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Scattering for Radial Defocusing Inhomogeneous Bi-Harmonic SchrÖDinger Equations

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Abstract
This note studies the asymptotic behavior of global solutions to the fourth-order Schrödinger equation

$$i\dot{u} + \Delta^2 u - F(x, u) = 0.$$

Indeed, for both cases, local and non-local source term, the scattering is obtained in the defocusing mass super-critical and energy sub-critical regimes, with radial setting.

Keywords Fourth-order Schrödinger equation · Decay · Scattering

Mathematics Subject Classification (2010) 35Q55

1 Introduction

This note is concerned with the energy scattering theory of the Cauchy problem for the following inhomogeneous defocusing Schrödinger equation

$$\begin{cases} i\dot{u} + \Delta^2 u + |x|^{2b}|u|^{2(q-1)}u = 0; \\ u(0, .) = u_0, \end{cases} \tag{1.1}$$

and the inhomogeneous defocusing Choquard equation

$$\begin{cases} i\dot{u} + \Delta^2 u + (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}u = 0; \\ u(0, .) = u_0. \end{cases} \tag{1.2}$$

Here and hereafter $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, for some $N \geq 1$. The unbounded inhomogeneous term is $|\cdot|^b$, for some $b \neq 0$. The source terms satisfy $q > 1$ and $p \geq 2$. The Riesz-potential is defined on $\mathbb{R}^N$ by

$$I_\alpha : x \to \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}}, \quad 0 < \alpha < N.$$

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The homogeneous case corresponding to $b = 0$ was considered first in [6, 7] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with a Kerr non-linearity.

The Eq. 1.1 is invariant under the scaling

$$u_\lambda = \lambda^{\frac{2+b}{q-1}} u(\lambda^4 ., \lambda .), \quad \lambda > 0.$$

This gives the critical Sobolev index

$$s_c := \frac{N}{2} - \frac{2+b}{q-1}.$$

Similarly, the Eq. 1.2 satisfies the scaling invariance

$$u_\lambda = \lambda^{\frac{4+2b+\alpha}{2(p-1)}} u(\lambda^4 ., \lambda .), \quad \lambda > 0.$$

This gives the critical Sobolev index

$$s_c' := \frac{N}{2} - \frac{4+2b+\alpha}{2(p-1)}.$$

In this note, one focus on the mass super-critical and energy sub-critical regimes $0 < s_c, s_c' < 2$.

In the context of scattering of non-linear Schrödinger equations, [8] is a fundamental paper where a new point of view is introduced to the scattering problems via the concentration-compactness/rigidity argument. This paper is devoted to the homogeneous case but it is challenging to see how far it can be pushed in the inhomogeneous setting. Some progress in this direction have been obtained in several papers, in between we mention some of them [1, 3, 9].

To the author knowledge, few works dealing with inhomogeneous fourth-order Schrödinger equations exist in the literature. Indeed, in the mass-critical case, the existence of non-global solutions with negative energy was investigated in [2]. Moreover, the local well-posedness in the energy space was treated recently in [5]. The inhomogeneous bi-harmonic Choquard problem was considered in the submitted paper [17].

It is the aim of this note, to investigate the asymptotic behavior of global solutions to both inhomogeneous fourth-order Schrödinger and Choquard equations. Indeed, by use of Morawetz estimates and some decay results, in the spirit of [19], one obtains the scattering in the energy space. Note that it is unclear how to get the decay of solutions in the L^2 sub-critical setting, established in this work, using the [8] technique.

The rest of this paper is organized as follows. In Section 3, one proves the scattering of global solutions to the inhomogeneous fourth-order Schrödinger (1.1). In the last section, one establishes the scattering of global solutions to the inhomogeneous fourth-order Choquard (1.2).

Here and hereafter, C denotes a constant which may vary from line to another. Denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the usual norm $\|\cdot\|_r := \|\cdot\|_{L^r}$ and $\|\cdot\| := \|\cdot\|_2$. The inhomogeneous Sobolev space $H^2 := H^2(\mathbb{R}^N)$ is endowed with the norm

$$\|\cdot\|_{H^2} := \Big(\|\cdot\|^2 + \|\Delta \cdot\|^2\Big)^{\frac{1}{2}}.$$

Let us denote also $C_T(X) := C([0, T], X)$ and X_{rd} the set of radial elements in X. Moreover, for an eventual solution to Eq. 1.1 or Eq. 1.2, $T^* > 0$ denotes it's lifespan. Finally, $x^\pm$ are two real numbers near to x satisfying $x^+ > x$ and $x^- < x$.

2 Background and Main Results

This section contains the statements of the main results of this paper and some standard estimates needed in the sequel.

2.1 Preliminary

The mass-critical and energy-critical exponents for the Schrödinger problem (1.1) are

$$q_* := 1 + \frac{4+2b}{N} \quad \text{and} \quad q^* := \begin{cases} 1 + \frac{4+2b}{N-4}, & \text{if} \quad N \geq 5; \\ \infty, & \text{if} \quad 1 \leq N \leq 4. \end{cases}$$

The mass-critical and energy-critical exponents for the Choquard problem (1.2) are

$$p_* := 1 + \frac{\alpha+4+2b}{N} \quad \text{and} \quad p^* := \begin{cases} 1 + \frac{4+2b+\alpha}{N-4}, & \text{if} \quad N \geq 5; \\ \infty, & \text{if} \quad 1 \leq N \leq 4. \end{cases}$$

Let us recall some local well-posedness results [5, 17] about the above inhomogeneous fourth-order Schrödinger problems.

Proposition 2.1 *Let $N \geq 3$, $\max\{-4, -\frac{N}{2}\} < 2b < 0$, $\max\{1, 1 + \frac{1+2b}{N}\} < q < q^*$ and $u_0 \in H^2$. Then, there exists $T^* = T^*(\|u_0\|_{H^2})$ such that Eq. 1.1 admits a unique maximal solution*

$$u \in C_{T^*}(H^2).$$

Moreover,

1. *the solution satisfies the mass and energy conservation laws*

$$Mass := M[u(t)] := \int_{\mathbb{R}^N} |u(t,x)|^2 dx = M[u_0];$$

$$Energy := \mathcal{E}[u(t)] := \|\Delta u(t)\|^2 + \frac{1}{q} \int_{\mathbb{R}^N} |x|^{2b} |u(t)|^{2q}\, dx = \mathcal{E}[u_0].$$

2. *$u \in L^q_{loc}(\mathbb{R}, W^{2,r})$ for any admissible pair (q, r) in the meaning of definition 2.13.*

Proposition 2.2 *Let $N \geq 3$, $0 < \alpha < N$ and $\max\{-(N+\alpha), -4(1+\frac{\alpha}{N}), N-8-\alpha\} < 2b < 0$.*

Assume that $N \geq 5$ or $3 \leq N \leq 4$ and $2\alpha + 4b + N > 0$. If $u_0 \in H^2$ and $2 \leq p < p^*$. Then, there exists $T^* = T^*(\|u_0\|_{H^2})$ such that Eq. 1.2 admits a unique maximal solution

$$u \in C_{T^*}(H^2).$$

Moreover,

1. *the solutions satisfies the mass and the following energy*

$$E[u(t)] := \|\Delta u(t)\|^2 + \frac{1}{p} \int_{\mathbb{R}^N} |x|^b (I_\alpha * |\cdot|^b |u(t)|^p) |u(t)|^p\, dx = E[u_0].$$

2. *$u \in L^q_{loc}((0, T^*), W^{2,r})$ for any admissible pair (q, r).*

Remark 2.3 The regularity condition $p \geq 2$, gives the restriction $N - 8 - \alpha < 2b$. This seems to be technical, because it don't appear in the energy.

2.2 Main Results

This sub-section contains the contribution of this note.

2.2.1 Results about the Schrödinger problem (1.1)

The first main goal of this manuscript is to prove the following scattering result.

Theorem 2.4 *Take $N \geq 5$, $\max\{-4, -\frac{N}{2}\} < 2b < 0$, $q_* < q < q^*$ and $u \in C(\mathbb{R}, H^2_{rd})$ be a global solution to Eq. 1.1. Then, there exists $u_\pm \in H^2$ such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta^2}u_\pm\|_{H^2} = 0.$$

In order to prove the scattering, one needs the following result about the decay of global solutions to the Schrödinger (1.1).

Proposition 2.5 *Take $N \geq 5$, $\max\{-4, -\frac{N}{2}\} < 2b < 0$, $\max\{1, 1 + \frac{2+2b}{N}\} < q < q^*$ and $u \in C(\mathbb{R}, H^2_{rd})$ be a global solution to Eq. 1.1. Then,*

$$\lim_{t\to\pm\infty} \|u(t)\|_r = 0, \quad \text{for all} \quad 2 < r < \frac{2N}{N-4}. \tag{2.1}$$

The following Morawetz estimate is a standard tools to prove the previous decay result.

Proposition 2.6 *Take $N \geq 5$, $\max\{-4, -\frac{N}{2}\} < 2b < 0$, $\max\{1, 1 + \frac{2+2b}{N}\} < q < q^*$ and $u \in C(\mathbb{R}, H^2_{rd})$ be a global solution to Eq. 1.1. Then,*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^N} |x|^{-1+2b}|u|^{2q}\,dx\,dt \lesssim \|u_0\|_{H^2}.$$

Remark 2.7 The condition $N \geq 5$ is required because of Morawetz estimate. Moreover, the radial assumption is needed is order to guarantee that if by the absurd there is no decay of the Lebesgue norm of the solution, then it has to concentrate in suitable ball of uniform size centered in the origin, and it is in contradiction with the Morawetz estimate.

Remark 2.8 the decay of solutions (2.1) is weaker than the scattering, but it is available in the mass-sub-critical case.

2.2.2 Results about the Choquard problem (1.2)

The second main goal of this manuscript is to prove the following scattering result.

Theorem 2.9 *Let $N \geq 5$, $0 < \alpha < N$, $\max\{-4(1 + \frac{\alpha}{N}), N - 8 - \alpha\} < 2b < 0$ and $p_* < p < p^*$ such that $p \geq 2$. Take $u \in C(\mathbb{R}, H^2_{rd})$ be a global solution to Eq. 1.2. Then, there exists $u_\pm \in H^2$ such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta^2}u_\pm\|_{H^2} = 0.$$

In order to prove the scattering, one needs the following result about the decay of global solutions to the Choquard (1.2).

Proposition 2.10 *Take* $N \geq 5$, $0 < \alpha < N$, $\max\{-4(1+\frac{\alpha}{N}), N-8-\alpha\} < 2b < 0$, $2 \leq p < p^*$ *and* $u \in C(\mathbb{R}, H^2_{rd})$ *be a global solution to Eq.* 1.2*. Then,*

$$\lim_{t\to\pm\infty} \|u(t)\|_r = 0, \quad \textit{for all} \quad 2 < r < \frac{2N}{N-4}.$$

The following Morawetz estimates stand for some standard tools to prove the previous decay result.

Proposition 2.11 *Take* $N \geq 5$, $0 < \alpha < N$ *and* $\max\{-4(1+\frac{\alpha}{N}), N-8-\alpha\} < 2b < 0$, $2 \leq p < p^*$ *and* $u \in C(\mathbb{R}, H^2_{rd})$ *be a global solution to Eq.* 1.2*. Then,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^N} |x|^{-1+2b} (I_\alpha * |u|^p)|u|^p \, dx \, dt \lesssim \|u_0\|_{H^2}.$$

Remark 2.12 It seems that the local and non-local source terms in the above Schrödinger problems have a similar asymptotic behavior.

2.3 Useful Estimates

Let us gather some classical tools needed in the sequel.

Definition 2.13 A couple of real numbers (q, r) is said to be admissible if

$$2 \leq r < \frac{2N}{N-4} \quad \text{and} \quad N\Big(\frac{1}{2} - \frac{1}{r}\Big) = \frac{4}{q},$$

where $\frac{2N}{N-4} = \infty$ if $1 \leq N \leq 4$. Denote the set of admissible pairs by Γ and $(q, r) \in \Gamma'$ if $(q', r') \in \Gamma$ and the Strichartz spaces

$$S(I) := \cap_{(q,r)\in\Gamma} L^q(I, L^r) \quad \text{and} \quad S'(I) := \cap_{(q,r)\in\Gamma'} L^q(I, L^r).$$

Recall the Strichartz estimates [4, 14].

Proposition 2.14 *Let* $N \geq 1$ *and* $t_0 \in I \subset \mathbb{R}$ *an interval. Then,*

$$\sup_{(q,r)\in\Gamma} \|u\|_{L^q(I,L^r)} \lesssim \|u(t_0)\| + \inf_{(\tilde{q},\tilde{r})\in\Gamma} \|i\dot{u} + \Delta^2 u\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'})};$$

$$\sup_{(q,r)\in\Gamma} \|\Delta u\|_{L^q(I,L^r)} \lesssim \|\Delta u(t_0)\| + \|i\dot{u} + \Delta^2 u\|_{L^2(I,\dot{W}^{1,\frac{2N}{2+N}})}, \quad \forall N \geq 3.$$

Let us recall a Hardy-Littlewood-Sobolev inequality [11].

Lemma 2.15 *Let* $N \geq 1$, $0 < \lambda < N$ *and* $1 < s, r < \infty$ *be such that* $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$*. Then,*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} \, dx \, dy \leq C(N, s, \lambda)\|f\|_r \|g\|_s, \quad \forall f \in L^r, \forall g \in L^s.$$

The next consequence [15], is adapted to the Choquard problem.

Corollary 2.16 *Let $N \geq 1$, $0 < \lambda < N$ and $1 < s, r, q < \infty$ be such that $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$. Then,*

$$\|(I_\alpha * f)g\|_{r'} \leq C(N, s, \alpha)\|f\|_s \|g\|_q, \quad \forall f \in L^s, \forall g \in L^q.$$

Sobolev injections [12] give a meaning to several computations done in this note.

Lemma 2.17 *Let $N \geq 2$, then*

1. *$H^2 \hookrightarrow L^q$ for any $q \in [2, \frac{2N}{N-4}]$ if $N \geq 5$ and any $2 \leq q < \infty$ if $N \leq 4$;*
2. *the following injection $H^2_{rd} \hookrightarrow\hookrightarrow L^q$ is compact for any $q \in (2, \frac{2N}{N-4})$ if $N \geq 5$ and any $2 < q < \infty$ if $N \leq 4$.*

Finally, let us give an abstract result.

Lemma 2.18 *Let $T > 0$ and $X \in C([0, T], \mathbb{R}_+)$ such that*

$$X \leq a + bX^\theta \text{ on } [0, T],$$

where $a, b > 0$, $\theta > 1$, $a < (1 - \frac{1}{\theta})(\theta b)^{\frac{-1}{\theta}}$ and $X(0) \leq (\theta b)^{\frac{-1}{\theta-1}}$. Then

$$X \leq \frac{\theta}{\theta - 1} a \text{ on } [0, T].$$

Proof The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (b\theta)^{\frac{1}{1-\theta}}]$ and increasing on $[(b\theta)^{\frac{1}{1-\theta}}, \infty)$. The assumptions imply that $f((b\theta)^{\frac{1}{1-\theta}}) < 0$ and $f(\frac{\theta}{\theta-1}a) \leq 0$. As $f(X(t)) \geq 0$, $f(0) > 0$ and $X(0) \leq (b\theta)^{\frac{1}{1-\theta}}$, we conclude the proof by a continuity argument. □

3 The Schrödinger Problem (1.1)

3.1 Morawetz Identity

In this sub-section, one proves Proposition 2.6 about a classical Morawetz estimate satisfied by the energy global solutions to the inhomogeneous Schrödinger problem (1.1). One adopts the convention that repeated indices are summed. Also, if f, g are two differentiable functions, one defines the momentum brackets by

$$\{f, g\}_p := \Re(f\nabla\bar{g} - g\nabla\bar{f}).$$

Let us start with an auxiliary result.

Lemma 3.1 *Take $N \geq 5$, $\max\{-\frac{N}{2}, -4\} < 2b < 0$, $\max\{1, 1 + \frac{2+2b}{N}\} < q < q^*$ and $u \in C_T(H^2)$ be a local solution to Eq. 1.1. Let $a : \mathbb{R}^N \to \mathbb{R}$ be a convex smooth function and the real function defined on $[0, T)$, by*

$$M : t \to 2\int_{\mathbb{R}^N} \nabla a(x)\Im(\nabla u(t, x)\bar{u}(t, x))\, dx.$$

Then, the following equality holds on $[0, T)$*,*

$$\begin{aligned} M' &= 2\int_{\mathbb{R}^N} \Big(2\partial_{jk}\Delta a \partial_j u \partial_k \bar{u} - \frac{1}{2}(\Delta^3 a)|u|^2 - 4\partial_{jk}a\partial_{ik}u\partial_{ij}\bar{u} \\ &+ \Delta^2 a|\nabla u|^2 - \partial_j a\{|x|^{2b}|u|^{2(q-1)}u, u\}_p^j\Big)\, dx \\ &= 2\int_{\mathbb{R}^N} \Big(2\partial_{jk}\Delta a \partial_j u \partial_k \bar{u} - \frac{1}{2}(\Delta^3 a)|u|^2 - 4\partial_{jk}a\partial_{ik}u\partial_{ij}\bar{u} \\ &+ \Delta^2 a|\nabla u|^2 - \frac{q-1}{q}(\Delta a)|x|^{2b}|u|^{2q} + \frac{1}{q}\nabla a \nabla(|x|^{2b})|u|^{2q}\Big)\, dx. \end{aligned}$$

Proof Denote the source term $\mathcal{N} := |x|^{2b}|u|^{2(q-1)}u$ and compute

$$\begin{aligned} \partial_t \Im(\partial_k u \bar{u}) &= \Im(\partial_k \dot{u}\bar{u}) + \Im(\partial_k u \dot{\bar{u}}) \\ &= \Re(i\dot{u}\partial_k \bar{u}) - \Re(i\partial_k \dot{u}\bar{u}) \\ &= \Re(\partial_k \bar{u}(-\Delta^2 u - \mathcal{N})) - \Re(\bar{u}\partial_k(-\Delta^2 u - \mathcal{N})) \\ &= \Re(\bar{u}\partial_k \Delta^2 u - \partial_k \bar{u}\Delta^2 u) + \Re(\bar{u}\partial_k \mathcal{N} - \partial_k \bar{u}\mathcal{N}). \end{aligned}$$

Thus,

$$\begin{aligned} M' &= 2\int_{\mathbb{R}^N} \partial_k a \Re(\bar{u}\partial_k\Delta^2 u - \partial_k\bar{u}\Delta^2 u)\, dx - 2\int_{\mathbb{R}^N} \partial_k a\{|x|^{2b}|u|^{2(q-1)}u, u\}_p^k\, dx \\ &= -2\int_{\mathbb{R}^N} \Delta a \Re(\bar{u}\Delta^2 u)\, dx - 4\int_{\mathbb{R}^N} \Re(\partial_k a \partial_k \bar{u}\Delta^2 u)\, dx \\ &\quad -2\int_{\mathbb{R}^N} \partial_k a\{|x|^{2b}|u|^{2(q-1)}u, u\}_p^k\, dx \end{aligned}$$

The first equality in the above Lemma follows as in Proposition 3.1 in [13]. For the second equality, it is sufficient to use the identity

$$\{|x|^{2b}|u|^{2(q-1)}u, u\}_p = -\frac{q-1}{q}\nabla(|x|^{2b}|u|^{2q}) - \frac{1}{q}\nabla(|x|^{2b})|u|^{2q}.$$

□

Now, one proves the Morawetz estimate.

Proof of Proposition 2.6 For a vector $e \in \mathbb{R}^N$, denote

$$\nabla_e u := (\frac{e}{|e|}.\nabla u)\frac{e}{|e|} \quad \text{and} \quad \nabla_e^{\perp} u := \nabla u - \nabla_e u.$$

Compute, taking account of [10],

$$\begin{aligned} 2\partial_{jk}\Delta a \partial_j u \partial_k \bar{u} &= \frac{2(N-1)}{|\cdot|^3}\Big(2|\nabla_e u|^2 - |\nabla_e^{\perp} u|\Big); \\ \partial_{jk}a\partial_{ij}\bar{u}a\partial_{ik}u &= \frac{1}{|\cdot|}\sum_i \Big(|\nabla\partial_i u|^2 - |\nabla_e \partial_i u|^2\Big) \geq \frac{N-1}{|\cdot|^3}|\nabla_e u|^2. \end{aligned}$$

Thus, if one assumes that $\Delta^2 a \le 0$, one gets

$$\begin{aligned} M' &= 2\int_{\mathbb{R}^N}\Big(2\partial_{jk}\Delta a\partial_j u\partial_k\bar u - \frac{1}{2}(\Delta^3 a)|u|^2 - 4\partial_{jk}a\partial_{ik}u\partial_{ij}\bar u \\ &+ \Delta^2 a|\nabla u|^2 - \frac{q-1}{q}(\Delta a)|x|^{2b}|u|^{2q} + \frac{1}{q}\nabla a\nabla(|x|^{2b})|u|^{2q}\Big)\,dx \\ &\le 2\int_{\mathbb{R}^N}\Big(\frac{2(N-1)}{|x|^3}\Big(2|\nabla_e u|^2 - |\nabla_e^{\perp}u|\Big) - 4\frac{N-1}{|x|^3}|\nabla_e u|^2 \\ &- \frac{1}{2}(\Delta^3 a)|u|^2 + \Delta^2 a|\nabla u|^2 - \frac{q-1}{q}(\Delta a)|x|^{2b}|u|^{2q} + \frac{1}{q}\nabla a\nabla(|x|^{2b})|u|^{2q}\Big)\,dx \\ &\le \int_{\mathbb{R}^N}\Big(-\frac{1}{2}(\Delta^3 a)|u|^2 - \frac{q-1}{q}(\Delta a)|x|^{2b}|u|^{2q} + \frac{1}{q}\nabla a\nabla(|x|^{2b})|u|^{2q}\Big)\,dx. \end{aligned}$$

This gives

$$\int_0^T\int_{\mathbb{R}^N}\Big(\frac{1}{2}(\Delta^3 a)|u|^2 + \frac{q-1}{q}(\Delta a)|x|^{2b}|u|^{2q} - \frac{1}{q}\nabla a\nabla(|x|^{2b})|u|^{2q}\Big)\,dx \lesssim \sup_{[0,T]}|M|.$$

Take the choice $a := |\cdot|$ and compute for $N \ge 5$, the derivatives

$$\begin{aligned} \nabla a = \frac{\cdot}{|\cdot|}, \quad \Delta a = \frac{N-1}{|\cdot|}; \\ \Delta^2 a = -\frac{(N-1)(N-3)}{|\cdot|^3}. \end{aligned}$$

Moreover,

$$\Delta^3 a = \begin{cases} C\delta_0, & \text{if} \quad N = 5; \\ \frac{3(N-1)(N-3)(N-5)}{|\cdot|^5}, & \text{if} \quad N \ge 6. \end{cases}$$

Finally,

$$\begin{aligned} \|u_0\|_{H^2} &\gtrsim \sup_{[0,T]}|M| \\ &\gtrsim \int_0^T\int_{\mathbb{R}^N}\Big((q-1)(\Delta a)|x|^{2b}|u|^{2q} - \nabla a\nabla(|x|^{2b})|u|^{2q}\Big)\,dx \\ &\gtrsim ((q-1)(N-1)-2b)\int_0^T\int_{\mathbb{R}^N}|x|^{2b-1}|u|^{2q}\,dx \\ &\gtrsim \int_0^T\int_{\mathbb{R}^N}|x|^{2b-1}|u|^{2q}\,dx. \end{aligned}$$

This ends the proof. □

3.2 Decay of Global Solutions

The goal of this subsection is to prove the long time decay of the energy global solutions to the inhomogeneous Schrödinger problem (1.1). Let us give an intermediate result in the spirit of [18, 19].

Lemma 3.2 *Take $N \ge 3$, $\max\{-4, -\frac{N}{2}\} < 2b < 0$ and $\max\{1, 1 + \frac{2+2b}{N}\} < q < q^*$. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ to be a cut-off function and (φ_n) be a sequence in H^2 satisfying*

$\sup_n \|\varphi_n\|_{H^2} < \infty$ *and* $\varphi_n \rightharpoonup \varphi$ *in* H^2. *Let* u_n *(respectively* u*) be the solution in* $C(\mathbb{R}, H^2)$ *to Eq. 1.1 with initial data* φ_n *(respectively* φ*). Then, for every* $\varepsilon > 0$*, there exist* $T_\varepsilon > 0$ *and* $n_\varepsilon \in \mathbb{N}$ *such that*

$$\|\chi(u_n - u)\|_{L^\infty_{T_\varepsilon}(L^2)} < \varepsilon, \quad \forall n > n_\varepsilon.$$

Proof The proof follows by arguing like [16] via Lemma 3.2 in [5]. □

Now, let us prove the long time decay for global solutions to Eq. 1.2.

Proof of Proposition 2.5 By an interpolation argument, it is sufficient to establish the equality

$$\lim_{t\to\infty} \|u(t)\|_{2+\frac{4}{N}} = 0.$$

Recall the localized Gagliardo-Nirenberg inequality [16],

$$\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} \lesssim \Big(\sup_{x\in\mathbb{R}^N} \|u\|_{L^2(Q_1(x))}\Big)^{1+\frac{4}{N}} \|u\|_{H^2}.$$

Here $Q_r(x)$ denotes the cubic in $\mathbb{R}^N$ with center x and radius $r > 0$.

One proceeds by contradiction. Assume that there exist a sequence (t_n) of positive real numbers and $\varepsilon > 0$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\|u(t_n)\|_{L^{2+\frac{4}{N}}} > \varepsilon, \quad \forall n \in \mathbb{N}.$$

Thanks to the conservation laws and the localized Gagliardo-Nirenberg inequality above, there exist a sequence (x_n) in $\mathbb{R}^N$ and a positive real number denoted also by $\varepsilon > 0$ such that

$$\|u(t_n)\|_{L^2(Q_1(x_n))} \geq \varepsilon \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Following the paper [16], via the previous Lemma, there exist $T, n_\epsilon > 0$ such that $t_{n+1} - t_n > T$ for $n \geq n_\varepsilon$ and

$$\|u(t)\|_{L^2(Q_2(x_n))} \geq \frac{\varepsilon}{4}, \quad \forall t \in [t_n, t_n + T], \quad \forall n \geq n_\varepsilon.$$

Moreover, the radial setting via (3.1) implies that (x_n) is bounded. So, thanks to Morawetz estimate in Proposition 2.6, one gets

$$\begin{aligned}
\|u_0\|_{H^2} &\gtrsim \int_{\mathbb{R}} \int_{\mathbb{R}^N} |x|^{2b-1} |u(t,x)|^{2q}\, dx\, dt \\
&\gtrsim \sum_n \int_{t_n}^{t_n+T} \int_{Q_2(x_n)} |x|^{2b-1} |u(t,x)|^{2q}\, dx\, dt \\
&\gtrsim \sum_n \int_{t_n}^{t_n+T} \|u(t)\|_{L^2(Q_2(x_n))}^{2q}\, dt \\
&\gtrsim \sum_n (\frac{\varepsilon}{4})^{2q} T = \infty.
\end{aligned}$$

This contradiction achieves the proof. □

3.3 Scattering

This subsection is concerned with the proof of the main result of this paper about scattering of energy global solutions to the inhomogeneous Schrödinger problem (1.2). Take the quantities

$$\|\cdot\|_{S(0,T)} := \sup_{(q,r)\in\Gamma} \|\cdot\|_{L^q_T(L^r)} \quad \text{and} \quad \langle\cdot\rangle := (1+\Delta)\cdot$$

Let us give an intermediate result.

Lemma 3.3 *Take $N \geq 3$, $\max\{-4, -\frac{N}{2}\} < 2b < 0$, $q_* < q < q^*$ and $u \in C(\mathbb{R}, H^2)$ be a global solution to Eq. 1.1. Then, there exist $2 < q_1, q_2 < \frac{2N}{N-4}$ and $0 < \theta_1, \theta_2 < 2(q-1)$ such that*

$$\|\left\langle u - e^{i.\Delta^2}u_0\right\rangle\|_{S(0,T)} \lesssim \|u\|^{\theta_1}_{L^\infty_T(L^{q_1})}\|\langle u\rangle\|^{2q-1-\theta_1}_{S(0,T)} + \|u\|^{\theta_2}_{L^\infty_T(L^{q_2})}\|\langle u\rangle\|^{2q-1-\theta_2}_{S(0,T)}$$

Proof With Duhamel formula and Strichartz estimates, one writes

$$\begin{aligned}
\|\left\langle u - e^{i.\Delta^2}u_0\right\rangle\|_{S(0,T)} &\lesssim \||x|^{2b}|u|^{2(q-1)}u\|_{S'_T(|x|<1)} + \||x|^{2b}|u|^{2(q-1)}u\|_{S'_T(|x|>1)} \\
&+ \|\nabla(|x|^{2b}|u|^{2(q-1)}u)\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))} \\
&+ \|\nabla(|x|^{2b}|u|^{2(q-1)}u)\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|>1))} \\
&:= (A)+(B)+(C)+(D).
\end{aligned}$$

Now, let us deal with the quantity (C).

$$\begin{aligned}
(C) &:= \|\nabla(|x|^{2b}|u|^{2(q-1)}u)\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))} \\
&\lesssim \||x|^{2b}|u|^{2(q-1)}\nabla u\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))} + \||x|^{2b-1}|u|^{2q-1}\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))} \\
&\lesssim (C_1)+(C_2).
\end{aligned}$$

Thanks to Hölder and Sobolev inequalities, one has

$$\begin{aligned}
(C_1) &:= \||x|^{2b}|u|^{2(q-1)}\nabla u\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))} \\
&\lesssim \||x|^{2b}\|_{\mu_1}\|\|u\|^{2(q-1)}_{r_1}\|\nabla u\|_{a_1}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_1}_{L^\infty_T(L^{r_1})}\|\|u\|^{2(q-1)-\theta_1}_{r_1}\|\Delta u\|_{r_1}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_1}_{L^\infty_T(L^{r_1})}\|\|u\|^{2q-1-\theta_1}_{W^{2,r_1}}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_1}_{L^\infty_T(L^{r_1})}\|u\|^{2q-1-\theta_1}_{L^{q_1}_T(W^{2,r_1})}.
\end{aligned}$$

Here $q_1 := 2(2q-1-\theta_1)$, $(q_1, r_1) \in \Gamma$, $\frac{1}{a_1} = \frac{1}{r_1} - \frac{1}{N}$ and

$$\frac{4+N}{2N} - \frac{2q-1}{r_1} = \frac{1}{\mu_1} > \frac{-2b}{N}.$$

Thus,

$$\frac{2q-1}{r_1} < \frac{4+4b+N}{2N}, \quad r_1 := \frac{2N(2q-1)\alpha}{4+4b+N}, \quad \alpha = 1^+;$$

$$N(\frac{1}{2} - \frac{1}{r_1}) = \frac{4}{q_1} = \frac{2}{2q-1-\theta_1}.$$

A computation gives that the condition $\theta_1 \in (0, 2(q-1))$ is equivalent to

$$2 < \frac{8(2q-1)}{N(2q-1) - (4 + N + 4b)} < 2(2q-1).$$

This is satisfied because $q_* < q < q^*$. Now, one estimates the second term by use of Hölder estimate ans Sobolev injections.

$$\begin{aligned}(C_2) &\lesssim \| |x|^{2b-1}|u|^{2(q-1)}u\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))} \\ &\lesssim \| \||x|^{2b-1}\|_{\mu_2}\|\|u\|_{r_2}^{2(q-1)}\|u\|_{a_2}\|_{L^2(0,T)} \\ &\lesssim \|u\|^{\theta_2}_{L^\infty_T(L^{r_2})}\|\|u\|_{r_2}^{2(q-1)-\theta_2}\|\Delta u\|_{r_2}\|_{L^2(0,T)} \\ &\lesssim \|u\|^{\theta_2}_{L^\infty_T(L^{r_2})}\|\|u\|_{W^{2,r_2}}^{2q-1-\theta_2}\|_{L^2(0,T)} \\ &\lesssim \|u\|^{\theta_2}_{L^\infty_T(L^{r_2})}\|u\|^{2q-1-\theta_2}_{L^{q_2}_T(W^{2,r_2})}.\end{aligned}$$

Here $(q_2, r_2) \in \Gamma$ and

$$\begin{aligned}\frac{2+N}{2N} &= \frac{1}{\mu_2} + \frac{2(q-1)}{r_2} + \frac{1}{a_2}; \\ \frac{1}{a_2} &:= \frac{1}{r_2} - \frac{2}{N}; \\ \mu_2 &< \frac{N}{1-2b}; \\ q_2 &:= 2(2q-1-\theta_2).\end{aligned}$$

Thus,

$$\begin{aligned} r_2 &> \frac{2N(2q-1)}{4+4b+N}; \\ N(\frac{1}{2} - \frac{1}{r_2}) &= \frac{4}{q_2} = \frac{2}{2q-1-\theta_2}; \\ 2 &< q_2 < 2(2q-1).\end{aligned}$$

These are the same conditions above. Thus, one takes $(q_2, r_2) := (q_1, r_1)$. The estimate of (A) follows as (C_1). Moreover, the control of the same terms on the the complementary of the ball follows similarly. This finishes the proof. □

Now, let us prove the main result of this section.

Proof of Theorem 2.4 Taking account of Lemma 4.3 via the decay of solutions and the absorption Lemma 2.18, one gets

$$\langle u\rangle \in S(\mathbb{R}) := \cap_{(q,r)\in\Gamma} L^q(\mathbb{R}, L^r(\mathbb{R}^N)).$$

This implies that, via Strichartz estimate via the proof of the previous Lemma, that when $s, t \to \infty$,

$$\begin{aligned}\|e^{-it\Delta^2}u(t) - e^{-is\Delta^2}u(s)\|_{H^2} &\lesssim \| |x|^{2b}|u|^{2(q-1)}u\|_{L^2((t,s),W^{1,\frac{2N}{2+N}})} \\ &\lesssim \|u\|^{\theta_1}_{L^\infty(\mathbb{R},L^{q_1})}\|\langle u\rangle\|^{2q-1-\theta_1}_{S(s,t)} + \|u\|^{\theta_2}_{L^\infty(\mathbb{R},L^{q_2})}\|\langle u\rangle\|^{2q-1-\theta_2}_{S(s,t)} \\ &\to 0.\end{aligned}$$

Take $u_\pm := \lim_{t\to\pm\infty} e^{-it\Delta^2}u(t)$ in H^2. Thus,

$$\lim_{t\to\pm\infty}\|u(t)-e^{it\Delta^2}u_\pm\|_{H^2}=0.$$

The scattering is proved. □

4 The Choqad Problem (1.2)

4.1 Morawetz Identity

In this subsection, one proves Proposition 2.11 about a classical Morawetz estimate satisfied by the energy global solutions to the inhomogeneous Choquard problem (1.2). Let us start with an auxiliary result.

Proposition 4.1 *Take* $N\geq 5$, $0<\alpha<N$, $\max\{-4(1+\frac{\alpha}{N}), N-8-\alpha\}<2b<0$, $2\leq p<p^*$ *and* $u\in C_T(H^2)$ *be a local solution to Eq. 1.1. Let* $a:\mathbb{R}^N\to\mathbb{R}$ *be a convex smooth function and the real function defined on* $[0,T)$*, by*

$$M:t\to 2\int_{\mathbb{R}^N}\nabla a(x)\Im(\nabla u(t,x)\bar{u}(t,x))\,dx.$$

Then, the following equality holds on $[0,T)$*,*

$$\begin{aligned}
M' &= 2\int_{\mathbb{R}^N}\Big(2\partial_{jk}\Delta a\partial_j u\partial_k\bar{u}-\frac{1}{2}(\Delta^3 a)|u|^2-4\partial_{jk}a\partial_{ik}u\partial_{ij}\bar{u}\\
&\quad+\Delta^2 a|\nabla u|^2-\partial_j a\{(I_\alpha*|\cdot|^b|u|^p)|x|^b|u|^{p-2}u,u\}_p^j\Big)\,dx\\
&=2\int_{\mathbb{R}^N}\Big(2\partial_{jk}\Delta a\partial_j u\partial_k\bar{u}-\frac{1}{2}(\Delta^3 a)|u|^2-4\partial_{jk}a\partial_{ik}u\partial_{ij}\bar{u}+\Delta^2 a|\nabla u|^2\Big)\\
&\quad-2\Big((-1+\frac{2}{p})\int_{\mathbb{R}^N}\Delta a(I_\alpha*|\cdot|^b|u|^p)|x|^b|u|^p\,dx\\
&\quad+\frac{2}{p}\int_{\mathbb{R}^N}\partial_k a\partial_k(|x|^b[I_\alpha*|\cdot|^b|u|^p])|u|^p\,dx\Big).
\end{aligned}$$

Proof Denote the source term $\mathcal{N}:=(I_\alpha*|\cdot|^b|u|^p)|x|^b|u|^{p-2}u$. By previous computation, one gets

$$M'=-2\int_{\mathbb{R}^N}\Delta a\Re(\bar{u}\Delta^2 u)\,dx-4\int_{\mathbb{R}^N}\Re(\partial_k a\partial_k\bar{u}\Delta^2 u)\,dx-2\int_{\mathbb{R}^N}\partial_k a\{\mathcal{N},u\}_p^k\,dx$$

On the other hand

$$\begin{aligned}(I) &:= \int_{\mathbb{R}^N} \partial_k a \Re(\bar{u}\partial_k \mathcal{N} - \partial_k \bar{u}\mathcal{N})\,dx\\ &= \int_{\mathbb{R}^N} \partial_k a \Re(\partial_k[\bar{u}\mathcal{N}] - 2\partial_k \bar{u}\mathcal{N})\,dx\\ &= -\int_{\mathbb{R}^N} \Big(\Delta a \bar{u}\mathcal{N} + 2\Re(\partial_k a \partial_k \bar{u}\mathcal{N})\Big)\,dx\\ &= -\int_{\mathbb{R}^N} \Big(\Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p + 2\Re(\partial_k a \partial_k \bar{u}\mathcal{N})\Big)\,dx\\ &= -\int_{\mathbb{R}^N} \Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p\,dx - \frac{2}{p}\int_{\mathbb{R}^N} \partial_k a \partial_k(|u|^p)(I_\alpha * |\cdot|^b|u|^p)|x|^b\,dx.\end{aligned}$$

Moreover,

$$\begin{aligned}(A) &:= \int_{\mathbb{R}^N} |x|^b \partial_k a \partial_k(|u|^p)(I_\alpha * |\cdot|^b|u|^p)\,dx\\ &= -\int_{\mathbb{R}^N} div(|x|^b\partial_k a(I_\alpha * |\cdot|^b|u|^p))|u|^p\,dx\\ &= -\int_{\mathbb{R}^N} \Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p\,dx - \int_{\mathbb{R}^N} \partial_k a \partial_k(|x|^b[I_\alpha * |\cdot|^b|u|^p])|u|^p\,dx.\end{aligned}$$

Then,

$$\begin{aligned}(I) &= -\int_{\mathbb{R}^N} \Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p\,dx - \frac{2}{p}(A)\\ &= -\int_{\mathbb{R}^N} \Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p\,dx\\ &- \frac{2}{p}(-\int_{\mathbb{R}^N} \Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p\,dx - \int_{\mathbb{R}^N} \partial_k a \partial_k(|x|^b[I_\alpha * |\cdot|^b|u|^p])|u|^p\,dx)\\ &= (-1 + \frac{2}{p})\int_{\mathbb{R}^N} \Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p\,d\\ &+ \frac{2}{p}\int_{\mathbb{R}^N} \partial_k a \partial_k(|x|^b[I_\alpha * |\cdot|^b|u|^p])|u|^p\,dx.\end{aligned}$$

This closes the proof. □

Now, one proves the Morawetz estimate.

Proof of Proposition 2.11 Arguing as in the proof of Proposition 2.6, if one assumes that $\Delta^2 a \le 0$ and $\Delta^3 a \ge 0$, one gets

$$\begin{aligned}M' &= 2\int_{\mathbb{R}^N} \Big(2\partial_{jk}\Delta a \partial_j u \partial_k \bar{u} - \frac{1}{2}(\Delta^3 a)|u|^2 - 4\partial_{jk} a \partial_{ik} u \partial_{ij}\bar{u} + \Delta^2 a|\nabla u|^2\Big)\,dx\\ &- 2(-1 + \frac{2}{p})\int_{\mathbb{R}^N} \Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p\,d\\ &- \frac{4}{p}\int_{\mathbb{R}^N} \partial_k a \partial_k(|x|^b[I_\alpha * |\cdot|^b|u|^p])|u|^p\,dx\\ &\le \int_{\mathbb{R}^N} \Big(-2(-1 + \frac{2}{p})\Delta a (I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^p - \frac{4}{p}\partial_k a \partial_k(|x|^b[I_\alpha * |\cdot|^b|u|^p])|u|^p\Big)\,dx\end{aligned}$$

This gives

$$\int_0^T\int_{\mathbb{R}^N}\Big(2(1-\frac{2}{p})\Delta a(I_\alpha*|\cdot|^b|u|^p)|x|^b|u|^p-\frac{4}{p}\partial_k a\partial_k(|x|^b[I_\alpha*|\cdot|^b|u|^p])|u|^p\Big)\,dx\lesssim\sup_{[0,T]}|M|.$$

Take the choice $a:=|\cdot|$.

Taking account of previous computations, since $b<0$, one gets

$$\begin{aligned}\|u_0\|_{H^2}&\gtrsim\sup_{[0,T]}|M|\\&\gtrsim\int_0^T\int_{\mathbb{R}^N}\Big(\Delta a(I_\alpha*|\cdot|^b|u|^p)|x|^b|u|^p-\partial_k a\partial_k(|x|^b[I_\alpha*|\cdot|^b|u|^p])|u|^p\Big)\,dx\,dt\\&\gtrsim\int_0^T\int_{\mathbb{R}^N}\Big((I_\alpha*|\cdot|^b|u|^p)|x|^{b-1}|u|^p-b|x|^{b-1}[I_\alpha*|\cdot|^b|u|^p])|u|^p\\&+(N-\alpha)\frac{x}{|x|}[\frac{\cdot}{|\cdot|^2}I_\alpha*|\cdot|^b|u|^p]|x|^b|u|^p\Big)\,dx\,dt\\&\gtrsim\int_0^T\int_{\mathbb{R}^N}\Big([I_\alpha*|\cdot|^b|u|^p])|x|^{b-1}|u|^p+\frac{x}{|x|}[\frac{\cdot}{|\cdot|^2}I_\alpha*|\cdot|^b|u|^p]|x|^b|u|^p\Big)\,dx\,dt.\end{aligned}$$

Now, write

$$\begin{aligned}(D)&:=\int_{\mathbb{R}^N}\frac{x}{|x|}|x|^b[\frac{\cdot}{|\cdot|^2}I_\alpha*|\cdot|^b|u|^p]|u(x)|^p\,dx\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{x}{|x|}|x|^b\frac{x-z}{|x-z|^2}I_\alpha(x-z)|z|^b|u(z)|^p|u(x)|^p\,dx\,dz\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{z}{|z|}|z|^b\frac{z-x}{|x-z|^2}I_\alpha(x-z)|z|^b|u(z)|^p|u(x)|^p\,dx\,dz\\&=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{I_\alpha(x-z)}{|x-z|^2}|x|^b|z|^b|u(z)|^p|u(x)|^p(x-z)\Big(\frac{x}{|x|}-\frac{z}{|z|}\Big)\,dx\,dz.\end{aligned}$$

Then, $(D)\geq 0$ because

$$(x-z)\Big(\frac{x}{|x|}-\frac{z}{|z|}\Big)=(|x||z|-xz)\Big(\frac{|x|+|z|}{|x||z|}\Big)\geq 0.$$

The proof is closed. □

4.2 Decay of Global Solutions

The goal of this subsection is to prove the long time decay of the energy global solutions to the inhomogeneous Choquard problem (1.1). Let us give an intermediate result.

Lemma 4.2 *Take $N\geq 5$, $0<\alpha<N$, $\max\{-4(1+\frac{\alpha}{N}),N-8-\alpha\}<2b<0$, $2\leq p<p^*$. Let $\chi\in C_0^\infty(\mathbb{R}^N)$ to be a cut-off function and (φ_n) be a sequence in H^2 satisfying $\sup_n\|\varphi_n\|_{H^2}<\infty$ and $\varphi_n\rightharpoonup\varphi$ in H^2. Let u_n (respectively u) be the solution in $C(\mathbb{R},H^2)$ to Eq. 1.1 with initial data φ_n (respectively φ). Then, for every $\varepsilon>0$, there exist $T_\varepsilon>0$ and $n_\varepsilon\in\mathbb{N}$ such that*

$$\|\chi(u_n-u)\|_{L^\infty_{T_\varepsilon}(L^2)}<\varepsilon,\quad\forall n>n_\varepsilon.$$

Proof Let $v_n:=\chi u_n$ and $v:=\chi u$.

Denote $w_n := v_n - v$ and $\mathcal{N}_u := (I_\alpha * |u|^p)|x|^b|u|^{p-2}u$.

Using Strichartz estimate and Corollary 2.16, assuming that $supp(\chi) \subset \{|x| < 1\}$, one has

$$\begin{aligned}\|\chi(\mathcal{N}_{u_n} - \mathcal{N}_u)\|_{S'(0,T)} &\lesssim \|(I_\alpha * [|\cdot|^b|u_n|^p - |\cdot|^b|u|^p)]|x|^b|u_n|^{p-2}v_n\|_{L_T^{q'}(L^{r'}(|x|<1))} \\ &+ \|(I_\alpha * |\cdot|^b|u|^p)(|x|^b|u_n|^{p-2}v_n - |x|^b|u|^{p-2}v)\|_{L_T^{q'}(L^{r'}(|x|<1))} \\ &\lesssim (I) + (II),\end{aligned}$$

where $(q, r) \in \Gamma$. Take $\mu := (\frac{N}{-b})^-$ and $r := \frac{2Np}{\alpha+N-\frac{2N}{\mu}}$. Then, $1 + \frac{\alpha}{N} = \frac{2}{\mu} + \frac{2p}{r}$ and using Hölder and Hardy-Littlewood-Sobolev inequalities, one gets

$$\begin{aligned}(II) &= \|(I_\alpha * |\cdot|^b|u|^p)(|x|^b|u_n|^{p-2}v_n - |x|^b|u|^{p-2}v)\|_{L_T^{q'}(L^{r'}(|x|<1))} \\ &\lesssim \|(I_\alpha * |\cdot|^b|u|^p)|x|^b(|u_n|^{p-2} + |u|^{p-2})w_n\|_{L_T^{q'}(L^{r'}(|x|<1))} \\ &\lesssim \||x|^b\|^2_{L^\mu(|x|<1)}\|(\|u_n\|_r^{2(p-1)} + \|u\|_r^{2(p-1)})\|w_n\|_r\|_{L^{q'}(0,T)} \\ &\lesssim \|(\|u_n\|_r^{2(p-1)} + \|u\|_r^{2(p-1)})\|w_n\|_r\|_{L^{q'}(0,T)}.\end{aligned}$$

Because $2 \le p < p^*$, there exists $\delta > 0$ such that $\frac{1}{q'} = \frac{1}{q} + \frac{1}{\delta}$ and $2 < r < \frac{2N}{N-4}$. Then, taking account of Sobolev embeddings and Hölder inequality, one obtains

$$\begin{aligned}(II) &\lesssim T^{\frac{1}{\delta}}\left(\|u_n\|^{2(p-1)}_{L_T^\infty(L^r)} + \|u\|^{2(p-1)}_{L_T^\infty(L^r)}\right)\|w_n\|_{S(0,T)} \\ &\lesssim T^{\frac{1}{\delta}}\left(\|u_n\|^{2(p-1)}_{L_T^\infty(H^2)} + \|u\|^{2(p-1)}_{L_T^\infty(H^2)}\right)\|w_n\|_{S(0,T)} \\ &\lesssim T^{\frac{1}{\delta}}\|w_n\|_{S(0,T)}.\end{aligned}$$

Similarly, one estimates (I). Now, taking account of computation done in the proof follows like the proof of Lemma 2.2 in [16], one gets

$$\|w_n\|_{S(0,T)} \lesssim \|\chi(\varphi_n - \varphi)\| + T + T^{\frac{1}{\delta}}\|w_n\|_{S(0,T)}.$$

The proof is achieved via Rellich Theorem. □

Now, let us prove the long time decay for global solutions to Eq. 1.2.

Proof of Proposition 2.10 The proof follows like the proof of Proposition 2.5. One gives only the last lines. Thanks to Morawetz estimate, one gets

$$\begin{aligned}\|u_0\|_{H^2} &\gtrsim \int_{\mathbb{R}}\int_{\mathbb{R}^N}(I_\alpha * |\cdot|^b|u(t)|^p)|x|^{b-1}|u(t,x)|^p\,dx\,dt \\ &\gtrsim \sum_n \int_{t_n}^{t_n+T}\int_{Q_2(x_n)}\int_{Q_2(x_n)}\frac{|x|^{b-1}|y|^b}{|x-y|^{N-\alpha}}|u(t,y)|^p|u(t,x)|^p\,dy\,dx\,dt.\end{aligned}$$

Now, with the radial assumption via the Eq. 3.1, the sequence (x_n) is bounded. Thus,

$$\begin{aligned}\|u_0\|_{H^2} &\gtrsim \sum_n \int_{t_n}^{t_n+T} \Big(\int_{Q_2(x_n)} |u(t,x)|^p\,dx\Big)^2 dt\\ &\gtrsim \sum_n \int_{t_n}^{t_n+T} \|u(t)\|^{2p}_{L^2(Q_2(x_n))}\,dt\\ &\gtrsim \sum_n (\frac{\varepsilon}{4})^{2p} T = \infty.\end{aligned}$$

This contradiction achieves the proof. □

4.3 Scattering

This subsection is concerned with the proof of the scattering of energy global solutions to the inhomogeneous Choquard problem (1.1).

Let us give an intermediate result.

Lemma 4.3 *Take* $N \geq 5$, $0 < \alpha < N$, $\max\{-4(1+\frac{\alpha}{N}), N-8-\alpha\} < 2b < 0$, $2 \leq p < p^*$ *and* $u \in C_T(H^2)$ *be a local solution to Eq. 1.1. Then, there exist* $2 < p_1, p_2 < \frac{2N}{N-4}$ *and* $0 < \theta_1, \theta_2 < 2(p-1)$ *such that*

$$\|\left\langle u - e^{i.\Delta^2}u_0\right\rangle\|_{S(0,T)} \lesssim \|u\|^{\theta_1}_{L^\infty_T(L^{p_1})}\|\left\langle u\right\rangle\|^{2p-1-\theta_1}_{S(0,T)} + \|u\|^{\theta_2}_{L^\infty_T(L^{p_2})}\|\left\langle u\right\rangle\|^{2p-1-\theta_2}_{S(0,T)}.$$

Proof With Duhamel formula and Strichartz estimates, one writes

$$\begin{aligned}&\|\left\langle u - e^{i.\Delta^2}u_0\right\rangle\|_{S(0,T)}\\ \lesssim\ & \|(I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}u\|_{S'_T(|x|<1)} + \|(I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}u\|_{S'_T(|x|>1)}\\ +\ & \|\nabla((I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}u)\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))}\\ +\ & \|\nabla((I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}u)\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|>1))}\\ :=\ & (A) + (B) + (C) + (D).\end{aligned}$$

Now, let us deal with the quantity (C).

$$\begin{aligned}(C) &:= \|\nabla((I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}u)\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))}\\ &\lesssim \|(I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}\nabla u\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))}\\ &+ \|(I_\alpha * |\cdot|^b|u|^p)|x|^{b-1}|u|^{p-1}\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))}\\ &+ \|(I_\alpha * |\cdot|^{b-1}|u|^p)|x|^b|u|^p\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))}\\ &+ \|(I_\alpha * |\cdot|^b|u|^{p-1}\nabla u)|x|^b|u|^{p-1}\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))}\\ &\lesssim (C_1) + (C_2) + (C_3) + (C_4).\end{aligned}$$

Thanks to Hölder, Hardy-Littlewood-Sobolev and Sobolev inequalities, one has

$$\begin{aligned}
(C_1) &:= \|(I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}\nabla u\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))} \\
&\lesssim \| \||x|^b\|^2_{\mu_1}\|u\|^{2(p-1)}_{r_1}\|\nabla u\|_{a_1}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_1}_{L^\infty_T(L^{r_1})}\| \|u\|^{2(p-1)-\theta_1}_{r_1}\|\Delta u\|_{r_1}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_1}_{L^\infty_T(L^{r_1})}\| \|u\|^{2p-1-\theta_1}_{W^{2,r_1}}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_1}_{L^\infty_T(L^{r_1})}\|u\|^{2p-1-\theta_1}_{L^{q_1}_T(W^{2,r_1})}.
\end{aligned}$$

Here $q_1 := 2(2p-1-\theta_1)$, $(q_1, r_1) \in \Gamma$, $\frac{1}{a_1} = \frac{1}{r_1} - \frac{1}{N}$ and

$$\frac{4+2\alpha+N}{2N} - \frac{2p-1}{r_1} = \frac{2}{\mu_1} > \frac{-2b}{N}.$$

Thus,

$$\frac{2p-1}{r_1} < \frac{4+2\alpha+4b+N}{2N}, \quad r_1 := \frac{2N(2p-1)\alpha}{4+2\alpha+4b+N}, \quad \alpha = 1^+;$$
$$N(\frac{1}{2} - \frac{1}{r_1}) = \frac{4}{q_1} = \frac{2}{2p-1-\theta_1}.$$

A computation gives that the condition $\theta_1 \in (0, 2(p-1))$ is equivalent to

$$2 < \frac{8(2p-1)}{N(2p-1)-(4+2\alpha+N+4b)} < 2(2p-1).$$

This is satisfied because $p_* < p < p^*$. Now, one estimates the second term by use of Hölder and Hardy-Littlewood-Sobolev estimates via Sobolev injections.

$$\begin{aligned}
(C_2) &\lesssim \|(I_\alpha * |\cdot|^b|u|^p)|x|^{b-1}|u|^{p-1}\|_{L^2_T(L^{\frac{2N}{2+N}}(|x|<1))} \\
&\lesssim \| \||x|^b\|_{\mu_3}\||x|^{b-1}\|_{\mu_2}\|u\|^{2(p-1)}_{r_2}\|u\|_{a_2}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_2}_{L^\infty_T(L^{r_1})}\| \|u\|^{2(p-1)-\theta_2}_{r_2}\|\Delta u\|_{r_2}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_2}_{L^\infty_T(L^{r_2})}\| \|u\|^{2p-1-\theta_2}_{W^{2,r_2}}\|_{L^2(0,T)} \\
&\lesssim \|u\|^{\theta_2}_{L^\infty_T(L^{r_2})}\|u\|^{2p-1-\theta_2}_{L^{q_2}_T(W^{2,r_2})}.
\end{aligned}$$

Here $(q_2, r_2) \in \Gamma$ and

$$\frac{2+2\alpha+N}{2N} = \frac{1}{\mu_2} + \frac{1}{\mu_3} + \frac{2(p-1)}{r_2} + \frac{1}{a_2};$$
$$\frac{1}{a_2} := \frac{1}{r_2} - \frac{2}{N};$$
$$\frac{1}{\mu_2} + \frac{1}{\mu_3} > \frac{1-2b}{N};$$
$$q_2 := 2(2p-1-\theta_2).$$

Thus,

$$r_2 > \frac{2N(2p-1)}{4+2\alpha+4b+N};$$
$$N(\frac{1}{2}-\frac{1}{r_2}) = \frac{4}{q_2} = \frac{2}{2p-1-\theta_2};$$
$$2 < q_2 < 2(2p-1).$$

These are the same conditions above. Thus, one takes $(q_2, r_2) := (q_1, r_1)$. The estimate of (A) follows as (C_1). Moreover, the control of the same terms on the the complementary of the ball follows similarly. This finishes the proof. □

Now, let us prove the main result of this section.

Proof of Theorem 2.4 Taking account of Lemma 4.3 via the decay of solutions and the absorption Lemma 2.18, one gets

$$\langle u\rangle \in S(\mathbb{R}) := \cap_{(q,r)\in\Gamma} L^q(\mathbb{R}, L^r(\mathbb{R}^N)).$$

This implies that, via Strichartz estimate via the proof of the previous Lemma, that when $s, t \to \infty$,

$$\begin{aligned}\|e^{-it\Delta^2}u(t) - e^{-is\Delta^2}u(s)\|_{H^2} &\lesssim \|(I_\alpha * |\cdot|^b|u|^p)|x|^b|u|^{p-2}u\|_{L^2((t,s),W^{1,\frac{2N}{2+N}})}\\ &\lesssim \|u\|^{\theta_1}_{L^\infty(\mathbb{R},L^{q_1})}\|\langle u\rangle\|^{2p-1-\theta_1}_{S(s,t)} + \|u\|^{\theta_2}_{L^\infty(\mathbb{R},L^{q_2})}\|\langle u\rangle\|^{2p-1-\theta_2}_{S(s,t)}\\ &\to 0.\end{aligned}$$

Take $u_\pm := \lim_{t\to\pm\infty} e^{-it\Delta^2}u(t)$ in H^2. Thus,

$$\lim_{t\to\pm\infty}\|u(t) - e^{it\Delta^2}u_\pm\|_{H^2} = 0.$$

The scattering is proved. □

Data availability statement The data that supports the findings of this study are available within the article.

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