{L}(x)| \leq C|x|^{1+\alpha'}, \tag{5.15}$$

where $\tilde{L}$ is the affine approximation of w at 0. We now state the relevant improvement of flatness result when $|p|$ is large.

Lemma 5.3 *With u, A, f, g, p, F as in Lemma 5.1, there exist universal $\epsilon_0 > 0$, $r \in (0,1)$ and $\alpha > 0$ such that if $|p| > L(\epsilon_0)$, $|D_q F| \le \frac{1}{L(\epsilon_0)}$, then there exists an affine function $\tilde{L}$ with universal bounds as in* Eq. 5.13 *such that*

$$||u - \tilde{L}||_{L^\infty(B_r^+)} \le r^{1+\alpha}. \tag{5.16}$$

Proof From Lemma 5.1, we have that given $\epsilon' > 0$, there exists $L(\epsilon') > 0$ such that if $|p| > L(\epsilon')$, $|D_q F| \le \frac{1}{L(\epsilon')}$, then there exists v which is a solution to an equation of the type (5.14) such that

$$||u - v||_{L^\infty\left(B_{1/2}^+\right)} \le \epsilon'. \tag{5.17}$$

Now from Eq. 5.15, we have

$$|v(x) - \tilde{L}(x)| \le C|x|^{1+\alpha'}. \tag{5.18}$$

where $\tilde{L}$ is the affine approximation of v at 0. We first choose

$$\alpha < \min\left\{\alpha_0, \alpha', \frac{1}{1+\beta}\right\}. \tag{5.19}$$

Subsequently we choose r small enough such that

$$Cr^{1+\alpha'} \le \frac{r^{\alpha+1}}{2}, \tag{5.20}$$

where C, α' are as in Eq. 5.15. Finally we let $\epsilon_0 = \epsilon' = \frac{r^{\alpha+1}}{2}$. Therefore, the desired estimate in Eq. 5.16 follows from Eqs. 5.17-5.20 by an application of triangle inequality provided $|p| > L(\epsilon_0)$ and $|D_q F| \le \frac{1}{L(\epsilon_0)}$. □

Before, proceeding further, we make the following important remark.

Remark 5.4 We note that although in the proof of Lemma 5.3, one only needs to take $\alpha < \alpha'$, however for subsequent iterative arguments which involves rescaling, we have to additionally ensure that $\alpha < \min\left\{\alpha_0, \frac{1}{1+\beta}\right\}$.

We now have the analogous improvement of flatness result when $|p| \le L(\epsilon_0)$.

Lemma 5.5 *Let u such that $|u| \le 1$ be a viscosity solution to* Eq. 5.29 *where $|p| \le L(\epsilon_0)$ with ϵ_0 as in Lemma 5.3. Then there exists $\eta > 0$ such that if $||f||_{L^\infty}, ||g||_{C^{\alpha_0}} \le \eta$, then there exists an affine function $\tilde{L} = \tilde{a} + <\tilde{b}, x>$ $\left(, \tilde{a} \in \mathbb{R}, \tilde{b} \in \mathbb{R}^n\right)$ with universal bounds such that*

$$||u - \tilde{L}||_{L^\infty(B_r^+)} \le r^{1+\alpha}, \tag{5.21}$$

where r, α are as in Lemma 5.3. Moreover we also additionally have that

$$<\tilde{b}, e_n> = 0 \tag{5.22}$$

Proof Step 1:. We first show that given $\epsilon > 0$, there exists $\eta = \eta(\epsilon) > 0$, such that if $||f||_{L^\infty}, ||g||_{C^\alpha} \le \eta$, then there exists a function v which solves

$$\begin{cases} F\left(D^2 v, Dv, x\right) = 0 \text{ in } B_{3/4}^+, \\ v_{x_n} = 0 \text{ on } B_{3/4}^0, \end{cases} \tag{5.23}$$

and

$$||v-u||_{L^\infty\left(B^+_{1/2}\right)}\le \epsilon.$$

If not, then there exists $\epsilon > 0$ for which the assertion is violated for a sequence u_k, f_k, g_k, p_k such that $f_k, g_k \to 0$, $|p_k| \le L(\epsilon_0)$ and where u_k solves the following problem

$$\begin{cases}\langle A(x)(Du_k+p_k), Du_k+p_k\rangle^{\beta/2} F\left(D^2u_k, Du_k, x\right) = f_k \text{ in } B^+_1,\\ (u_k)_{x_n} = g_k \text{ on } B^0_1.\end{cases} \tag{5.24}$$

Now, since $|p_k|\le L(\epsilon_0)$, we find that the equation is uniformly elliptic when $|Du_k| > 2L(\epsilon_0)$ (say in the viscosity sense). We also note that Eq. 5.24 can be rewritten as:

$$\begin{cases}F\left(D^2u_k, Du_k, x\right) = \frac{f_k}{\langle A(x)(Du_k+p_k), Du_k+p_k\rangle^{\beta/2}} \text{ in } B^+_1,\\ (u_k)_{x_n} = g_k \text{ on } B^0_1,\end{cases} \tag{5.25}$$

where for $|Du_k| > 2L(\epsilon_0)$, one has

$$\frac{|f_k|}{\langle A(x)(Du_k+p_k), Du_k+p_k\rangle^{\beta/2}} \le \frac{|f_k|}{\langle A(x)L(\epsilon_0), L(\epsilon_0)\rangle^{\beta/2}} \to 0 \text{ as } k\to\infty.$$

Consequently, from the uniform boundary Hölder estimates as in Theorem 3.9, we have that up to a subsequence, $u_k \to v_0$ in $\overline{B^+_{3/4}}$, $p_k \to p_0$ such that v_0 is a viscosity solution to

$$\begin{cases}\langle A(x)(Dv_0+p_0), Dv_0+p_0\rangle^{\beta/2} F\left(D^2v_0, Dv_0, x\right) = 0 \text{ in } B^+_{3/4},\\ (v_0)_{x_n} = 0 \text{ on } B^0_{3/4}.\end{cases} \tag{5.26}$$

Such a stability result follows from an argument as in Proposition 2.1 in [24]. Now, by arguing as in the proof of Lemma 6 in [22], we can assert that v_0 in fact solves

$$\begin{cases}F\left(D^2v_0, Dv_0, x\right) = 0 \text{ in } B^+_{3/4},\\ (v_0)_{x_n} = 0 \text{ on } B^0_{3/4}.\end{cases} \tag{5.27}$$

This leads to a contradiction for large $k's$.

Step 2: (Conclusion)

Now, we take η corresponding to $\epsilon = \epsilon_0$, where ϵ_0 is as in Lemma 5.3. The rest of the arguments are the same as in Lemma 5.3 because the universal estimate in Eq. 5.15 also holds for v_0. Also (5.22) follows because $\tilde{L}$ corresponds to the affine approximation of v_0 at 0 which satisfies homogeneous Neumann condition as in Eq. 5.27. □

Now, we let

$$\eta_0 = \min\{\eta, 1/L(\epsilon_0)\}, \tag{5.28}$$

where η, ϵ_0 are as in Lemma 5.5 and $L(\epsilon_0)$ is as in Lemma 5.3 corresponding to ϵ_0. Finally as a consequence of Lemmas 5.3 and 5.5, we obtain that the following uniform improvement of flatness which doesn't take into account the size of $|p|$.

Lemma 5.6 *Let u be such that $|u|\le 1$ and is a viscosity solution to the following Neumann problem,*

$$\begin{cases}\langle A(x)(Du+p), (Du+p)\rangle^{\beta/2} F\left(D^2u, Du, x\right) = f \textit{ in } B^+_1,\\ u_{x_n} = g \textit{ on } B^0_1,\end{cases} \tag{5.29}$$

where $p \in \mathbb{R}^n$, A is Lipschitz and uniformly elliptic and F is uniformly elliptic in M with ellipticity bounds λ and Λ, Lipschitz in the gradient variable q and continuous in x with a modulus of continuity ω. Also suppose $|F(0,0,0)| \leq 1$. Then with η_0 as in Eq. 5.28 *above, we have that if $||f||_{L^\infty}, ||g||_{C^{\alpha_0}}, |D_qF| \leq \eta_0$, then there exists an affine function $\tilde{L} = \tilde{a} + <\tilde{b}, x>$ with universal bounds such that*

$$\begin{cases} ||u - \tilde{L}||_{L^\infty(B_r^+)} \leq r^{1+\alpha}, \\ <\tilde{b}, e_n> = 0 \end{cases} \tag{5.30}$$

where $r, \alpha \in (0,1)$ are universal constants. Furthermore, we can additionally ensure that α satisfies (5.19).

With Lemma 5.6 in hand, we now prove our main result Theorem 2.1.

5.3 Proof of Theorem 2.1

Proof Step 1: (Basic reductions) In view of our discussion in Section 5.1, we may first assume that $\partial\Omega = \{x_n = 0\}$ and u solves an equation of the type (5.1). Then, by letting

$$u_s(x) = \frac{u(sx)}{s},$$

we have that u_s solves

$$\begin{cases} \langle A(sx)Du_s(x), Du_s(x)\rangle^{\beta/2} sF\left(\frac{D^2u_s}{s}, Du_s, sx\right) = sf(sx). \\ (u_s)_{x_n}(x) = g(sx) \text{ on } \{x_n = 0\}. \end{cases} \tag{5.31}$$

Now, by choosing s sufficiently small, we can ensure that the operator

$$F_s(M, q, x) = sF\left(\frac{M}{s}, q, sx\right)$$

satisfies $|D_qF_s| \leq \eta_0$ and also that

$$|F_s(0,0,0)| \leq \frac{1}{2}. \tag{5.32}$$

Subsequently we let u_s as our new u and F_s as our new F which now additionally satisfies $|D_qF| \leq \eta_0$. Then by letting $v = u - g(0)x_n - u(0)$, we have that $v(0) = 0$ and it solves

$$\begin{cases} \langle A(x)(Dv + g(0)e_n), Dv + g(0)e_n)\rangle^{\beta/2} F(D^2v, Dv + g(0)e_n, x) = f \text{ in } B_1^+, \\ v_{x_n} = g - g(0) \text{ on } B_1^0, \end{cases} \tag{5.33}$$

We now define

$$\tilde{v} = \frac{v}{\kappa}$$

where

$$\kappa = \left(1 + ||v||_{L^\infty} + \left(\frac{||f||_{L^\infty}}{2\eta_0}\right)^{\frac{1}{1+\beta}} + \frac{||g||_{C^{\alpha_0}}}{2\eta_0}\right)$$

with η_0 as in Lemma 5.6. Then we observe that $\tilde{v}$ solves

$$\begin{cases} \langle A(x)(D\tilde{v} + \frac{g(0)}{\kappa}e_n), D\tilde{v} + \frac{g(0)}{\kappa}e_n\rangle^{\beta/2}\kappa^{-1}F(\kappa D^2\tilde{v}, \kappa D\tilde{v}, x) = \kappa^{-(1+\beta)}f(x) = \tilde{f} \text{ in } B_1^+, \\ (\tilde{v})_{x_n} = \tilde{g} = \frac{g}{\kappa} \text{ on } B_1^0. \end{cases} \tag{5.34}$$

Now since $\kappa > 1$, we find that the new operator in Eq. 5.34 satisfies similar structural conditions as F. Moreover, we additionally have that $\|\tilde{v}\| \leq 1$, $||\tilde{f}||_{L^\infty} \leq \eta_0$, $||\tilde{g}||_{C^{\alpha_0}} \leq \eta_0$. Thus by letting $\tilde{v}$ as our new v, $\tilde{g}$ as our new g and so on, we may assume without loss of generality that v satisfies an equation of the type (5.29) such that the following holds,

$$||D_q F||, \|f\|_{L^\infty}, ||g||_{C^{\alpha_0}} \leq \eta_0.$$

Moreover, we also have that for our new g that $g(0) = 0$ holds.

Step 2: We now show that for all r, α as in Lemma 5.6, we have that for every $k = 0, 1, 2...$, there exists $L_k = \langle b_k, x\rangle$ such that

$$\begin{cases} ||v - L_k||_{L^\infty(B^+_{r^k})} \leq r^{k(1+\alpha)}, \\ \langle b_k, e_n\rangle = 0, \\ |b_k - b_{k+1}| \leq C r^{k\alpha}. \end{cases} \tag{5.35}$$

We prove the claim in Eq. 5.35 by induction. For $k = 1$, it follows from Lemma 5.6 in view of our reductions as in Step 1. Also note that since $v(0) = 0$, by keeping track of the arguments that leads to Lemma 5.6, we can additionally ensure that $L_1(0) = 0$. We now assume that the assertion in Eq. 5.35 holds up to some k. For such a k, we let

$$w = \frac{(v - L_k)\left(r^k x\right)}{r^{k(1+\alpha)}}.$$

Then, we have that $|w| \leq 1$ in B_1^+ and it satisfies the following inequalities in the viscosity sense

$$\begin{cases} \langle A\left(r^k x\right)(Dw + p_k), Dw + p_k\rangle^{\beta/2} r^{k(1-\alpha)} F(r^{k(\alpha-1)} D^2 w, r^{k\alpha} Dw + b_k, r^k x) = f_k(x) \text{ in } B_1^+ \\ (w)_{x_n} = g_k(x) \text{ on } T_1, \end{cases} \tag{5.36}$$

where $p_k = r^{-k\alpha} p + r^{-k\alpha} b_k$, $f_k(x) = r^{k(1-\alpha(1+\beta))} f\left(r^k x\right)$ and $g_k(x) = r^{-k\alpha} g\left(r^k x\right)$. Now, since $||g||_{C^{\alpha_0}} \leq \eta_0$, $g(0) = 0$ and $\alpha_0 > \alpha$, therefore, one can deduce easily that $||g_k||_{C^{\alpha_0}} \leq \eta_0$. Also since $\alpha < \frac{1}{1+\beta}$ and $||f||_{L^\infty} \leq \eta_0$, therefore we can infer that $||f_k||_{L^\infty} \leq \eta_0$. Moreover, it also follows that the operator $F_{r,k}$ in Eq. 5.36 defined as

$$F_{r,k}(M, q, x) = r^{k(1-\alpha)} F(r^{k(\alpha-1)} M, r^{k\alpha} q + b_k, r^k x),$$

has the same ellipticity bounds as F. Moreover, $||D_q F_{r,k}|| \leq r^k ||D_q F|| \leq \eta_0$ since $r < 1$. Also using Eq. 5.32 we have that

$$|F_{r,k}(0, 0, 0)| \leq r^{k(1-\alpha)} |F(0, 0, 0)| + \eta_0 r^{k(1-\alpha)} |b_k| \leq \frac{1}{2} + C\eta_0 \leq 1 \tag{5.37}$$

provided η_0 is further adjusted in the beginning.

Therefore, we can again apply Lemma 5.6 to obain for some $\tilde{L}(x) =< \tilde{b}, x >$ satisfying $< \tilde{b}, e_n >= 0$ that the following inequality holds,

$$||w - < \tilde{b}, x > ||_{L^\infty(B_r^+)} \leq r^{1+\alpha}$$

Over here, we crucially used the fact that since $w(0) = 0$, therefore as for $k = 1$, we also additionally obtain that $\tilde{L}(0) = \tilde{a} = 0$ by applying Lemma 5.6 in this specific situation. Scaling back to v, we deduce that Eq. 5.35 holds for $k + 1$ with $L_{k+1}(x) = L_k(x) + r^{k(1+\alpha)} \tilde{L}(r^{-k} x)$. This verifies the induction step and finishes the proof of Step 2.

Step 3 (Conclusion)

It follows from Eq. 5.35 by a standard analysis argument that $L_0 = \lim_{k\to\infty} L_k$ is the affine approximation of order $1+\alpha$ at 0 for v and consequently $L_0 + g(0)x_n$ is the $1+\alpha$ order affine approximation for u at 0. Likewise we have an affine approximation of order $1+\alpha$ at all boundary points. Now going back to the original domain Ω, we can assert that there exists an affine approximation for u of order $1+\alpha$ at all points of $\partial\Omega \cap B_1$. At this point, by a standard argument as in [26], one can combine the boundary $C^{1,\alpha}$ estimate with the interior ones as in [22] to conclude that $u \in C^{1,\alpha}(\overline{\Omega \cap B_{1/2}})$. Over here we note that although the interior regularity result in [22] is stated for

$$|Du|^\beta F(D^2u) = f$$

nevertheless, the proof works exactly the same way for equations of the type

$$|Du + L|^\beta F\left(D^2u, x\right) = f,$$

when F depends continuously on x. This finishes the proof of the theorem. □

Proof of Corollary 2.3 We first rewrite the boundary condition in Corollary 2.3 as follows

$$\begin{cases} |Du|^\beta F\left(D^2u, x\right) = f, \text{ in } \Omega \cap B_1(0), 0 \in \partial\Omega, \beta \geq 0, \\ u_\nu = \tilde{g}, \text{ on } \partial\Omega \cap B_1(0), \end{cases} \tag{5.38}$$

where $\tilde{g} = g - h(x)u$. Then by flattening and by applying the Hölder regularity result Theorem 3.9, we obtain that u is C^α up to the boundary. This in turn implies that $\tilde{g}$ is Hölder continuous and consequently the conclusion follows from Theorem 2.1. □

In closing, we make the following remark.

Remark 5.7 It seems plausible that the techniques in this paper can be modified to yield $C^{1,\alpha}$ regularity results for Neumann boundary problems of the type

$$\begin{cases} |Du|^\beta(\Delta u + (p-2)\Delta_\infty^N u) = f, \text{ in } \Omega \cap B_1(0), 0 \in \partial\Omega, \beta \geq 0, \\ u_\nu = \tilde{g}, \text{ on } \partial\Omega \cap B_1(0), \end{cases} \tag{5.39}$$

where $\Delta_\infty^N u$ is the normalized infinity laplacian operator. The case when $\beta = 0$ corresponds to the Poisson problem for the normalized $p-$laplacian operator and this has been studied in various contexts in a number of papers. See for instance [4, 8, 10] and one can find the references therein. For general $\beta > 0$, we refer to [3] for the interior $C^{1,\alpha}$ regularity result for such equations and also to [21] and [1] for the parabolic counterpart of such results.

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