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Blaschke Products and Zero Sets in Weighted Dirichlet Spaces

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Abstract

In this paper, we deal with superharmonically weighted Dirichlet spaces $\mathcal{D}_\omega$. First, we prove that the classical Dirichlet space is the largest, among all these spaces, which contains no infinite Blaschke product. Next, we give new sufficient conditions on a Blaschke sequence to be a zero set for $\mathcal{D}_\omega$. Our conditions improve Shapiro-Shields condition for $\mathcal{D}_\alpha$, when $\alpha \in (0, 1)$.

Keywords Blaschke product · Dirichlet space · Capacity

Mathematics Subject Classification (2010) 31C25 · 30J10 · 31C15

1 Introduction

Let $\mathbb{D}$ be the unit disc of the complex plane $\mathbb{C}$ and let $\mathbb{T} := \partial\mathbb{D}$ be the unit circle. Let dA (resp. dm) be the normalized Lebesgue measure on $\mathbb{D}$ (resp. $\mathbb{T}$). The Hardy space H^2 is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log(1/|z|)dA(z) < \infty.$$

A weight ω is a function $\omega : \mathbb{D} \to (0, +\infty]$ which is integrable on $\mathbb{D}$ with respect to dA. The weighted Dirichlet space $\mathcal{D}_\omega$ associated with ω is defined by

$$\mathcal{D}_\omega := \left\{ f \in H^2 : \mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty \right\}.$$

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The space $\mathcal{D}_\omega$ will be endowed by the hilbertian norm $\|f\|_\omega^2 := \|f\|_{H^2}^2 + \mathcal{D}_\omega(f)$. Let $0 \leq \alpha < 1$ and denote by $\mathcal{D}_\alpha$ the standard Dirichlet space which corresponds to the weight $\omega_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha$. The classical Dirichlet space is $\mathcal{D}_0$ and will be denoted by $\mathcal{D}$.

In this paper, we are mainly interested in superhamonically weighted Dirichlet spaces. Several results on these spaces can be found in [3, 6, 15–17, 32]. In general the description of zero sets remains an open problem, even for the standard Dirichlet spaces $\mathcal{D}_\alpha$ for $\alpha \in [0, 1)$. Recall that a sequence $\mathcal{Z} = (z_n)_{n\geq 1} \subset \mathbb{D}$ is a zero set for $\mathcal{D}_\omega$ if there is a function in $\mathcal{D}_\omega$ that vanishes on $\mathcal{Z}$ and nowhere else in $\mathbb{D}$.

We say that $\mathcal{Z} = (z_n)_{n\geq 1} \subset \mathbb{D}$ is a Blaschke sequence if $\sum_{n\geq 1}(1-|z_n|) < \infty$. The associated Blaschke product B is given by

$$B(z) = \prod_{n\geq 1} \frac{|z_n|}{z_n} \frac{z_n - z}{1-\overline{z_n}z},$$

with the convention $|z_n|/z_n = -1$, if $z_n = 0$. It is known [12] that zero sets for H^2 are the Blaschke sequences and that every Blaschke product is bounded, so it belongs in H^2. For the classical Dirichlet space, L. Carleson [10] proved that if $\mathcal{Z}$ satisfies

$$\sum_{n\geq 1} \frac{1}{\log^{1-\varepsilon} 1/(1-|z_n|)} < \infty$$

for some $\varepsilon \in (0, 1)$, then $\mathcal{Z}$ is a zero set for $\mathcal{D}$. In [31], H.S. Shapiro and A.L. Shields improve this result by proving that if

$$\sum_{n\geq 1} \frac{1}{\log 1/(1-|z_n|)} < \infty,$$

then $\mathcal{Z}$ is a zero set for $\mathcal{D}$. This last result is sharp in the following sense, if $(r_n)_n \subset (0, 1)$ such that $\sum 1/|\log(1-r_n)| = \infty$, then there exists a sequence $(\theta_n)_n$ such that $(r_n e^{i\theta_n})_n$ is not a zero set for $\mathcal{D}$ (see [24]). Similar results are obtained for all standard Dirichlet spaces $\mathcal{D}_\alpha$ (see for instance [25]).

D. Marshall and C. Sundberg [23] observed that the argument used by Shapiro-Shields works for all Hilbert spaces of analytic functions on $\mathbb{D}$ that enjoy Pick property (see also [1, 30]). Note that S. Shimorin [32] proved that every superharmonically weighted Dirichlet space $\mathcal{D}_\omega$ possesses Pick property. So, if $\mathcal{Z} = (z_n)_{n\geq 1}$ satisfies the Shapiro-Shields condition, namely

$$\sum_{n\geq 1} \frac{1}{K^\omega(z_n, z_n)} < \infty, \tag{1}$$

where K^ω is the reproducing kernel of $\mathcal{D}_\omega$, then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$. For $\alpha \in (0, 1)$, it is known that the reproducing kernel of $\mathcal{D}_\alpha$ is given by $K^\alpha(z, w) = \frac{1}{(1-z\bar{w})^\alpha}$ and the Shapiro-Shields condition for $\mathcal{D}_\alpha$ becomes

$$\sum_{n\geq 1} \left(1-|z_n|^2\right)^\alpha < \infty. \tag{2}$$

In fact, condition (2) implies that $B \in \mathcal{D}_\alpha$ [9]. Moreover, if $\mathcal{Z}$ is uniformly separated then condition (2) is also necessary [13, 34]. It is worth mentioning that this result remains true if $\mathcal{Z}$ is only separated [4, 26].

For the classical Dirichlet space the situation is quite different. L. Carleson proved in [11] (see also [17]) the following formula

$$\mathcal{D}(Bf) = \mathcal{D}(f) + \sum_{n\geq 1} \int_{\mathbb{T}} \frac{1-|z_n|^2}{|1-\overline{z_n}\zeta|^2} |f(\zeta)|^2 dm(\zeta) \quad (f \in H^2). \tag{3}$$

As a consequence, the only Blaschke products in $\mathcal{D}$ are finite Blaschke products.

Our first goal in this paper is to determine all superharmonically weighted Dirichlet spaces which contain no infinite Blaschke product. We prove that $\mathcal{D}$ is the largest space among all superharmonically weighted Dirichlet spaces which contains no infinite Blaschke product. To state our result let us denote by P_ν the Poisson transform of the positive measure ν on $\mathbb{T}$.

Theorem 1.1 *Let ω be a superharmonic weight and let P_ν be the harmonic part of ω. Let h be the derivative of ν with respect to m. The following are equivalent.*

i) *$\mathcal{D}_\omega$ contains no infinite Blaschke product.*
ii) *There exists $c > 0$, such that $h \geq c$ a.e. on $\mathbb{T}$.*
iii) $\liminf_{|z|\to 1^-} \omega(z) > 0.$
iv) $\mathcal{D}_\omega \subset \mathcal{D}$.

The second goal in this paper is to give some sufficient conditions which ensure that a sequence $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$. Observe that since $\mathcal{D}_\omega \subset H^2$, then each zero set for $\mathcal{D}_\omega$ satisfies the Blaschke condition. The converse is in general not true (see Section 6). For the standard Dirichlet spaces there are several papers that deal with this problem (see for instance [10, 22, 25, 31]). In Section 4, we give two ways to construct functions $f \in \mathcal{D}_\omega$ such that $fB \in \mathcal{D}_\omega$. The first one is the classical Carleson's construction of smooth outer functions (see Theorem 4.1). While the second construction is based on potential theory induced by $\mathcal{D}_\omega$. We will construct outer functions in $\mathcal{D}_\omega$ with large real part on a given sequence of closed subsets of $\mathbb{T}$. Namely, if c_ω denotes the capacity associated with $\mathcal{D}_\omega$ (see Section 4.2), we have the following result which is in the spirit of Theorem 5.1 of [18] (see also [14, 29]).

Theorem 1.2 *Let ω be a superharmonic weight. Let $(F_n)_n$ be a family of closed subsets of $\mathbb{T}$ and let $E_n = \cup_{k\geq n} F_k$. There exists a function $f \in \mathcal{D}_\omega$ such that* $\mathrm{Re} f \geq 0$ *on $\mathbb{D}$ and*

$$\mathrm{Re} f \geq \log \frac{1}{c_\omega(E_n)} \; c_\omega - q.e. \text{ on } F_n \text{ for all } n \geq 1.$$

Our main result on zero sets deals with general superharmonic weights ω (radial or non-radial). To state our theorem let us introduce some notations. Let Ψ_ω be the function given by

$$\Psi_\omega(w) = (1-|w|^2) \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{w}z|^2} d\mu(z) + P_\nu(w),$$

where $\mu = -\Delta\omega$ and P_ν is the harmonic part of ω. Let $\gamma > 0$ and let $z \in \mathbb{D} \setminus \{0\}$. The closed arc of $\mathbb{T}$ centered at $z/|z|$ and of length $(1-|z|)^\gamma$ will be denoted by $I(z, \gamma)$. We also write $I(0, \gamma) = \mathbb{T}$.

Theorem 1.3 *Let ω be a superharmonic weight on $\mathbb{D}$. Suppose that there exists $\alpha \in [0, 1)$ such that $\omega(z) = O((1-|z|^2)^\alpha)$. Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a sequence of $\mathbb{D}$ such that $\sum_{n\geq 1}(1 - |z_n|^2)^{1-\varepsilon} < \infty$, for some $\varepsilon \in (0, 1)$. Let $\gamma = \frac{\varepsilon}{1-\alpha}$ and let $E_n = \cup_{k\geq n} I(z_k, \gamma)$. If there exists $A > 0$ such that*

$$\sum_{n\geq 1} \Psi_\omega(z_n) c_\omega^A(E_n) < \infty,$$

then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$.

Note that this result allows to give new examples of zero sets for $\mathcal{D}_\omega$, even for the standard spaces $\mathcal{D}_\alpha$. For $\alpha \in (0, 1)$, one can see easily that $\Psi_{\omega_\alpha} \asymp \omega_\alpha$. By Theorem 1.3, if

$$\sum_{n\geq 1}(1-|z_n|)^\alpha c_\alpha^A(E_n) < \infty,$$

then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\alpha$. So, our condition improves Shapiro-Shields condition and takes into account Arg z_n. A more general result for $\mathcal{D}_\alpha$ is obtained in Theorem 4.6.

The paper is organized as follows. In Section 2, we give an upper estimate of $\|Bf\|_\omega$, where $f \in H^2$ and B is a general Blaschke product. Section 3 is devoted to Blaschke products in $\mathcal{D}_\omega$. We also give in this section the proof of Theorem 1.1. In Section 4, we state and prove our main theorems on zero sets for $\mathcal{D}_\omega$. In Section 5, we consider general weighted Dirichlet spaces and give extensions of some previous results. The last section contains some remarks and two problems.

Throughout the paper, we use the following notations

- $A \lesssim B$ means that there is a constant C such that $A \leq CB$.
- $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

2 Norm Estimates

In this section we give an upper estimate of $\|Bf\|_\omega$ which will be useful in the proofs of our main results.

Let ω be a superharmonic weight on $\mathbb{D}$. By Jensen-Riesz representation theorem (see for instance [5, 27]), there exist a positive Borel measure μ on $\mathbb{D}$ and a finite positive Borel measure ν on $\mathbb{T}$ such that

$$\omega(z) = U_\mu(z) + P_\nu(z) \quad (z \in \mathbb{D}), \tag{4}$$

where U_μ is the Green potential of μ defined by

$$U_\mu(z) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{w-z}\right| d\mu(w)$$

and P_ν is the Poisson transform of ν defined by

$$P_\nu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{\zeta}z|^2} d\nu(\zeta).$$

Recall that $\mu = -\Delta\omega$ where Δ is the distributional laplacian operator. It is known [5] that U_μ is not identically infinite if and only if

$$\int_{\mathbb{D}} (1-|z|) d\mu(z) < \infty. \tag{5}$$

So, in the sequel we suppose that Eq. 5 is satisfied. Note that U_μ (resp. P_ν) is called the pure superharmonic (resp. the harmonic) part of ω. We say that ω is purely superharmonic if $\omega = U_\mu$.

S. Richter and C. Sundberg extended Carleson's formula (3) to all harmonically weighted Dirichlet spaces [17, 28]. In particular, we have

$$\mathcal{D}_{P_\nu}(Bf) = \mathcal{D}_{P_\nu}(f) + \sum_{n\geq 1}(1-|z_n|^2)\int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1-\overline{z_n}\zeta|^2}d\nu(\zeta), \tag{6}$$

where B is the Blaschke product associated with $(z_n)_{n\geq 1}$ and $f \in H^2$. On the other hand, using Green's formula [20], it is clear that

$$\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right| dA(w) = P(|f|^2)(z) - |f(z)|^2 \quad (z \in \mathbb{D}).$$

So, if $\omega = U_\mu$ is a purely superharmonic weight then we have the following formula

$$\mathcal{D}_\omega(f) = \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right| dA(w)d\mu(z) = \int_{\mathbb{D}} \left(P(|f|^2)(z) - |f(z)|^2\right) d\mu(z) \tag{7}$$

In particular, $f \in \mathcal{D}_\omega$ if and only if Eq. 7 is finite. The latter result was obtained by A. Aleman [2, 3]. See also [6].

Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a Blaschke sequence of $\mathbb{D}$ and let μ be a positive Borel measure on $\mathbb{D}$. Let $\mathcal{Z}_\mu$ be the function defined by

$$\mathcal{Z}_\mu(\zeta) = \sum_{n\geq 1}(1-|z_n|^2)\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2}d\mu(z) \quad (\zeta \in \mathbb{T}).$$

The following estimate will be useful in the sequel.

Theorem 2.1 *Let $\omega = U_\mu$ be a purely superharmonic weight. Let $f \in \mathcal{D}_\omega$ and let B be the Blaschke product associated with the sequence $\mathcal{Z} = (z_n)_{n\geq 1}$ of $\mathbb{D}$. Then*

$$\|Bf\|_\omega^2 \lesssim \|f\|_\omega^2 + \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta)dm(\zeta). \tag{8}$$

Proof By equation (7), we have

$$\|Bf\|_\omega^2 \asymp \|f\|_\omega^2 + \int_{\mathbb{D}} |f(z)|^2(1-|B(z)|^2)d\mu(z). \tag{9}$$

Let B_k denotes the finite Blaschke product associated with $(z_n)_{1\leq n\leq k}$ and let $B_0 = 1$. We have

$$1-|B_k(z)|^2 = \sum_{n=1}^{k}|B_{n-1}(z)|^2 - |B_n(z)|^2 = \sum_{n=1}^{k}\frac{|B_{n-1}(z)|^2(1-|z_n|^2)(1-|z|^2)}{|1-\bar{z_n}z|^2}.$$

Then

$$1-|B(z)|^2 = \sum_{n=1}^{\infty}\frac{|B_{n-1}(z)|^2(1-|z_n|^2)(1-|z|^2)}{|1-\bar{z_n}z|^2} \leq \sum_{n=1}^{\infty}\frac{(1-|z_n|^2)(1-|z|^2)}{|1-\bar{z_n}z|^2}$$

Combining this inequality with Eq. 9, we obtain

$$\begin{aligned}\|Bf\|_\omega^2 &\lesssim \|f\|_\omega^2 + \sum_{n=1}^{\infty}(1-|z_n|^2)\int_{\mathbb{D}} \frac{|f(z)|^2(1-|z|^2)}{|1-\bar{z_n}z|^2}d\mu(z)\\ &\le \|f\|_\omega^2 + \sum_{n=1}^{\infty}(1-|z_n|^2)\int_{\mathbb{T}} |f(\zeta)|^2 \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|\zeta-z|^2|1-\bar{z_n}z|^2}d\mu(z)dm(\zeta)\\ &= \|f\|_\omega^2 + \int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta)dm(\zeta).\end{aligned}$$

The proof is complete. □

Note that if $\omega = U_\mu + P_\nu$ is a general superharmonic weight we have

$$\|Bf\|_\omega^2 \lesssim \|f\|_\omega^2 + \int_{\mathbb{T}} |f(\zeta)|^2\mathcal{Z}_\mu(\zeta)dm(\zeta) + \sum_{n\ge 1}(1-|z_n|^2)\int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1-\overline{z_n}\zeta|^2}d\nu(\zeta), \quad (10)$$

by Eqs. 6 and 8.

3 Blaschke Products

Our aim in this section is to prove Theorem 1.1. First, we give a sufficient condition which ensures that an infinite Blaschke product belongs to $\mathcal{D}_\omega$.

Let μ be a positive Borel measure on $\mathbb{D}$ satisfying Eq. 5. The function Ψ_μ is given by

$$\Psi_\mu(w) = (1-|w|^2)\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{w}z|^2}d\mu(z) \quad (w\in\mathbb{D}).$$

Let $\mathcal{Z} = (z_n)_{n\ge 1} \subset \mathbb{D}$ and recall that $\mathcal{Z}$ is said to be uniformly separated if there exists $\delta > 0$ such that

$$\prod_{n:\, n\ne k} \left|\frac{z_n - z_k}{1-\overline{z}_n z_k}\right| \ge \delta, \quad (k\ge 1).$$

Theorem 3.1 *Let $\omega = U_\mu + P_\nu$ be a superharmonic weight, where μ is a positive Borel measure on $\mathbb{D}$ satisfying* Eq. 5 *and ν be a finite positive Borel measure on $\mathbb{T}$. Let B be the Blaschke product associated with $\mathcal{Z} = (z_n)_{n\ge 1} \subset \mathbb{D}$. Then $B \in \mathcal{D}_\omega$, if*

$$\sum_{n\ge 1} \Psi_\mu(z_n) + P_\nu(z_n) < \infty. \quad (11)$$

If in addition, $\mathcal{Z}$ is uniformly separated then Eq. 11 *is also necessary.*

Proof Applying Eq. 10, with $f = 1$, we get

$$\begin{aligned}\|B\|_\omega^2 &\le 1 + \int_{\mathbb{T}} \mathcal{Z}_\mu(\zeta)dm(\zeta) + \sum_{n\ge 1}\int_{\mathbb{T}} \frac{1-|z_n|^2}{|1-\overline{z_n}\zeta|^2}d\nu(\zeta)\\ &= 1 + \sum_{n\ge 1} \Psi_\mu(z_n) + P_\nu(z_n) < \infty.\end{aligned}$$

This proves that $B \in \mathcal{D}_\omega$. If $\mathcal{Z}$ is uniformly separated, by [13], we have

$$1 - |B(z)|^2 \asymp \sum_{n \geq 1} \frac{(1 - |z_n|^2)(1 - |z|^2)}{|1 - \bar{z_n} z|^2},$$

The second assertion comes from Eqs. 6 and 7. □

The following lemma will be useful in the proof of the main result.

Lemma 3.2 *Let $\omega = U_\mu + P_\nu$ be a superharmonic weight, where μ is a positive Borel measure on $\mathbb{D}$ satisfying* Eq. 5 *and ν is a finite positive Borel measure on $\mathbb{T}$. The following are equivalent.*

i) *$\mathcal{D}_\omega$ contains an infinite Blaschke product.*

ii) $\liminf_{|z| \to 1^-} \big(\Psi_\mu(z) + P_\nu(z)\big) = 0.$

Proof Suppose that there exists an infinite Blaschke sequence $\mathcal{Z} = (z_n)_{n \geq 1} \subset \mathbb{D}$ such that the associated Blaschke product B belongs to $\mathcal{D}_\omega$. Let $(a_n)_{n \geq 1}$ be a uniformly separated infinite subsequence of $\mathcal{Z}$ and let B_1 be the associated Blaschke product. Clearly, from Eqs. 6 and 7, $B_1 \in \mathcal{D}_\omega$. By Theorem 3.1, we have $\sum_{n \geq 1} \Psi_\mu(a_n) + P_\nu(a_n) < \infty$. In particular, $\liminf_{|z| \to 1^-} \big(\Psi_\mu(z) + P_\nu(z)\big) = 0$.

Conversely, suppose that $\liminf_{|z| \to 1^-} \big(\Psi_\mu(z) + P_\nu(z)\big) = 0$. Then there exists $(z_n)_{n \geq 1}$ such that $\Psi_\mu(z_n) + P_\nu(z_n) < 1/2^n$. By Theorem 3.1, the Blaschke product associated with $(z_n)_{n \geq 1}$ belongs to $\mathcal{D}_\omega$. □

Proof of Theorem 1.1 The following implications $ii) \implies iii) \implies iv)$ are obvious. $iv) \implies i)$ comes from the fact that $\mathcal{D}$ contains no infinite Blaschke product.

It remains to prove that $i) \implies ii)$. To this end, suppose that $ii)$ is not satisfied and suppose that $\omega = U_\mu + P_\nu$. For $\varepsilon > 0$, we have $m(\{\zeta \in \mathbb{T} : \ h(\zeta) < \varepsilon\}) := a > 0$. Let $(r_n)_{n \geq 1} \in (0, 1)$ be a sequence which converges to 1. By Fatou's lemma, we have $\lim_n P_\nu(r_n \zeta) = h(\zeta)$ *a.e.*. Then for n, large enough, we have

$$m(\{\zeta \in \mathbb{T} : \ P_\nu(r_n \zeta) < \varepsilon\}) \geq a/2.$$

Let $u_n(\zeta) = \Psi_\mu(r_n \zeta)$. We have

$$\|u_n\|_1 := \int_{\mathbb{T}} u_n(\zeta) dm(\zeta) \asymp \int_{\mathbb{D}} \frac{(1 - r_n)}{1 - r_n |z|} (1 - |z|) d\mu(z).$$

Since the measure $(1 - |z|)d\mu(z)$ is finite, $\lim_n \|u_n\|_1 = 0$. In particular we have, for large n,

$$m(\{\zeta \in \mathbb{T}\, u_n(\zeta) \geq \varepsilon\}) < a/2.$$

Consequently,

$$m(\{\zeta \in \mathbb{T} : \ P_\nu(r_n \zeta) < \varepsilon\} \cap \{\zeta \in \mathbb{T}\, u_n(\zeta) < \varepsilon\}) > 0.$$

This implies that $\liminf_{|z| \to 1^-} \Psi_\mu(z) + P_\nu(z) = 0$. By applying Lemma 3.2 we obtain the result.

□

4 Zero Sets for $\mathcal{D}_\omega$

Since $\mathcal{D}_\omega$ possesses a division property it is clear that a Blaschke sequence $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$ if and only if there exists an outer function $f \in \mathcal{D}_\omega$ such that $Bf \in \mathcal{D}_\omega$. In this section we give two ways to construct such functions.

4.1 Regularization Using Smooth Functions

We begin this subsection by recalling some standard facts. Let ρ be an increasing function such that $\rho(0) = 0$. We say that a closed subset E of $\mathbb{T}$ is a $\rho-$Carleson set if

$$\int_{\mathbb{T}} \log \frac{1}{\rho(d(\zeta, E))} dm(\zeta) < \infty.$$

It is known (see for instance [8, 11, 33]) that if $\rho(t) = O(t^\varepsilon)$, for some $\varepsilon > 0$, then there exists an analytic function f on $\mathbb{D}$ with bounded derivatives and such that $|f(z)| = O(\rho(d(z, E)))$.

The aim of the following theorem is to exhibit conditions ensuring the membership of fB to $\mathcal{D}_\omega$. For simplicity, we state our result only in the radial case.

Theorem 4.1 *Let ρ be an increasing function such that $\rho(t) = O(t^\varepsilon)$ for some $\varepsilon > 0$. Let E be a $\rho-$Carleson set. Let $\omega = U_\mu$ be a purely superharmonic radial weight on $\mathbb{D}$. Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a Blaschke sequence such that*

$$\sum_{n\geq 1} \Psi_\mu(z_n)\rho(16d(z_n, E)) < \infty. \tag{12}$$

Then $\mathcal{Z}$ is a zero set for $\mathcal{D}_\omega$.

We will need the following lemma, which is due to K. Kellay and J. Mashreghi [22].

Lemma 4.2 *Let ρ, E satisfying the conditions of Theorem 4.1. Then*

$$\int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \overline{z}\zeta|^2} \rho(d(\zeta, E)) dm(\zeta) \lesssim \rho(2d(z, E)) + (1 - |z|^2) \int_{2d(z,E)}^{2} \frac{\rho(t)}{t^2} dt \quad (z \in \mathbb{D}).$$

Proof of Theorem 4.1 Since $\rho-$Carleson sets and $\rho^{2/\varepsilon}-$Carleson sets are the same, we suppose that $\rho(t) = O(t^2)$. By the above discussion, there exists a function $f \in \mathcal{D}_\omega$ such that $|f(z)|^2 \leq \rho(d(z, E))$. We have

$$\begin{aligned}\int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta) dm(\zeta) &\leq \int_{\mathbb{T}} \rho(d(\zeta, E)) \mathcal{Z}_\mu(\zeta) dm(\zeta) \\ &= \sum_{n\geq 1} (1 - |z_n|^2) \int_{\mathbb{D}} \frac{(1 - |z|^2)^2}{|1 - \overline{z_n} z|^2} \int_{\mathbb{T}} \frac{\rho(d(\zeta, E))}{|1 - \overline{z}\zeta|^2} dm(\zeta) d\mu(z).\end{aligned}$$

By Lemma 4.2 we have

$$\int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \overline{z}\zeta|^2} \rho(d(\zeta, E)) dm(\zeta) \lesssim \rho(2d(z, E)) + (1 - |z|^2).$$

Then

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{z_n}z|^2} \int_{\mathbb{T}} \frac{\rho(d(\zeta, E))}{|1-\overline{z}\zeta|^2} dm(\zeta)d\mu(z) \lesssim \int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-\overline{z_n}z|^2}\rho(2d(z,E))d\mu(z) + \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{z_n}z|^2} d\mu(z).$$

Since ω is radial, $d\mu(r\zeta) = d\lambda(r)dm(\zeta)$, where λ is a positive Borel measure on [0, 1]. Recall that $\omega \neq +\infty$ is equivalent to

$$\int_{\mathbb{D}} (1-|z|)d\mu(z) = \int_0^1 (1-r)d\lambda(r) < \infty.$$

Then we have

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{z_n}z|^2} d\mu(z) = \int_0^1 \int_{\mathbb{T}} \frac{(1-r^2)^2}{|1-r\overline{z_n}\zeta|^2} dm(\zeta)d\lambda(r) \lesssim \int_0^1 (1-r)d\lambda(r) < \infty.$$

Note that $d(z, E) \leq \max(2d(z/|z|, E), 2(1-|z|))$. Using Lemma 4.2 again, we get

$$\begin{aligned}\int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-\overline{z_n}z|^2}\rho(2d(z,E))d\mu(z) &\leq \int_0^1 \int_{\mathbb{T}} \frac{(1-r^2)}{|1-\overline{z_n}r\zeta|^2}\rho(4d(\zeta,E))dm(\zeta)d\lambda(r) \\ &\quad + \int_0^1 \int_{\mathbb{T}} \frac{(1-r^2)}{|1-\overline{z_n}r\zeta|^2}\rho(4(1-r))dm(\zeta)d\lambda(r) \\ &\lesssim \int_0^1 \frac{1-r}{1-r|z_n|}\rho(8d(rz_n,E))d\lambda(r) + \int_0^1 (1-r)d\lambda(r).\end{aligned}$$

Combining these inequalities, we obtain

$$\int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta)dm(\zeta) \lesssim \sum_{n\geq 1}(1-|z_n|^2)\int_0^1 \frac{1-r}{1-r|z_n|}\rho(8d(rz_n,E))d\lambda(r) + \sum_{n\geq 1}(1-|z_n|^2).$$

Since ρ is increasing,

$$\rho(8d(rz_n,E)) \leq \rho(16d(z_n,E)) + \rho(8(1-r)) \lesssim \rho(16d(z_n,E)) + (1-r)^2.$$

We get

$$\begin{aligned}\int_{\mathbb{T}} |f(\zeta)|^2 \mathcal{Z}_\mu(\zeta)dm(\zeta) &\lesssim \sum_{n\geq 1}(1-|z_n|^2)\left(\int_0^1 \frac{1-r}{1-r|z_n|}d\lambda(r)\right)\rho(16d(z_n,E)) \\ &\quad + \sum_{n\geq 1}(1-|z_n|^2) \\ &\asymp \sum_{n\geq 1}\Psi_\mu(z_n)\rho(16d(z_n,E)) + \sum_{n\geq 1}(1-|z_n|^2).\end{aligned}$$

By Theorem 2.1 and Eq. 12, $Bf \in \mathcal{D}_\omega$. The proof is complete. □

Let $\Lambda = \{(\alpha, \beta) : \alpha \in (0,1), \beta \in (-\infty, +\infty)\} \cup \{0\} \times (0, +\infty) \cup \{1\} \times (-\infty, 0)$ and let

$$\omega_{\alpha,\beta}(z) = \frac{(1-|z|^2)^\alpha}{\log^\beta(1/1-|z|)}, \quad (\alpha, \beta) \in \Lambda. \tag{13}$$

Corollary 1 *Let ρ be an increasing function such that $\rho(t) = O(t^\varepsilon)$ for some $\varepsilon > 0$ and let E be a $\rho-$Carleson set. Set $(\alpha, \beta) \in \Lambda$ and suppose that a Blaschke sequence $\mathcal{Z} = (z_n)_{n\geq 1}$ of $\mathbb{D}$ satisfies*

$$\sum_{n\geq 1} \omega_{\alpha,\beta}(z_n)\rho(16d(z_n, E)) < \infty.$$

Then $\mathcal{Z}$ is a zero set for $\mathcal{D}_{\omega_{\alpha,\beta}}$.

Proof By a straightforward computation we get, for all $(\alpha, \beta) \in \Lambda$, that $\Psi_{\omega_{\alpha,\beta}} \asymp \omega_{\alpha,\beta}$. The desired result comes from Theorem 4.1. □

4.2 Regularization Using Potential Theory

Let $\omega = U_\mu + P_\nu$ be a superharmonic weight on $\mathbb{D}$. Now we introduce the notion of capacity associated with $\mathcal{D}_\omega$. The space $\mathcal{D}^h_\omega$ is given by

$$\mathcal{D}^h_\omega := \{f \in L^2(\mu) : \ \|f\|^2_\omega := \|f\|^2_{L^2(m)} + \mathcal{D}_\omega(f) < \infty\},$$

where

$$\mathcal{D}_\omega(f) = \int_{\mathbb{T}}\int_{\mathbb{T}} |f(\zeta) - f(\eta)|^2 A_\mu(\zeta, \eta)dm(\zeta)dm(\eta) + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|2} d\nu(\zeta)dm(\eta)$$

and

$$A_\mu(\zeta, \eta) = \int_{\mathbb{D}} \frac{(1 - |z|)}{|1 - \overline{\zeta}z|^2} \frac{(1 - |z|)}{|1 - \overline{z}\eta|^2} d\mu(z).$$

Note that, the real part of $\mathcal{D}^h_\omega$ is a Dirichlet space in the sense of Beurling-Deny (see [7, 19]). So, the capacity c_ω associated with $\mathcal{D}^h_\omega$ is defined by

$$c_\omega(E) = \inf\{\|f\|^2_\omega : \ f \in \mathcal{D}^h_\omega \text{ and } f \geq 1 \ a.e \text{ on a neighborhood of } E \}.$$

By the definition, c_ω is inner, that is

$$c_\omega(E) = \inf\{c_\omega(U) : U \text{ is open and } E \subset U\}.$$

We also have the following properties

- $c_\omega(\emptyset) = 0$ and $c_\omega(E_1) \leq c_\omega(E_2)$, whenever $E_1 \subset E_2 \subset \mathbb{T}$.
- Let $(A_n)_n$ be an increasing sequence of subsets of $\mathbb{T}$. Then $c_\omega(\cup_n A_n) = \lim_{n\to\infty} c_\omega(A_n)$.
- Let (K_n) be a decreasing sequence of compacts subset of $\mathbb{T}$. Then $c_\omega(\cap_n K_n) = \lim_{n\to\infty} c_\omega(K_n)$.

Then by Choquet's theorem we have

$$c_\omega(E) = \sup\{c_\omega(K) : K \text{ is compact and } K \subset E\},$$

where E is a Borelian subset of $\mathbb{T}$. A property is said to be satisfied $c_\omega-$ quasi everywhere, if there exists a subset $N \subset \mathbb{T}$ with $c_\omega(N) = 0$ and such that the property is satisfied on $\mathbb{T} \setminus N$. We say that a function f defined on $\mathbb{T}$ is quasi continuous if for every $\varepsilon > 0$ there exists an open subset U with $c_\omega(U) < \varepsilon$ and such that f is continuous on $\mathbb{T} \setminus U$. It is known, from the general potential theory associated with Dirichlet forms, that for each function $f \in \mathcal{D}^h_\omega$, there exists a quasi continuous function $\tilde{f} \in \mathcal{D}^h_\omega$ such that $\tilde{f} = f$ a.e.. Note also that if f and g are two quasi continuous functions of $\mathcal{D}^h_\omega$ such that $f = g$ a.e. on an open subset U then $f = g \ c_\omega - q.e.$ on U. Using these facts one can see that for every subset E of $\mathbb{T}$ we have

$$c_\omega(E) = \inf\{\|f\|^2_\omega : \ f \in \mathcal{D}^h_\omega \text{ and } \tilde{f} \geq 1 \ c_\omega - q.e. \text{on } E \}.$$

Now, we introduce the notion of potential which will play an important role in our construction. We say that a positive Borel measure λ on $\mathbb{T}$ is of finite energy (with respect to $\mathcal{D}^h_\omega$) if there exists a constant $C > 0$ such that

$$\int |f(\zeta)|d\lambda(\zeta) \leq C\|f\|_\omega, \quad (f \in \mathcal{D}^h_\omega \cap \mathcal{C}(\mathbb{T})),$$

where $\mathcal{C}(\mathbb{T})$ is the space of continuous functions on $\mathbb{T}$. In particular the linear form $f \to \int f(\zeta)d\lambda(\zeta)$ can be extended to the whole space $\mathcal{D}^h_\omega$. Consequently, there exists a unique (since $\mathcal{D}^h_\omega \cap \mathcal{C}(\mathbb{T})$ is dense in $\mathcal{D}^h_\omega$) function $g_\lambda \in \mathcal{D}^h_\omega$ such that

$$\langle g_\lambda, f\rangle = \int f(\zeta)d\lambda(\zeta) \quad (f \in \mathcal{D}^h_\omega \cap \mathcal{C}(\mathbb{T})).$$

Let E be a closed subset of $\mathbb{T}$. The set of positive Borel measures supported by E and with finite energy with respect to $\mathcal{D}^h_\omega$, will be denoted by $S_\omega(E)$. It is known that if $\lambda \in S_\omega(\mathbb{T})$, then λ charges no sets of $c_\omega-$ capacity zero. In particular if $f \in \mathcal{D}^h_\omega$, then $\tilde{f} \in L^1(\lambda)$ and

$$\langle g_\lambda, f\rangle = \int \tilde{f}(\zeta)d\lambda(\zeta) \quad (f \in \mathcal{D}^h_\omega). \tag{14}$$

The function g_λ is called the potential of λ with respect $\mathcal{D}^h_\omega$. We have the following fundamental theorem.

Theorem 4.3 *Let E be a closed subset of $\mathbb{T}$ such that $c_\omega(E) > 0$. There exists a unique measure $\lambda_E \in S_\omega(E)$ such that*

$$c_\omega(E) = \|g_{\lambda_E}\|^2_\omega = \lambda_E(E).$$

Moreover, we have $0 \leq g_{\lambda_E} \leq 1$ and $\tilde{g}_{\lambda_E} = 1\ c_\omega - q.e.$ on E.

A direct and important consequence of this theorem is the following expression of the capacity of closed sets in terms of measures with finite energy

$$c_\omega(E) = \sup\{\lambda(E) :\ \lambda \in S_\omega(E) \text{ and } \tilde{g_\lambda} \leq 1\ c_\omega - q.e.\},$$

where E is a closed subset of $\mathbb{T}$. For more details we refer to [19].

Let K^ω be the reproducing kernel of $\mathcal{D}_\omega$. The Poisson transform of a function $f \in \mathcal{D}^h_\omega$ will be also denoted by f. Let $f^+ = \langle f, K^\omega\rangle$ the analytic part of f. Since $\overline{f^-} =: \overline{f - f^+} \in \mathcal{D}_\omega$, we have

$$f(z) = \langle f, 2\mathrm{Re}K^\omega_z - 1\rangle \quad (z \in \mathbb{D}).$$

Let $\lambda \in S_\omega(\mathbb{T})$ and let $f_\lambda =: g^+_\lambda$. We have

$$g_\lambda(z) = \int_{\mathbb{T}} (2\mathrm{Re}K^\omega_z(\zeta) - 1)d\lambda(\zeta) \text{ and } f_\lambda(z) = \int_{\mathbb{T}} K^\omega_z(\zeta)d\lambda(\zeta) \quad (z \in \mathbb{D}).$$

From the above discussion we have the following result.

Lemma 4.4 *Let ω be a superharmonic weight and let E be a closed subset of $\mathbb{T}$ such that $c_\omega(E) > 0$. Then $\mathrm{Re} f_{\lambda_E} \geq 0$, $\mathrm{Re} f_{\lambda_E} = 1\ c_\omega - q.e.$ on E and $\|f_{\lambda_E}\|^2_\omega \asymp c_\omega(E)$.*

The following lemma will be used in the proof of Theorem 1.2.

Lemma 4.5 *Let ω be a superharmonic weight. Let $(E_n)_n$ be a (finite or infinite) decreasing sequence of closed subsets of $\mathbb{T}$ such that $c_\omega(E_n) > 0$ for each n. There exists a function $f \in \mathcal{D}_\omega$ such that* $\mathrm{Re} f \geq 0$ *on* $\mathbb{D}$, $\|f\|_\omega \leq 1$ *and*

$$\mathrm{Re} f \geq \frac{1}{4}\log\frac{1}{c_\omega(E_n)} \quad c_\omega - q.e. \text{ on } \ E_n \text{ for all } n.$$

Proof Let $\Lambda_p = \{n : \ 2^p \leq \log(1/c_\omega(E_n)) < 2^{p+1}\}$. Write $\{p : \ \Lambda_p \neq \emptyset\} = \{p_j, \ j \geq 1\}$ such that $p_j < p_{j+1}$ and put $n_j = \min \Lambda_{p_j}$. Then we have

$$2^{p_j} \leq \log\frac{1}{c_\omega(E_n)} < 2^{p_j+1}, \quad n_j \leq n < n_{j+1}.$$

Let $f = \sum_{j\geq 1} 2^{p_j-1} f_j$, where $f_j = f_{\lambda_{E_{n_j}}}$. We have

$$\|f\|_\omega \leq \sum_{j\geq 1} 2^{p_j-1}\|f_j\|_\omega \leq \sum_{j\geq 1} 2^{p_j-1} c_\omega^{1/2}(E_j) \leq \sum_{j\geq 1}\frac{2^{p_j-1}}{2^{2^{p_j-1}}} \leq 1.$$

Let $n \geq 1$ and let j be such that $n_j \leq n < n_{j+1}$. Since $E_n \subset E_{n_j}$ we have $\mathrm{Re} f_j = 1$, $c_\omega-$ q.e. on E_n. We get

$$\mathrm{Re} f \geq 2^{p_j-1} \geq \frac{1}{4}\log\frac{1}{c_\omega(E_n)}, \quad c_\omega\text{–q.e. on } E_n.$$

□

Proof of Theorem 1.2 Obviously, one can suppose that $c_\omega(F_n) > 0$, for all $n \geq 1$. Let $E_j^n = \cup_{j\leq k\leq n} F_k$. By Lemma 4.5, there exists a sequences $(f_n)_n \subset \mathcal{D}_\omega$ such that $\mathrm{Re} f_n \geq 0$ $a.e.$ on $\mathbb{T}$,

$$\sup_n \|f_n\|_\omega < \infty \quad \text{and} \quad \mathrm{Re} f_n \geq \log\frac{1}{c_\omega(E_j^n)}\ c_\omega - q.e. \text{ on } F_j \ (1 \leq j \leq n).$$

Since $E_j^n \subset E_j$, Re $f_n \geq \log\frac{1}{c_\omega(E_j)}$ $c_\omega - q.e.$ on F_j $(1 \leq j \leq n)$. One can extract from (f_n) a subsequence which converges weakly to a quasi-continuous function $f \in \mathcal{D}_\omega$. Since weak convergence implies uniform convergence on compact subsets of $\mathbb{D}$, we have $\mathrm{Re} f \geq 0$ on $\mathbb{D}$. By a standard argument, it is easy to see that $\mathrm{Re} f \geq \log\frac{1}{c_\omega(E_j)}$ $c_\omega - q.e.$ on F_j for $j \geq 1$. Hence the proof is complete. □

Before giving the proof of Theorem 1.3 remark that the condition $\omega(z) = U_\mu(z) + P_\nu(z) = O((1-|z|)^\alpha)$ implies that

$$\nu(I) \lesssim (1-|z|)P_\nu(z) = O(m(I)^{1+\alpha}),$$

where I is an arbitrary arc of $\mathbb{T}$ and $z \in \mathbb{D}\setminus\{0\}$ such that $z/|z|$ is the center of I and $1-|z|$ is the length of I. This implies that $\nu = 0$ if $\alpha > 0$ and $d\nu = \phi dm$ for some bounded function ϕ, if $\alpha = 0$.

Proof of Theorem 1.3 By Theorem 1.2, there exists a function $f \in \mathcal{D}_\omega$ such that $\mathrm{Re} f \geq 0$ $a.e.$ on $\mathbb{T}$ and $\mathrm{Re} f \geq \log(1/c_\omega(E_n))$ $c_\omega - q.e.$ on $I(z_n, \gamma)$. Clearly, the bounded analytic function $h =: e^{-Af} \in \mathcal{D}_\omega$ and $|h| \le c_\omega(E_n)^A\ c_\omega - q.e.$ on $I(z_n, \gamma)$. Our goal is to prove that $Bh \in \mathcal{D}_\omega$. By Theorem 2.1 and equality (7) it suffices to prove that

$$\sum_{n\ge 1}\int_{\mathbb{T}}\int_{\mathbb{D}} |h(\zeta)|^2 \frac{(1-|z_n|^2)((1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} d\mu(z)dm(\zeta) < \infty, \tag{15}$$

and

$$\sum_{n\ge 1}\int_{\mathbb{T}} |h(\zeta)|^2 \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} d\nu(\zeta) < \infty. \tag{16}$$

Since $|h| \le c_\omega(E_n)^A\ c_\omega - q.e.$ on $I(z_n, \gamma)$, we have

$$\int_{I(z_n,\gamma)}\int_{\mathbb{D}} |h(\zeta)|^2 \frac{(1-|z_n|^2)((1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} d\mu(z)dm(\zeta) \le \Psi_\mu(z_n)c_\omega^{2A}(E_n)$$

On the other hand, we have

$$\begin{aligned} U_\mu(w) &= \int_{\mathbb{D}} \log\frac{1}{|\varphi_z(w)|} d\mu(z) \\ &\ge \int_{\mathbb{D}} (1-|\varphi_z(w)|) d\mu(z) \\ &\asymp \int_{\mathbb{D}} \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} d\mu(z). \end{aligned}$$

Let D_z be the disc centered at z and of radius $\frac{1-|z|}{4}$. We get

$$\begin{aligned} \int_{\mathbb{D}} \frac{(1-|z_n|^2)U_\mu(w)}{|1-\overline{z_n}w|^2|1-\overline{w}\zeta|^2} dA(w) &\gtrsim \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|z_n|^2)}{|1-\overline{z_n}w|^2|1-\overline{w}\zeta|^2} \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w)d\mu(z) \\ &\ge \int_{\mathbb{D}}\int_{D_z} \frac{(1-|z_n|^2)}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w)d\mu(z) \\ &\asymp \int_{\mathbb{D}} \frac{(1-|z_n|^2)(1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} d\mu(z). \end{aligned}$$

Since h is bounded and $U_\mu(z) \le \omega(z) = O((1-|z|)^\alpha)$, we deduce that

$$\begin{aligned} \int_{\mathbb{D}} |h(\zeta)|^2 \frac{(1-|z_n|^2)(1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} d\mu(z) &\lesssim \int_{\mathbb{D}} \frac{(1-|z_n|^2)((1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} d\mu(z) \\ &\lesssim \int_{\mathbb{D}} \frac{(1-|z_n|^2)U_\mu(z)}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} dA(z) \\ &\lesssim \frac{(1-|z_n|^2)}{|1-\overline{z_n}\zeta|^{2-\alpha}} \end{aligned}$$

Let $J_n = \mathbb{T} \setminus I(z_n, \gamma)$. We have

$$\begin{aligned}\int_{J_n}\int_{\mathbb{D}} |h(\zeta)|^2 \frac{(1-|z_n|^2)((1-|z|^2)^2}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} d\mu(z)dm(\zeta) &\lesssim \int_{J_n} \frac{(1-|z_n|^2)}{|1-\overline{z_n}\zeta|^{2-\alpha}} dm(\zeta)\\ &\asymp \frac{1-|z_n|^2}{dist^{1-\alpha}(z_n, J_n)}\\ &\asymp (1-|z_n|^2)^{1-\gamma(1-\alpha)}\\ &= (1-|z_n|^2)^{1-\varepsilon},\end{aligned}$$

which completes the proof of Eq. 15.

To prove Eq. 16 we suppose that $\alpha = 0$, otherwise $\nu = 0$. Let $d\nu = \phi dm$, where ϕ is a bounded function. We have

$$\begin{aligned}\sum_{n\geq 1}\int_{\mathbb{T}} |h(\zeta)|^2 \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} d\nu(\zeta) &\lesssim \sum_{n\geq 1}\left(\int_{I(z_n,\gamma)} |h(\zeta)|^2 \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} d\nu(\zeta) + \int_{J_n} \frac{1-|z_n|^2}{|1-\overline{z}_n\zeta|^2} dm(\zeta)\right)\\ &\lesssim \sum_{n\geq 1} P_\nu(z_n) c_\omega^A(E_n) + \sum_{n\geq 1}(1-|z_n|)^{1-\varepsilon},\end{aligned}$$

and Eq. 16 is proved. □

Applying the latter result to $\omega_{\alpha,\beta}$, which is given by Eq. 13, we get

Corollary 2 *Let* $(\alpha, \beta) \in \Lambda$. *Let* $\mathcal{Z} = (z_n)_{n\geq 1}$ *be a sequence of* $\mathbb{D}$ *such that* $\sum_{n\geq 1}(1 - |z_n|^2)^{1-\varepsilon} < \infty$, *for some* $\varepsilon \in (0, 1)$. *Let* $\gamma = \frac{\varepsilon}{1-\alpha}$ *and let* $E_n = \cup_{k\geq n} I(z_k, \gamma)$. *If there exists* $A > 0$ *such that*

$$\sum_{n\geq 1} \omega_{\alpha,\beta}(z_n) c^A_{\omega_{\alpha,\beta}}(E_n) < \infty,$$

then $\mathcal{Z}$ *is a zero set for* $\mathcal{D}_{\omega_{\alpha,\beta}}$.

Theorem 4.6 *Let* $\alpha, \varepsilon \in (0, 1)$ *and let* $\gamma = \frac{\varepsilon}{1-\alpha}$. *Let* $\mathcal{Z} = (z_n)_{n\geq 1}$ *be a sequence of* $\mathbb{D}$ *and let* $E_n = \cup_{k\geq n} I(z_k, \gamma)$. *Suppose that* $\sum_{n\geq 1}(1 - |z_n|)^{1-\varepsilon} < \infty$. *If*

$$\sum_{n\geq 1}(1-|z_n|)^\alpha \exp\left(-\log^{1/\beta}(1/c_\alpha(E_n))\right) < \infty,$$

for some $\beta \in (\alpha, 1)$, *then* $\mathcal{Z}$ *is a zero set for* $\mathcal{D}_\alpha$.

Proof Recall that the reproducing kernel of $\mathcal{D}_\alpha$ is given by $K^\alpha(z, w) = \frac{1}{(1-z\overline{w})^\alpha}$. Let $f \in \mathcal{D}_\alpha$ be the function constructed in Theorem 1.2. We have Re$f \geq 0$ *a.e.* on $\mathbb{T}$ and Re$f \geq \log(1/c_\alpha(\cup_{j\geq n} I_j))$ $c_\alpha - q.e.$ on I_n. Since $|\mathrm{Arg}(K^\alpha(z, w))| < \frac{\pi}{2}\alpha$, it is easy to see that Re$f^{1/\beta} \geq 0$. This implies that $h =: e^{-f^{1/\beta}} \in \mathcal{D}_\alpha$. We get the result by following the proof of Theorem 1.3. □

Note that for the classical Dirichlet space $\mathcal{D}$ the associated capacity c_0 is comparable with the logarithmic capacity and the reproducing kernel is given by

$$K(z,w) = \frac{1}{z\overline{w}} \log \frac{1}{1 - z\overline{w}}.$$

Clearly, we have $|\mathrm{Im} K(z,w)| \leq \frac{\pi}{2}$. Our method implies the following result.

Theorem 4.7 *Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a sequence of $\mathbb{D} \setminus \{0\}$. Let $\varepsilon \in (0,1)$ and suppose that $\sum_{n\geq 1}(1-|z_n|)^{1-\varepsilon} < \infty$. Let $E_n = \cup_{j\geq n} I_j(z_n, \varepsilon)$. If*

$$\sum_{n\geq 1} \exp\left(-\frac{A}{c_0(E_n)}\right) < \infty,$$

for some $A > 0$, then $\mathcal{Z}$ is a zero set for $\mathcal{D}$.

Proof It suffices to replace h in the proof of Theorem 1.3 by $\exp(-Ae^f)$. □

5 General Weights

In this section we discuss briefly the case of general positive weights. Let ω be an integrable positive weight on $\mathbb{D}$. Let B be the Blaschke product associated with the sequence $\mathcal{Z} = (z_n)_{n\geq 1}$. Using the fact that $(1-|z|^2)|B'(z)| \leq 1$, we get

$$(1-|z|^2)|B'(z)|^2 \leq \sum_{n\geq 1} \frac{1-|z_n|^2}{|1-\overline{z_n}\zeta|^2}.$$

Now one can show easily the following proposition.

Proposition 5.1 *Let $f \in H^2$, and let B be the Blaschke product associated with the Blaschke sequence $\mathcal{Z} = (z_n)_{n\geq 1}$ of $\mathbb{D}$. Then*

$$\|Bf\|_\omega^2 \lesssim \|f\|_\omega^2 + \int_{\mathbb{T}} |f(\zeta)|^2 \tilde{\mathcal{Z}}_\omega(\zeta) dm(\zeta). \tag{17}$$

where

$$\tilde{\mathcal{Z}}_\omega(\zeta) = \sum_{n\geq 1}(1-|z_n|^2) \int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{z_n}z|^2|1-\overline{z}\zeta|^2} dA(z) \quad (\zeta \in \mathbb{T}).$$

Note that for ω_α we have

$$\tilde{\mathcal{Z}}_{\omega_\alpha}(\zeta) \asymp \sum_{n\geq 1} \frac{1-|z_n|^2}{|1-\overline{z_n}\zeta|^{2-\alpha}} \quad (\zeta \in \mathbb{T}).$$

A comparison between Proposition 5.1 and Theorem 2.1 is needed. Let $\omega = U_\mu$ be a purely superharmonic weight. To get $\mathcal{Z}_\mu \asymp \tilde{\mathcal{Z}}_\omega$ it suffices to have $(1-|z|^2)^2|\Delta\omega(z)| \asymp \omega(z)$. Note that this condition is satisfied for a large class of regular radial concave weights. For example, it is satisfied by the weights $\omega_{\alpha,\beta}$, when $\alpha \in (0,1)$ and $\beta \in (-\infty, +\infty)$. However, if $\alpha = 0$ and $\beta > 0$ (which means that $\mathcal{D}_{\omega_{\alpha,\beta}}$ is close to the classical Dirichlet space) the situation is quite different. Indeed, we have

$$(1-|z|^2)^2|\Delta\omega_{0,\beta}(z)| \asymp \frac{\omega_{0,\beta}(z)}{\log\left(\frac{1}{1-|z|}\right)} = \omega_{0,\beta+1}(z), \quad (|z| \to 1^-).$$

and Theorem 2.1 is more precise than Proposition 5.1. The same phenomenon happens when $\mathcal{D}_\omega$ is close to H^2. It is the case for $\mathcal{D}_{\omega_{\alpha,\beta}}$ with $\alpha = 1$ and $\beta < 0$.

By Proposition 5.1, it is clear that $B \in \mathcal{D}_\omega$ if $\sum_{n\geq 1} \Phi_\omega(z_n) < \infty$, where Φ_ω is given by

$$\Phi_\omega(w) = (1-|w|^2)\int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{z}w|^2(1-|z|^2)}dA(z).$$

An analogue of Theorem 4.1 can be stated as follow.

Theorem 5.2 *Let ω be a radial weight and let E, ρ satisfying conditions of Theorem 4.1. Let $\mathcal{Z} = (z_n)_{n\geq 1}$ be a Blaschke sequence of $\mathbb{D}$. If*

$$\sum_{n\geq 1} \Phi_\omega(z_n)\rho(16d(z_n, E)) < \infty,$$

then $\mathcal{Z}$ is a zero set of $\mathcal{D}_\omega$.

6 Final Remarks

Let ω be a superharmonic weight. As mentioned in the introduction, a Blaschke sequence $\mathcal{Z} = (z_n)_{n\geq 1}$ is a zero set for $\mathcal{D}_\omega$ if $\sum_{n\geq 1} \frac{1}{K^\omega(z_n,z_n)} < \infty$. To take advantage from this result we need to estimate $K^\omega(z,z)$.

Problem 1: Give an estimate of $K^\omega(z,z)$, where $\omega = U_\mu + P_\nu$ is a general superharmonic weight.

The estimate of $K^\omega(z,z)$, when ω is a "regular" radial weight is not difficult to obtain. Indeed, for example if $\omega = \omega_{\alpha,\beta}$, with $(\alpha,\beta) \in \Lambda$, then

$$K^\omega(z,z) \asymp 1/\tilde{\omega}_{\alpha\beta}(z), \quad (z \in \mathbb{D}),$$

where

$$\tilde{\omega}_{\alpha,\beta} = \begin{cases} \omega_{\alpha,\beta} & \text{if } \alpha \in (0,1] \\ \omega_{\alpha,\beta+1} & \text{if } \alpha = 0. \end{cases}$$

Note also that for the harmonic case $\omega = P_\nu$ it was proved in [15] that

$$K^\omega(z,z) \asymp 1 + \int_0^{|z|} \frac{dr}{(1-r)P_\nu(rz/|z|) + (1-r)^2} \quad (z \in \mathbb{D}\setminus\{0\}).$$

We remark that an answer to this problem will allow to get estimates of the capacity of some borelian subsets of $\mathbb{T}$ (see [15] for the harmonic case).

In [21], D. Guillot considered harmonic weights $\omega = P_\nu$, where $\nu = \sum_{n\geq 1} c_n \delta_{\zeta_n}$ is a positive finite atomic measure on $\mathbb{T}$. He proved that every Blaschke sequence is a zero set for $\mathcal{D}_\omega$ if and only if

$$\int_{\mathbb{T}} \log V_2(\nu)(\zeta) dm(\zeta) < \infty,$$

where $V_2(\nu)(\zeta) = \int_{\mathbb{T}} \frac{d\nu(\lambda)}{|\zeta - \lambda|^2}$ is the newtonian potential. A natural and interesting problem raised from this result is the following.

Problem 2: Determine all superharmonicaly weighted Dirichlet spaces for which every Blaschke sequence is a zero set.

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