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Overconvergence Properties of Dirichlet Series

Mayya Golitsyna[1]

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Abstract

In this paper we use potential theoretic arguments to establish new results concerning the overconvergence of Dirichlet series. Let $\sum_{j=0}^{\infty} a_j e^{-\lambda_j s}$ converge on the half-plane $\{\mathrm{Re}(s) > 0\}$ to a holomorphic function f. Our first result gives sufficient conditions for a subsequence of partial sums of the series to converge at every regular point of f. The second result shows, in particular, that if a subsequence of the partial sums of the series is uniformly bounded on a nonpolar compact set $K \subset \{\mathrm{Re}(s) < 0\}$ and $\xi \in \{\mathrm{Re}(s) = 0\}$ is a regular point of f, then this subsequence converges on a neighbourhood of ξ.

Keywords Dirichlet series · Overconvergence · Minimally thin sets

Mathematics Subject Classification (2010) 31B50

1 Introduction

Recently, potential theoretic techniques have been applied to obtain new results regarding convergence of sequences of polynomials in $\mathbb{C}$ (for a survey, see Gardiner [4]). In particular, the following theorem of Müller and Yavrian (Theorem 1 of [5]) describes properties of a convergent sequence of polynomials given its behaviour on a non-thin set at ∞. We recall that a set E is *non-thin* at a point $\zeta \in \mathbb{C}$ if

$$\limsup_{s \to \zeta, s \in E} u(s) = u(\zeta)$$

for each function u that is subharmonic on a neighbourhood of ζ. Further, a set is said to be non-thin at infinity if its image under inversion in the unit circle is non-thin at 0. We write $|| \cdot ||_K$ for the supremum norm over a compact set K.

Theorem A *Let Γ be a continuum in $\mathbb{C}$, and let $E \subset \mathbb{C}$ be a closed set which is non-thin at ∞. Suppose that (P_n) is a sequence of polynomials with $\deg(P_n) \leq d_n$, where (d_n) is*

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✉ Mayya Golitsyna
mayya.golitsyna@ucdconnect.ie

[1] School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland

increasing, that

$$\limsup_{n\to\infty} ||f - P_n||_\Gamma^{1/d_n} < 1,$$

for some analytic function $f : \Gamma \to \mathbb{C}$, *and that*

$$\limsup_{n\to\infty} |P_n(z)|^{1/d_n} \le 1 \quad (z \in E).$$

(i) *Then the function* f *extends holomorphically to a function that has a simply connected domain of existence* $G_f \subset \mathbb{C}$, *and* (P_n) *converges locally uniformly to* f *there.*
(ii) *If, in addition,* (d_{n+1}/d_n) *is bounded, then* f *extends to an entire function.*

This result has found particular application to the study of Taylor expansions of holomorphic functions. It is thus natural to consider its analogue for Dirichlet series.

We consider a general Dirichlet series of the form

$$\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}, \tag{1}$$

where $a_j \in \mathbb{C}$, $\lambda_j > 0$, the sequence (λ_j) is strictly increasing and $\lambda_j \to \infty$. We write S_q for the sum of its first q terms. The abscissa of convergence and the abscissa of absolute convergence of the series are defined respectively by

$$\sigma_c = \inf\{\mathrm{Re}(s) : \sum_{j=1}^{\infty} a_j e^{-\lambda_j s} \text{ is convergent}\},$$
$$\sigma_a = \inf\{\mathrm{Re}(s) : \sum_{j=1}^{\infty} a_j e^{-\lambda_j s} \text{ is absolutely convergent}\}.$$

Then the Dirichlet series converges for $s \in \mathbb{C}^+_{\sigma_c}$ and converges uniformly absolutely on $\mathbb{C}^+_\alpha$ for $\alpha > \sigma_a$, where $\mathbb{C}^+_a = \{\mathrm{Re}(s) > a\}$. (We also write $\mathbb{C}^-_a = \{\mathrm{Re}(s) < a\}$.) The abscissae assume values in $[-\infty, +\infty]$, and their difference, where defined, satisfies the inequality

$$0 \le \sigma_a - \sigma_c \le \limsup_{j\to\infty} \frac{\log j}{\lambda_j}.$$

The first aim of this paper is to establish an analogue of Theorem A for partial sums of Dirichlet series. In this case a new phenomenon is that the notion of minimal thinness replaces thinness. We will say that a set $E \subset \mathbb{C}^-_0$ is *minimally thin at* $-\infty$ *as a subset of* $\mathbb{C}^-_0$ if there is a positive superharmonic function u on $\mathbb{C}^-_0$ such that

$$\liminf_{|s|\to\infty, s\in E} \frac{u(s)}{-\mathrm{Re}(s)} > \inf_{s\in\mathbb{C}^-_0} \frac{u(s)}{-\mathrm{Re}(s)}. \tag{2}$$

By considering the case where $u \equiv 1$, it is easy to see that any strip of the form $S = \{a < \mathrm{Re}(s) < 0\}$ is minimally thin at $-\infty$ as a subset of $\mathbb{C}^-_0$. On the other hand, the half-line $\{\mathrm{Re}(s) < t, \mathrm{Im}(s) = 0\}$ is not minimally thin at $-\infty$ as a subset of $\mathbb{C}^-_0$ for any $t < 0$ (see Section 9.7 of [1]). We will discuss the concept of minimal thinness in more detail in Section 2. We can now formulate our first result.

Theorem 1 *Let* $E \subset \mathbb{C}^-_0$ *be a set which is not minimally thin at* $-\infty$ *as a subset of* $\mathbb{C}^-_0$. *Suppose that* $\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$ *converges to a holomorphic function* f *on the half-plane* $\mathbb{C}^+_0$, *that the abscissa of absolute convergence is less than* $+\infty$, *and that a subsequence* (S_{q_n}) *of the partial sums satisfies*

$$\limsup_{n\to\infty} \frac{1}{\lambda_{q_n}} \log |S_{q_n}(s)| \le 0 \quad (s \in E).$$

(i) *If* Ω *is a domain containing* $\mathbb{C}_0^+$ *and* f *has a holomorphic extension to* Ω, *then the subsequence* (S_{q_n}) *converges locally uniformly on* Ω *to* f.
(ii) *If* $(\lambda_{q_n}/\lambda_{q_{n-1}})$ *is bounded, then the function* f *is entire.*

It follows that, under the hypotheses of Theorem 1(i), there is a largest domain Ω to which f has a holomorphic extension, and Ω is simply connected.

In the next theorem we show, in particular, that if a subsequence of partial sums is bounded on a nonpolar compact set lying outside the half-plane of convergence, then it is convergent on a neighbourhood of any point ξ on the line of convergence provided that ξ is a regular point of f. In fact, it is enough to require that f has a holomorphic extension to some domain Ω such that $\mathbb{C}\backslash\Omega$ is thin (in the ordinary sense) at ξ. This result is inspired by Theorem 1 of [3].

Theorem 2 *Suppose that* $\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$ *is convergent on* $\mathbb{C}_0^+$ *to a holomorphic function* f *and the abscissa of absolute convergence is less than* $+\infty$. *Let* Ω *be a domain containing* $\mathbb{C}_0^+$ *such that* $\mathbb{C}\backslash\Omega$ *is thin at a point* $\xi \in \partial\Omega \cap \{\mathrm{Re}(s) = 0\}$, *and suppose that* f *has a holomorphic extension to* Ω. *If a subsequence* (S_{q_n}) *of the partial sums is uniformly bounded on a nonpolar compact set* $K \subset \mathbb{C}_0^-$, *then* (S_{q_n}) *converges uniformly on a neighbourhood of* ξ; *in particular,* f *has a holomorphic extension to a neighbourhood of* ξ.

2 Potential Theoretical Background

We write $\mathscr{U}_+(\Omega)$ for the set of positive superharmonic functions on Ω. A positive harmonic function h on a domain Ω is called *minimal* if any other harmonic function h' on Ω that satisfies $0 \le h' \le h$ is a constant multiple of h. We make use of the Martin boundary of a domain Ω, which we denote by Δ (or Δ_Ω). There is a Martin kernel $M_\Omega(\cdot,\cdot)$, defined on $\Omega \times \Delta$, which plays a role analogous to the Poisson kernel for a disc. When a point $y \in \Delta$ is fixed, $M_\Omega(\cdot, y)$ is called a Martin function. An account of the Martin boundary may be found in Chapter 8 of [1].

If $\Omega = \mathbb{C}_0^+$, the Martin boundary has a one-to-one correspondence with $\partial^\infty\mathbb{C}_0^+$, which is the boundary of $\mathbb{C}_0^+$ in the one-point compactification of the complex plane. We use $+\infty$ to denote the point of the Martin boundary of $\mathbb{C}_0^+$ that is associated with infinity. We can normalize the kernel so that $M_{\mathbb{C}_0^+}(s, +\infty) = \mathrm{Re}(s)$.

Given $u \in \mathscr{U}_+(\Omega)$ and $E \subset \mathbb{C}$, we define the *reduced function of* u *relative to* E *in* Ω by

$$\mathscr{R}_u^E(s) = \inf\{v(s) : v \in \mathscr{U}_+(\Omega) \text{ and } v \ge u \text{ on } E \cap \Omega\} \quad (s \in \Omega).$$

Its lower semicontinuous regularization $\widehat{\mathscr{R}}_u^E$ is a superharmonic function on Ω. Reduction in a different domain will result in a different function even when the set E and the function u stay unchanged. Each time we use reduction we will state the domain explicitly. We note that modifying the set E by a polar set does not affect the regularized reduction; see Theorem 5.3.4 (iv) of [1].

Let Ω^c denote $\mathbb{C}\backslash\Omega$. If a domain Ω satisfies $\mathbb{C}_b^+ \subset \Omega \subset \mathbb{C}_a^+$, where $a < b$, then Theorem 9.5.5 of [1] shows that the function

$$h(s) = \mathrm{Re}(s-a) - \widehat{\mathscr{R}}^{\Omega^c}_{\mathrm{Re}(\cdot - a)}(s), \tag{3}$$

where the reduction is in $\mathbb{C}_a^+$, is a minimal harmonic function on Ω, and so is a multiple of a Martin function. We denote the Martin boundary point of Ω associated with h by $+\infty$, and normalize the Martin kernel so that $M_\Omega(\cdot, +\infty) = h$. We note that

$$\mathrm{Re}(s-b) \le M_\Omega(s, +\infty) \le \mathrm{Re}(s-a). \tag{4}$$

We say that E is *minimally thin at* $+\infty$ *with respect to* Ω, for Ω as above, if there exists $u \in \mathscr{U}_+(\Omega)$ such that

$$\liminf_{s\in E, |s|\to\infty} \frac{u(s)}{M_\Omega(s,+\infty)} > \inf_{s\in\Omega} \frac{u(s)}{M_\Omega(s,+\infty)}. \tag{5}$$

Clearly, a subset of a minimally thin set is minimally thin. Modifying a set by a polar set does not affect whether it is minimally thin. By Theorem 9.5.5 (iii), the half-line $\{\mathrm{Re}(s) > b,\ \mathrm{Im}(s) = 0\}$ is not minimally thin at $+\infty$ as a subset of Ω.

A similar construction applies when $\mathbb{C}_a^- \subset \Omega \subset \mathbb{C}_b^-$, and we can define minimal thinness at $-\infty$ in this case as well.

3 Preliminary Results

In this section we make several observations regarding sequences of the form $((1/\lambda_{q_n}) \log |S_{q_n}|)$. These will be used in the proofs of Theorem 1 and Theorem 2. We recall that $\lambda_j > 0$, the sequence (λ_j) is strictly increasing and $\lambda_j \to \infty$. We need the following lemma.

Lemma 3 *Let $a \in \mathbb{R}$ and $P(s) = \sum_{j=1}^n a_j e^{-\lambda_j s}$ be a Dirichlet polynomial. Then*

$$\frac{1}{\lambda_n}\log|P(s)| \le -\mathrm{Re}(s-a) + \frac{1}{\lambda_n}\log\left(\sup_{s\in\mathbb{C}_a^+}|P(s)|\right) \quad (s\in\mathbb{C}_a^-).$$

Proof Without loss of generality we can suppose that $a = 0$, and we dismiss the trivial case where $P \equiv 0$. The function

$$u(s) = \frac{1}{\lambda_n}\log\frac{|P(s)|}{\sup_{s\in\mathbb{C}_0^+}|P(s)|}$$

is subharmonic on $\mathbb{C}$ and non-positive on $\overline{\mathbb{C}_0^+}$. When $s \in \mathbb{C}_0^-$ we have

$$u(s) = \frac{1}{\lambda_n}\log\left(|e^{-\lambda_n s}| \cdot \left|\sum_{j=1}^n \frac{a_j}{\sup_{s\in\mathbb{C}_0^+}|P(s)|} e^{(\lambda_n-\lambda_j)s}\right|\right) \le -\mathrm{Re}(s) + c,$$

where

$$c = \frac{1}{\lambda_n}\log\left(\sum_{j=1}^n \frac{|a_j|}{\sup_{s\in\mathbb{C}_0^+}|P(s)|}\right).$$

Since u is bounded above on every vertical strip $\{\sigma_1 \le \mathrm{Re}(s) \le \sigma_2\}$, we can apply Theorem 3.5.9 in [1] to conclude that the function $\sigma \mapsto \sup\{u(\sigma + it) : t \in \mathbb{R}\}$ is convex on $[\sigma_1, \sigma_2]$. Choosing $\sigma_1 < 0$ and $\sigma_2 = 0$ we have

$$\begin{aligned}\sup_{t\in\mathbb{R}} u(\sigma + it) &\le \frac{\sigma}{\sigma_1} \sup_{t\in\mathbb{R}} u(\sigma_1 + it) - \frac{\sigma - \sigma_1}{\sigma_1} \sup_{t\in\mathbb{R}} u(it) \\ &\le \frac{\sigma}{\sigma_1}(-\sigma_1 + c) = -\sigma + \frac{\sigma c}{\sigma_1}.\end{aligned}$$

Since σ_1 can be arbitrarily large, we conclude that $u(s) \le -\mathrm{Re}(s)$, as required. □

We now suppose that the series $\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$ converges on $\mathbb{C}_0^+$ to a function f, that the abscissa of absolute convergence is less than $+\infty$, and that (q_n) is a strictly increasing sequence of positive integers.

The sequence $((1/\lambda_{q_n}) \log |S_{q_n}|)$ is locally uniformly convergent on $\mathbb{C}_0^+$, and so locally uniformly bounded there. Let the number $\alpha > 0$ be greater than the abscissa of absolute convergence. By Lemma 3,

$$\frac{1}{\lambda_{q_n}} \log |S_{q_n}(s)| \le -\mathrm{Re}(s - \alpha) + \frac{1}{\lambda_{q_n}} \log \left(\sup_{s\in\mathbb{C}_\alpha^+} |S_{q_n}(s)| \right) \quad (s \in \mathbb{C}_\alpha^-). \tag{6}$$

Hence, the sequence $((1/\lambda_{q_n}) \log |S_{q_n}|)$ is locally uniformly bounded above on $\mathbb{C}$. Therefore, the upper-semicontinuous regularization w^* of the function

$$w = \limsup_{n\to\infty} \frac{1}{\lambda_{q_n}} \log |S_{q_n}|$$

is subharmonic on $\mathbb{C}$ by Theorem 5.7.1 of [1], and differs from w on a polar set. Passing to the limit in Eq. 6, we have

$$w^*(s) \le -\mathrm{Re}(s - \alpha) \quad (s \in \mathbb{C}_\alpha^-).$$

Moreover, $w^* \le 0$ on $\mathbb{C}_0^+$, so

$$\limsup_{s\to y} \frac{w^*(s) + \mathrm{Re}(s)}{-\mathrm{Re}(s - \alpha)} \le 0 \quad (y \in \partial^\infty \mathbb{C}_0^-).$$

Consequently, by Theorem 3.1.6 of [1],

$$w^*(s) \le -\mathrm{Re}(s) \quad (s \in \mathbb{C}_0^-). \tag{7}$$

Let Ω be a domain containing $\mathbb{C}_0^+$, and suppose the function f has a holomorphic extension to Ω. The sequence $((1/\lambda_{q_n}) \log |S_{q_n} - f|)$ is locally uniformly bounded above on Ω since

$$\frac{1}{\lambda_{q_n}} \log |S_{q_n}(s) - f(s)| \le \frac{1}{\lambda_{q_n}} \log(2 \max\{|S_{q_n}(s)|, |f(s)|\}) \quad (s \in \Omega). \tag{8}$$

Thus, the upper-semicontinuous regularization u^* of the function

$$u(s) = \limsup_{n\to\infty} \frac{1}{\lambda_{q_n}} \log |S_{q_n}(s) - f(s)|$$

is subharmonic on Ω by Theorem 5.7.1 of [1]. The estimate (8) implies that

$$u^*(s) \le \max\{w^*(s), 0\} \quad (s \in \Omega). \tag{9}$$

On $\mathbb{C}_\alpha^+$ we have

$$\begin{aligned}\frac{1}{\lambda_{q_n}}\log\left|\sum_{j=q_n+1}^{\infty}a_je^{-\lambda_js}\right| &= \frac{1}{\lambda_{q_n}}\log\left|\sum_{j=q_n+1}^{\infty}a_je^{-\lambda_j(s-\alpha)}e^{-\lambda_j\alpha}\right| \\ &\le -\mathrm{Re}(s-\alpha)+\frac{1}{\lambda_{q_n}}\log\left(\sum_{j=q_n+1}^{\infty}|a_j|e^{-\lambda_j\alpha}e^{(\lambda_{q_n}-\lambda_j)\mathrm{Re}(s-\alpha)}\right) \\ &\le -\mathrm{Re}(s-\alpha)+\frac{1}{\lambda_{q_n}}\log\left(\sum_{j=q_n+1}^{\infty}|a_j|e^{-\lambda_j\alpha}\right).\end{aligned}$$

The series in the last line converges. Hence

$$u^*(s)\le -\mathrm{Re}(s-\alpha)\quad (s\in\mathbb{C}_\alpha^+). \tag{10}$$

4 Proof of Theorem 1

Let the functions w^* and u^*, and the number α, be as defined in Section 3. The hypothesis of the theorem implies that $w^*\le 0$ on a set $\widetilde{E}$, which differs from E by at most a polar set. So $\widetilde{E}$ is also not minimally thin at $-\infty$ as a subset of $\mathbb{C}_0^-$. From Eq. 7, the function $-\mathrm{Re}(s)-w^*(s)$ is a positive superharmonic function on $\mathbb{C}_0^-$. Therefore, by the definition of minimal thinness,

$$\inf_{\mathbb{C}_0^-}\frac{-\mathrm{Re}(s)-w^*(s)}{-\mathrm{Re}(s)}=\liminf_{s\to\infty,s\in\widetilde{E}}\frac{-\mathrm{Re}(s)-w^*(s)}{-\mathrm{Re}(s)}\ge\liminf_{s\to\infty,s\in\widetilde{E}}\frac{-\mathrm{Re}(s)}{-\mathrm{Re}(s)}=1,$$

and so $w^*\le 0$ on $\mathbb{C}_0^-$. Hence $w^*\le 0$ on $\mathbb{C}$.

Proof of part (i) From Eq. 9, we see that $u^*\le 0$ on Ω. Further, Eq. 10 implies that $u^*<0$ on $\mathbb{C}_\alpha^+$. Thus, by the maximum principle, $u^*<0$ on Ω. For any bounded subdomain W of Ω such that $\overline{W}\subset\Omega$, there exists $\varepsilon>0$ such that $u<-\varepsilon$ on W. By Corollary 5.7.2 of [1], for any compact subset K of W there exists n_0 such that

$$\frac{1}{\lambda_{q_n}}\log|f(s)-S_{q_n}(s)|<-\varepsilon/2\quad (s\in K,\ n>n_0).$$

Thus $|f(s)-S_{q_n}(s)|<(e^{-\varepsilon/2})^{\lambda_{q_n}}$ on K, whence $||f-S_{q_n}||_K\to 0$ as $n\to\infty$. Hence (S_{q_n}) converges locally uniformly to f on W and, consequently, on Ω.

Proof of part (ii) Since there exists $c>1$ such that $\sup_n\lambda_{q_n}/\lambda_{q_{n-1}}<c$, we can pass to a subsequence of (S_{q_n}) to ensure that $\lambda_{q_n}\in[c^n,c^{n+1}]$ for all large enough n. Then the series $\sum_{n=1}^{\infty}e^{-\lambda_{q_n}}$ converges, and $(\lambda_{q_n}/\lambda_{q_{n-1}})$ is bounded. We consider the function v^*, which is the upper-semicontinuous regularization of

$$v(s)=\limsup_{n\to\infty}\frac{1}{\lambda_{q_n}}\log|S_{q_n}(s)-S_{q_{n-1}}(s)|.$$

We note that

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{1}{\lambda_{q_n}}\log|S_{q_n}-S_{q_{n-1}}| &\le \limsup_{n\to\infty}\frac{1}{\lambda_{q_n}}\log(2\max\{|S_{q_n}|,|S_{q_{n-1}}|\})\\
&\le \max\left\{\limsup_{n\to\infty}\frac{1}{\lambda_{q_n}}\log|S_{q_n}|,\limsup_{n\to\infty}\frac{1}{\lambda_{q_n}}\log^+|S_{q_{n-1}}|\right\}\\
&\le \max\left\{w,\limsup_{n\to\infty}\frac{1}{\lambda_{q_{n-1}}}\log^+|S_{q_{n-1}}|\right\}\\
&\le \max\{w,\max\{w,0\}\}\\
&= \max\{w,0\}.
\end{aligned}
$$

Hence $v^* \le \max\{w^*,0\} = 0$. Let $d > 1$ be such that $\lambda_{q_n}/\lambda_{q_{n-1}} \le d$ for all n. Then, on $\mathbb{C}_\alpha^+$,

$$
\begin{aligned}
v &= \limsup_{n\to\infty}\frac{1}{\lambda_{q_n}}\log|(S_{q_n}-f)+(f-S_{q_{n-1}})|\\
&\le \limsup_{n\to\infty}\frac{1}{\lambda_{q_n}}\max\{\log(2|S_{q_n}-f|),\log(2|f-S_{q_{n-1}}|)\}\\
&\le \limsup_{n\to\infty}\max\left\{\frac{1}{\lambda_{q_n}}\log(|S_{q_n}-f|),\frac{1}{d}\frac{1}{\lambda_{q_{n-1}}}\log(|f-S_{q_{n-1}}|)\right\}\\
&\le \frac{1}{d}u,
\end{aligned}
$$

where the final step holds because

$$
\frac{1}{\lambda_{q_n}}\log|f(s)-S_{q_n}(s)|<0 \quad \text{for large } n \text{ and } s\in\mathbb{C}_\alpha^+,
$$

by Eq. 10. Therefore, $v^*(s) \le d^{-1}u^*(s) \le -d^{-1}\mathrm{Re}(s-\alpha)$ for $s \in \mathbb{C}_\alpha^+$. The function $\sigma \mapsto \sup_{t\in\mathbb{R}} v^*(\sigma+it)$ is convex and less than $\min\{0, d^{-1}(\alpha-\sigma)\}$. Hence, $v^* \equiv -\infty$.

By Corollary 5.7.2 of [1], for any compact set $K \subset \mathbb{C}$ and number $M > 0$ there exists $N > 0$ such that for $n > N$ we have

$$
\frac{1}{\lambda_{q_n}}\log|S_{q_n}(s)-S_{q_{n-1}}(s)| \le -M \quad (s\in K).
$$

Therefore,

$$
|S_{q_n}(s)-S_{q_{n-1}}(s)| \le e^{-M\lambda_{q_n}} \quad (s\in K).
$$

Writing $S_{q_n} = S_{q_N} + \sum_{j=N+1}^{n}(S_{q_j}-S_{q_{j-1}})$, for $n > N$, we see that (S_{q_n}) converges on K. Since K is arbitrary, it converges on the whole plane and the function f is entire.

5 Proof of Theorem 2

The functions w^*, u^* and the number α are as defined in Section 3. Without loss of generality we suppose that $\xi = 0$, and by the maximum principle we may suppose that the complement of K has no bounded components. Let U be the largest domain containing $\mathbb{C}_0^+$ on which the sequence (S_{q_n}) converges. Then U is simply connected and consequently non-thin at every point of ∂U. Clearly, f has extension to U, so we can suppose that $U \subset \Omega$, by enlarging Ω if necessary. Then the function u^* is defined and negative on U. We see that $u^*(s) \le 0$ for $s \in \Omega \cap \partial U$ since U is non-thin at s. Moreover, if $u^*(s) < 0$, the argument used to prove Theorem 1 (i) would show that $s \in U$, so

$$
u^*(s) = 0 \quad (s \in \Omega \cap \partial U). \tag{11}
$$

We fix $a > 0$ small enough so that $K \cap \overline{D(0,a)} = \emptyset$. Let U_a be the component of $U \cap (\mathbb{C}_0^+ \cup D(0,a))$ that contains $\mathbb{C}_0^+$. Since U is simply connected, so is U_a. We define the closed set $L = \overline{U}_a \cup K$.

By Eq. 7, $-\mathrm{Re}(\cdot) - w^*$ is a positive superharmonic function on $\mathbb{C}_0^-$. The inequality $w^* \leq 0$ and, consequently, $-\mathrm{Re}(\cdot) - w^* \geq -\mathrm{Re}(\cdot)$ holds on L with the exception of at most a polar set. Hence, for $s \in \mathbb{C} \backslash L$,

$$-\mathrm{Re}(s) - w^*(s) \geq \widehat{\mathscr{R}}^L_{-\mathrm{Re}(\cdot)}(s),$$

where the reduction is in $\mathbb{C}_0^-$, and so

$$w^*(s) \leq -\mathrm{Re}(s) - \widehat{\mathscr{R}}^L_{-\mathrm{Re}(\cdot)}(s) = M_{\mathbb{C}\backslash L}(s, -\infty),$$

and, by Eq. 9,

$$u^*(s) \leq M_{\mathbb{C}\backslash L}(s, -\infty) \quad (s \in \Omega \backslash L). \tag{12}$$

The estimate (9) also provides that

$$u^*(s) \leq 0 \quad (s \in \Omega \cap L), \tag{13}$$

and we will now further estimate u^* on U_a. Since $-u^* \in \mathscr{U}_+(U_a)$ and the half-line $\{\mathrm{Re}(s) > \alpha,\ \mathrm{Im}(s) = 0\}$ is not minimally thin at $+\infty$ as a subset of U_a, and taking into account (10) and (4), we have

$$\inf_{s \in U_a} \frac{-u^*(s)}{M_{U_a}(s, +\infty)} = \liminf_{t \to +\infty, t \in \mathbb{R}} \frac{-u^*(t)}{M_{U_a}(t, +\infty)} \geq \liminf_{t \to +\infty, t \in \mathbb{R}} \frac{t - \alpha}{t + a} = 1,$$

whence

$$u^*(s) \leq -M_{U_a}(s, +\infty) \quad (s \in U_a). \tag{14}$$

We define the function v^*, which is the upper-semicontinuous regularization of

$$v(s) = \begin{cases} M_{\mathbb{C}\backslash L}(s, -\infty) & (s \in \mathbb{C} \backslash L), \\ -M_{U_a}(s, +\infty) & (s \in U_a), \\ 0 & (\text{elsewhere on } \mathbb{C}). \end{cases}$$

We note that $v^*(s)$ is negative if and only if $s \in U_a$, so to prove the theorem we only need to show that $v^*(0) < 0$. We borrow arguments from the proofs of Theorems 1 and 3 in [3]. The first step is to show that v^* is subharmonic on $\Omega \cap D(0,a)$. Then we construct a harmonic majorant for v^* on a subdomain of Ω and show that this majorant is negative on a circle centered at 0.

The estimates (12), (13) and (14) show that $u^* \leq v^*$ on Ω. The function v^* is subharmonic on $D(0,a) \backslash \partial U_a$ by construction. Let $s \in \partial U_a \cap \Omega \cap D(0,a)$. Then $s \in \partial U$ since it lies inside $D(0,a)$, and so $u^*(s) = 0$ by Eq. 11. For small r the disc $D(s,r) \subset \Omega$, so

$$v^*(s) = 0 = u^*(s) \leq A(u^*; s, r) \leq A(v^*; s, r),$$

where $A(g; s, r)$ denotes the area mean value of a function g over $D(s,r)$. Hence v^* satisfies the mean value inequality and, consequently, it is subharmonic on $\Omega \cap D(0,a)$.

When $s \in \mathbb{C}_0^+$,

$$v^*(s) = -M_{U_a}(s, +\infty) \leq -M_{\mathbb{C}_0^+}(s, +\infty) = -\mathrm{Re}(s).$$

Further, since $L \cap \mathbb{C}_0^-$ contains a nonpolar set K, we see that

$$M_{\mathbb{C}\backslash L}(s, -\infty) < M_{\mathbb{C}_0^-}(s, -\infty) = -\mathrm{Re}(s) \quad (s \in \mathbb{C}_0^- \backslash L).$$

Hence

$$v^*(s) < -\mathrm{Re}(s) \quad (s \in \mathbb{C}_0^-),$$

as v^* is non-positive on L. Let Ω_0 be the component of $\Omega \cap D(0, a/2)$ that meets $\mathbb{C}_0^+$, and let $H_{v^*}^{\Omega_0}$ denote the solution to the Dirichlet problem on Ω_0 with boundary data v^*. Since the set $\partial\Omega_0 \cap \mathbb{C}_0^-$ has positive harmonic measure for Ω_0, we obtain a strictly positive harmonic function on Ω_0 by defining

$$g(s) = -\mathrm{Re}(s) - H_{v^*}^{\Omega_0}(s) \quad (s \in \Omega_0).$$

The thinness of $\mathbb{C}\backslash\Omega$ at 0 tells us that 0 is an irregular boundary point for the Dirichlet problem on Ω_0, so, by Theorem 7.5.5 of [1], there exists a set A, thin at 0, and a number $l \geq 0$, such that

$$g(s) \to l \quad (s \to 0, s \notin A).$$

Further, by the continuity of g on Ω_0, we may assume that the set A is closed. We must have $l > 0$, for otherwise g would be a barrier for the open set $\Omega_0\backslash A$ at the irregular boundary point 0, which is impossible. By Theorem 7.3.9 of [1], we can choose $r_0 \in (0, l/2)$ such that $\partial D(0, r_0) \subset \Omega_0\backslash A$ and $g > l/2$ on $\partial D(0, r_0)$. We now use the subharmonicity of v^* on Ω to see that

$$v^*(s) \leq H_{v^*}^{\Omega_0}(s) = -\mathrm{Re}(s) - g(s) < l/2 - l/2 = 0 \quad (s \in \partial D(0, r_0)),$$

whence $\partial D(0, r_0) \subset U_a$. Since U_a is simply connected, we conclude that (S_{q_n}) converges uniformly on $\overline{D}(0, r_0)$ as required.

Definition 1 Let (q_n) be an increasing sequence of integers, where $q_n \to \infty$ as $n \to \infty$. We say that a Dirichlet series $\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$ has Hadamard-Ostrowski gaps (q_n) if there exists $\theta > 0$ such that

$$\frac{\lambda_{q_n+1}}{\lambda_{q_n}} \geq 1 + \theta \quad (n \in \mathbb{N}).$$

Corollary 1 *Let $\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$ be a Dirichlet series that converges on $\mathbb{C}_0^+$ and is analytically continuable to a domain Ω strictly larger than $\mathbb{C}_0^+$, but not to a neighbourhood of a given point $\xi \in \{\mathrm{Re}(s) = 0\}$. If $\mathbb{C}\backslash\Omega$ is thin at ξ, then the series cannot possess Hadamard Ostrowski gaps.*

Proof We argue by contradiction. Suppose the series has Hadamard-Ostrowski gaps. Since Ω is strictly larger than $\mathbb{C}_0^+$, it contains a full neighbourhood of a point ζ on the imaginary axis. Theorem 1 of [2] implies that the sequence (S_{q_n}) converges on a, possibly smaller, neighbourhood of ζ, and, in particular, on some disc lying entirely in $\mathbb{C}_0^-$. A disc has a non-zero capacity, so by Theorem 2, f has analytic extension to a neighbourhood of ξ, contradicting the hypothesis. □

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