Potential Analysis (2022) 57:181–199
https://doi.org/10.1007/s11118-021-09911-6

Stationary Solutions of Fokker-Planck Equations with Nonlinear Reaction Terms in Bounded Domains

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Received: 8 January 2020 / Accepted: 9 February 2021 / Published online: 27 February 2021

Abstract
Using an operator approach, we discuss stationary solutions to Fokker-Planck equations and systems with nonlinear reaction terms. The existence of solutions is obtained by using Banach, Schauder and Schaefer fixed point theorems, and for systems by means of Perov's fixed point theorem. Using the Ekeland variational principle, it is proved that the unique solution of the problem minimizes the energy functional, and in case of a system that it is the Nash equilibrium of the energy functionals associated to the component equations.

Keywords Elliptic equation · Reaction-diffusion equation · Semi-linear Fokker-Planck equation · Fixed point · Variational method · Nash type equilibrium

Mathematics Subject Classification (2010) 35J60 · 47H30 · 47J05

1 Introduction

The Fokker-Planck equation arises as a mathematical model in many areas of physics and biology, mostly connected with the analysis of random phenomena (see, e.g., [1, 2, 4, 15, 19, 20]). It has the form

$$w_t - \text{ div } (\mathbf{D}\nabla w + w\mathbf{F}) = h,$$

where $\mathbf{D} = \mathbf{D}(x)$ is a symmetric (diffusion) matrix, $\mathbf{F} = \mathbf{F}(x)$ is a given vector field, $h = h(t, x)$ is the source term, and $w = w(t, x)$ is a probability distribution. It is the continuity equation

$$w_t + \text{div } J = h,$$

for the flux density $J = -\mathbf{D}\nabla w - w\mathbf{F}$ involving both diffusion, by the term $\mathbf{D}\nabla w$, and drift, by $w\mathbf{F}$. In case that $\mathbf{D} = DI$, where I is the identity matrix and D is a constant, the

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equation reads

$$w_t - D\Delta w - \text{div } (w\mathbf{F}) = h.$$

We consider the semi-linear case, where the source term h is a reaction term $h(w)$ depending on the state, namely equations of the form

$$w_t - D\Delta w - \text{div } (w\mathbf{F}) = h(w).$$

The reaction term $h(w)$ can be very general and nonlocal both in time and space. In particular, the cases of the equations with memory and of integral-differential equations are included. The reaction term can involve power-type nonlinearities, or rational functions simulating the saturation effect and making the equation a self-limiting model (see [2]).

In this paper, we consider only the stationary equation in a smooth bounded domain $\Omega \subset \mathbb{R}^d$, with a potential flow $\mathbf{F} = -\varepsilon D\nabla H$, where $\varepsilon > 0$ and H is a sufficiently smooth function, let it be in $C^1(\Omega)$, having a number of properties as shown below, but which could be very irregular near the boundary. Thus we consider the following semi-linear problem

$$\begin{cases} -\Delta w + \varepsilon \text{ div } (w\nabla H) = \Phi(w) \text{ in } \Omega \\ \int_\Omega w = 1. \end{cases} \tag{1.1}$$

Assuming that $\exp H \in L^\infty(\Omega)$ and making the substitution

$$w = (u+1)\rho,$$

with

$$\rho := \exp(\varepsilon H) / |\exp(\varepsilon H)|_{L^1(\Omega)},$$

we replace the average condition $\int_\Omega w = 1$ by the equality $\int_\Omega \rho u = 0$, and the above problem becomes

$$\begin{cases} -\Delta u - \varepsilon \nabla u \cdot \nabla H = \Psi(u) \text{ in } \Omega \\ \int_\Omega \rho u = 0, \end{cases} \tag{1.2}$$

where

$$\Psi(u) = \frac{1}{\rho}\Phi(\rho(u+1)),$$

leading to the study of renormalized solutions u in a special weighted space.

We also consider systems of such equations modeling the evolution of many randomly diffusing particles. This is the case of chemical reactions involving several reagents that react and diffuse simultaneously.

Compared to other approaches in the literature (see [4, 7, 9, 11, 13, 19]), our approach is essentially based on the theory of nonlinear operators and by this, the specificity of the subject is brought inside the unifying nonlinear functional analysis. We first consider the solution operator associated to the non-homogeneous problem, which is defined by using the general theory of positive-define self-adjoint linear operators, and next its composition with the nonlinear mapping giving the right-hand side (nonlinearity) of the semi-linear problem. Then, joint suitable properties of the solution operator and nonlinearity allow us to make use of several fixed point principles: Banach's fixed point theorem, which guarantees the existence and uniqueness of the solution, and its property of being a global minimum of the energy functional; Schauder's and Schaefer's fixed point theorems, which not only guarantee the existence of a solution, but also give its localization in terms of the energetic norm.

Our approach to reaction-diffusion systems of Fokker-Planck equations is based on the vector method that uses matrices instead of constants, vector-valued norms and Perov's fixed point theorem (for the vector approach to nonlinear systems, see [5, 6, 16, 18]). In this case, the obtained solution is a Nash equilibrium of the energy functionals associated to the equations of the system. The variational properties of solutions are obtained by means of Ekeland's principle.

2 Preliminaries. Linear Fokker-Planck Equations

Our approach to linear Fokker-Planck equations makes use of the variational theory of positive-define symmetric linear operators (see [14, Ch. 4], or [21, Ch. 5]). The application of this theory to linear Fokker-Planck equations is detailed in this section.

2.1 The Fokker-Planck Operator

Consider the Banach normalized weighted spaces

$$L^q_\rho = \left\{ u : \ \rho^{1/q} u \in L^q(\Omega), \ \int_\Omega \rho u = 0 \right\} \quad (1 \le q < +\infty)$$

with norm

$$|u|^q_{L^q_\rho} = \int_\Omega \rho \, |u|^q .$$

For $q = 2$, we endow L^2_ρ with the inner product and norm

$$(u, v)_\rho = \int_\Omega \rho u v, \quad |u|_\rho = \left(\int_\Omega \rho u^2 \right)^{\frac{1}{2}} .$$

Consider the linear operator in $L^2_{\rho,}$ defined by

$$\mathcal{L}u = -\Delta u - \varepsilon \nabla u \cdot \nabla H$$

with the domain

$$D(\mathcal{L}) = \{ u \in C^2_0(\Omega) : \ \int_\Omega \rho u = 0 \},$$

where $C^2_0(\Omega)$ is the space of all functions in $C^2(\Omega)$ with compact support included in Ω. For any $u \in C^2_0(\Omega)$, $\Delta u \in C_0(\Omega)$, and since $H \in C^1(\Omega)$, one has $\nabla H \in C\left(\Omega, \mathbb{R}^d\right)$. Hence $\mathcal{L}u \in C_0(\Omega) \subset L^2_\rho$, that is $\mathcal{L}$ is well-defined. Also $D(\mathcal{L})$ is *dense* in L^2_ρ. Indeed, if $u \in L^2_\rho$, then $v := \sqrt{\rho} u \in L^2(\Omega)$ and in view of the density of $C^\infty_0(\Omega)$ into $L^2(\Omega)$, there exists in $C^\infty_0(\Omega)$ a sequence (v_k) with $v_k \to v$ in $L^2(\Omega)$. Let $\varphi_k \in C^\infty_0(\Omega)$ be such that $\varphi_k \to 1$ in $L^2(\Omega)$, and let

$$u_k := \frac{1}{\sqrt{\rho}} (v_k - c_k \varphi_k),$$

where $c_k = \int_\Omega \sqrt{\rho} v_k / \int_\Omega \sqrt{\rho} \varphi_k$. Clearly $u_k \in D(\mathcal{L})$. Also

$$|u_k - u|_\rho = |v_k - c_k \varphi_k - v|_{L^2(\Omega)} \le |v_k - v|_{L^2(\Omega)} + c_k |\varphi_k|_{L^2(\Omega)} . \tag{2.1}$$

Hence $u_k \to u$ in L^2_ρ if $c_k \to 0$. To show this, first note that

$$0 < \int_\Omega \sqrt{\rho} \le \int_\Omega \sqrt{\rho} \, |1 - \varphi_k| + \int_\Omega \sqrt{\rho} \varphi_k \le |\rho|_{L^1(\Omega)} |1 - \varphi_k|_{L^2(\Omega)} + \int_\Omega \sqrt{\rho} \varphi_k$$

whence we have that the sequence $\left(\int_\Omega \sqrt{\rho} \varphi_k \right)$ is bounded from below by a positive number C. Then $c_k \le (1/C) \int_\Omega \sqrt{\rho} v_k$. Next, in view of $\int_\Omega \sqrt{\rho} v = 0$, one has

$$\left| \int_\Omega \sqrt{\rho} v_k \right| = \left| \int_\Omega \sqrt{\rho} (v_k - v) \right| \le |\rho|_{L^1(\Omega)} |v_k - v|_{L^2(\Omega)} \to 0.$$

Hence $c_k \to 0$ and from (2.1) it follows that $u_k \to u$ in L^2_ρ. Therefore $\mathcal{L}$ is densely defined on L^2_ρ.

The operator $\mathcal{L}$ is *symmetric*. Indeed, since $\nabla\rho = \varepsilon\rho\nabla H$, we have

$$\begin{aligned}(\mathcal{L}u, v)_\rho &= -\int_\Omega \rho v\,(\Delta u + \varepsilon\nabla u\cdot\nabla H) = \int_\Omega \nabla(\rho v)\cdot\nabla u - \int_\Omega v\nabla u\cdot\nabla\rho \\ &= \int_\Omega \rho\nabla u\cdot\nabla v = (u, \mathcal{L}v)_\rho\,.\end{aligned}$$

Finally note that

$$(\mathcal{L}u, u)_\rho = \int_\Omega \rho\,|\nabla u|^2 > 0$$

for every $u \in D(\mathcal{L})\setminus\{0\}$, that is the operator $\mathcal{L}$ is *strictly positive*.

2.2 The Energetic Space

We may endow $D(\mathcal{L})$ with two inner products

$$\langle u, v\rangle = (u, v)_\rho + (\mathcal{L}u, v)_\rho\,, \quad [u, v] = (\mathcal{L}u, v)_\rho$$

and the corresponding norms

$$\|u\|^2 = |u|^2_\rho + (\mathcal{L}u, u)_\rho\,, \quad [u]^2 = (\mathcal{L}u, u)_\rho\,.$$

Let $E_\mathcal{L}$ (called the *energetic space* of $\mathcal{L}$) be the completion of the prehilbertian space $(D(\mathcal{L}),\ \|\cdot\|)$ and let us use the same notations $\langle\cdot,\cdot\rangle$, $[\cdot,\cdot]$, $\|\cdot\|$ and $[\cdot]$ for the corresponding maps extended by density to $E_\mathcal{L}$. Since $|u|_\rho \le \|u\|$ for all $u \in D(\mathcal{L})$, we have $D(\mathcal{L}) \subset E_\mathcal{L} \subset L^2_\rho$ with dense and continuous embeddings. Recall that, from the construction of the completion, any element u of $E_\mathcal{L}$ can be seen as the limit in L^2_ρ of a sequence of functions from $D(\mathcal{L})$ which is fundamental with respect the norm $\|\cdot\|$, and that this limit is common for all such sequences (u_k), (v_k) which are equivalent in the sense that $\|u_k - v_k\| \to 0$. If (u_k) is a fundamental sequence in $D(\mathcal{L})$, then there exist $v, v_i \in L^2(\Omega)$, $i = 1, \cdot\cdot, d$ such that

$$\sqrt{\rho}u_k \to v, \quad \sqrt{\rho}\frac{\partial u_k}{\partial x_i} \to v_i \quad (i = 1, \cdot\cdot, d) \ \text{ in } L^2(\Omega)\,.$$

Thus, if we denote

$$u := \frac{v}{\sqrt{\rho}}, \quad \frac{\partial u}{\partial x_i} := \frac{v_i}{\sqrt{\rho}} \quad (i = 1, \cdot\cdot, d)\,,$$

then we may say that for every $u, v \in E_\mathcal{L}$,

$$\langle u, v\rangle = \int_\Omega \rho\,(uv + \nabla u\cdot\nabla v)\,, \quad [u, v] = \int_\Omega \rho\nabla u\cdot\nabla v, \tag{2.2}$$

$$\|u\|^2 = \int_\Omega \rho\left(u^2 + |\nabla u|^2\right), \quad [u]^2 = \int_\Omega \rho\,|\nabla u|^2\,. \tag{2.3}$$

Notice that the functional $[\cdot]$ is only a semi-norm on $E_\mathcal{L}$. To make it a norm, equivalent to the norm $\|\cdot\|$ on $E_\mathcal{L}$, we need a compactness assumption. To this aim, we state the following condition:

(C_q) The embedding $D(\mathcal{L}) \subset L^q_\rho$ is compact, i.e., any sequence of functions in $D(\mathcal{L})$ which is bounded with respect to the norm $\|\cdot\|$ has a subsequence that converges in L^q_ρ.

Clearly condition (C_q) implies that the embedding $E_\mathcal{L} \subset L^q_\rho$ is also compact.

The next condition (H) gives an exact representation of the space $E_{\mathcal{L}}$, and consequently, it is sufficient for (C_q) to hold for some values of q.

(H) There exists a constant $c > 0$ such that

$$|\nabla H|\,\delta_\Omega \leq c \text{ in } \Omega,$$

where δ_Ω gives the distance to the boundary $\partial\Omega$, i.e.

$$\delta_\Omega(x) = \min_{y\in\partial\Omega} |x-y| \ (x \in \Omega).$$

Proposition 2.1 *If condition (H) is satisfied for a constant c sufficiently small, then*

$$E_{\mathcal{L}} = \frac{1}{\sqrt{\rho}} H^1_{0,\rho}(\Omega) \tag{2.4}$$

where $H^1_{0,\rho}(\Omega) = \{v \in H^1_0(\Omega) : \int_\Omega \sqrt{\rho} v = 0\}$, *and* $[u]$, $\left|\sqrt{\rho}u\right|_{H^1_0(\Omega)}$ *and* $\|u\|$ *are equivalent norms on* $E_{\mathcal{L}}$.

Proof For any $u \in D(\mathcal{L})$, one has

$$\nabla\left(\sqrt{\rho}u\right) = \sqrt{\rho}\nabla u + \frac{1}{2\sqrt{\rho}} u\nabla\rho = \sqrt{\rho}\nabla u + \frac{1}{2}\sqrt{\rho}u\frac{\nabla\rho}{\rho}.$$

Since $\rho^{-1}\nabla\rho = \varepsilon\nabla H$, from (H) and Hardy's inequality [3], we can estimate the last addendum of the previous identity as follows

$$\left|\frac{1}{2}\sqrt{\rho}u\varepsilon\nabla H\right|_{L^2(\Omega)} \leq \varepsilon\frac{c}{2}\left|\frac{\sqrt{\rho}u}{\delta_\Omega}\right|_{L^2(\Omega)} \leq \varepsilon\frac{\widetilde{c}}{2}\left|\nabla\left(\sqrt{\rho}u\right)\right|_{L^2(\Omega)}.$$

Consequently

$$\left(\int_\Omega \left|\nabla\left(\sqrt{\rho}u\right)\right|^2\right)^{\frac{1}{2}} \leq \left(\int_\Omega \rho\left|\nabla u\right|^2\right)^{\frac{1}{2}} + \varepsilon\frac{\widetilde{c}}{2}\left(\int_\Omega \left|\nabla\left(\sqrt{\rho}u\right)\right|^2\right)^{\frac{1}{2}},$$

$$\left(\int_\Omega \rho\left|\nabla u\right|^2\right)^{\frac{1}{2}} \leq \left(\int_\Omega \left|\nabla\left(\sqrt{\rho}u\right)\right|^2\right)^{\frac{1}{2}} + \varepsilon\frac{\widetilde{c}}{2}\left(\int_\Omega \left|\nabla\left(\sqrt{\rho}u\right)\right|^2\right)^{\frac{1}{2}},$$

whence, if $c > 0$ is small enough that $1 - \varepsilon\widetilde{c}/2 > 0$, we obtain

$$\left|\sqrt{\rho}u\right|_{H^1_0(\Omega)} \leq c_0[u], \quad [u] \leq c_1\left|\sqrt{\rho}u\right|_{H^1_0(\Omega)}, \tag{2.5}$$

where $c_0 = 1/(1-\varepsilon\widetilde{c}/2)$ and $c_1 = 1 + \varepsilon\widetilde{c}/2$. As a result, for any sequence of functions $u_k \in D(\mathcal{L})$ which is fundamental with respect to the norm $\|\cdot\|$, the sequence $\sqrt{\rho}u_k$ is fundamental with respect to the norm $|\cdot|_{H^1_0(\Omega)}$, and conversely, if v_k is fundamental with respect to the norm $|\cdot|_{H^1_0(\Omega)}$, then the sequence $u_k = v_k/\sqrt{\rho}$ is fundamental with respect to the norm $\|\cdot\|$. This proves (2.4), while (2.5) together with the continuous embedding $\left(1/\sqrt{\rho}\right) H^1_{0,\rho}(\Omega) \subset L^2_\rho$ shows that $[u]$, $\left|\sqrt{\rho}u\right|_{H^1_0(\Omega)}$ and $\|u\|$ are equivalent norms on $E_{\mathcal{L}}$. □

Remark 2.1 The behavior on the boundary $\partial\Omega$ and the regularity of the solution $w = (u+1)\rho$ with $u \in E_{\mathcal{L}}$ strongly depend on the behavior and regularity of H. Thus, under assumption *(H)*, if $H \in H^1(\Omega)$, then $w \in W^{1,1}(\Omega)$. Indeed, from *(H)* (cf. Proposition 2.1) one has $w = \rho + u\rho = \rho + \sqrt{\rho}v$, where $v \in H^1_0(\Omega)$. Now, since ρ was assumed in $L^\infty(\Omega)$,

$$\partial_{x_i}\rho = \varepsilon\rho\partial_{x_i}H \in L^2(\Omega),$$

hence $\rho \in H^1(\Omega)$. Furthermore,

$$\partial_{x_i}\left(\sqrt{\rho}v\right) = \sqrt{\rho}\partial_{x_i}v + \frac{\varepsilon}{2}\sqrt{\rho}v\partial_{x_i}H \in L^1(\Omega).$$

Therefore $w \in W^{1,1}(\Omega)$. For an exhaustive discussion of regularity of solutions we refer the reader to [4, Chapter 1].

2.3 The Poincaré Inequality

Assume that condition (C_2) holds. The space $E_{\mathcal{L}}$ being reflexive (as a Hilbert space), one deduces from a result in paper [12] that

$$\mu := \inf_{u \in E_{\mathcal{L}},\, u \neq 0} \frac{[u]^2}{|u|_\rho^2} > 0$$

and the infimum is reached. From this, we have the Poincaré inequality

$$\mu\, |u|_\rho^2 \leq [u]^2 \quad \text{for } u \in E_{\mathcal{L}}$$

which ensures that $[\cdot]$ is a norm on $E_{\mathcal{L}}$, equivalent to the norm $\|\cdot\|$. Let $E'_{\mathcal{L}}$ be the dual of $(E_{\mathcal{L}}, [\cdot])$. If we identify L^2_ρ to its dual, then we have

$$D(\mathcal{L}) \subset E_{\mathcal{L}} \subset L^2_\rho \subset E'_{\mathcal{L}} \tag{2.6}$$

where the last embedding is compact too. For $f \in E'_{\mathcal{L}}$ and $u \in E_{\mathcal{L}}$, let (f, u) be the value of the linear functional f at u. In case that $f \in L^2_\rho$, one has $(f, u) = (f, u)_\rho$.

Throughout the paper we assume that condition (C_2) holds.

2.4 The Solution Operator

Returning to the operator $\mathcal{L}$, for a fixed $f \in E'_{\mathcal{L}}$, we define the *weak solution* of the stationary problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ \int_\Omega \rho u = 0 \end{cases} \tag{2.7}$$

as being a function $u \in E_{\mathcal{L}}$ such that for every $v \in E_{\mathcal{L}}$, one has that $[u, v] = (f, v)$. In particular, if $f \in L^2_\rho$, this identity becomes

$$\int_\Omega \rho \nabla u \cdot \nabla v = \int_\Omega \rho f v \quad \text{for } v \in E_{\mathcal{L}}.$$

From Riesz's representation theorem, since $(f, \cdot)$ is a continuous linear functional on $(E_{\mathcal{L}}, [\cdot])$, it follows that problem (2.7) has a unique weak solution u_f. Thus we may define the *solution operator*

$$\mathcal{L}^{-1} : E'_{\mathcal{L}} \to E_{\mathcal{L}}, \quad \mathcal{L}^{-1} f := u_f.$$

Recall that under condition (C_2), the operator $\mathcal{L}$ has a sequence of eigenvalues (λ_k) with $0 < \lambda_1 = \mu \leq \cdot\cdot \leq \lambda_k \leq \cdot\cdot, \lambda_k \to +\infty$, and correspondingly a sequence (ϕ_k) of eigenfunctions, which is orthonormal and complete in $L^2_{\mathcal{L}}$. Also the sequence $\left(\phi_k/\sqrt{\lambda_k}\right)$ is orthonormal and complete in $(E_{\mathcal{L}}, [\cdot])$. This yields the Fourier representation of the solution operator:

$$\mathcal{L}^{-1} f = \sum \left[u_f, \frac{\phi_k}{\sqrt{\lambda_k}}\right] \frac{\phi_k}{\sqrt{\lambda_k}} = \sum \frac{(f, \phi_k)}{\lambda_k} \phi_k,$$

where the series converges in $E_{\mathcal{L}}$ and L^2_ρ.

Also note that $\mathcal{L}^{-1}$ is an isometry between $E'_{\mathcal{L}}$ and $E_{\mathcal{L}}$, i.e., $\left[\mathcal{L}^{-1} f\right] = |f|_{E'_{\mathcal{L}}}$ for every $f \in E'_{\mathcal{L}}$, and that the exact Poincaré inequality

$$\lambda_1 \, |u|_\rho^2 \leq [u]^2 \quad \text{for } u \in E_{\mathcal{L}} \tag{2.8}$$

is accompanied by the Poincaré inequality for the dual, namely

$$\lambda_1 \, |f|_{E'_{\mathcal{L}}}^2 \leq |f|_\rho^2 \quad \text{for } f \in L^2_\rho. \tag{2.9}$$

Indeed, if $f \in L^2_\rho$, then using (2.8) we have

$$\begin{aligned} |f|_{E'_{\mathcal{L}}} &= \sup_{u \in E_{\mathcal{L}},\, u \neq 0} \frac{|(f,u)|}{[u]} = \sup_{u \in E_{\mathcal{L}},\, u \neq 0} \frac{\left|(f,u)_\rho\right|}{[u]} \\ &\leq \sup_{u \in E_{\mathcal{L}},\, u \neq 0} \frac{|f|_\rho \, |u|_\rho}{[u]} \leq \sup_{u \in E_{\mathcal{L}},\, u \neq 0} \frac{1}{\sqrt{\lambda_1}} \frac{|f|_\rho \, [u]}{[u]} = \frac{|f|_\rho}{\sqrt{\lambda_1}}, \end{aligned}$$

that is (2.9).

2.5 The Energy Functional

According to the variational theory of positive-define symmetric linear operators, for each fixed $f \in E'_{\mathcal{L}}$, the functional $J : E_{\mathcal{L}} \to \mathbb{R}$,

$$Ju = \frac{1}{2} [u]^2 - (f, u)$$

is C^1 and $J'u = \mathcal{L}u - f$, more exactly

$$\left(J'u, v\right) = [u, v] - (f, v) \quad \text{for all } \; u, v \in E_{\mathcal{L}}.$$

Therefore the weak solution of problem (2.7) is the critical point of the *energy functional* J.

3 Semilinear Fokker-Planck Equations

We now turn back to the semi-linear problems (1.1) and (1.2).

3.1 Existence and Uniqueness via Banach's Fixed Point Theorem

Our first result is about the existence and uniqueness of the solution to the semilinear problem (1.2) and consequently to (1.1).

Let j_0 and j be the canonical injections of the embeddings $E_{\mathcal{L}} \subset L^2_\rho$ and $L^2_\rho \subset E'_{\mathcal{L}}$, respectively.

Notice that problem (1.2) is equivalent with the fixed point equation $u = \mathcal{L}^{-1}\Psi(u)$ in $E_{\mathcal{L}}$. In view of embeddings (2.6), we may discuss three cases:

- Ψ maps $E_{\mathcal{L}}$ into $E'_{\mathcal{L}}$;
- Ψ maps $E_{\mathcal{L}}$ into L^2_ρ; here by the composition $\mathcal{L}^{-1}\Psi$ we mean $\mathcal{L}^{-1} j \Psi$;
- Ψ maps L^2_ρ into L^2_ρ; here by $\mathcal{L}^{-1}\Psi$ we mean $\mathcal{L}^{-1} j \Psi j_0$.

Our first results are existence and uniqueness theorems, the first in terms of Ψ and the second in terms of Φ.

Theorem 3.1 *Problems* (1.2) *and* (1.1) *have unique weak solution* $u \in E_{\mathcal{L}}$ *and* $w = (u+1)\,\rho \in L^2(\Omega)$, *respectively, if one of the following conditions holds:*

(a) $\Psi : E_{\mathcal{L}} \to E'_{\mathcal{L}}$ *and there is a constant* $0 \leq a_0 < 1$ *such that*

$$|\Psi(u) - \Psi(v)|_{E'_{\mathcal{L}}} \leq a_0 [u - v] \quad \text{for } u, v \in E_{\mathcal{L}}.$$

(b) $\Psi : E_{\mathcal{L}} \to L^2_\rho$ *and there is a constant* $0 \leq a_1 < \sqrt{\lambda_1}$ *such that*

$$|\Psi(u) - \Psi(v)|_\rho \leq a_1 [u - v] \quad \text{for } u, v \in E_{\mathcal{L}}.$$

(c) $\Psi : L^2_\rho \to L^2_\rho$ *and there is a constant* $0 \leq a_2 < \lambda_1$ *such that*

$$|\Psi(u) - \Psi(v)|_\rho \leq a_2 |u - v|_\rho \quad \text{for } u, v \in L^2_\rho.$$

Proof (a) Under condition (a), for any $u, v \in E_{\mathcal{L}}$, one has

$$\left[\mathcal{L}^{-1}\Psi u - \mathcal{L}^{-1}\Psi v\right] = |\Psi u - \Psi v|_{E'_{\mathcal{L}}} \leq a_0 [u - v], \tag{3.1}$$

and the conclusion follows from Banach's contraction principle.
(b) In case that Ψ takes values in L^2_ρ, using the Poincaré inequality (2.9), we have

$$|\Psi u - \Psi v|_{E'_{\mathcal{L}}} \leq \frac{1}{\sqrt{\lambda_1}} |\Psi u - \Psi v|_\rho \leq \frac{a_1}{\sqrt{\lambda_1}} [u - v]$$

and the result follows from case (a) where $a_0 = a_1/\sqrt{\lambda_1}$.
(c) This case reduces to (b) with $a_1 = a_2/\sqrt{\lambda_1}$ since in virtue of (2.8), one has

$$|\Psi(u) - \Psi(v)|_\rho \leq a_2 |u - v|_\rho \leq \frac{a_2}{\sqrt{\lambda_1}} [u - v].$$

□

Theorem 3.2 *Problems* (1.2) *and* (1.1) *have a unique weak solution* $u \in E_{\mathcal{L}}$ *and* $w = (u+1)\,\rho \in L^2(\Omega)$, *respectively, if one of the following conditions holds:*

(d) $\Phi : L^2(\Omega) \to L^2(\Omega)$, $\Phi(\rho) \in L^2_{\rho^{-1}}(\Omega)$, *and there is a constant* $0 \leq a < \lambda_1$ *such that*

$$|\Phi(u)(x) - \Phi(v)(x)| \leq a\,|u(x) - v(x)| \quad \text{for } u, v \in L^2(\Omega) \text{ and a.a. } x \in \Omega. \tag{3.2}$$

(e) $\Phi(u)(x) = f(x, u(x))$, *where* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory conditions,* $f(\cdot, 0) \in L^2(\Omega)$, $f(\cdot, \rho(\cdot)) \in L^2_{\rho^{-1}}$, *and there exists* $0 \leq a < \lambda_1$ *such that*

$$|f(x, u) - f(x, v)| \leq a\,|u - v| \quad \text{for all } u, v \in \mathbb{R} \text{ and a.a. } x \in \Omega. \tag{3.3}$$

Proof (d) First, if $u \in L^2_\rho$, then $\sqrt{\rho}u \in L^2(\Omega)$ and

$$\begin{aligned}
\left|\sqrt{\rho}\Psi(u)\right|_{L^2(\Omega)} &= \left|\frac{1}{\sqrt{\rho}}\Phi((u+1)\,\rho)\right|_{L^2(\Omega)} \\
&\leq \left|\frac{1}{\sqrt{\rho}}(\Phi((u+1)\,\rho) - \Phi(\rho))\right|_{L^2(\Omega)} + \left|\frac{1}{\sqrt{\rho}}\Phi(\rho)\right|_{L^2(\Omega)} \\
&\leq a\left|\sqrt{\rho}u\right|_{L^2(\Omega)} + \left|\frac{1}{\sqrt{\rho}}\Phi(\rho)\right|_{L^2(\Omega)} = a\,|u|_\rho + \left|\frac{1}{\sqrt{\rho}}\Phi(\rho)\right|_{L^2(\Omega)} < +\infty.
\end{aligned}$$

Hence $\Psi(u) \in L^2_\rho$. Also, for any $u, v \in L^2_\rho$, we have

$$\begin{aligned}|\Psi(u) - \Psi(v)|_\rho &= \left|\frac{1}{\rho}\left(\Phi((u+1)\rho) - \Phi((v+1)\rho)\right)\right|_\rho \\ &= \left|\frac{1}{\sqrt{\rho}}\left(\Phi((u+1)\rho) - \Phi((v+1)\rho)\right)\right|_{L^2(\Omega)} \\ &\le a\left|\sqrt{\rho}(u-v)\right|_{L^2(\Omega)} = a|u-v|_\rho .\end{aligned}$$

Thus we are in case (c) of Theorem 3.1.
(e) Under the assumptions of f, the Nemytskii operator Φ maps $L^2(\Omega)$ into itself. In addition (3.3) immediately yields (3.2). Hence we are in case (d). □

3.2 Variational Characterization of the Solution

The next result gives a variational characterization of the solution guaranteed by the previous theorems.

Theorem 3.3 *(j) Assume that Ψ is in case (a) of Theorem 3.1 and there is a C^1-functional $\Theta : E_{\mathcal{L}} \to \mathbb{R}$ bounded from above on bounded sets and such that $\Psi = \Theta'$. Then the unique solution u of problem (1.2) is the unique minimum point of the energy functional*

$$Jv = \frac{1}{2}[v]^2 - \Theta(v). \tag{3.4}$$

(jj) Assume that Φ is in case (d) of Theorem 3.2 and there is a C^1-functional $\theta : L^2(\Omega) \to \mathbb{R}$ bounded from above on bounded sets and such that $\Phi(v) = \rho\theta'(v-\rho)$ for $v \in L^2(\Omega)$. Then the unique solution u of problem (1.2) is the unique minimum point of the energy functional (3.4) for $\Theta(v) = \theta(\rho v)$, $v \in E_{\mathcal{L}}$.

(jjj) Assume that f is in case (e) of Theorem 3.2 and in addition that there exists $h \in L^2(\Omega)$ and a small enough $c \ge 0$ such that

$$|f(x,s)| \le \rho(x)(c|s| + h(x)) \quad \textit{for } s \in \mathbb{R} \textit{ and a.a. } x \in \Omega. \tag{3.5}$$

Then the unique solution u of problem (1.2) is the unique minimum point of the energy functional (3.4) for

$$\Theta(v) = \int_\Omega \left(\int_0^{\rho v} \frac{1}{\rho} f(x, \tau + \rho)\, d\tau\right).$$

Proof (j) Since $J' = \mathcal{L} - \Psi$, one has that the unique solution of (1.2) is the unique critical point of J. Let B be a closed ball of the space $E_{\mathcal{L}}$ with center at the origin and positive radius $R \ge |\Psi(0)|_{E'_{\mathcal{L}}}/(1-a_0)$. Then, by (3.1) we immediately see that

$$\mathcal{L}^{-1}\Psi(B) \subset B. \tag{3.6}$$

To prove that the solution u minimizes J we use the weak form of Ekeland's variational principle [8, 10]. Note that the boundedness of Θ guarantees the functional J to be bounded from below on B. Indeed, for any $v \in B$, one has $Jv \ge -\Theta(v) \ge -c > -\infty$, where $\Theta(v) \le c$ for every $v \in B$. Then, using the weak form of Ekeland's variational principle, there is a minimizing sequence (u_k) of elements in B such that

$$Ju_k \le \inf_B J + \frac{1}{k} \tag{3.7}$$

and

$$Ju_k \leq Jv + \frac{1}{k}[v - v_k] \tag{3.8}$$

for all $v \in B$. For any fixed index k, choose

$$v_t = u_k - t\mathcal{L}^{-1}J'u_k, \quad 0 < t < 1.$$

Using $\mathcal{L}^{-1}\Psi u_k = u_k - \mathcal{L}^{-1}J'u_k$, one has

$$v_t = (1-t)\,u_k + t\mathcal{L}^{-1}\Psi u_k.$$

Here, one has $u_k \in B$, and by the invariance property (3.6), $\mathcal{L}^{-1}\Psi u_k$ also belongs to B. Since B is convex, it follows that $v_t \in B$ for every $t \in (0,1)$. Replacing v by v_t into (3.15) and then dividing by t, yields

$$t^{-1}\left(Ju_k - J\left(u_k - t\mathcal{L}^{-1}J'u_k\right)\right) \leq \frac{1}{k}\left[\mathcal{L}^{-1}J'u_k\right],$$

whence letting t go to zero, one finds

$$\left(J'u_k, \mathcal{L}^{-1}J'u_k\right) \leq \frac{1}{k}\left[\mathcal{L}^{-1}J'u_k\right],$$

that is

$$\left[\mathcal{L}^{-1}J'u_k\right] \leq \frac{1}{k}.$$

Hence

$$Ju_k \to \inf_B J \quad \text{and} \quad \mathcal{L}^{-1}J'u_k \to 0 \ \text{ in } \ E_{\mathcal{L}}. \tag{3.9}$$

Let $v_k := \mathcal{L}^{-1}J'u_k$. From

$$\left[u_{k+p} - u_k\right] \leq \left[v_{k+p} - v_k\right] + \left[\mathcal{L}^{-1}\Psi u_{k+p} - \mathcal{L}^{-1}\Psi u_k\right]$$

and the contraction condition (3.1), we obtain

$$\left[u_{k+p} - u_k\right] \leq (1-a_0)^{-1}\left[v_{k+p} - v_k\right].$$

The sequence (v_k) is a Cauchy sequence (as a convergent sequence). Consequently (u_k) is a Cauchy sequence, hence convergent to some $u \in B$. Now passing to the limit in (3.9) gives

$$Ju = \inf_B J \quad \text{and} \quad J'u = 0.$$

Due to the fact that R was taken arbitrary bigger than $|\Psi(0)|_{E'_{\mathcal{L}}}/(1-a_0)$ and to the uniqueness of the critical point of J, we may conclude that u minimizes J on the whole space $E_{\mathcal{L}}$, that is $Ju = \inf_{E_{\mathcal{L}}} J$.

(jj) For $u, v \in E_{\mathcal{L}}$, we have $\rho u, \rho v \in L^2(\Omega)$ and

$$\begin{aligned}
\Theta(u+tv) - \Theta(u) &= \theta(\rho(u+tv)) - \theta(\rho u) \\
&= t\left(\theta'(\rho u),\ \rho v\right)_{L^2(\Omega)} + o(|t|) \\
&= t\left(\frac{1}{\rho}\Phi(\rho(u+1)),\ \rho v\right)_{L^2(\Omega)} + o(|t|) \\
&= t\left(\frac{1}{\rho}\Phi(\rho(u+1)),\ v\right)_{\rho} + o(|t|) \\
&= t\left(\Psi(u), v\right)_{\rho} + o(|t|).
\end{aligned}$$

It follows that $\Theta'(u) = \Psi(u)$. Hence we are in case (j).

(jjj) Under the assumptions on f, the function $g(x,s) = \rho(x)^{-1} f(x, s + \rho(x))$ satisfies the Carathéodory conditions and the growth inequality $|g(x,s)| \leq c|s| + \widetilde{h}(x)$, with

$\widetilde{h}(x) = h(x) + c\rho(x)$, which makes the Nemytskii operator N_g associated to g to act in $L^2(\Omega)$. In addition the functional

$$\theta(v) = \int_\Omega \left(\int_0^v g(x,\tau)\, d\tau \right)$$

is C^1 on $L^2(\Omega)$ and $\theta'(v) = N_g(v)$ for $v \in L^2(\Omega)$. On the other hand, $\Phi(v) = f(\cdot, v)$. Now it is easy to check the equality $\Phi(v) = \rho\theta'(v-\rho)$. Further, the functional J is bounded from below on B (where B is like in case (j)) provided c is small enough. In fact, thanks to (3.5), after some computations including the use of Poincaré's inequality leading a constant c_1, we obtain an estimate of the type

$$Jv \geq \left(\frac{1}{2} - cc_1 \right) [v]^2 + c_2[v] + c_3.$$

This shows that J is bounded from below if $c < 1/(2c_1)$. Therefore Ekeland's principle applies as well in this case. □

The next theorem gives the variational characterization of the solution of problem (1.1) and it is a direct consequence of the previous result just by making the change of variable $w = (u+1)\rho$.

Theorem 3.4 *(k) Under the assumptions of Theorem 3.1, if in addition $\Phi(v) = \rho\Theta'\left(\frac{v}{\rho} - 1\right)$, where $\Theta : E_{\mathcal{L}} \to \mathbb{R}$ is a C^1-functional, then the unique solution w of problem (1.1) is the unique minimum point of the energy functional*

$$J_0 v = \frac{1}{2} \left[\frac{v}{\rho} \right]^2 - \Theta_0(v), \tag{3.10}$$

where $\Theta_0(v) = \Theta\left(\frac{v}{\rho} - 1\right)$.

(kk) Assume that Φ is in case (d) of Theorem 3.2 and there is a C^1-functional $\theta : L^2(\Omega) \to \mathbb{R}$ such that $\Phi(v) = \rho\theta'(v-\rho)$ for $v \in L^2(\Omega)$. Then the unique solution w of problem (1.1) is the unique minimum point of the energy functional (3.10) for $\Theta_0(v) = \theta(v-\rho)$, $v \in E_{\mathcal{L}}$.

(kkk) Under the assumptions of case (e) of Theorem 3.2 the unique solution w of problem (1.1) is the unique minimum point of the functional

$$J_0(v) = \int_\Omega \left(\frac{1}{2}\rho \left| \nabla\left(\frac{v}{\rho}\right) \right|^2 - \int_0^{v-\rho} \frac{1}{\rho} f(s, \tau+\rho)\, d\tau \right). \tag{3.11}$$

Remark 3.1 In particular, in the absence of drift, that is when $H = 0$, one has $\rho = \rho_0 := 1/meas(\Omega)$ and the functional (3.11) reduces to

$$\begin{aligned} J_0(v) &= \frac{1}{\rho_0} \int_\Omega \left(\frac{1}{2} |\nabla v|^2 - \int_0^{v-\rho_0} f(s, \tau+\rho_0)\, d\tau \right) \\ &= \frac{1}{\rho_0} \int_\Omega \left(\frac{1}{2} |\nabla v|^2 - \int_{\rho_0}^{v} f(x,\xi) d\xi \right) \\ &= \frac{1}{\rho_0} \int_\Omega \left(\frac{1}{2} |\nabla v|^2 - \int_0^{v} f(x,\xi) d\xi \right) + \frac{1}{\rho_0} \int_\Omega \left(\int_0^{\rho_0} f(x,\xi) d\xi \right). \end{aligned}$$

Since the last term is a constant, we may say that in this case, the solution w of problem (1.1) minimizes the functional

$$\int_{\Omega}\left(\frac{1}{2}|\nabla v|^2-\int_0^v f(x,\xi)d\xi\right),$$

which is the energy functional of the Dirichlet problem for the equation $-\Delta w=f(x,w)$.

3.3 Existence via Schauder's Fixed Point Theorem

If instead of Lipschitz continuity, we only assume a linear growth condition, and we add compactness, then the existence of solutions still holds based on Schauder's fixed point theorem.

As in the case of Section 3.1, we give results first in terms of Ψ and next in terms of Φ.

Theorem 3.5 *The problems* (1.2) *and* (1.1) *have at least one weak solution* $u\in E_{\mathcal{L}}$ *and* $w=(u+1)\rho\in L^2(\Omega)$, *respectively, if one of the following conditions holds:*

(a) $\Psi:E_{\mathcal{L}}\to E'_{\mathcal{L}}$ *is completely continuous and there are constants* $0\le a_0<1$ *and* $b_0\ge 0$ *such that*

$$|\Psi(u)|_{E'_{\mathcal{L}}}\le a_0[u]+b_0 \ \ \textit{for } u\in E_{\mathcal{L}}. \tag{3.12}$$

(b) $\Psi:E_{\mathcal{L}}\to L^2_\rho$ *is continuous, and there are constants* $0\le a_1<\sqrt{\lambda_1}$ *and* $b_1\ge 0$ *such that*

$$|\Psi(u)|_\rho\le a_1[u]+b_1 \ \ \textit{for } u\in E_{\mathcal{L}}. \tag{3.13}$$

(c) $\Psi:L^2_\rho\to L^2_\rho$ *is continuous, and there are constants* $0\le a_2<\lambda_1$ *and* $b_2\ge 0$ *such that*

$$|\Psi(u)|_\rho\le a_2|u|_\rho+b_2 \ \ \textit{for } u\in L^2_\rho. \tag{3.14}$$

Proof (a) The operator $\mathcal{L}^{-1}\Psi$ is completely continuous. In addition if $R\ge b_0/(1-a_0)$, then for every $u\in E_{\mathcal{L}}$ with $[u]\le R$, one has

$$\left[\mathcal{L}^{-1}\Psi u\right]=|\Psi u|_{E'_{\mathcal{L}}}\le a_0[u]+b_0\le a_0R+b_0\le R.$$

Hence the operator $\mathcal{L}^{-1}\Psi$ maps the closed ball of $E_{\mathcal{L}}$ with center at the origin and radius R into itself. The conclusion follows now from Schauder's fixed point theorem.
(b) The condition (C_2) implies that the embedding $L^2_\rho\subset E'_{\mathcal{L}}$ is compact, i.e. the injection $j:L^2_\rho\to E'_{\mathcal{L}}$ is completely continuous. Also (3.13) shows that Ψ is bounded (maps bounded sets into bounded sets). Hence the operator $j\Psi$ is completely continuous as a composition of two bounded and continuous operators where one of them, namely j, is completely continuous. In addition, from Poincaré's inequality and (3.13),

$$|\Psi(u)|_{E'_{\mathcal{L}}}\le\frac{1}{\sqrt{\lambda_1}}|\Psi(u)|_\rho\le\frac{1}{\sqrt{\lambda_1}}(a_1[u]+b_1).$$

Hence (3.12) holds with $a_0=a_1/\sqrt{\lambda_1}$ and $b_0=b_1/\sqrt{\lambda_1}$. Thus we are in case (a).
(c) In this case, the operator Ψj_0 is continuous from $E_{\mathcal{L}}$ to L^2_ρ. In addition

$$|\Psi(u)|_\rho\le a_2|u|_\rho+b_2\le\frac{a_2}{\sqrt{\lambda_1}}[u]+b_2 \quad \text{for } u\in E_{\mathcal{L}}.$$

Hence we are in case (b) with $a_1=a_2/\sqrt{\lambda_1}$ and $b_1=b_2$. □

Theorem 3.6 *The problems* (1.2) *and* (1.1) *have at least one weak solution* $u \in E_{\mathcal{L}}$ *and* $w = (u+1)\,\rho \in L^2(\Omega)$, *respectively, if one of the following conditions holds:*

(d) $\Phi : L^2(\Omega) \to L^2(\Omega)$ *is continuous from* $L^2_{\rho^{-1}}$ *to* $L^2_{\rho^{-1}}$, *and there are constants* $0 \le a < \lambda_1$ *and* $h \in L^2_{\rho^{-1}}$ *such that*

$$|\Phi(u)(x)| \le a\,|u(x)| + h(x) \quad \text{for } u \in L^2(\Omega) \text{ and a.a. } x \in \Omega. \tag{3.15}$$

(e) $\Phi(u)(x) = f(x, u(x))$, *where* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory conditions, and there exist* $0 \le a < \lambda_1$ *and* $h \in L^2_{\rho^{-1}}$ *such that*

$$|f(x,s)| \le a\,|s| + h(x) \quad \text{for every } s \in \mathbb{R} \text{ and a.a. } x \in \Omega. \tag{3.16}$$

Proof (d) The case reduces to case (c) in Theorem 3.5. As in the proof of Theorem 3.2, we can show that Ψ maps L^2_ρ into itself and that (3.14) holds. Next we prove that Ψ is continuous from L^2_ρ to itself. Indeed, if $u_k \to u$ in L^2_ρ, the $\sqrt{\rho}u_k \to \sqrt{\rho}u$ in $L^2(\Omega)$, so $\rho u_k \to \rho u$ in $L^2_{\rho^{-1}}$. Consequently, $\Phi((u_k+1)\,\rho) \to \Phi((u+1)\,\rho)$ in $L^2_{\rho^{-1}}$, that is $\Psi(u_k) \to \Psi(u)$ in L^2_ρ, as wished.
(e) We reduce this case to (d). Under the above conditions on f, the Nemytskii operator associated to f maps continuously $L^2(\Omega)$ into $L^2(\Omega)$ and $|f(x,v)|_{L^2(\Omega)} \le a\,|v|_{L^2(\Omega)} + |h|_{L^2(\Omega)}$. The same is true for the function

$$g(x,s) = \frac{1}{\sqrt{\rho(x)}} f\left(x, s\sqrt{\rho(x)}\right)$$

which also satisfies the Carathéodory conditions and the growth condition

$$|g(x,s)| \le a\,|s| + h(x) \quad \text{for every } s \in \mathbb{R} \text{ and a.a. } x \in \Omega.$$

Thus the Nemytskii operator associated to g is continuous from $L^2(\Omega)$ to $L^2(\Omega)$. This implies that Φ is continuous from $L^2_{\rho^{-1}}$ to $L^2_{\rho^{-1}}$. Indeed, if $u_k \to u$ in $L^2_{\rho^{-1}}$, then

$$u_k/\sqrt{\rho} \to u/\sqrt{\rho} \ \text{ in } \ L^2(\Omega), \quad \text{whence} \quad g\left(\cdot, u_k/\sqrt{\rho}\right) \to g\left(\cdot, u/\sqrt{\rho}\right) \ \text{ in } \ L^2(\Omega).$$

Hence $f(\cdot, u_k)/\sqrt{\rho} \to f(\cdot, u)/\sqrt{\rho}$ in $L^2(\Omega)$, that is $f(\cdot, u_k) = \Phi(u_k) \to f(\cdot, u) = \Phi(u)$ in $L^2_{\rho^{-1}}$. Also (3.16) clearly gives (3.15). Hence we are in case (d). □

3.4 Existence via Schaefer's Fixed Point Theorem

In the next result, the linear growth of the nonlinear reaction term is relaxed in case that in compensation, a sign condition holds for a part of the reaction term.

Let $\Phi = \Phi_0 + \Phi_1$ and correspondingly $\Psi = \Psi_0 + \Psi_1$ and $f = f_0 + f_1$.

We first state a general existence principle.

Theorem 3.7 *Let* Ψ_0 *be as in Theorem 3.5 (a). If in addition* $\Psi_1 : E_{\mathcal{L}} \to E'_{\mathcal{L}}$ *is completely continuous and*

$$(\Psi_1(u), u) \le 0 \quad \text{for } u \in E_{\mathcal{L}},$$

then the problems (1.2) *and* (1.1) *have at least one weak solution* $u \in E_{\mathcal{L}}$ *and* $w = (u+1)\,\rho \in L^2(\Omega)$, *respectively. In addition* $[u] \le b_0/(1-a_0)$.

Proof The operator $\mathcal{L}^{-1}\Psi$ is completely continuous and for every solution $u \in E_{\mathcal{L}} \setminus \{0\}$ of the equation $u = \lambda \mathcal{L}^{-1}\Psi(u)$ and any $\lambda \in (0,1)$, one has

$$[u] = \lambda \frac{(\Psi(u), u)}{[u]} = \lambda \frac{(\Psi_0(u), u)}{[u]} + \lambda \frac{(\Psi_1(u), u)}{[u]}$$
$$\leq \lambda \frac{(\Psi_0(u), u)}{[u]} \leq \lambda |\Psi_0(u)|_{E'_{\mathcal{L}}} < a_0[u] + b_0.$$

Here we have assumed without loss of generality that $b_0 > 0$. Hence $[u] < b_0/(1-a_0)$, that is, the set of all solutions of the equations $u = \lambda \mathcal{L}^{-1}\Psi(u)$ for $\lambda \in (0,1)$, is bounded in $E_{\mathcal{L}}$. Now Schaefer's fixed point theorem guarantees the existence of a fixed point $u \in E_{\mathcal{L}}$ of $\mathcal{L}^{-1}\Psi$ with $[u] \leq b_0/(1-a_0)$. □

The next theorem gives us some sufficient conditions for the complete continuity of Ψ_1.

Theorem 3.8 *The operator Ψ_1 is completely continuous from $E_{\mathcal{L}}$ to $E'_{\mathcal{L}}$ if the compactness condition (C_q) holds for some $q \geq 2$, and one of the following conditions is satisfied:*

(i) $\Psi_1 : E_{\mathcal{L}} \to L^p_\rho$ *is continuous and bounded for some* $p \geq q/(q-1)$*;*
(ii) $\Psi_1 : L^q_\rho \to L^p_\rho$ *is continuous and bounded for some* $p \geq q/(q-1)$*;*
(iii) $\Phi_1 : L^q_{\rho^{1-q}} \to L^p_{\rho^{1-p}}$ *is continuous and bounded for some* $p \geq q/(q-1)$*;*
(iv) $\Phi_1(u)(x) = f_1(x, u(x))$, *where* $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory conditions, and there exist* $a \in \mathbb{R}_+$ *and* $h \in L^p_{\rho^{1-p}}$ *such that*

$$|f_1(x,s)| \leq a\rho^{\frac{p-q}{p}} |s|^{\frac{q}{p}} + h(x) \quad \textit{for every } s \in \mathbb{R} \textit{ and a.a. } x \in \Omega. \tag{3.17}$$

Proof (i) From (C_q), the embedding $E_{\mathcal{L}} \subseteq L^q_\rho$ is compact, and so is the embedding $L^{q'}_\rho \subseteq E'_{\mathcal{L}}$. For $p \geq q/(q-1) = q'$, since $\rho \in L^\infty(\Omega)$, one has $L^p_\rho \subseteq L^{q'}_\rho$, whence the compact inclusion $L^p_\rho \subseteq E'_{\mathcal{L}}$.
(ii) Use the embedding $E_{\mathcal{L}} \subseteq L^q_\rho$ to reduce the case to (i).
(iii) Let $u_k \to u$ in L^q_ρ. Then $\rho u_k \to \rho u$ in $L^q_{\rho^{1-q}}$. Then, $\rho(u_k+1) \to \rho(u+1)$ in $L^q_{\rho^{1-q}}$, and by the assumption, $\Phi_1(\rho(u_k+1)) \to \Phi_1(\rho(u+1))$ in $L^p_{\rho^{1-p}}$. This yields $\Psi_1(u_k) = \rho^{-1}\Phi_1(\rho(u_k+1)) \to \Psi_1(u)$ in L^p_ρ. Thus we are in case (ii).
(iv) We are in case (iii). First observe that $\Phi_1(u)(x) = \rho(x)^{1-1/p} N_g(v)(x)$, where for each $u \in L^q_{\rho^{1-q}}$, $v := \rho^{1/q-1}u \in L^q(\Omega)$, and

$$g(x,s) = \rho(x)^{\frac{1}{p}-1} f_1\left(x, \rho(x)^{1-\frac{1}{q}} s\right).$$

The problem reduces to show that the Nemytskii operator N_g is well-defined from $L^q(\Omega)$ to $L^p(\Omega)$. Indeed, using (3.17) we have

$$|g(x,s)| = \rho(x)^{\frac{1}{p}-1} \left| f_1\left(x, \rho(x)^{1-\frac{1}{q}} s\right)\right| \leq a|s|^{\frac{q}{p}} + h_0(x),$$

where $h_0 = \rho^{\frac{1-p}{p}} h \in L^p(\Omega)$. □

Remark 3.2 (Positive solutions) As it is well-known, the existence of nonnegative solutions of boundary values problems is closely connected with maximum principles (see [4, Chapter 2]). For our elliptic operator $\mathcal{L}u = -\Delta u - \varepsilon \nabla u \cdot \nabla H$, the maximum principle holds, more exactly, if $u \in C^2(\Omega)$, $\mathcal{L}u \geq 0$ and there is $x_0 \in \Omega$ with $u(x_0) = \inf_\Omega u$, then u is constant on the connected component of Ω that contains x_0. Consequently, assuming that

$\Phi(v) \geq 0$ for every function v, and that a solution u of problem (1.2) is regular belonging to $D(\mathcal{L})$, then $u \geq -1$. Indeed, otherwise, we would have $u = 0$ around $\partial\Omega$, $u \neq 0$ on a compact subset of Ω and $u(x_0) = \min_\Omega u < -1$ for some $x_0 \in \Omega$. This implies that u is constant $u(x_0)$ on the connected component of Ω that contains x_0. But this is impossible since u is zero on $\partial\Omega$. The case of generalized solutions can be discussed similarly using the maximum principle for weak solutions.

4 Fokker-Planck Reaction-Diffusion Systems

Under the assumptions of Theorem 3.3, the unique stationary solution of a single equation is a global minimum of the associated energy function. We now prove that in case of a system of equations and under suitable conditions, the stationary solution is a Nash type equilibrium with respect to the couple of energy functionals associated to the component equations.

For simplicity, we shall consider only systems of two equations, that is

$$\begin{cases} -\Delta w_1 + \varepsilon_1 div\,(w_1 \nabla H_1) = \Phi_1(w_1, w_2) \\ -\Delta w_2 + \varepsilon_2 div\,(w_2 \nabla H_2) = \Phi_2(w_1, w_2) \\ \int_\Omega w_1 = 1\ ,\ \int_\Omega w_2 = 1. \end{cases} \tag{4.1}$$

In this case, denoting by

$$\rho_i = e^{\varepsilon_i H_i} / |e^{\varepsilon_i H_i}|_{L^1(\Omega)}$$

and making the substitution

$$w_i = (u_i + 1)\rho_i,$$

for $i = 1, 2$, we arrive to the system

$$\begin{cases} -\Delta u_1 - \varepsilon_1 \nabla u_1 \cdot \nabla H_1 = \Psi_1(u_1, u_2) \\ -\Delta u_2 - \varepsilon_2 \nabla u_2 \cdot \nabla H_2 = \Psi_2(u_1, u_2) \\ \int_\Omega \rho_1 u_1 = 0\ ,\ \int_\Omega \rho_2 u_2 = 0 \end{cases} \tag{4.2}$$

where

$$\Psi_i(u_1, u_2) = \frac{1}{\rho_i} \Phi_i((w_1 + 1)\rho_1, (w_2 + 1)\rho_2), \quad i = 1, 2.$$

All the elements defined in Section 2 for one equation, $\mathcal{L}$, $E_{\mathcal{L}}$, J and the scalar products and norms given by (2.2), are now duplicated for the two equations of the system, and we show it by an index i, $i = 1, 2$. We point out that we do not assume a variational structure on the whole system, but only for each component equation; thus, more exactly, we assume that there exist functions $\Theta_i : E_{\mathcal{L}_1} \times E_{\mathcal{L}_2} \to \mathbb{R}$, $i = 1, 2$, bounded from above on bounded sets and such that for each i, Ψ_i is the Fréchet derivative of $\Theta_i(u_1, u_2)$ with respect to the variable u_i. Hence, the energy functionals are

$$J_i(u_1, u_2) = \frac{1}{2}[u_i]_i^2 - \Theta_i(u_1, u_2), \quad i = 1, 2.$$

The analogue for systems of Theorem 3.1 case (a) is the following result. The reader can easily obtain the analogues for the cases (b)-(e).

Theorem 4.1 *Assume that* $\Psi_i : E_{\mathcal{L}_1} \times E_{\mathcal{L}_2} \to E'_{\mathcal{L}_1} \times E'_{\mathcal{L}_2}$ *and there are nonnegative constants* a_{ij}, $i, j = 1, 2$, *such that*

$$|\Psi_i(u_1, u_2) - \Psi_i(v_1, v_2)|_{E'_{\mathcal{L}_i}} \leq a_{i1}[u_1 - v_1]_1 + a_{i2}[u_2 - v_2]_2\ ,\ \ i = 1, 2, \tag{4.3}$$

for all $u_1, v_1 \in E_{\mathcal{L}_1}$ *and* $u_2, v_2 \in E_{\mathcal{L}_2}$, *and the spectral radius of the matrix* $M = (a_{ij})_{i,j=1,2}$ *is strictly less than 1. Then, problems* (4.2) *and* (4.1) *have unique weak solutions* $(u_1, u_2) \in E_{\mathcal{L}_1} \times E_{\mathcal{L}_2}$ *and* $(w_1, w_2) \in L^2(\Omega) \times L^2(\Omega)$, $w_i = (u_i + 1)\rho_i$ $(i = 1, 2)$, *respectively, and* (u_1, u_2) *is a Nash equilibrium of the pair of functionals* (J_1, J_2), *namely*

$$J_1(u_1, u_2) = \min_{E_{\mathcal{L}_1}} J_1(\cdot, u_2), \quad J_2(u_1, u_2) = \min_{E_{\mathcal{L}_2}} J_2(u_1, \cdot). \tag{4.4}$$

Proof First we prove the existence and the uniqueness of the solution to problem (4.2). Using (4.3) and the same arguments as in the proof of Theorem 3.1(a), we arrive to

$$\left[\mathcal{L}_i^{-1}\Psi_i(u_1, u_2) - \mathcal{L}_i^{-1}\Psi_i(v_1, v_2)\right]_i \leq a_{i1}[u_1 - v_1]_1 + a_{i2}[u_2 - v_2]_2, \quad i = 1, 2, \tag{4.5}$$

which, using the matrix M, can be written in the matrix form

$$\begin{pmatrix} \left[\mathcal{L}_1^{-1}\Psi_1(u_1, u_2) - \mathcal{L}_1^{-1}\Psi_1(v_1, v_2)\right]_1 \\ \left[\mathcal{L}_2^{-1}\Psi_2(u_1, u_2) - \mathcal{L}_2^{-1}\Psi_2(v_1, v_2)\right]_2 \end{pmatrix} \leq M \begin{pmatrix} [u_1 - v_1]_1 \\ [u_2 - v_2]_2 \end{pmatrix}.$$

Now, since the spectral radius of matrix M is strictly less than 1, the existence and uniqueness of the solution (u_1, u_2) follow from Perov's fixed point theorem (see [18, Theorem 1]).

In order to use the weak form of Ekeland's principle, we look for two balls $B_i \subset E_{\mathcal{L}_i}$ of positive radius R_i, $i = 1, 2$, with the property that

$$\mathcal{L}_i^{-1}\Psi_i(B_1 \times B_2) \subset B_i \quad \text{for } i = 1, 2. \tag{4.6}$$

Taking $v_1 = v_2 = 0$ in (4.5) and assuming that $[u_i]_i \leq R_i$, $i = 1, 2$, we obtain

$$\left[\mathcal{L}_i^{-1}\Psi_i(u_1, u_2)\right]_i \leq \gamma_i + a_{i1}R_1 + a_{i2}R_2, \quad i = 1, 2,$$

where $\gamma_i = \left[\mathcal{L}_i^{-1}\Psi_i(0, 0)\right]_i$. Hence, in order to obtain the desired inclusions (4.6), it is enough to have $\gamma_i + a_{i1}R_1 + a_{i2}R_2 \leq R_i$, $i = 1, 2$, or in the matrix form

$$(I - M)\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \geq \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

Multiplying on the left by $(I - M)^{-1}$ (which is a positive matrix since the spectral radius of M is less than 1, see [18]) yields

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \geq (I - M)^{-1}\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

which shows that the desired numbers R_1, R_2 exist.

Next, we prove that the solution (u_1, u_2) is the Nash equilibrium of the pair of functionals (J_1, J_2). To this aim, we use an iterative procedure. Denote by J_{ii} the Fréchet derivative of the functional $J_i(u_1, u_2)$ with respect to u_i. To begin the iterative procedure, we fix an arbitrary element $u_{2,0} \in B_2$. At each step $k \geq 1$, $u_{2,k-1} \in B_2$ been found at the previous step $k - 1$, first as in the proof of Theorem 3.3, we apply Ekeland's principle in B_1 to the functional $J_1\left(\cdot, u_{2,k-1}\right)$ and find an element $u_{1,k} \in B_1$ such that

$$J_1\left(u_{1,k}, u_{2,k-1}\right) \leq \inf_{B_1} J_1(\cdot, u_{2,k-1}) + \frac{1}{k}, \quad \left[\mathcal{L}_1^{-1}J_{11}(u_{1,k}, u_{2,k-1})\right]_1 \leq \frac{1}{k}. \tag{4.7}$$

Next, we apply Ekeland's principle in B_2 to the functional $J_2(u_{1,k},\cdot)$ and obtain an element $u_{2,k}\in B_2$ with

$$J_2\left(u_{1,k},u_{2,k}\right)\le \inf_{B_2} J_2(u_{1,k},\cdot)+\frac{1}{k},\quad \left[\mathcal{L}_2^{-1}J_{22}(u_{1,k},u_{2,k})\right]_2\le\frac{1}{k}. \tag{4.8}$$

Our aim now is to prove that the two sequences $\left(u_{i,k}\right)_k$, $i=1,2$ are Cauchy and so convergent. Let

$$\alpha_k=\mathcal{L}_1^{-1}J_{11}(u_{1,k},u_{2,k-1})\quad\text{and}\quad \beta_k=\mathcal{L}_2^{-1}J_{22}(u_{1,k},u_{2,k}).$$

Clearly $\alpha_k\to 0$ and $\beta_k\to 0$ in $E_{\mathcal{L}_1}$ and $E_{\mathcal{L}_2}$, respectively. As in the case of only one equation, we have

$$u_{1,k}-\mathcal{L}_1^{-1}\Psi_1(u_{1,k},u_{2,k-1})=\alpha_k, \tag{4.9}$$

$$u_{2,k}-\mathcal{L}_2^{-1}\Psi_2(u_{1,k},u_{2,k})=\beta_k. \tag{4.10}$$

By (4.9), we deduce

$$\begin{aligned}\left[u_{1,k+p}-u_{1,k}\right]_1 &\le \left[\mathcal{L}_1^{-1}\Psi_1(u_{1,k+p},u_{2,k+p-1})-\mathcal{L}_1^{-1}\Psi_1(u_{1,k},u_{2,k-1})\right]_1+[\alpha_{k+p}-\alpha_k]_1\\ &\le a_{11}\left[u_{1,k+p}-u_{1,k}\right]_1+a_{12}\left[u_{2,k+p-1}-u_{2,k-1}\right]_2+[\alpha_{k+p}-\alpha_k]_1\\ &\le a_{11}\left[u_{1,k+p}-u_{1,k}\right]_1+a_{12}\left[u_{2,k+p}-u_{2,k}\right]_2\\ &\quad +a_{12}\left(\left[u_{2,k+p-1}-u_{2,k-1}\right]_2-\left[u_{2,k+p}-u_{2,k}\right]_2\right)+[\alpha_{k+p}-\alpha_k]_1.\end{aligned} \tag{4.11}$$

By (4.10) we have

$$\left[u_{2,k+p}-u_{2,k}\right]_2\le a_{21}\left[u_{1,k+p}-u_{1,k}\right]_1+a_{22}\left[u_{2,k+p}-u_{2,k}\right]_2+[\beta_{k+p}-\beta_k]_2. \tag{4.12}$$

Denote

$$\begin{aligned}\delta_{k,p}&=\left[u_{1,k+p}-u_{1,k}\right]_1, & \eta_{k,p}&=\left[u_{2,k+p}-u_{2,k}\right]_2,\\ \xi_{k,p}&=[\alpha_{k+p}-\alpha_k]_1, & \chi_{k,p}&=[\beta_{k+p}-\beta_k]_2.\end{aligned}$$

Obviously, $\xi_{k,p}\to 0$ and $\chi_{k,p}\to 0$ as $k\to\infty$, uniformly with respect to p. Using the above notations, the inequalities (4.11) and (4.12) become

$$\begin{aligned}\delta_{k,p}&\le a_{11}\delta_{k,p}+a_{12}\eta_{k,p}+\xi_{k,p}+a_{12}\left(\eta_{k-1,p}-\eta_{k,p}\right),\\ \eta_{k,p}&\le a_{21}\delta_{k,p}+a_{22}\eta_{k,p}+\chi_{k,p}.\end{aligned}$$

These can be put under the following matrix form

$$\begin{pmatrix}\delta_{k,p}\\ \eta_{k,p}\end{pmatrix}\le M\begin{pmatrix}\delta_{k,p}\\ \eta_{k,p}\end{pmatrix}+\begin{pmatrix}\xi_{k,p}+a_{12}\left(\eta_{k-1,p}-\eta_{k,p}\right)\\ \chi_{k,p}\end{pmatrix}.$$

Hence

$$\begin{pmatrix}\delta_{k,p}\\ \eta_{k,p}\end{pmatrix}\le (I-M)^{-1}\begin{pmatrix}\xi_{k,p}+a_{12}\left(\eta_{k-1,p}-\eta_{k,p}\right)\\ \chi_{k,p}\end{pmatrix}.$$

Let $(I-M)^{-1}=(\mu_{ij})$. Then

$$\begin{aligned}\delta_{k,p}&\le \mu_{11}\left(\xi_{k,p}+a_{12}\left(\eta_{k-1,p}-\eta_{k,p}\right)\right)+\mu_{12}\chi_{k,p},\\ \eta_{k,p}&\le \mu_{21}\left(\xi_{k,p}+a_{12}\left(\eta_{k-1,p}-\eta_{k,p}\right)\right)+\mu_{22}\chi_{k,p}.\end{aligned}$$

The second inequality yields

$$\eta_{k,p}\le\frac{\mu_{21}a_{12}}{1+\mu_{21}a_{12}}\eta_{k-1,p}+\frac{\mu_{21}\xi_{k,p}+\mu_{22}\chi_{k,p}}{1+\mu_{21}a_{12}}. \tag{4.13}$$

Note that the sequence $(u_{k,p})_k$ is bounded uniformly with respect to p as follows from (4.10) (recall that $u_{1,k} \in B_1$ and $u_{2,k} \in B_2$). Consequently, by its definition, the sequence $(\eta_{k,p})_k$ is also bounded uniformly with respect to p. Thus we can apply to it the following lemma proved in [16, Lemma 3.2].

Lemma 4.1 *Let $(x_{k,p})_k$ and $(y_{k,p})_k$ be two sequences of real numbers depending on a parameter p such that the sequence $(x_{k,p})_k$ is bounded uniformly with respect to p, and*

$$0 \leq x_{k,p} \leq \lambda x_{k-1,p} + y_{k,p} \tag{4.14}$$

for all k, p and some $\lambda \in [0, 1[$. If $y_{k,p} \to 0$ as $k \to \infty$ uniformly with respect to p, then $x_{k,p} \to 0$ uniformly with respect to p.

Indeed, (4.13) reads as (4.14). Therefore $\eta_{k,p} \to 0$ uniformly with respect to p, which proves that the sequence $(u_{2,k})_k$ is Cauchy. Next, inequality (4.11) together with $a_{11} < 1$ (which is a consequence of the fact that the spectral radius of matrix M is less that 1) implies that $(u_{1,k})_k$ is Cauchy too. Let v_1, v_2 be the limits of the sequences $(u_{1,k})_k$ and $(u_{2,k})_k$, respectively. Passing to the limit in (4.7) and (4.8), we obtain that (v_1, v_2) solves system (4.2). The uniqueness of the solution and the arbitrariness of R_1, R_2 imply that $(v_1, v_2) = (u_1, u_2)$ and that (u_1, u_2) satisfies (4.4). □

Acknowledgments The authors would like to express their gratitude to the anonymous reviewers for their time and effort for reading the manuscript and valuable comments and suggestions.

This research is carried out within the national group GNAMPA of INdAM and supported by the INdAM-GNAMPA Project 2019 *Metodi topologici per problemi differenziali nonlineari ed applicazioni*.

References

1. Barbu, V.: Generalized solutions to nonlinear Fokker-Planck equations. J. Differ. Equat. **261**, 2446–2471 (2016)
2. Bengfort, M., Malchow, H., Hilker, F.M.: The Fokker–Planck law of diffusion and pattern formation in heterogeneous environments. J. Math. Biol. **73**, 683–704 (2016)
3. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011)
4. Bogachev, V.I., Krylov, N.V., Röckner, M., Shaposhnikov, S.V.: Fokker–Planck–Kolmogorov Equations. Amer. Math. Soc., Providence, Rhode Island (2015)
5. Cardinali, T., Precup, R., Rubbioni, P.: A unified existence theory for evolution equations and systems under nonlocal conditions. J. Math. Anal. Appl. **432**, 1039–1057 (2015)
6. Cardinali, T., Precup, R., Rubbioni, P.: Heterogeneous vectorial fixed point theorems. Mediterr. J. Math. **14**(2), Art. 83, 12 (2017)
7. Chupin, L.: Fokker-planck equation in bounded domain. Ann. Inst. Fourier, Grenoble **60**, 217–255 (2010)
8. de Figueiredo, D.G.: Lectures on the Ekeland variational principle with applications and detours, Tata Institute of Fundamental Research, Bombay Lectures on Mathematics and Physics-Mathematics, vol. 81. Springer, Berlin (1989)
9. Droniou, J.: Non-coercive elliptic problems. Potential Anal. **17**, 181–203 (2002)
10. Ekeland, I.: On the variational principle. J. Math. Anal. Appl. **47**, 324–353 (1974)
11. Franck, T.D.: Nonlinear Fokker-Planck Equations. Fundamentals and Applications. Springer, Berlin (2005)
12. Jebelean, P., Precup, R.: Poincaré inequalities in reflexive cones. Appl. Math. Lett. **24**, 359–363 (2011)
13. Markowich, P.A., Villani, C.: On the trend to equilibrium for the Fokker-Planck equation: an interplay between physics and functional analysis. Matematica Contemporanea (SBM) **19**, 1–31 (2000)
14. Mikhlin, S.G.: Linear Partial Differential Equations. Vysshaya Shkola, Moscow (1977). [in Russian]

15. Okubo, A., Levin, S.A. Diffusion and Ecological Problems: Modern Perspectives, Interdisciplinary Applied Mathematics, 2nd edn., vol. 14. Springer, New York (2001)
16. Precup, R.: Nash-type equilibria and periodic solutions to nonvariational systems. Adv. Nonlinear Anal. **3**, 197–207 (2014)
17. Precup, R.: Linear and Semilinear Partial Differential Equations. De Gruyter, Berlin (2013)
18. Precup, R.: The role of matrices that are convergent to zero in the study of semilinear operator systems. Math. Comput. Modelling **49**(3-4), 703–708 (2009)
19. Risken, H.: The Fokker-Planck Equation: Methods of Solution and Applications. Springer, Berlin (1996)
20. Sattin, F.: Fick's law and Fokker-Planck equation in inhomogeneous environments. Phys. Lett. A **372**(22), 3941–3945 (2008)
21. Zeidler, E.: Applied Functional Analysis: Applications to Mathematical Physics. Springer, Berlin (1995)