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Uniform Shapiro-Lopatinski Conditions and Boundary Value Problems on Manifolds with Bounded Geometry

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Abstract
We study the regularity of the solutions of second order boundary value problems on *manifolds with boundary and bounded geometry*. We first show that the regularity property of a given boundary value problem (P, C) is equivalent to the *uniform* regularity of the natural family (P_x, C_x) of associated boundary value problems in local coordinates. We verify that this property is satisfied for the Dirichlet boundary conditions and strongly elliptic operators via a compactness argument. We then introduce a *uniform Shapiro-Lopatinski regularity condition,* which is a modification of the classical one, and we prove that it characterizes the boundary value problems that satisfy the usual regularity property. We also show that the natural Robin boundary conditions always satisfy the uniform Shapiro-Lopatinski regularity condition, provided that our operator satisfies the strong Legendre condition. This is achieved by proving that "well-posedness implies regularity" via a modification of the classical "Nirenberg trick". When combining our regularity results with the Poincaré inequality of (Ammann-Große-Nistor, preprint 2015), one obtains the usual well-posedness results for the classical boundary value problems in the usual scale of Sobolev spaces, thus extending these important, well-known theorems from smooth, bounded domains, to manifolds with boundary and bounded geometry. As we show in several examples, these results do not hold true anymore if one drops the bounded geometry assumption. We also introduce a *uniform Agmon condition* and show that it is equivalent to the coerciveness. Consequently, we prove a well-posedness result for parabolic equations whose elliptic generator satisfies the uniform Agmon condition.

Keywords Manifolds of bounded geometry · Boundary value problems · Shapiro-Lopatinski condition

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1 Introduction

We study the regularity of the solutions of general second order boundary value problems on *manifolds with boundary and bounded geometry.* We obtain several sufficient and several necessary conditions for the regularity of the solution in the usual scale of Sobolev spaces. To state these results in more detail, we need to introduce some notation.

One of the reasons for our interest in manifolds with bounded geometry is due to the fact that they provide a very convenient setting to study evolution equations in a geometric setting [15, 17, 21, 24, 25, 30, 31, 64, 65]. For instance, maximal regularity results on manifolds with bounded geometry were obtained by H. Amann [6, 7] and A. Mazzucato and V. Nistor [49]. Applications to Quantum Field Theory were obtained C. Bär and A. Strohmaier [16], by C. Gérard [29], and by W. Junker and E. Schrohe [41], to mention just a few of the many papers on the subject. In this paper, however, we establish some basic results for *elliptic* equations, which are useful in the study of evolution equation and for many other problems.

Let M be a manifold with boundary and bounded geometry, see Definition 2.5, and consider a boundary value problem of the form

$$\begin{cases} Pu = f & \text{in } M \\ Cu = h & \text{on } \partial M. \end{cases} \tag{1}$$

More precisely, we are interested in *regularity results*, which generally say that if u has a minimal regularity, but f and h have a good regularity, then u also has a good regularity. See Definition 4.11 for a precise definition.

We provide *three main methods* to obtain a good regularity for u based on:

(1) the uniform regularity of the local problems;
(2) the uniform Shapiro-Lopatinski conditions; and
(3) the well-posedness in energy spaces.

The first two methods provide a characterization of the boundary value problems that satisfy regularity, but are not always easy to use in practice. Most of the examples of boundary value problems that we know that are satisfying regularity are obtained from a well-posedness result in spaces of minimal regularity (energy spaces). In turn, these well-posedness results are obtained from coercivity.

Let us discuss now each of the three methods and the specific results and applications contained in this paper.

Uniform Regularity of the Local Problems This method is discussed in Section 5. In that section, we associate to each point x of M a local operator P_x. This operator is associated to a boundary value problem (P_x, C_x) if x is on the boundary. These operators are defined on small–but uniform size–coordinate patches around x (some care is needed close to the boundary). We introduce then *uniform regularity conditions* for these local operators. This amounts to regularity conditions for each of the local operators in such a way that the resulting constants are independent of x (see Definition 5.1). We then show that the initial operator satisfies regularity if, and only if, the associated family of local operators satisfies a uniform regularity condition. This is Theorem 5.6. We further show that if the family of local operators is compact and satisfies regularity, then it satisfies a *uniform* regularity condition. The compactness condition is satisfied, for instance, by strongly elliptic operators endowed with *Dirichlet boundary conditions*. This then yields right away a regularity result for these operators (Theorem 5.12).

Uniform Shapiro-Lopatinski Conditions This method is discussed in Section 6. In analogy with the classical case, we introduce a *uniform Shapiro-Lopatinski regularity condition* and we show that it characterizes the boundary value problems that satisfy regularity (Theorem 6.9). The usual Shapiro-Lopatinski conditions are not expressed in a quantitative way that will make them amenable to generalize right away to uniform conditions. For this reason, we go back to a more basic formulation of the Shapiro-Lopatinski condition in terms of the left invertibility of suitable model problems $(P_x^{(0)}, C_x^{(0)})$, one for each point of the boundary, and we require the norms of inverses of these problems to be bounded. If the problems (P_x, C_x) are the *local* versions of the initial problem (P, C), the problems $(P_x^{(0)}, C_x^{(0)})$ are the *micro-local* versions of (P, C).

Well-Posedness in Energy Spaces This method is discussed in Section 7. While intellectually satisfying (especially in view of the classical results), the uniform Shapiro-Lopatinski is not very easy to use in practice. For this reason, we extend a result of Nirenberg [56] to prove that the problems that are well-posed in energy spaces satisfy regularity (Theorem 7.2). We then combine this method with the other two to check that the Robin boundary conditions $e_1\partial_\nu^a + Q$ (for suitable Q, see Remark 4.4) satisfy, first, the classical (i.e. not necessarily uniform) Shapiro-Lopatinski boundary conditions, then that they satisfy the compactness condition, and hence, the uniform regularity for the local problems. This allows us to conclude a regularity result for mixed Dirichlet/Robin boundary conditions (with suitable Q). Let us explain this result in more detail.

For the rest of this introduction, (M, g) will be a Riemannian manifold with boundary and bounded geometry and $E \to M$ will be a vector bundle with bounded geometry. We assume that we are given a decomposition $E|_{\partial M} = F_0 \oplus F_1$ as an orthogonal sum of vector bundles with bounded geometry and we let e_j denote the orthogonal projection $E \to F_j$. Let a be a sesquilinear form on $T^*M \otimes E$ and let Q be a first order differential operator acting on the sections of $F_1 \to \partial M$. To this data we associate the operator $\tilde{P}_{(a,Q)} \colon H^1(M; E) \to H^1(M; E)^*$ defined by

$$\langle \tilde{P}_{(a,Q)} u, v\rangle = \int_M a(\nabla u, \nabla v) d\operatorname{vol}_g + \int_{\partial M} (Qu, v) d\operatorname{vol}_{\partial g} . \tag{2}$$

We shall be interested in the second order operator $\tilde{P} := \tilde{P}_{(a,Q)} + Q_1$, where Q_1 is a first order differential operator. An operator of this form will be called a *second order differential operator in divergence form.* Let ν be the outer normal vector field to the boundary and r be the distance to the boundary, then $dr(\nu) = 1$ and $\partial_\nu^a u := a(dr, \nabla u)$ is the associated conormal derivative, see Remarks 4.3, 4.4 and Example 4.8. We let P be obtained from $\tilde{P}$ using the restriction to H_0^1. See Section 4.1 for more details. Our result is a regularity result for the problem

$$\begin{cases} Pu = f & \text{in } M \\ e_0 u = h_0 & \text{on } \partial M \\ e_1\partial_\nu^a u + Qu = h_1 & \text{on } \partial M\,. \end{cases} \tag{3}$$

Recall that a sesquilinear form a on a hermitian vector bundle $V \to X$ is *called strongly* coercive (or *strictly positive*) if there is some $c > 0$ such that $\Re a(\xi, \xi) \geq c|\xi|^2$ for all $x \in X$ and $\xi \in V_x$. If the sesquilinear form a on $T^*M \otimes E$ used to define P is strongly coercive, then P is said to satisfy the *strong Legendre condition*. See also Definition 6.11.

Theorem 1.1 *Let M be a manifold with boundary and bounded geometry, $E \to M$ be a vector bundle with bounded geometry, and $\tilde{P} := \tilde{P}_{(a,Q)} + Q_1$ be second order differential* *operator in divergence form acting on* (M, E)*, as above. Assume that* $\tilde{P}$ *has coefficients in* $W^{\ell+1,\infty}$*, that* a *that is strongly coercive, and that* $Q + Q^*$ *is a zero order operator. Then there is* $C > 0$ *such that, if* $u \in H^1(M; E)$*,* $f := Pu \in H^{\ell-1}(M; E)$*,* $h_0 := e_0 u|_{\partial M} \in H^{\ell+1/2}(\partial M; F_0)$*, and* $h_1 := e_1 \partial_\nu^a u + Qu \in H^{\ell-1/2}(\partial M; F_1)$*, then* $u \in H^{\ell+1}(M; E)$ *and*

$$\|u\|_{H^{\ell+1}} \leq C\left(\|f\|_{H^{\ell-1}} + \|h_0\|_{H^{\ell+1/2}} + \|h_1\|_{H^{\ell-1/2}} + \|u\|_{H^1}\right).$$

In Section 7.4, we will see why, in general, the bounded geometry assumption is useful. Many of our results extend to higher order equations, however, including this would have greatly extended the length of the paper. See nevertheless Remarks 4.13 and 7.3.

Contents of the Paper The results in this paper are a natural continuation of our joint paper with Bernd Ammann [10]. In that paper, we established some general geometric results on manifolds with boundary and bounded geometry and dealt almost exclusively with Dirichlet boundary conditions. Also, in [10] we restricted ourselves to the case of the Laplace operator. The main contribution of this paper is the fact that we consider general uniformly elliptic operators and we work under general boundary conditions. This requires some new, specific ideas and results. Further results are included in [9].

Here are the contents of the paper. Section 2 contains some basic definitions and some background results. Several of these results are from [10]. For instance, in this section we review the needed facts on manifolds with bounded geometry and we recall the definition of Sobolev spaces using partitions of unity. Section 3 contains some preliminary results on Sobolev spaces and differential operators. In particular, we provide a description of Sobolev on manifolds with bounded geometry spaces using vector fields. The first two sections can be skipped at a first reading by the seasoned researcher. Section 4 introduces our differential operators and boundary value problems, both of which are given in a variational (i.e. weak) form. We show that, under some mild conditions, all non-degenerate boundary conditions are equivalent to variational ones, which justifies us to concentrate on the later. Also in this section we formulate the regularity condition. The last three sections describe the three methods explained above and their applications. In the last section, in addition to discussing the third method mentioned above ("Well-posedness in energy spaces"), we also introduce a *uniform Agmon condition*, which, we show, is equivalent to the coercivity of the operator. This then leads to an application to the well-posedness of some parabolic equations on manifolds with bounded geometry.

2 Background Material and Notation

We begin with some background material, for the benefit of the reader. More precisely, we introduce here the needed function spaces and we recall the needed results from [10] as well as from other sources. We also use this opportunity to fix the notation, which is standard, so this section can be skipped at a first reading. This section contains no new results. *Throughout this paper,* M *will be a (usually non-compact) connected smooth manifold, possibly with boundary.*

2.1 General Notations and Definitions

We begin with the most standard concepts and some notation.

2.1.1 Continuous Operators

Let X and Y be Banach spaces and $A\colon X \to Y$ be a linear map. Recall that A is *continuous* (or *bounded*) if, and only if, $\|A\|_{X,Y} := \inf_{x\neq 0} \frac{\|Ax\|}{\|x\|} < \infty$. When the spaces on which A acts are clear, we shall drop them from the notation of the norm, thus write $\|A\| = \|A\|_{X,Y}$. We say that A is an *isomorphism* if it is a continuous bijection (in which case the inverse will also be continuous, by the open mapping theorem).

2.1.2 The Conjugate Dual Spaces

For complex vector spaces V and W, a *sesquilinear* map $V \times W \to \mathbb{C}$ will always be (complex) linear in V and anti-linear in W. A *Hermitian form* on V is a positive definite sesquilinear map $(\cdot\,,\,\cdot)\colon V \times V \to \mathbb{C}$, which implies $(w, v) = \overline{(v, w)}$. In order to deal with the complex version of the Lax–Milgram lemma, we introduce the following notation and conventions.

Notation 2.1 Let V be a complex vector space, usually a Hilbert space. We shall denote by $\overline{V}$ the complex conjugate vector space to V. If V is endowed with a topology, we denote by V' the (topological) dual of V. It will be convenient to denote $V^* := (\overline{V})' \simeq \overline{V'}$. We can thus regard a continuous sesquilinear form $B\colon V \times W \to \mathbb{C}$ as a continuous bilinear form on $V \times \overline{W}$, or, moreover, as a map $V \to W^*$. Let us assume that V is a Hilbert space with inner product $(\ ,\ )\colon V \times V \to \mathbb{C}$. If $T\colon V \to W$ is a continuous map of *Hilbert* spaces, then we denote by $T^*\colon W^* \to V^*$ the *adjoint* of T, as usual.

2.1.3 Vector Bundles

Let $E \to M$ be a smooth real or complex vector bundle endowed with metric $(\ ,\ )_E$. We denote by $\Gamma(M; E)$ the set of *smooth* sections of E. Let E be endowed with a connection

$$\nabla^E\colon \Gamma(M; E) \to \Gamma(M; E \otimes T^*M).$$

We assume that ∇^E is metric preserving, which means that

$$X(\xi, \eta)_E = (\nabla_X \xi, \eta)_E + (\xi, \nabla_X \eta)_E.$$

If $F \to M$ is another vector bundle with connection ∇^F, then we endow $E \otimes F$ with the induced product connection $\nabla^{E\otimes F}(\xi \otimes \eta) := (\nabla^E \xi) \otimes \eta + \xi \otimes \nabla^F \eta$.

We endow the tangent bundle $TM \to M$ with the Levi-Civita connection ∇^M, which is the unique torsion free, metric preserving connection on TM. We endow T^*M and all the tensor product bundles $E \otimes T^{*\otimes k}M := E \otimes (T^*M)^{\otimes k}$ with the induced tensor product connections (we write $V^{\otimes k} := V \otimes V \otimes \ldots \otimes V$, k-times).

2.2 Manifolds with Boundary and Bounded Geometry

We now recall some basic material on manifolds with boundary and bounded geometry, mostly from [10], to which we refer for more details and references.

Definition 2.2 A vector bundle $E \to M$ with given connection is said to *have totally bounded curvature* if its curvature and all its covariant derivatives are bounded (that is, $\|\nabla^k R^E\|_\infty < \infty$ for all k). If TM has totally bounded curvature, we shall then say that M has *totally bounded curvature*.

Let $\exp_p^M : T_pM \to M$ be the exponential map at p associated to the metric and

$$r_{\mathrm{inj}}(p) := \sup\{r \mid \exp_p^M : B_r^{T_pM}(0) \to M \text{ is a diffeomorphism onto its image}\}$$
$$r_{\mathrm{inj}}(M) := \inf_{p\in M} r_{\mathrm{inj}}(p).$$

The following concept is classical and fundamental.

Definition 2.3 A Riemannian manifold without boundary (M, g) is said to be of *bounded geometry* if $r_{\mathrm{inj}}(M) > 0$ and if M has *totally bounded curvature*.

If M has boundary, clearly $r_{\mathrm{inj}}(M) = 0$, so a manifold with non-empty boundary will never have bounded geometry in the sense of the above definition. However, Schick has found a way around this difficulty [63]. Let us recall a definition equivalent to his following [10]. The main point is to describe the boundary as a suitable submanifold of a manifold without boundary and with bounded geometry. Let us consider then a hypersurface $\subset M$, i.e. a submanifold with $\dim H = \dim M - 1$. Assume that H carries a globally defined unit normal vector field ν and let $\exp^\perp(x,t) := \exp_x^M(t\nu_x)$ be the exponential in the direction of the chosen unit normal vector. By II^N we denote the *second fundamental form* of N (in M: $\mathrm{II}^N(X,Y)\nu := \nabla_X Y - \nabla_X^N Y$).

Definition 2.4 Let (M^m, g) be a Riemannian manifold of bounded geometry with a hypersurface $H = H^{m-1} \subset M$ and a unit normal field ν on H. We say that H is a *bounded geometry hypersurface* in M if the following conditions are fulfilled:

(i) H is a closed subset of M;
(ii) $\|(\nabla^H)^k \mathrm{II}^H\|_{L^\infty} < \infty$ for all $k \geq 0$;
(iii) $\exp^\perp : H \times (-\delta, \delta) \to M$ is a diffeomorphism onto its image for some $\delta > 0$.

We shall denote by r_∂ the largest value of δ satisfying this definition.

We have shown in [10] that $(H, g|_H)$ is then a manifold of bounded geometry in its own right. See also [24, 25] for a larger class of submanifolds of manifolds with bounded geometry.

Definition 2.5 A Riemannian manifold M with (smooth) boundary has *bounded geometry* if there is a Riemannian manifold $\widehat{M}$ with bounded geometry satisfying

(i) M is contained in $\widehat{M}$;
(ii) ∂M is a bounded geometry hypersurface in $\widehat{M}$.

Example 2.6 Lie manifolds have bounded geometry [11, 12]. It follows that, Lie manifolds with boundary are manifolds with boundary and bounded geometry.

For $x, y \in M$, we let $\mathrm{dist}(x, y)$ denote the distance between x and y with respect to the metric g.

2.3 Coverings, Partitions of Unity, and Sobolev Spaces

We now introduce some basic notation and constructions for manifolds with boundary and bounded geometry. The boundary ∂M of a manifold with boundary M will always be a subset of M: i.e. $\partial M \subset M$. *For the rest of the paper, we will always assume that $E \to M$ has totally bounded curvature, and, for the rest of this section, we shall assume that M is a manifold with boundary and bounded geometry.*

Let ν be the inner unit normal vector field of ∂M and let r_∂ as in the bounded geometry condition for ∂M in M, Definition 2.4. Moreover, for a metric space X, we shall denote by $B_r^X(p)$ the open ball of radius r centered at p and set $B_r^m(p) := B_r^{\mathbb{R}^m}(p)$. We shall identify T_pM with $\mathbb{R}^{m+1}$ and, respectively, $T_p\partial M$ with $\mathbb{R}^m$, using an orthonormal basis, thus obtaining a diffeomorphism $\exp_p^M : B_r^{m+1}(0) \to B_r^M(p)$. For $r < \frac{1}{2}\min\{r_{\mathrm{inj}}(\partial M), r_{\mathrm{inj}}(M), r_\partial\}$ we define the maps

$$\begin{cases} \kappa_p : B_{2r}^m(0) \times [0, 2r) \to M, & \kappa_p(x,t) := \exp_q^M(t\nu_q), \ \text{if} p \in \partial M,\ q := \exp_p^{\partial M}(x) \\ \kappa_p : B_r^{m+1}(0) \to M, & \kappa_p(v) := \exp_p^M(v), \quad \text{if} \operatorname{dist}(p, \partial M) \geq r, \end{cases}$$

with range

$$U_p(r) := \begin{cases} \kappa_p(B_{2r}^m(0) \times [0, 2r)) \subset M & \text{if } p \in \partial M \\ \kappa_p(B_r^{m+1}(0)) \ = \ \exp_p^M(B_r^{m+1}(0)) \subset M & \text{otherwise.} \end{cases} \tag{4}$$

Definition 2.7 Let $r_{FC} := \min\left\{ \frac{1}{2} r_{\mathrm{inj}}(\partial M),\ \frac{1}{4} r_{\mathrm{inj}}(M),\ \frac{1}{2} r_\partial \right\}$ and $0 < r \leq r_{FC}$. Then $\kappa_p : B_r^m(0) \times [0, r) \to U_p(r)$ is called a *Fermi coordinate chart* at $p \in \partial M$. The charts κ_p for $\operatorname{dist}(p, \partial M) \geq r$ are called *geodesic normal coordinates*.

To define our Sobolev spaces, we need suitable coverings of our manifold. For the sets in the covering that are away from the boundary, we will use geodesic normal coordinates, whereas for the sets that intersect the boundary, we will use the Fermi coordinates introduced in Definition 2.7.

Definition 2.8 Let M be a manifold with boundary and bounded geometry and let $0 < r \leq r_{FC} := \min\left\{ \frac{1}{2} r_{\mathrm{inj}}(\partial M),\ \frac{1}{4} r_{\mathrm{inj}}(M),\ \frac{1}{2} r_\partial \right\}$, as in Definition 2.7. A subset $\{p_\gamma\}_{\gamma\in\mathbb{N}}$ is called an *r-covering subset of M* if the following conditions are satisfied:

(i) For each $R > 0$, there exists $N_R \in \mathbb{N}$ such that, for each $p \in M$, the set $\{\gamma \in \mathbb{N} | \operatorname{dist}(p_\gamma, p) < R\}$ has at most N_R elements.
(ii) For each $\gamma \in \mathbb{N}$, we have either $p_\gamma \in \partial M$ or $\operatorname{dist}(p_\gamma, \partial M) \geq r$, so that $U_\gamma := U_{p_\gamma}(r)$ is defined, compare to Eq. 4.
(iii) $M \subset \cup_{\gamma=1}^\infty U_\gamma$.

Remark 2.9 If $0 < r < r_{FC}$, then we can always find an r-covering subset of M, since M is a manifold with boundary and bounded geometry [36, Remark 4.6]. Moreover, it then follows from (i) of Definition 2.8 that the coverings $\{U_\gamma\}$ of M and $\{U_\gamma \cap \partial M\}$ of ∂M are uniformly locally finite.

We shall need the following class of partitions of unity defined using r-covering sets. Recall the definition of the sets $U_\gamma := U_{p_\gamma}(r)$ from Eq. 4.

Definition 2.10 A partition of unity $\{\phi_\gamma\}_{\gamma\in\mathbb{N}}$ of M is called *an r-uniform partition of unity associated to the r-covering set $\{p_\gamma\} \subset M$*, see Definition 2.8, if the support of each ϕ_γ is contained in U_γ and $\sup_\gamma \|\phi_\gamma\|_{W^{\ell,\infty}(M)} < \infty$ for each fixed $\ell \in \mathbb{Z}_+$.

In order to deal with boundary value problems with values in a vector bundle (that is, with boundary value problems for systems), we will also need the concept of synchronous trivializations, which we briefly recall here:

Definition 2.11 Let M be a Riemannian manifold with boundary and bounded geometry, and let $E \to M$ be a Hermitian vector bundle with metric connection. Let $(U_\gamma, \kappa_\gamma, \phi_\gamma)$ be Fermi and geodesic normal coordinates on M together with an associated r-uniform partition of unity as in the definitions above. If $p_\gamma \in M \setminus U_r(\partial M)$, then $E|_{U_\gamma}$ is trivialized by parallel transport along radial geodesics emanating from p_γ. If $p_\gamma \in \partial M$, then we trivialize $E|_{U_\gamma}$ as follows: First we trivialize $E|_{U_\gamma \cap \partial M}$ along the underlying geodesic normal coordinates on ∂M. Then, we trivialize by parallel transport along geodesics emanating from ∂M and being normal to ∂M. The resulting trivializations are called *synchronous trivializations along Fermi coordinates* and are maps

$$\xi_\gamma : \kappa_\gamma^{-1}(U_\gamma) \times \mathbb{C}^t \to E|_{U_\gamma} \tag{5}$$

where t is the rank of E.

We shall need a definition of Sobolev spaces using partitions of unity and "Fermi coordinates" [36] and a few standard results. In the scalar case, these results were stated in [10]. Here we stress the vector valued case. First, we have the following proposition that is a direct consequence of Theorems 14 and 26 in [36].

Proposition 2.12 *Let M be a Riemannian manifold with boundary and bounded geometry. Let $\{\phi_\gamma\}$ be an r-uniform partition of unity associated to an r-covering set $\{p_\gamma\} \subset M$ and let $\kappa_\gamma = \kappa_{p_\gamma}$ be as in Definition 2.7. Let $E \to M$ be a vector bundle with totally bounded curvature with trivializations ξ_γ as in Definition 2.11. Then*

$$|||u|||^p := \sum_\gamma \|\xi_\gamma^*(\phi_\gamma u)\|^p_{W^{s,p}}$$

defines a norm equivalent to the standard norm on $W^{s,p}(M; E)$, $s \in \mathbb{R}$, $1 < p < \infty$.

As in the scalar case [10], the space $\Gamma_c(M; E)$ of smooth, compactly supported sections of E is dense in $W^{s,p}(M; E)$, for $s \in \mathbb{R}$ and $1 < p < \infty$. This is obtained by truncating the sum. As usual, we shall let $H^\ell(M; E) := W^{\ell,2}(M; E)$, $s \in \mathbb{R}$. Similarly, we shall need the following extension of the trace theorem to the case of manifolds with boundary and bounded geometry, see Theorem 27 in [36] (see [10] for more references).

Theorem 2.13 (Trace theorem) *Let M be a manifold with boundary and bounded geometry and let $E \to M$ have totally bounded curvature. Then, for every $s > 1/2$, the restriction to the Dirichlet part of the boundary res: $\mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(\partial_D M)$ extends to a continuous, surjective map*

$$\mathrm{res}\colon H^s(M; E) \to H^{s-\frac{1}{2}}(\partial_D M; E).$$

Let ∂_ν be the normal derivative at the boundary. We then denote by $H_0^m(M; E)$ the kernel of the restrictions maps $\mathrm{res} \circ \partial_\nu^j$, $0 \le j \le m-1$. It is known that $H^{-m}(M; E^*)$ identifies with $H_0^m(M; E)^*$.

See also [12, 13, 37, 38, 44, 45, 65, 68, 69] for related results, in particular, for the use of the partitions of unity. See [26] and its outgrowth [27] for an introduction to manifolds of bounded geometry. See [20, 40, 46, 66] for the general results on Sobolev spaces not proved above.

3 Preliminary Results

We now include some preliminary results. This section consists mostly new results, but advanced reader can nevertheless skip this section at a first reading. *In this section we assume that M is a manifold with boundary and bounded geometry and that $E \to M$ is a vector bundle with totally bounded curvature (and, hence, with bounded geometry).*

3.1 Alternative Characterizations of Sobolev Spaces

We now provide an alternative description of Sobolev spaces using vector fields. If $A\colon V_1 \to V_2$ is a linear map of normed spaces, we define its *minimum reduced modulus* $\gamma(A)$ by

$$\gamma(A) := \inf_{\xi \in V_1/\ker(A)} \sup_{\eta \in \ker(A)} \frac{\|A\xi\|}{\|\xi + \eta\|},$$

i.e. $\gamma(A)$ is the largest number γ satisfying $\|A\xi\| \geq \gamma \|\xi + \ker(A)\|_{V_1/\ker(A)}$ for all $\xi \in V_1$ where $\|\cdot\|_{V_1/\ker(A)}$ is the quotient norm on $V_1/\ker(A)$. As it is well known, if V_1 is complete, a standard application of the Open Mapping Theorem gives that the image of A is closed if, and only if, $\gamma(A) > 0$. If V_1 and V_2 are Hilbert spaces and A is surjective, we have $\gamma(A)^{-2} = \|(AA^*)^{-1}\|$.

We have the following alternative characterization of Sobolev spaces in terms of vector fields. This alternative definition is more intuitive and easier to use in analysis. It is based on a the choice of a suitable finite family of vector fields, whose existence is assured by the following lemma.

Lemma 3.1 *Let (M, g) be a manifold with boundary and of bounded geometry. Then there exist vector fields $X_1, X_2, \ldots, X_N \in W^{\infty,\infty}(M; TM)$ such that for any $x \in M$ the map $\Phi_x\colon \mathbb{R}^N \ni (\lambda_1, \ldots, \lambda_N) \mapsto \sum \lambda_i X_i(x) \in T_xM$ is onto at any x and $\inf_x \gamma(\Phi_x) > 0$. Then we have that $[X_i, X_j] = \sum_{k=1}^N C_{ij}^k X_k$ and $\nabla_{X_i} X_j = \sum_{k=1}^N G_{ij}^k X_k$ for some functions $G_{ij}^k, C_{ij}^k \in W^{\infty,\infty}$ and any $1 \leq i, j \leq N$. Moreover, we can choose X_1 of length one and normal to the boundary of M and X_j tangent to the boundary for $j > 1$.*

Proof We use a covering by Fermi coordinates resp. geodesic normal coordinates (U_β, κ_β) as in Definition 2.8 together with an associated partition of unity ϕ_β as in Definition 2.10. We recall that all transition function $\kappa_\alpha \circ \kappa_\beta^{-1}$, the ϕ_β and all their derivatives are uniformly bounded, see [36, 65]. Let x_β^i be the coordinates corresponding to the chart κ_β. Note that the Christoffel symbols in each chart and their derivatives are also uniformly bounded due to the bounded geometry, [36, Lemma 3.10].

Let those coordinates be ordered such that $x_\beta^1 = r$, the distance to the boundary, for $U_\beta \cap \partial M \neq \emptyset$. Let $X_1 := \sum_{U_\beta \cap \partial M \neq \emptyset} \phi_\beta \partial_r$. Then, X_1 is normal to ∂M.

To define X_j for $j > 1$ we divide $\{U_\beta\}_\beta$ into finitely many disjoints subsets V_i, $i = 1, \ldots, d$, such that the U_β's in any V_i are pairwise disjoint. This is always possible since the covering $\{U_\beta\}_\beta$ is uniformly locally finite. We set

$$X_{i,j} := \sum_{U_\beta \in V_i} \phi_\beta \partial_{x_\beta^j} \quad \text{for} j \in \{2, \ldots, m\}$$

$$X_{i,1} := \sum_{U_\beta \in V_i, U_\beta \cap \partial M = \varnothing} \phi_\beta \partial_{x_\beta^1}$$

Then, $X_1, X_{i,j} \in W^{\infty,\infty}(M, TM)$. Moreover, all $X_{i,j}$ are tangent to the boundary by construction. By renaming the vector fields we obtain $X_1, \ldots, X_N$. Then Φ_x is onto by construction for all $x \in M$.

Next we show that the reduced minimum modulus is uniformly bounded from below: Let $x \in M$. Since the covering is uniformly locally finite, there is a chart κ_α around x with $\phi_\alpha(x) > c > 0$, c independent of x. Since $\partial^i_{x_\alpha}$ span $T_x M$ for all $(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$ there is a $(\mu_1, \ldots, \mu_N) \in \mathbb{R}^N$ with $\mu_j = 0$ if X_j was not build from κ_β. Thus,

$$\gamma(\Phi_x) \geq \inf_{(\lambda_1,\ldots,\lambda_N)\in\mathbb{R}^N} \frac{|\sum_i \mu_i X_i(x)|}{\sum_i |\mu_i|^2} \geq \inf_{(\lambda_1,\ldots,\lambda_N)\in\mathbb{R}^N} \frac{c\sum_i \mu_i^2}{\sum_i \mu_i^2} = c.$$

We have $\partial_{x_\beta^j} = \sum_i a^i_j \partial_{x^i_\alpha}$ with $a^i_j \in W^{\infty,\infty}$ with bounds independent of α and β since this is true for the transition functions $\kappa_\alpha \circ \kappa_\beta^{-1}$. Thus, together with the uniform bounds on the corresponding Christoffel symbols we get $\nabla_{\partial_{x^i_\alpha}} \partial_{x^j_\beta} = \sum_k b^k_{ij} \partial_{x^k_\alpha}$ with $b^k_{ij} \in W^{\infty,\infty}$. Altogether, we have

$$\begin{aligned} \nabla_{X_i} X_j &= \sum_{U_\beta \in V_i, U_\alpha \in V_j} \phi_\beta \nabla_{\partial_{x^i_\beta}} (\phi_\alpha \partial_{x^j_\alpha}) = \sum_{U_\beta \in V_i, U_\alpha \in V_j, k} \phi_\beta (\delta^j_k \partial_{x^i_\beta} \phi_\alpha + \phi_\alpha b^k_{ij}) \partial_{x^k_\beta} \\ &= \sum_k G^k_{ij} X_k \quad \text{with} G^k_{ij} = \sum_{U_\beta \in V_i, U_\alpha \in V_j} (\delta^j_k \partial_{x^i_\beta} \phi_\alpha + \phi_\alpha b^k_{ij}) \in W^{\infty,\infty} \end{aligned}$$

where we used in the last step that the supports of any two U_β in V_i are disjoint.

By, $[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i$ the remaining claim follows. □

Proposition 3.2 *Let $X_1, X_2, \ldots, X_N \in W^{\infty,\infty}(M; TM)$ be such that for any $x \in M$ the map $\Phi_x : \mathbb{R}^N \ni (\lambda_1, \ldots, \lambda_N) \mapsto \sum \lambda_i X_i(x) \in T_x M$ is onto at any x and* $\inf_x \gamma(\Phi_x) > 0$. *Then*

$$W^{\ell,p}(M; E) = \{ u \mid \nabla^E_{X_{k_1}} \nabla^E_{X_{k_2}} \ldots \nabla^E_{X_{k_j}} u \in L^p(M; E),\ j \leq \ell,\ k_1, \ldots, k_j \leq N \},$$

$1 \leq p \leq \infty$. *If E has furthermore totally bounded curvature, we have also*

$$W^{\ell,p}(M; E) = \left\{ u \mid \nabla^E_{X_{k_1}} \nabla^E_{X_{k_2}} \ldots \nabla^E_{X_{k_j}} u \in L^p(M; E) \right\},$$

where, this time, $1 \leq k_1 \leq k_2 \leq \ldots \leq k_j \leq N$, $j \leq \ell$.

We notice that the condition that $\Phi_x : \mathbb{R}^N \to T_x M$ be surjective implies that $\dim \ker \Phi_x = N - \dim M$ is constant. For compact M then $\inf_{x\in M} \gamma(\Phi_x) > 0$ is automatically satisfied.

Proof Let

$$\mathcal{X} := \left\{ u \mid \nabla^E_{X_{k_1}} \nabla^E_{X_{k_2}} \dots \nabla^E_{X_{k_j}} u \in L^p(M; E),\ j \le \ell,\ k_1, \dots, k_j \le N \right\}.$$

The fact that the vector fields X_k are bounded with bounded covariant derivatives gives that $\nabla_{X_k}\colon W^{j+1,p}(M; E) \to W^{j,p}(M; E)$ is bounded for any $j \ge 0$ and any p. This gives $W^{\ell,p}(M; E) \subset \mathcal{X}$.

To prove the converse, we proceed by induction as follows. First of all, the assumption that $\inf_x \gamma(\Phi_x) > 0$ together with the fact that Φ_x is surjective for all x gives that $\Phi_x^*(\Phi_x \Phi_x^*)^{-1}$ is a (pointwise bounded) right inverse to Φ_x, regarded as a map $(\underline{\mathbb{R}}^N)^{\otimes \ell} \to TM^{\otimes \ell}$. (Here $\underline{\mathbb{R}}^N$ is the *trivial* vector bundle of rank N.) Let us denote by Ψ the extension of this map to sections of the corresponding vector bundles, and, by abuse of notation, also by

$$\Psi\colon L^p(M; T^*M^{\otimes \ell} \otimes E) \to L^p(M; (\underline{\mathbb{R}}^N)^{\otimes \ell} \otimes E)$$

the induced maps. Then Ψ is continuous and a right inverse of (the map defined by) Φ. Therefore Ψ is a homeomorphism onto its image. That gives that $\Psi(\xi) \in L^p(M; (\underline{\mathbb{R}}^N)^{\otimes \ell} \otimes E)$ if, and only if, $\xi \in L^p(M; T^*M^{\otimes \ell} \otimes E)$. By taking the $(k_1, k_2, \dots, k_\ell)$ component of $\Psi(\xi)$, we obtain that $(\nabla^E)^\ell u \in L^p(M; T^*M^{\otimes \ell} \otimes E)$ if, and only if, $\langle (\nabla^E)^\ell u, X_{k_1} \otimes X_{k_2} \otimes \dots \otimes X_{k_\ell} \rangle \in L^p(M; E)$ for all $k_1, k_2, \dots, k_\ell \in \{1, 2, \dots, N\}$. To use induction, we notice that $\langle (\nabla^E)^\ell u, X_{k_1} \otimes X_{k_2} \otimes \dots \otimes X_{k_\ell} \rangle - \nabla^E_{X_{k_1}} \nabla^E_{X_{k_2}} \dots \nabla^E_{X_{k_\ell}} u$ is given by lower order terms of the same kind.

For the last part, we also notice that we can commute $\nabla^E_{X_i}$ with $\nabla^E_{X_j}$ up to totally bounded terms using Lemma 3.1. □

3.2 Sobolev Spaces Without Using Connections

Recall that we assume that $E \to M$ has totally bounded curvature. One way to think of such vector bundles is given by the following lemma:

Lemma 3.3 *Let us assume that M is a manifold with bounded geometry (possibly with boundary) and that $E \to M$ is a Hermitian vector bundle of totally bounded curvature. Then there exists a fibrewise isometric embedding $E \subset M \times \mathbb{C}^N$ into the trivial N-dimensional vector bundle with the standard metric such that, if e denotes the orthogonal projection onto E, then $e \in M_N(W^{\infty,\infty}(M))$ and the connection of E is equivalent to the Grassmann (projection) connection of the embedding (i.e. the difference of both connections is in $W^{\infty,\infty}$). Conversely, if $e \in M_N(W^{\infty,\infty}(M))$ and $E := e(M \times \mathbb{C}^N)$, then E with the Grassmann connection has totally bounded curvature.*

Proof Let us consider for each open subset U_γ as above the synchronous trivialization $\xi_\gamma\colon E|_{U_\gamma} \to \kappa_\gamma^{-1}(U_\gamma) \times \mathbb{C}^t$ from Eq. 5, with $\mathbb{C}^t$ the typical fiber above p_γ. Then $\phi_\gamma^{1/2} \xi_\gamma$ extends to a vector bundle map $E \to M \times \mathbb{C}^t$ that is in $W^{\infty,\infty}$ since M has bounded geometry and the connection on E has totally bounded curvature. Let $N = N_{5r}$, with N_{5r} as in Definition 2.8. By the construction of the sets U_γ, we can divide the set of all γ's into $N + 1$ disjoint subsets Γ_k, such that, for each fixed k and any $\gamma, \gamma' \in \Gamma_k$, the sets U_γ and $U_{\gamma'}$ are disjoint, by the construction of the sets U_γ. Let $\Psi_k := \sum_{\gamma \in \Gamma_k} \phi_\gamma^{1/2} \xi_\gamma$ and $\Psi := (\Psi_1, \Psi_2, \dots, \Psi_{N+1})\colon E \to M \times \mathbb{C}^{t(N+1)}$ be the resulting bundle morphism. Then Ψ is isometric, it is in $W^{\infty,\infty}$, and hence $e := \Psi\Psi^*$ is the desired projection. The equivalence of the connections follows from the uniform boundedness of the Christoffel symbols and their derivatives associated to the synchronous trivialization [36, Remark 5.3]. □

This lemma may be used to reduce differential operators acting on vector bundles to matrices of scalar differential operators. It also gives the following characterization of Sobolev spaces of sections of a vector bundle.

Proposition 3.4 *We use the notation of the last lemma. Then*

$$W^{k,p}(M; E) = eW^{k,p}(M)^N, \quad 1 \leq p \leq \infty.$$

We can now use this proposition to derive a description of Sobolev spaces on manifolds with bounded geometry that is completely independent of the use of connections.

Remark 3.5 The standard definition of the norm on Sobolev spaces is using powers of ∇ [10, 36]. For instance $W^{k,\infty}(M) := \{u \mid \nabla^j u \in L^\infty(M), 0 \leq j \leq k\}$ (alternatively, it is the space of functions with uniformly bounded derivatives of order $\leq k$ in any normal geodesic coordinate chart on B_r^m, for any fixed r less than the injectivity radius $r_{\mathrm{inj}}(M)$ of M). We can define $W^{k,\infty}(M; TM)$ similarly. Let then $X_1, X_2, \ldots, X_N$ as in Proposition 3.2 and Lemma 3.1, $1 \leq p < \infty$. Then

$$W^{\ell,p}(M) = \{u \mid X_{k_1} X_{k_2} \ldots X_{k_j} u \in L^p(M),\ j \leq \ell,\ 1 \leq k_1 \leq \ldots \leq k_j \leq N\}.$$

Together with Proposition 3.4, this gives a description of Sobolev spaces without using connections.

3.3 Differential Operators and Partitions of Unity

A *differential operator on* E is an expression of the form $Pu := \sum_{j=0}^k a_j \nabla^j u$,with a_j a section of $\mathrm{End}(E) \otimes TM^{\otimes j}$. It can thus simply be regarded as a formal collection of coefficients. In particular, we do not identify the differential operator with the maps that it induces (since it induces many). A differential operator $Pu = \sum_{j=0}^k a_j \nabla^j u$ will be said to *have coefficients in* $W^{\ell,\infty}$ if $a_j \in W^{\ell,\infty}(M; \mathrm{End}(E) \otimes TM^{\otimes j})$. If $\ell = 0$, we shall say that P has *bounded* coefficients. If $\ell = \infty$, we shall say that P has *totally bounded* coefficients. The continuity of the contraction map

$$W^{\ell,\infty}(M; \mathrm{End}(E) \otimes TM^{\otimes j}) \otimes W^{\ell,p}(M; T^*M^{\otimes j} \otimes E) \to W^{\ell,p}(M; E),$$

gives that a differential operator $P = \sum_{j=0}^k a_j \nabla^j$ with coefficients in $W^{\ell,\infty}$ defines a *continuous map*

$$P = \sum_{j=0}^k a_j \nabla^j : W^{\ell+k,p}(M; E) \to W^{\ell,p}(M; E), \quad \ell \geq 0.$$

Lemma 3.6 *Let* $k \geq 0$ *and let* P *be an order* ℓ *differential operator with coefficients* $a_j \in W^{k+1,\infty}(M; \mathrm{End}(E) \otimes TM^{\otimes j})$. *Let* $\phi \in W^{k+\ell+1,\infty}(M)$. *Then the commutator* $[P, \phi]$ *defines a continuous linear map* $H^{k+\ell}(M; E) \to H^{k+1}(M; E)$. *Moreover, if* $\{\phi_\gamma\}_\gamma$ *is a bounded family in* $W^{k+\ell+1,\infty}(M)$, *then operator norms of* $[P, \phi_\gamma] : H^{k+\ell}(M; E) \to H^{k+1}(M; E)$ *are bounded uniformly in* γ.

Proof We have $[P, \phi]u = \sum_{j=1}^{\ell} a_j \sum_{s=0}^{j-1} \binom{j}{s} \nabla^{j-s}\phi\nabla^s u$. Thus,

$$\begin{aligned}\|[P,\phi]u\|_{H^{k+1}} &\le C\sum_{r=0}^{k+1}\sum_{j=0}^{\ell}\|\nabla^r\left(a_j\sum_{s=0}^{j-1}\nabla^{j-s}\phi\nabla^s u\right)\|_{L^2}\\ &\le C\sum_{r=0}^{k+1}\sum_{t=0}^{r}\sum_{j=0}^{\ell}\|\nabla^{r-t}a_j\nabla^t\left(\sum_{s=0}^{j-1}\nabla^{j-s}\phi\nabla^s u\right)\|_{L^2}\end{aligned}$$

and the claim follows by the regularity assumptions on a_j and ϕ. □

4 Variational Boundary Conditions and Regularity

We now introduce "differential operators in divergence form" from a global point of view. The natural boundary value problem associated to differential operators in divergence form will be called *variational boundary value problems.* In this subsection, we introduce and take a first look at these variational boundary value problems. We will see that, under some mild assumptions on our differential operator P, any non-degenerate boundary value problem is equivalent to a variational one. This allows to reduce the study of the former to that of the latter, for which several general regularity results will be obtained in the following sections. On the other hand, the degenerate boundary value problems are known to behave in a significantly different way than the non-degenerate ones (see, for instance, [54] and the references therein). The possibility of reducing non-degenerate boundary value problems to variational ones seems not to have been explored too much in the literature.

We will continue to assume that M is a smooth manifold with smooth boundary. Moreover, we will assume that TM has totally bounded curvature, but *we will not assume that M has bounded geometry,* since we want to allow M to be a domain in a Euclidean space. Recall that all differential operators are assumed to have bounded coefficients and $E \to M$ has totally bounded curvature.

4.1 Sesquilinear Forms and Operators in Divergence Form

It is important in applications to consider operators "in divergence form," which we will define below shortly. They provide a slightly different class of differential operators than the operators with coefficients in L^∞ considered above and will be useful in order to treat the Dirichlet and Robin problems on the same footing. To introduce second order differential operators in divergence form, we shall need the following data and assumptions:

Assumption 4.1 Let M be a smooth manifold with smooth boundary, $E \to M$ a vector bundle, a a sesquilinear form on $T^*M \otimes E$, and first order differential operators Q and Q_1 satisfying

(A1) TM and E have totally bounded curvature.

(A2) $E|_{\partial M} = F_0 \oplus F_1$, with F_0 and F_1 with totally bounded curvature and F_0 is the orthogonal complement of F_1.

(A3) $a = (a_x)_{x\in M}$ is a measurable, bounded family of hermitian sesquilinear forms

$$a = (a_x)_{x\in M}, \quad a_x : T_x^*M \otimes E_x \times T_x^*M \otimes E_x \to \mathbb{C}.$$

(A4) Q_1 has L^∞ coefficients and acts on (M, E).

(A5) Q has $W^{1,\infty}$ coefficients and acts on $(\partial M, F_1)$.
(A6) V is the closed subspace $H_0^1(M; E) \subset V \subset H^1(M; E)$ defined by

$$V := \{u \in H^1(M; E) \mid u|_{\partial M} \in \Gamma(\partial M; F_1)\}.$$

The family (a_x) defines a section a of the bundle $((T^*M \otimes E) \otimes (T^*M \otimes \overline{E}))'$. In general, we say that a section $a = (a_x)_{x \in M}$ is a *bounded sesquilinear form on* $T^*M \otimes E$ if it is an L^∞-section of $((T^*M \otimes E) \otimes (T^*M \otimes \overline{E}))'$.

4.1.1 The Dirichlet (Sesquilinear) Form

Using our assumptions 4.1, we first define

$$B_a(u, v) := \int_M a(\nabla u, \nabla v) \, \mathrm{dvol}_g, \tag{6}$$

which is the *Dirichlet form* associated to $a = (a_x)_{x \in M}$. Then the *Dirichlet form* $B\colon V \times V \to \mathbb{C}$ associated to a, Q, and Q_1 is

$$B(u, v) := B_a(u, v) + (Q_1 u, v)_{L^2(M;E)} + (Qu|_{\partial M}, v|_{\partial M})_{L^2(\partial M; F_1)}, \tag{7}$$

where, initially, $u, v \in V \cap H^\infty(M; E)$, and then we then extend B to a sesquilinear linear form $B\colon V \times V \to \mathbb{C}$ by continuity. In the future, we shall usually write $(Qu, v)_{L^2(\partial M)}$ instead of $(Qu|_{\partial M}, v|_{\partial M})_{L^2(\partial M; F_1)}$.

4.1.2 The Induced Operator $\tilde{P}$

The continuous, sesquilinear form $B\colon V \times V \to \mathbb{C}$ defines a linear map (or operator)

$$\begin{aligned} \tilde{P}\colon V &\longrightarrow V^* \\ \langle \tilde{P}(v), w \rangle &:= B(v, w), \quad v, w \in V. \end{aligned} \tag{8}$$

If $B = B_a$, we shall denote by $\tilde{P}_a$ the associated operator. We note that B and $\tilde{P}$ depend on the choice of the metric g, although this will typically *not* be shown in the notation, since the metric will be fixed.

Definition 4.2 We shall say that a differential operator $\tilde{P}\colon V \to V^*$ obtained as in Eq. 8 is a *second order differential operator in divergence form.*

Recall that $H^{-1}(M; E^*) \simeq H_0^1(M; E)^*$. Using the metric on E, we shall identify $H^{-1}(M; E^*) \simeq H^{-1}(M; E)$. Since $H_0^1(M; E) \subset V$, we obtain the natural map $V^* \to H_0^1(M; E)^* \simeq H^{-1}(M; E^*)$, and hence $\tilde{P}$ gives rise to a map $P\colon V \to H^{-1}(M; E^*)$. Clearly, P does not depend on Q, whereas this is in general not the case for $\tilde{P}$. In fact, Q only enters in the boundary conditions, see Example 4.8. Similarly, $\tilde{P}$ extends to a map $\tilde{P} : H^1(M; E) \to H^1(M; E)^*$. We notice that P (and hence also $\tilde{P}$) determines the form a, which is the principal symbol of P. We shall say that a is the sesquilinear form associated to P. More precisely, let us identify $((T^*M \otimes E) \otimes (T^*M \otimes \overline{E}))' \simeq (T^*M \otimes T^*M)' \otimes \mathrm{End}(E)$ using the metric on E. Then the quadratic function

$$T^*M \ni \xi \to a(\xi, \xi) \in \mathrm{End}(E) \tag{9}$$

is the principal symbol of P. In fact, it would be actually more natural to start with $a \in TM \otimes TM \otimes \mathrm{End}(E)$. However, as we *always* consider a metric on E, this makes not difference for us.

In a certain way, P is the "true" differential operator associated to B, whereas $\tilde{P}$ includes, in addition to P, also boundary terms. To better understand this statement as well as the difference between P and $\tilde{P}$, a calculation in local coordinates is contained in Example 4.8 below, see also [22, 46].

4.2 Variational Boundary Value Problems

We now examine the relation between the operators in divergence form (i.e. of the form $\tilde{P}$) and boundary value problems. In particular, we discuss the weak formulation of the Robin problem. See also [4, 43, 46, 51, 57, 58] for the weak formulation of boundary value problems. We assume that we are given a decomposition

$$E|_{\partial M} = F_0 \oplus F_1 \tag{10}$$

of the restriction of E to the boundary into a direct sum of two vector bundles *with totally bounded curvature.* We consider boundary differentials operators

$$C_j : H^\infty(M; E) \to L^2(\partial M; F_j), \tag{11}$$

where C_0 is of order zero and $C_1 = C_{10} + C_{11}\partial_\nu$, with C_{11} and C_{00} of order zero, ∂_ν the covariant normal derivative at the boundary in the direction of the outer unit normal vector ν, and C_{10} only including derivatives tangential to ∂M. Each of the operators C_0, C_{10}, and C_{11} factors through a map $H^\infty(\partial M; E_{|\partial M}) \to L^2(\partial M; F_j)$, which will be denoted with the same symbol and we will assume to be differential operators with bounded coefficients. If $C_0^{-1} \in L^\infty(\partial M; \mathrm{End}(F_0))$ and $C_{11}^{-1} \in L^\infty(\partial M; \mathrm{End}(F_1))$ we shall say that the boundary conditions $C = (C_0, C_1)$ are *non-degenerate*.

We are interested in boundary value problems of the form (1) where $h = (h_0, h_1)$, $h_j \in \Gamma(\partial M; F_j)$, and the relation $Cu = h$ means $C_0 u = h_0$ and $C_1 u = h_1$. We shall regard the boundary conditions (C_0, C_1) and (C_0', C_1') as *equivalent* if $(C_0', C_1') = (D_0 C_0, D_1 C_1)$, where D_j, for each j, is a bounded automorphism of F_j with bounded inverse. The two sets of solutions of two equivalent boundary value problems are in a natural (continuous) bijection, which justifies looking at equivalence classes of boundary value problems. More precisely, let C and C' be equivalent boundary value problems and (D_0, D_1) be the automorphisms implementing the equivalence. Assume C and C' have totally bounded coefficients, for simplicity, and let $h_0 \in H^{k+1/2}(\partial M; F_0)$, $h_1 \in H^{k-1/2}(\partial M; F_1)$, $h := (h_0, h_1)$ and $h' := (D_0 h_0, D_1 h_1)$. Then the boundary value problems $Pu = f$, $Cu = h$ and $Pu = f$, $C'u = h'$ have the same solutions $u \in H^{k+1}(M; E)$.

4.2.1 Definition of Variational Boundary Conditions

Let e_j be the orthogonal projection onto F_j at the boundary. Also, let B be the basic sesquilinear form defined in Eq. 7 of the previous subsection. Thus consider *for the rest of this paper*

$$V := \{u \in H^1(M; E) \mid e_0 u = 0 \text{ on } \partial M\}. \tag{12}$$

Spaces V of this form with e_0 non-trivial (i.e. different from the restriction to some components of the boundary) arises in the study of the Hodge-Laplacian and in the study of the Ricci flow [61]. Let $k \geq 1$. For any $h \in H^{k-1/2}(\partial M; F_1)$ and $f \in H^{k-1}(M; E)$, we let

$$F(v) := \int_M (f, v) \; \mathrm{dvol}_g + \int_{\partial M} (h, v) \, d\,\mathrm{vol}_{\partial g} \, ,$$

where $d\,\mathrm{vol}_{\partial g}$ is the induced volume form on ∂M. Then F defines a linear functional on $\overline{V}$ and hence an element of V^*. We denote $j_{k-1}(f, h) := F$ the induced map

$$j_{k-1} \colon H^{k-1}(M; E) \oplus H^{k-1/2}(\partial M; F_1) \to V^*. \tag{13}$$

The formula for F makes sense also for $k = 0$ and $f = 0$, in which case it is just the dual map to the restriction at the boundary. Recall that all our differential operators, including $\tilde{P}$, have bounded coefficients.

Remark 4.3 We identify $(T^*M \otimes E \otimes T^*M \otimes E)'$ with $(T^*M \otimes T^*M)' \otimes \mathrm{End}(E)$ using the metrics on TM and E, respectively, as above. Then $a(dr, dr)$ can be regarded as a section of $\mathrm{End}(E)|_{\partial M}$ and we have on V that $a(dr, \nabla.) = a(dr, dr)\partial_\nu + Q'$, where Q' is a differential operator on $\Gamma(\partial M; E|_{\partial M})$ (that is, it does not involve normal derivatives at the boundary; it involves only tangential derivatives).

Note that we are not assuming the bundles F_i to be orthogonal.

Remark 4.4 Let us assume, in the definition of $B(u, v)$, Eq. 7, that $u \in H^2(M; E) \cap V$. Then we obtain

$$B(u, v) := (Pu, v) + \int_{\partial M} (\partial_\nu^P u, v) d\,\mathrm{vol}_{\partial g}, \tag{14}$$

where $d\,\mathrm{vol}_{\partial g}$ is the volume form on ∂M, as before. We have $\partial_\nu^P u = e_1 a(dr, dr \otimes \nabla_\nu u) + Q$, a first order differential operator. We shall also denote $\partial_\nu^a u := a(dr, dr \otimes \nabla_\nu u)$, with u a section of F_1 over ∂M. For reasons of symmetry (to have a class of operators stable under adjoints), one may want to consider in the minimal regularity case the following bilinear form

$$B(u, v) := B_a(u, v) + (Q_1 u, v)_{L^2(M)} + (u, Q_2 v)_{L^2(M)} + (Qu, v)_{L^2(\partial M)}.$$

In that case, the boundary operator $\partial_\nu^P u$ will depend also on Q_2. However, if our operators have coefficients in $W^{1,\infty}$, which is the case when dealing with regularity estimates, as in this paper, then we can absorb Q_2 into Q_1 by taking adjoints, with the price of obtaining an additional boundary term, which, nevertheless, can then be absorbed into Q. See also Example 4.8, where the term Q_2 was kept in the formula.

Lemma 4.5 *Assume $u \in H^2(M; E)$ and let $\tilde{P}$ be as in* Eq. 8. *The equation*

$$\tilde{P}(u) = j_{k-1}(f, h) \tag{15}$$

is then equivalent to the mixed Dirichlet/Robin boundary value problem

$$\begin{cases} Pu = f & \text{in } M \\ e_0 u = 0 & \text{on } \partial M \\ e_1 \partial_\nu^a u + Qu = h & \text{on } \partial M, \end{cases} \tag{16}$$

where $\partial_\nu^a u := a(dr, \nabla u)$ as above.

Proof Indeed, $e_0 u = 0$ on ∂M since u is in V, which is, by definition the domain of $\tilde{P}$. Let $F := j_{k-1}(f, h)$. The rest follows from Eq. 7, which gives

$$\begin{aligned}\langle \tilde{P}(u), w\rangle - F(w) &= B(u, w) - \int_M (f, w)\, \mathrm{dvol}_g - \int_{\partial M} (h, w)\, d\, \mathrm{vol}_{\partial g} \\ &= (Pu - f, w)_M - \int_{\partial M} (\partial_\nu^a u + Qu - h, w)\, d\, \mathrm{vol}_{\partial g}\,. \quad \square\end{aligned}$$

The boundary conditions of this lemma will be the main object of study for us. As we will see below in Proposition 4.7, these boundary conditions turn out to be, in fact, quite general. Recall the following standard terminology.

Definition 4.6 We keep the notation of the Lemma 4.5. We shall say, as usual, that $e_0 u = h_0$ are the *Dirichlet* boundary conditions and $e_1 \partial_\nu^a u + Qu = h_1$ are the *natural* (or *Robin*) boundary conditions. Also, we shall say that P and the boundary conditions $(e_0, e_1\partial_\nu^a + Q)$ of Eq. 16 are obtained from *a variational formulation.* We shall also say that $(e_0, e_1\partial_\nu^a + Q)$ are *variational boundary conditions associated to* $\tilde{P}$.

Non-degenerate boundary conditions are equivalent to variational ones for suitable a, as we will see below.

Proposition 4.7 *Let $\tilde{P}$ be a second order differential operator in divergence form associated to a, regarded as a bounded, measurable section of $T^*M \otimes T^*M \otimes \mathrm{End}(E)$, such that $e_1 a(dr, dr) e_1$ is invertible in $L^\infty(\partial M; \mathrm{End}(F_1))$. Let $C = (C_0, C_1)$ be non-degenerate boundary conditions. Then there is a first order differential operator Q with bounded coefficients such that the boundary conditions C and $(e_0, e_1\partial_\nu^a + Q)$ are equivalent. In particular, C is equivalent to some variational boundary conditions associated to $\tilde{P}$.*

This proposition justifies our choice to consider only boundary value problems of the form (16) instead of the general form (1). Indeed, let $\tilde{P}$ be the differential operator in divergence form with boundary conditions $(e_0, e_1\partial_\nu^a + Q)$. Then the solutions of the equation $\tilde{P}(u) = F$ are in a natural bijections to the solutions of the boundary value problem $Pu = f$ and $Cu = h$, where f and h depend linearly and continuously on F.

Proof Let $C_1 = C_{10} + C_{11}\partial_\nu$ as explained after Eq. 11. We have $C_0 e_0 = C_0$ with C_0 invertible. Then, $e_1\partial_\nu^a + Q = e_1 a(dr, dr) e_1 \partial_\nu + Q'$ and $e_1 a(dr, dr) e_1 C_{11}^{-1} C_1 = e_1 a(dr, dr) e_1 \partial_\nu + Q''$, where Q' and Q'' are first order differential operators on F_1 with bounded coefficients. Since Q (and hence also Q') can be chosen arbitrarily with these properties, we can certainly arrange that $Q' = Q''$ within the class of operators considered. $\square$

There is no good reason to chose C_0 other than e_0. On the other hand, there is no reason to expect, in general, that C_{11} be invertible. However, if C_{11} is *not* invertible, the behavior of the problem becomes completely different and, to the best of our knowledge, it is not fully understood at this time (see, however, [54] and the references therein).

Example 4.8 Recall the operator $\tilde{P}$ from Eq. 8 and the related form B from Eq. 7. Let us assume that $M = U \subset \mathbb{R}^m$ is a submanifold with boundary of dimension m. (Here ∂U denotes the boundary of U as a manifold with boundary, not as a subset of $\mathbb{R}^m$!) Let $E = \underline{\mathbb{C}}^N$, $F_0 = \underline{\mathbb{C}}^{N_0}$, and $F_1 = \underline{\mathbb{C}}^{N_1}$ be trivial bundles with $N = N_0 + N_1$—all equipped with the standard metric. Finally, we assume that are matrix valued functions $a_{ij}, b_i, b_j^*, c, d \in$

$L^\infty(U; M_N)$. We will assume that the metric on U is the euclidean metric, since this will not really decrease the generality, but will simplify our notation. In this example, we choose

$$V := \{u \in H^1(U)^N \mid u_1 = u_2 = \ldots = u_{N_1} = 0 \text{ on } \partial U\}. \tag{17}$$

We let a denote the bilinear form on $\underline{\mathbb{C}}^{mN}$ associated to the matrix $(a_{ij} \in (\underline{\mathbb{C}}^N)^* \otimes \underline{\mathbb{C}}^N)_{ij}$. We have $\nabla_i = \partial_i$ since we are dealing with the trivial bundles with standard metric over the euclidean space. Let $Q = (Q^{kl})_{k,l=N_1+1}^N$ be a matrix first order differential operator on ∂U and $(Pu)_k$ be the kth component of Pu. This gives for all $w \in V$

$$\begin{aligned}\langle \tilde{P}_{(a,Q)} u, w\rangle &= \int_U a(\nabla u, \nabla w) dx + \int_{\partial U} (Qu, w) dS \\ &= \sum_{i,j=1}^m \sum_{k,l=1}^N \int_U a_{ij}^{kl} \partial_j u_l \partial_i \overline{w_k} dx + \int_{\partial U} (Qu, w) dS,\end{aligned}$$

where dS is the induced volume form on ∂U. In particular,

$$(P_{(a,Q)}u)_k = (P_a u)_k = -\sum_{i,j=1}^m \sum_{l=1}^N \partial_i (a_{ij}^{kl} \partial_j u_l).$$

Let $Q_1 u := \sum_{j=1}^m b_j \partial_j u + cu$ and $Q_2 u := \sum_{i=1}^m \overline{b_i^*} \partial_i u$, let $u, v \in \mathcal{C}^\infty(\overline{U})^N \cap V$, and let ν be the *outer, unit* normal to ∂U. The formula $\int_U (\partial_j u)\, dx = \int_{\partial U} \nu_j u\, dS$ gives

$$\begin{aligned}\langle \tilde{P}(u), w\rangle &:= \int_U a(\nabla u, \nabla w) dx + (Q_1 u, w)_U + (u, Q_2 w)_U \\ &+ \sum_{k,l=N_1+1}^N \int_{\partial U} Q^{kl} u_k(x) \overline{w_l(x)} dS = \int_U Pu(x)\overline{w(x)} dx + \int_{\partial U} \partial_\nu^P u(x) \overline{w(x)} dS.\end{aligned}$$

Let us extend Q to an $N \times N$-matrix by completing it with zeroes. In turn, this gives that

$$Pu = P_a u + \sum_{j=1}^m b_j \partial_j u + cu - \sum_{i=1}^m \partial_i (b_i^* u)$$

and

$$(\partial_\nu^P u)_k := \begin{cases} \sum_{l=N_1+1}^N \left(\sum_{ij=1}^m \nu_i a_{ij}^{kl} \partial_j u_l + \sum_i \nu_i b_i^{*kl} u_l \right) + (Qu)_k & \text{if } k > N_1 \\ 0 & \text{otherwise.} \end{cases}$$

If the coefficients $\overline{b_i^*}$ are differentiable *and* $F_1 = 0$, we can get rid of the operator Q_2 by absorbing it into Q_1. However, if these coefficients are not differentiable and we want a class of operators that is closed under adjoints, then we need to include the Q_2 term into the definition of P (or, rather, $\tilde{P}$). Moreover, if $F_1 \neq 0$, we see that $\tilde{P}u$ contains an additional term compared to Pu, meaning that $\tilde{P}u - Pu$ is a distribution supported on ∂U. Also, we see that $\tilde{P}$ determines P, but not the other way around.

4.2.2 Uniformly Strongly Elliptic Operators

There are some classes of operators for which the conditions of Proposition 4.7 are almost automatically satisfied. Recall the following standard terminology.

Definition 4.9 We shall say that the operator $\tilde{P}\colon V \to V^*$ defined using the sesquilinear form B, see Eq. 8, is *uniformly strongly elliptic* if there exists $c_a > 0$ such that

$$\Re\big(a(\eta \otimes \xi, \eta \otimes \xi)\big) \geq c_a \|\eta\|^2 \|\xi\|^2$$

for all x and all $\xi \in E_x$ and $\eta \in T_x^* M$. The associated operator P will be called *uniformly strongly elliptic*. A family of operators is called *uniformly strongly elliptic* if each operator is uniformly strongly elliptic and we can choose the same constant c_a for all operators in the family. We shall say that $\tilde{P}$ is *uniformly elliptic* if there exists $c_e > 0$ such that, for every $\xi \in E_x$, there exists $\xi_1 \in E_x$, $\|\xi_1\| = \|\xi\|$, such that

$$\big|a(\eta \otimes \xi, \eta \otimes \xi_1)\big| \geq c_e \|\eta\|^2 \|\xi\|^2$$

for all x and $\eta \in T_x^* M$.

Note that in the above definition, we have taken advantage of the fact that our operator acts on sections of the same bundle. For operators acting between sections of different bundles, the definition will change in an obvious way (replacing the injectivity of the principal symbol with its invertibility). We see that if P is uniformly strongly elliptic, then $e_1 a(\nu, \nu) e_1$ is invertible in $L^\infty(\partial M; \mathrm{End}(F_1))$. (If $e_1 = 1$, i.e. if $F = E|_{\partial M}$, it is enough to assume that a is uniformly elliptic. We agree that if F_1 is the zero bundle on some component of ∂M, then we consider every endomorphism of it to be invertible on that set.)

4.2.3 The Scale of Regularity for Boundary Value Problems

Let P be associated to B as in Eq. 8 and V be as in Eq. 12. We define

$$\check{H}^{\ell-1}(M; E) := \begin{cases} H^{\ell-1}(M; E) \oplus H^{\ell-1/2}(\partial M; F_1) & \text{for } \ell \geq 1 \\ V^* & \text{for } \ell = 0. \end{cases}$$

The natural exact sequence $0 \to H^{-1/2}(\partial M; F_1) \to V^* \to H^{-1}(M; E) \to 0$, where the second map is induced by the trace map, see Theorem 2.13, shows that we have a natural scale of regularity spaces. In particular, $\check{H}^{\ell+1}(M; E) \subset \check{H}^{\ell}(M; E)$, $\ell \geq 0$. In general, the natural inclusion $\check{H}^{\ell}(M; E) \to \check{H}^{-1}(M; E) := V^*$ is given by the operators j_ℓ defined in Eq. 13. Let $\tilde{P}_\ell \colon H^{\ell+1}(M; E) \cap V \to \check{H}^{\ell-1}(M; E)$ be given by

$$\tilde{P}_\ell(u) := (Pu,\ e_1 \partial_\nu^a u + Qu) \ \text{for} \ell \geq 1. \tag{18}$$

Then the relation between $\tilde{P}_\ell$ and $\tilde{P}$ is by Eq. 15 expressed in the commutativity of the diagram

$$\begin{array}{ccc} H^{\ell+1}(M; E) \cap V & \xrightarrow{\tilde{P}_\ell} & \check{H}^{\ell-1}(M; E) \\ \big\downarrow & & \big\downarrow {\scriptstyle j_{\ell-1}} \\ H^1(M; E) \cap V & \xrightarrow{\tilde{P}} & \check{H}^{-1}(M; E) \end{array} \tag{19}$$

where the vertical arrows are the natural inclusions. Thus, although the definition of $\tilde{P}_0 := \tilde{P}\colon H^1(M; E) \cap V \to \check{H}^{-1}(M; E)$ is different from that of $\tilde{P}_\ell$ for $\ell > 0$, it fits into a scale of regularity spaces.

4.3 Regularity Conditions

The scale of regularity spaces provides a good setting to study regularity conditions

Notation 4.10 We shall denote by $\mathcal{D}^{\ell,j}(M; E)$ the set of pairs (D, C), where D is a second order differential operator defined on sections of $E \to M$ and C is an order j boundary condition, with both D and C assumed to have coefficients in $W^{\ell,\infty}$. In case M has no boundary (and thus there are no boundary conditions), we shall denote the resulting space $\mathcal{D}^{\ell,\emptyset}(M; E)$. If $E := \underline{\mathbb{C}}$ (that is, if we are dealing with *scalar* boundary value problems), then we shall drop the vector bundle from the notation.

The assumption that C is an order j boundary condition implies that $E|_{\partial M} = F_j$, and hence that $F_{1-j} = 0$. The general case just involves a more complicated notation. We endow the space $\mathcal{D}^{\ell,j}(M; E)$ with the norm defined by the maximum of the $W^{\ell,\infty}$-norms of the coefficients. Recall the following definition:

Definition 4.11 We say that $(D, C) \in \mathcal{D}^{\ell,j}(M; E)$, $\ell \geq j+1$, satisfies an $H^{\ell+1}$*-regularity estimate* on M if there exists $c_R > 0$ with the following property: For any $w \in H^k(M; E)$, $\ell \geq k \geq j + 1$, with compact support in M such that $Dw \in H^{k-1}(M; E)$ and $Cw \in H^{k-j+1/2}(\partial M; E)$, we have $w \in H^{k+1}(M; E)$ and

$$\|w\|_{H^{k+1}(M;E)} \leq c_R \left(\|Dw\|_{H^{k-1}(M;E)} + \|w\|_{H^k(M;E)} + \|Cw\|_{H^{k-j+1/2}(\partial M;E)}\right).$$

If $D \in \mathcal{D}^{\ell,\emptyset}(M; E)$, we just drop the term $\|Cw\|_{H^{k-j+1/2}(\partial M;E)}$. If $j = 1$ and if (D, C) are obtained from a variational formulation, then we allow also $\ell = 1$, with the relations $Dw \in L^2(M; E)$ and $Cw \in H^{1/2}(\partial M; E)$ being replaced with $\tilde{D}w = j_0(f, h)$, with $(f, h) \in L^2(M; E) \oplus H^{1/2}(\partial M; E)$ and the estimate is replaced with

$$\|w\|_{H^2(M;E)} \leq c_R \left(\|f\|_{L^2(M;E)} + \|w\|_{H^1(M;E)} + \|h\|_{H^{1/2}(\partial M;E)}\right),$$

in which case we also say that $\tilde{D}$ satisfies an $H^{\ell+1}$*-regularity estimate* on M.

The following comments on this definition are in order.

Remark 4.12 (i) The condition that w have compact support in M *does not* imply that it vanishes near the boundary ∂M.

(ii) Typically, we shall consider boundary value problems coming from a variational formulation, but, at this time, it is not necessary to make this assumption.

(iii) The conditions that D and C have coefficients in $W^{\ell,\infty}$ are needed so that $Dw \in H^{k-1}(M; E)$ and $Cw \in H^{k-j+1/2}(\partial M; E)$ for $w \in H^{k+1}(M; E)$, $k \leq \ell$.

(iv) Let $(D', C') \in \mathcal{D}^{\ell,j}(M; E)$ be a second boundary value problem such that $D - D'$ has order < 2 and $C - C'$ has order $< j$. Then (D, C) satisfies an $H^{\ell+1}$-regularity estimate on M if, and only if, (D', C') does so.

Remark 4.13 The careful reader might have noticed already that, apart from notation, we have not really used the fact that our operator is second order, except maybe in Proposition 4.7. For instance, for an order $2m$ problem, one would consider

$$a \in L^\infty(M; T^{\otimes m}M \otimes T^{\otimes m}M \otimes \mathrm{End}(E)),$$

but the form B would involve several boundary terms relating the normal derivatives at the boundary. For V we would take an intermediate subspace $H_0^m(M; E) \subset V \subset H^m(M; E)$. This could be given by a sub-bundle $F_1 \subset E^m_{\partial M}$ representing the possible values of $(u, \partial_\nu u, \ldots, \partial_\nu^{m-1}u)$ at the boundary:

$$V := \{u \in H^m(M; E) \mid (u, \partial_\nu u, \ldots, \partial_\nu^{m-1}u) \in L^\infty(\partial M; F_1)\}.$$

Nevertheless, non-local conditions are also important [47]. One could then proceed to deal with higher order problems as in [46], for instance. Since Proposition 4.7 plays an important role in justifying our choice of *essentially always* using variational boundary value conditions and in view of the many complications that arise when dealing with higher order problems (see [2, 46, 60, 62]), we decided to restrict ourselves to the case of second order operators. See also Remark 7.3.

5 Uniform Regularity Estimates for Families

We shall consider the same framework as in the previous section, in particular, (M, g) will continue to be a Riemannian manifold with smooth boundary ∂M such that TM has totally bounded curvature. Also, we continue to assume that $E \to M$ has totally bounded curvature.

5.1 Compact Families of Boundary Value Problems

We use the same notation as in the previous section. In particular, B, $\tilde{P}$, P, a, $e_1\partial_\nu^a + Q$, $C = (C_0, C_1)$, and $F_0 \oplus F_1 = E|_{\partial M}$ are as in the previous section. In particular, $\tilde{P}$ will always represent a second order differential operator in divergence form associated to the sesquilinear form a; in particular, (the quadratic map associated to) a is the principal symbol of P.

We shall need a *uniform* version of Definition 4.11.

Definition 5.1 Let $S \subset \mathcal{D}^{\ell,j}(M; E)$ and $N \subset M$ a submanifold with boundary $\partial N = \partial M \cap N$ of M. We shall say that S satisfies a *uniform $H^{\ell+1}$-regularity estimate on $N \subset M$* if each $(D, C) \in S$ satisfies an $H^{\ell+1}$-regularity estimate on $N \subset M$ with a bound c_R (in Definition 4.11) independent of $(D, C) \in S$. If $S \subset \mathcal{D}^{\ell,\emptyset}(M; E)$, we just consider $D \in S$.

Note that S in the above definition is *not* assumed to be bounded. However, typically, we shall obtain the independence of the bound c_R by assuming also that S is compact (or, sometimes, just precompact).

Proposition 5.2 *Assume that $S \subset \mathcal{D}^{\ell,j}(M; E)$ is compact and that each (D, C) satisfies an $H^{\ell+1}$-regularity estimate on M. Also, assume that the restriction $H^j(M; E) \to H^{j-1/2}(\partial M; E)$ is continuous for all $j \geq 1$. Then S satisfies a uniform $H^{\ell+1}$-regularity estimate on M. The same result holds if $S \subset \mathcal{D}^{\ell,\emptyset}(M; E)$.*

The assumption that the restriction (trace map) $H^j(M; E) \to H^{j-1/2}(\partial M; E)$ is continuous for all $j \geq 1$ is, of course, satisfied if M is an open subset of a manifold with boundary and bounded geometry M_0 [36] (see Theorem 2.13). In particular, this is the case if M is the domain of a Fermi coordinate chart. Also, note that here the boundary is in the sense of manifolds with boundary, and not in the sense of point-set topology.

Proof Consider first the case $S \subset \mathcal{D}^{\ell,j}(M; E)$. Let us assume the contrary and show that we obtain a contradiction. That is, let us assume that there exist sequences $(D_i, C_i) \in S$ and $w_i \in H^{k+1}(M; E)$, with compact support in M, such that

$$\|w_i\|_{H^{k+1}(M;E)} > 2^i \left(\|D_i w_i\|_{H^{k-1}(M;E)} + \|w_i\|_{H^k(M;E)} + \|C_i w_i\|_{H^{k-j+1/2}(\partial M;E)}\right).$$

Since S forms a compact subset in $\mathcal{D}^{\ell,j}(M;E)$, by replacing (D_i, C_i) with a subsequence, if necessary, we can assume that (D_i, C_i) converges. Let us denote the limit with $(D, C) \in S$. Thus, there is a sequence $\epsilon_i \to 0$ with

$$\begin{aligned} \|D_i w\|_{H^{k-1}(M;E)} &\geq \|Dw\|_{H^{k-1}(M;E)} - \epsilon_i \|w\|_{H^{k+1}(M;E)}, \\ \|C_i w\|_{H^{k-j+\frac{1}{2}}(\partial M;E)} &\geq \|Cw\|_{H^{k-j+\frac{1}{2}}(\partial M;E)} - \epsilon_i \|w\|_{H^{k+\frac{1}{2}}(\partial M;E)}. \end{aligned}$$

Together with the assumed continuity of the trace map, this implies that there exists $c' > 0$ such that

$$\begin{aligned} \|w_i\|_{H^{k+1}(M;E)} > 2^i \big(&\|Dw_i\|_{H^{k-1}(M;E)} + \|w_i\|_{H^k(M;E)} + \|Cw_i\|_{H^{k-j+1/2}(\partial M;E)} \\ &- c'\epsilon_i \|w_i\|_{H^{k+1}(M;E)} \big), \end{aligned}$$

for all i. On the other hand, (D, C) satisfies, by assumption, an H^{k+1}- regularity estimate. Consequently, there is a $c > 0$ such that

$$\|Dw_i\|_{H^{k-1}(M;E)} + \|w_i\|_{H^k(M;E)} + \|Cw_i\|_{H^{k-j+1/2}(\partial M;E)} \geq c^{-1} \|w_i\|_{H^{k+1}(M;E)},$$

and hence we obtain

$$\|w_i\|_{H^{k+1}(M;E)} \geq 2^i (c^{-1} - c'\epsilon_i) \|w_i\|_{H^{k+1}(M;E)}.$$

For $i \to \infty$, this gives the desired contradiction since $2^i(c^{-1} - c'\epsilon_i) \to \infty$, for $i \to \infty$, and $\|w_i\|_{H^{k+1}(M;E)} \neq 0$. This completes the proof if $S \subset \mathcal{D}^{\ell,j}(M;E)$.

If $S \subset \mathcal{D}^{\ell,\emptyset}(M;E)$, the proof is obtained by simply dropping the terms that contain Cw from the above proof. (We can further replace $2\epsilon_i$ with ϵ_i, but that is not essential.) □

Recall that a *relatively compact* subset is a subset whose closure is compact.

Proposition 5.3 *Let $N \subset M$ be a relatively compact open subset and let $S \subset \mathcal{D}^{\ell+1,j}(M;E)$ be a bounded subset. Assume that every $\mathcal{D}^{\ell,j}(N;E)$–limit $(\tilde{D}, \tilde{C})$ of a sequence $(D_i, C_i) \in S$ satisfies an $H^{\ell+1}$-regularity estimate on N. Then S satisfies a uniform $H^{\ell+1}$-regularity estimate on N. The same result holds if $S \subset \mathcal{D}^{\ell+1,\emptyset}(M;E)$.*

We remark that in this proposition the compactness condition of Proposition 5.2 is replaced by a *higher regularity assumption* on the coefficients. This is needed in order to use the Arzela-Ascoli Theorem. Moreover, we note that by choosing a constant sequence, we see that the assumptions imply that each element in S satisfies an $H^{\ell+1}$-regularity estimate on N. We also note that $(\tilde{D}, \tilde{C})$ in the statement automatically satisfies $(\tilde{D}, \tilde{C}) \in \mathcal{D}^{\ell,j}(N;E)$.

Proof of Proposition 5.3 We treat explicitly only the case $S \subset \mathcal{D}^{\ell+1,j}(M;E)$, the case $S \subset \mathcal{D}^{\ell+1,\emptyset}(M;E)$ being completely similar. Since the coefficients of all boundary value problems in S are bounded in $W^{\ell+1,\infty}(M)$, the set of coefficients of the operators $(D, C) \in S$ is precompact in $W^{\ell,\infty}(N)$, by the Arzela-Ascoli theorem. Let K be the closure of S in $\mathcal{D}^{\ell,j}(N, E)$, which is therefore a compact set. Moreover, our assumptions imply that every element in K satisfies an $H^{\ell+1}$-regularity estimate. Proposition 5.2 then implies the result. □

We shall need the following lemma.

Lemma 5.4 *Let $S \subset \mathcal{D}^{\ell,j}(M; E)$ satisfy a uniform $H^{\ell+1}$-regularity estimate on M. If $N \subset M$ is an open subset, then S satisfies a uniform $H^{\ell+1}$-regularity estimate on N. The same result holds in the boundaryless case.*

Proof Let $w \in H^k(N; E)$ with compact support. Then $w \in H^k(M; E)$ and the uniform regularity estimate for w on M yields the desired result. □

Remark 5.5 To deal with the general case when C is not a boundary condition of fixed order j, but rather combines a boundary condition C_1 of order one and a boundary condition C_0 of order zero, we just replace $\|Cw\|_{H^{k-j+1/2}(\partial M;E)}$ with $\|C_1 w\|_{H^{k-1/2}(\partial M;E)} + \|C_0 w\|_{H^{k+1/2}(\partial M;E)}$. In this case, we may have that both $F_0 \neq 0$ and $F_1 \neq 0$.

5.2 Higher Regularity and Bounded Geometry

The relevance of uniform regularity conditions introduced in Section 5 is that it allows us to obtain higher regularity on manifolds with boundary and bounded geometry and suitable boundary conditions as follows.

Let (M, g) be a Riemannian manifold with boundary and bounded geometry, as before. Let (P, C) be a boundary value problem on M. We shall assume that (P, C) comes from a variational formulation, since most of the results will require this assumption. (This means that if $C = (C_0, C_1)$, then $\tilde{P}$ can be identified with (P, C_1) and C_0 is simply the projection onto F_0 at the boundary.) We assume, for notational simplicity, that C has constant order j at the boundary. For the same reasons, we also assume P acts and takes values in the same vector bundle E (that is, $E_1 = E$), as before. The results of this subsection hold, however, in full generality (when the vector bundles E, E_1, and F of Section 5 are distinct), but with some obvious changes. Let $0 < r \leq r_{FC}$, as in Definition 2.7. Recall that U_p and κ_p from Eq. 4 and ξ_p from Definition 2.11 are such that either $p \in \partial M$ or $\operatorname{dist}(p, \partial M) \geq r$. We denote by (P_p, C_p) the induced boundary value problems on $\kappa_p^{-1}(U_p) = B_{2r}^m(0) \times [0, 2r)$, if $p \in \partial M$. Then $P_p = \xi_p^* \circ P \circ (\xi_p)_*$ and $C_p = \xi_p^* \circ C \circ (\xi_p)_*$, with the obvious notation, meaning that the operators correspond through the diffeomorphisms ξ_p. If $\operatorname{dist}(p, \partial M) \geq r$, there is no C_p and we obtain a differential operator P_p on $B_r^{m+1}(0)$. Let t denote the rank of E and let

$$\begin{aligned} \mathcal{F}_b &:= \{(P_p, C_p) |\ p \in \partial M\} \subset \mathcal{D}^{0,j}(B_{2r}^m(0) \times [0, 2r); \mathbb{C}^t) \\ \mathcal{F}_i &:= \{(P_p) |\ \operatorname{dist}(p, \partial M) \geq r\} \subset \mathcal{D}^{0,\emptyset}(B_r^{m+1}(0); \mathbb{C}^t), \end{aligned} \tag{20}$$

be the induced *boundary* and *interior* families of operators. Note that we always equip $B_{2r}^m(0) \times [0, 2r)$, (respectively, $B_r^{m+1}(0)$) with the euclidean metric.

Theorem 5.6 *Let (M, g) be a Riemannian manifold with boundary and bounded geometry and let $E \to M$ be a Hermitian vector bundle with totally bounded curvature. Let $(P, C) \in \mathcal{D}^{\ell,j}(M; E)$. If each of the families $\mathcal{F}_b := \{(P_p, C_p)|\ p \in \partial M\}$ and $\mathcal{F}_i := \{P_p|\ \operatorname{dist}(p, \partial M) \geq r\}$ (see* Eq. 20 *and above) satisfy a uniform $H^{\ell+1}$-regularity estimate, then (P, C) satisfies an $H^{\ell+1}$-regularity estimate. The converse is also true.*

Proof This follows from Definition 4.11 of uniform order k regularity estimates, from Proposition 2.12, and from Lemma 3.6. Let us choose the r-uniform partition of unity ϕ_γ

used in those results such that $\partial_\nu \phi_\gamma$ vanishes at the boundary. This can be done first by choosing an r-uniform partition of unity on ∂M.

$$\begin{aligned}
\|u\|^2_{H^{k+1}} &\lesssim \sum_\gamma \|\xi_\gamma^*(\phi_\gamma u)\|^2_{H^{k+1}} \\
&\lesssim \sum_\gamma \Big(\|P_\gamma \xi_\gamma^*(\phi_\gamma u)\|_{H^{k-1}} + \|C_\gamma \xi_\gamma^*(\phi_\gamma u)\|_{H^{k-j+\frac{1}{2}}} + \|\xi_\gamma^*(\phi_\gamma u)\|_{H^k}\Big)^2 \\
&\lesssim \sum_\gamma \left(\|\xi_\gamma^* P(\phi_\gamma u)\|^2_{H^{k-1}} + \|\xi_\gamma^* C(\phi_\gamma u)\|^2_{H^{k-j+\frac{1}{2}}} + \|\xi_\gamma^*(\phi_\gamma u)\|^2_{H^k}\right) \\
&\lesssim (\|Pu\|^2_{H^{k-1}} + \|Cu\|^2_{H^{k-j+\frac{1}{2}}} + \|u\|_{H^k})^2 + \sum_\gamma \|[P, \phi_\gamma]u\|^2_{H^{k-1}} \\
&\quad + \sum_\gamma \|[C, \phi_\gamma]u\|^2_{H^{k-j+\frac{1}{2}}},
\end{aligned}$$

since the trivializations ξ_γ have uniformly bounded norms. Next we notice that $\sum_\gamma \|[P, \phi_\gamma]u\|^2_{H^{k-1}} \lesssim \|u\|^2_{H^k}$ since the family ϕ_γ is uniformly locally finite and by Lemma 3.6. The boundary term is treated similarly: $\sum_\gamma \|[C, \phi_\gamma]u\|^2_{H^{k-j-\frac{1}{2}}} \lesssim \|u\|^2_{H^{k-\frac{1}{2}}} \lesssim \|u\|^2_{H^k}$. Here the last inequality is given by the trace theorem. For the first inequality we note that for $j = 0$ the commutator is actually zero. For $j = 1$ we have $[C, \phi_\gamma] = [C_{10}, \phi_\gamma]$ since multiplication by ϕ_γ and ∂_ν commute, given the product form of ϕ_γ near the boundary. Together with Lemma 3.6, this completes the proof. □

Remark 5.7 The method of proof of Theorem 5.6 will yield similar global results in other classes of spaces, as long as the local regularity results are available and as long as a local description of these spaces using partitions of unity is available. This is the case for the L^p-Sobolev spaces, $1 < p < \infty$, for which we have both the local description using partitions of unity (Proposition 2.12) and the local regularity results [33].

5.3 Regularity for Dirichlet Boundary Conditions

The results of the previous two sections were tailored to deal with the Dirichlet boundary conditions. It is known [3, 4, 66] that strongly elliptic operators with Dirichlet boundary conditions satisfy regularity conditions. We formulate this well-known result as a lemma for further use. As before, $E \to M$ will be a vector bundle with bounded geometry.

Lemma 5.8 *Let $(P, C) \in \mathcal{D}^{\ell,0}(B_r^m(0) \times [0, r); E)$, $0 < r \leq \infty$, be a uniformly strongly elliptic boundary value problem with Dirichlet boundary conditions. Then P satisfies an $H^{\ell+1}$-regularity estimate on M. The same result holds for a uniformly elliptic operator $P \in \mathcal{D}^{\ell,\emptyset}(B_r^m(0); E)$, $0 < r \leq \infty$.*

From this we obtain

Corollary 5.9 *Let $S \subset \mathcal{D}^{\ell+1,0}(B_r^m(0) \times [0, r); E)$ be a bounded, uniformly strongly elliptic family of boundary value problems on $B_r^m(0) \times [0, r) \subset \mathbb{R}^{m+1}$ equipped with the euclidean metric, $r \leq \infty$. We assume that the boundary conditions are all Dirichlet. Then the family S satisfies a uniform $H^{\ell+1}$-regularity estimate on $B_{r'}^m(0) \times [0, r')$, $r' < r$.*

Proof Let $(D_n, C_n) \in S$ converge to $(D, C) \in \mathcal{D}^{\ell,j}(B^m_{r'}(0) \times [0, r'); E)$. Then D is a uniformly strongly elliptic operator because the parameter c_a stays away from 0 on S, in view of Definition 4.9. Lemma 5.8, then gives that (D, C) satisfies an $H^{\ell+1}$-regularity estimate, since the type of boundary conditions (Dirichlet or Neumann) do not change by taking limits. This allows us to use Proposition 5.3 for the relatively compact subset $N := B^m_{r'}(0) \times [0, r')$ of $M := B^m_r(0) \times [0, r)$ to obtain the result. □

Analogously, (in fact, even more directly, since we do not have to take boundary conditions into account), we obtain

Corollary 5.10 *Let $S \subset \mathcal{D}^{\ell+1,\emptyset}(B^{m+1}_r(0); E)$ be a bounded uniformly elliptic family of differential operators on $B^{m+1}_r(0) \subset \mathbb{R}^{m+1}$, for some $0 < r \leq \infty$. Then the family S satisfies a uniform $H^{\ell+1}$-regularity estimate on $B^{m+1}_{r'}(0) \subset \mathbb{R}^{m+1}$ for any $r' < r$.*

Remark 5.11 The regularity results of this section extend to the L^p-Sobolev spaces $W^{\ell,p}$, $1 < p < \infty$, with essentially the same proofs by using also the results in [33].

Note that in Corollaries 5.9 and 5.10 we use a slightly stronger assumption on the coefficients of our operators than usually, namely we require them to have $W^{\ell+1,\infty}$-regularity (usually we require only $W^{\ell,\infty}$-regularity). This is required since we will use Proposition 5.3. Combining these result, we obtain the following.

Theorem 5.12 *Let P be a uniformly strongly elliptic second order differential operator with coefficients in $W^{\ell+1,\infty}$ acting on sections of $E \to M$. Then there exists $c > 0$ such that, if $u \in H^\ell(M; E)$, $Pu \in H^{\ell-1}(M; E)$, and $u|_{\partial M} \in H^{\ell+1/2}(\partial M; E)$, then $u \in H^{\ell+1}(M; E)$ and*

$$\|u\|_{H^{\ell+1}(M;E)} \leq c \left(\|Pu\|_{H^{\ell-1}(M;E)} + \|u\|_{H^\ell(M;E)} + \|u\|_{H^{\ell+1/2}(\partial M;E)} \right).$$

Proof Corollaries 5.9 and 5.10 show that the assumptions of Theorem 5.6 are satisfied. That theorem immediately gives our result. □

An alternative proof of this result is obtained using the methods of Section 7. The advantage of the method used in this section is that it applies right away to higher order equations.

6 A Uniform Shapiro-Lopatinski Regularity Condition

The case of Neumann boundary conditions seems to be different (at least for systems, see [3, 4, 66, 67]). This case, as well as that of Robin boundary conditions motivates, in part, the results of this section. In particular, we introduce a uniform version of the Shapiro-Lopatinski condition [2–4, 23, 39, 48, 55, 66], which turns out to characterize the operators satisfying regularity. Our approach has several points in common to the ones in [46, 60]. To deal with the concrete case of Robin (and Neumann) boundary conditions, we use positivity to check that the uniform Shapiro-Lopatinski regularity conditions are satisfied.

6.1 Homogeneous Sobolev Spaces and Regularity Conditions

We need first the following homogeneous (with respect to dilations) versions of the usual Sobolev spaces. This setting was used for similar purposes in [60]. For simplicity, we work in $\mathbb{R}^n$, but the same considerations apply to any vector space V endowed with a metric (or a half-vector space $V_+ \subset V$). (See, however, Eq. 29 for the dependence of the norms on the choice of the metric.)

Let $\hat{u}$ be the Fourier transform of u, regarded as a tempered distribution (the normalizations are not important here). Consider the *semi-norm*

$$|u|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi, \tag{21}$$

and $\dot{H}^s(\mathbb{R}^n) := \{u \mid |u|_{H^s(\mathbb{R}^n)} < \infty\}$. Here u is such that $|\xi|^s|\hat{u}(\xi)|$ is a function (but $|\hat{u}(\xi)|$ is *not assumed* to be a function, which allows us to include polynomials of low degree in $\dot{H}^s(\mathbb{R}^n)$). When $s \in \mathbb{Z}_+$, the seminorm $|u|_{H^s(\mathbb{R}^n)}$ is equivalent to the (usual) $\sum_{|\alpha|=s} \|\partial^\alpha u\|_{L^2(\mathbb{R}^n)}$, which allows us to define in this case also $|u|_{H^s(\mathbb{R}^n_+)}$ for a function u defined only on a half-space of the form $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times [0, \infty)$. In what follows, we shall write simply $|\cdot|_{H^s}$ for the above semi-norms. The reason for considering the semi-norms $|\cdot|_{H^s}$ and the spaces $\dot{H}^s(\mathbb{R}^n)$ is that they have good dilation properties (see the next lemma). These definitions extend right away to functions with values in $\mathbb{C}^N$ yielding the spaces $\dot{H}^s(\mathbb{R}^n; \mathbb{C}^N) = \dot{H}^s(\mathbb{R}^n)^N$.

Lemma 6.1 *Let $s > 0$.*

(i) *Let $\alpha_t(f)(s) = f(ts)$, then $|\alpha_t(f)|_{H^s} = t^{s-n/2}|f|_{H^s}$.*
(ii) $\lim_{t\to\infty} t^{-s+n/2}\|\alpha_t(f)\|_{H^s} = |f|_{H^s}$.
(iii) *If T is an order k, homogeneous, constant coefficient differential operator, then it defines continuous maps $T\colon \dot{H}^s(\mathbb{R}^n) \to \dot{H}^{s-k}(\mathbb{R}^n)$.*
(iv) *If $s \in \mathbb{N}$, the restrictions $\dot{H}^s(\mathbb{R}^n) \to \dot{H}^s(\mathbb{R}^n_+)$ and $\dot{H}^s(\mathbb{R}^n) \to \dot{H}^{s-1/2}(\mathbb{R}^{n-1})$ are continuous and surjective and, if also T is as in (iii) and has order $k \leq s$, $T\colon \dot{H}^s(\mathbb{R}^n_+) \to \dot{H}^{s-k}(\mathbb{R}^n_+)$ is continuous.*

Proof The proof is standard [28, 40, 46, 66] and easy. We include a few details for the benefit of the reader.

(i), (ii), and (iii) are direct calculations based on the definition and the formulas for the norms in terms of the Fourier transform.

(iv) is slightly less trivial, but it is known in the classical case and our case can either be proved directly following the same method as in the case of the classical Sobolev spaces or it can be reduced to the classical Sobolev spaces using (ii). Indeed, for the continuity, this is immediate, as the restriction maps commute with α_t. For the surjectivity, one has to argue also that there exist right inverses for the restriction that are α_t-invariant. This is done by choosing a partition of unity that is invariant with respect to α_2 and then choosing a right inverse to the restriction on one coordinate patch that is continuous for the classical norms. We replicate this right inverse for all patches using α_2. This yields a continuous right inverse $H^{s-1/2}(\mathbb{R}^{n-1/2}) \to H^s(\mathbb{R}^n)$ for the restriction $H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1/2})$ that is, moreover, invariant for α_2. We obtain a fully $\mathbb{R}^*_+$-invariant inverse by integrating over the compact group $\mathbb{R}^*_+/2^{\mathbb{Z}}$. This right inverse will work also for the homogeneous spaces. □

Let D be an $N \times N$ matrix of second order differential operators on $B_r^m(0) \times [0, r)$, and let C be a boundary differential operator of order j. Assume the coefficients have $W^{\ell,\infty}$ smoothness, that is, that $(D, C) \in \mathcal{D}^{\ell,j}(B_r^m(0) \times [0, r); \mathbb{C}^N)$. Here $r > 0$, and the case $r = \infty$ is not excluded.

Definition 6.2 Let $\ell \geq j + 1$. We shall say that $(D, C) \in \mathcal{D}^{\ell,j}(B_r^m(0) \times [0, r); \mathbb{C}^N)$ satisfies an $\overset{\bullet}{H}{}^{\ell+1}$*-regularity estimate* on $B_r^m(0) \times [0, r)$ if there exists c_{SL} such that for all $\ell \geq k \geq j + 1$, we have

$$|w|_{H^{k+1}(\mathbb{R}_+^{m+1})} \leq c_{SL} \left(|Dw|_{H^{k-1}(\mathbb{R}_+^{m+1})} + |Cw|_{H^{k-j+1/2}(\mathbb{R}^m)} \right),$$

for all w smooth with compact support in $B_r^m(0) \times [0, r)$, where we have removed the vector bundle from the notation for the norms. If (D, C) are obtained from a variational formulation, then we allow also $\ell = k = j = 1$ and in that case we assume that $\tilde{D}w = j_0(f, h)$ and we replace the right hand side with $|f|_{L^2(\mathbb{R}_+^{m+1})} + |h|_{H^{1/2}(\mathbb{R}^m)}$. This case is, in fact, crucial in applications, since it is the one obtained using coercivity to prove well-posedness.

This definition is very similar to Definition 4.11, except that we consider *semi-norms* instead of norms. Also, we only require w to be smooth. However, an operator satisfying the conditions in Definition 6.2, will satisfy also those of Definition 4.11. We formulate this result as a lemma, for further use.

Lemma 6.3 *Assume* $(D, C) \in \mathcal{D}^{\ell,j}(B_r^m(0) \times [0, r); \mathbb{C}^N)$ *satisfies an* $\overset{\bullet}{H}{}^{\ell+1}(B_r^m(0) \times [0, r))^N$*-regularity estimate, then it satisfies an* $H^{\ell+1}$*-regularity estimate for all* $\ell \geq j + 1$ *(*$\ell \geq j$ *if* C *is a variational boundary condition). This result extends to uniform conditions.*

Proof Denote $M = B_r^m(0) \times [0, r)$ for the simplicity of the notation. Assume first that $k \geq 2$. Let $w \in \Gamma(M; E)$ be smooth with compact support in M such that $Dw \in H^{k-1}(M; E)$ and $Cw \in H^{k-j+1/2}(\partial M; E)$. We need to show that $w \in H^{k+1}(M; E)$. We have

$$\begin{aligned}
\|w\|_{H^{k+1}(M;E)} &\leq |w|_{H^{k+1}(M;E)} + \|w\|_{H^k(M;E)} \\
&\leq c_{SL} \left(|Dw|_{H^{k-1}} + |Cw|_{H^{k-j+1/2}(\partial M;E)} \right) + \|w\|_{H^k(M;E)} \\
&\leq c \left(\|Dw\|_{H^{k-1}(M;E)} + \|w\|_{H^k(M;E)} + \|Cw\|_{H^{k-j+1/2}(\partial M;E)} \right).
\end{aligned}$$

This result is extended to $w \in H^k(M; E)$ by a continuity and density argument using mollifying functions. The proof for $k = j = 1$ and variational boundary conditions is similar. □

Remark 6.4 Let us assume that $r = \infty$ and consider the map

$$(D, C) \colon \overset{\bullet}{H}{}^{k+1}(\mathbb{R}_+^{m+1})^N \to \overset{\bullet}{H}{}^{k-1}(\mathbb{R}_+^{m+1})^N \oplus \overset{\bullet}{H}{}^{k-j+1/2}(\mathbb{R}^m)^N, \tag{22}$$

given by $(D, C)(u) = (Du, Cu)$. Then (D, C) satisfies an $\overset{\bullet}{H}{}^{\ell+1}(\mathbb{R}_+^{m+1})^N$–regularity estimate if, and only if, (D, C) is injective with closed range for all $j+1 \leq k \leq \ell$. Equivalently, we have that (D, C) is injective and its minimal reduced module $\gamma(D, C) > 0$, see Section 3.1. (In this case, the number $\gamma(D, C)$ is the least c_{SL} in Definition 6.2.) In case D and C have constant coefficients, this is similar to the Shapiro-Lopatinski condition.

6.2 A Global Shapiro-Lopatinski Regularity Condition

Let us assume first that we are on a Euclidean space and that $D = \sum_{|\alpha|\leq 2} a_\alpha \partial^\alpha$, with a_α smooth, matrix valued functions. Recall that j is the order of the boundary conditions and that we assume for simplicity, that only one of the vector bundles F_0 and F_1 (from $E|_{\partial M} = F_0 \oplus F_1$) is non-zero. Thus, according to our notational convention, if $j = 0$, $C = C_0$ is a smooth, matrix valued function on $B_r^m(0)$ and, if $j = 1$, $C = C_1 = C_{10} + C_{11}\partial_\nu$, with $C_{10} = \sum_{|\alpha|\leq 1} c_\alpha \partial^\alpha$ a *first order* differential operator on $B_r^m(0)$ and C_{11} a smooth, matrix valued function on $B_r^m(0)$. Here $\partial_\nu = -\partial_n$ is the outward pointing normal derivative, where ∂_n is the partial derivative with respect to the last variable. We denote by $(D^{(0)}, C^{(0)})$ the *principal part* of (D, C) with *coefficients frozen at* 0, that is,

$$D^{(0)} := \sum_{|\alpha|=2} a_\alpha(0)\partial^\alpha \tag{23}$$

$$C_0^{(0)} := C_0(0), \quad \text{if}\, j = 0, \tag{24}$$

$$C_1^{(0)} := \sum_{|\alpha|=1} c_\alpha(0)\partial^\alpha + C_0(0)\partial_\nu, \quad \text{if}\, j = 1. \tag{25}$$

Thus $(D^{(0)}, C^{(0)})$ is a homogeneous, constant coefficient boundary value problem on $\mathbb{R}_+^{m+1}$. In case both F_0 and F_1 are non-zero, we let $C^{(0)} = (C_0^{(0)}, C_1^{(0)})$.

Motivated by Remark 6.4, we introduce the following definition.

Definition 6.5 We shall say that $(D, C) \in \mathcal{D}^{\ell,j}(B_r^m(0) \times [0, r); \mathbb{C}^N)$ satisfies the $H^{\ell+1}$-*Shapiro-Lopatinski regularity condition at 0* if $(D^{(0)}, C^{(0)})$ satisfies an $\dot{H}^{\ell+1}(\mathbb{R}_+^{m+1})^N$-regularity estimate.

Let us turn now to the case of a manifold with boundary. As noticed already, all the needed definitions and concepts (homogeneous Sobolev spaces $\dot{H}^{\ell+1}(V; E)$ and $\dot{H}^{\ell+1}(V_+; E)$, regularity conditions, ...) extend to a vector space (respectively, half-vector space) endowed with a metric. For instance, we define the principal part with coefficients frozen at a boundary point as follows.

Notation 6.6 Let $T_x^+ M$ be the half-space of $T_x M$ corresponding to the inward pointing vectors at $x \in \partial M$. Let (D_x, C_x) be the induced operator (defined only a neighborhood of 0) on $T_x^+ M$. The *principal part* $(D_x^{(0)}, C_x^{(0)})$ of (D_x, C_x) with *coefficients frozen at* x will then be a matrix of constant coefficient differential operators on $T_x^+ M$.

Most importantly, the above definition (Definition 6.5) generalizes to $(D, C) \in \mathcal{D}^{\ell,j}(M; E)$ and any point $x \in \partial M$.

Definition 6.7 We shall say that $(D, C) \in \mathcal{D}^{\ell,j}(M; E)$ satisfies the $H^{\ell+1}$-*Shapiro-Lopatinski regularity condition at* $x \in \partial M$ if $(D_x^{(0)}, C_x^{(0)})$ satisfies the $H^{\ell+1}$-Shapiro-Lopatinski regularity condition at 0 on $T_x^+ M$.

The above condition is closely related to the condition of "regularity upon freezing the coefficients" introduced in [66, Equation (11.30)] and used, for instance, in [50]. We are ready now to globalize the Shapiro-Lopatinski regularity condition.

Definition 6.8 We shall say that $(D, C) \in \mathcal{D}^{\ell,j}(M; E)$ satisfies a *uniform $H^{\ell+1}$-Shapiro-Lopatinski regularity condition (at ∂M)* if it satisfies the $H^{\ell+1}$-Shapiro-Lopatinski regularity condition at x for each $x \in \partial M$ and the constant c_{SL} of Definition 6.2 can be chosen independently of $x \in \partial M$.

In particular, the constant c_{SL} depends and scales with the metric; see Eq. 29 for the precise dependence on the metric. We thus see that $(D, C) \in \mathcal{D}^{\ell,j}(M; E)$ satisfies the uniform, $H^{\ell+1}$-Shapiro-Lopatinski regularity condition (at ∂M) if there exists $c_{SL} > 0$ such that, for all $x \in \partial M$ and all $1 \leq k \leq \ell$, we have

$$|w|_{H^{k+1}(T_x^+M)} \leq c_{SL} \left(|D_x^{(0)} w|_{H^{k-1}(T_x^+M)} + |C_x^{(0)} w|_{H^{k-j+1/2}(T_x\partial M)} \right). \tag{26}$$

We now apply these notions to a manifold M with boundary and bounded geometry. For any $x \in M$, we denote by (D_x, C_x) (or simply by D_x, if $x \notin \partial M$) the operators (respectively, operator) on $B_r^m(0) \times [0, r)$ (respectively, on $B_r^m(0)$) induced by (D, C) (respectively, by D) in Fermi coordinates around x. We let $\mathcal{F}_b = \{(D_x, C_x) \mid x \in \partial M\}$ and $\mathcal{F}_i = \{D_x \mid \operatorname{dist}(x, \partial M) \geq r\}$, as in Eq. 20. We have the following theorem.

Theorem 6.9 *Assume that M is a manifold with boundary and bounded geometry and that $E \to M$ is a vector bundle with bounded geometry. Let $(D, C) \in \mathcal{D}^{\ell+1,j}(M; E)$. The following are equivalent.*

(i) *(D, C) satisfies an $H^{\ell+1}$-regularity estimate on M.*
(ii) *The family $\mathcal{F}_b \cup \mathcal{F}_i = \{(D_x, C_x) \mid x \in \partial M\} \cup \{D_x \mid \operatorname{dist}(x, \partial M) \geq r\}$ satisfies a uniform $H^{\ell+1}$-regularity estimate.*
(iii) *D is uniformly elliptic on M and (D, C) satisfies a uniform $H^{\ell+1}$-Shapiro-Lopatinski regularity condition (at ∂M).*
(iv) *D is uniformly elliptic and (D, C) satisfies a uniform H^2-Shapiro-Lopatinski regularity condition (at ∂M).*

If (D, C) are obtained from a variational formulation, then the above conditions are equivalent also to

(v) *D is uniformly elliptic and (D, C) satisfies a uniform H^1-Shapiro-Lopatinski regularity condition (at ∂M).*

Note that usually we assume $(D, C) \in \mathcal{D}^{\ell,j}(M; E)$, whereas here we assume $(D, C) \in \mathcal{D}^{\ell+1,j}(M; E)$. This is needed for the proof of (iii) $\Rightarrow$ (i).

Proof The implication (i) $\Rightarrow$ (ii) follows from the definitions by localization. (This is the converse of Theorem 5.6.)

To obtain (ii) $\Rightarrow$ (iii) for each $x \in \partial M$, we consider ξ with compact support on T_x^+M and $\xi_\epsilon(v) = \xi(\epsilon^{-1}v)$. Using a chart around x with Fermi coordinates as in Section 2.3 and using the corresponding bounds of the transition function, the uniform $H^{\ell+1}$-regularity estimate for each $\epsilon > 0$ from Definition 4.11 implies

$$\begin{aligned} \epsilon^{k+1-n/2} \ \|\xi_\epsilon\|_{H^{k+1}(T_x^+M;E)} \leq \epsilon^{k+1-n/2} \bar{c}_R \Big(&\|\xi_\epsilon\|_{H^k(T_x^+M;E)} \\ &+ \epsilon^{k+1-n/2} \|D_x \xi_\epsilon\|_{H^{k-1}(T_x^+M;E)} + \|C_x \xi_\epsilon\|_{H^{k-j+1/2}(\partial T_x^+M;E)} \Big) \end{aligned}$$

for some $\bar{c}_R$ independent on x. Passing to the limit as $\epsilon \to 0$ and using Lemma 6.1(ii) we obtain

$$|\xi|_{H^{k+1}(T_x^+M;E)} \leq \bar{c}_R \left(|D_x^{(0)}\xi|_{H^{k-1}(T_x^+M;E)} + |C_x^{(0)}\xi|_{H^{k-j+1/2}(\partial T_x^+M;|E)}\right).$$

This gives right away that (D, C) satisfies a uniform, H^{k+1}-Shapiro-Lopatinski regularity condition (at ∂M). The same argument combined with a Fourier transform at an arbitrary interior point (and $0 < r < r_{FC}$ arbitrary) and a perturbation argument [39, 66] gives that D is uniformly elliptic on $M \setminus \partial M$. Hence D is uniformly elliptic on M.

The implications (iii) ⇒ (iv) ⇒ (v) are trivial. We have that, in fact, they are equivalent. This is seen in the same way in which one proves higher regularity for boundary value problems using divided differences. See any textbook, in particular [28, 40, 46, 66].

To complete the proof, it is enough then to show that (iii) ⇒ (i). We want to estimate $\|u\|_{H^{k+1}}$ in terms of $\|Du\|_{H^{k-1}}$, $\|Cu\|_{H^{k-j-1/2}}$ and $\|u\|_{H^k}$. This is done using an r-partition of unity ϕ_γ as in Definition 2.10 and following then almost word for word the proof of Theorem 5.6, but choosing $r >$ small enough. Let us use the notation of the proof of that theorem. Then the only difference in the estimate of the proof is that we need to replace P_γ and C_γ with their principal parts $P_\gamma^{(0)}$ and $C_\gamma^{(0)}$ and with coefficients frozen at p_γ. We note that the family $(P_\gamma^{(0)}, C_\gamma^{(0)})$ satisfies a uniform $H^{\ell+1}$-regularity estimate, in view of Lemma 6.3. The lower order terms can be absorbed into the weaker norm $\|u\|_{H^k}$. We then use the fact that $\|(P - P_\gamma)u\|_{H^{k-1}} \leq C(r)\|u\|_{H^{k+1}}$, for u with support in the ball of radius r centered at γ and with $C(r)$ independent of γ and with $C(r) \to 0$ as $r \to 0$. We have $C(r) \to 0$ when $r \to 0$ since P has coefficients in $W^{\ell+1,\infty}$. To obtain regularity estimates in the interior (away from the boundary), we use the uniform ellipticity of the operator. For more details, one can consult also Proposition 11.2 of [66], which is a similar result with a similar proof. □

In particular, the sequence (ii) ⇒ (iii) ⇒ (i) gives a new proof of Theorem 5.6. We obtain the following consequence.

Corollary 6.10 *Let $(D, C) \in \mathcal{D}^{\ell,j}(M; E)$. If (D, C) satisfies an order H^2-regularity estimate on M, then (D, C) satisfies an $H^{\ell+1}$-regularity estimate on M.*

It would be interesting to investigate the relation between the results of this paper and those of Karsten Bohlen [18, 19].

6.3 A Uniform Agmon Condition

In view of the results on the uniform Shapiro-Lopatinski conditions and of the usefulness of positivity apparent in the next section, we now consider the coercivity of our operators, in the same spirit as the uniform Shapiro-Lopatinski conditions. The notation and the approach are very similar. We keep the notation of Section 4. In particular, $\tilde{P}$ will be a second order differential operator in divergence form with associated boundary conditions $C = (C_0, C_1)$, as in Section 5.2. *From now on, M will be a manifold with boundary and bounded geometry.*

Recall the following standard terminology.

Definition 6.11 A sesquilinear form a on a hermitian vector bundle $V \to X$ is *called strongly* coercive (or *strictly positive*) if there is some $c > 0$ such that $\Re a(\xi, \xi) \geq c|\xi|^2$ for all $x \in X$ and $\xi \in V_x$. If the sesquilinear form a on $T^*M \otimes E$ used to define P is strongly coercive, then P is said to satisfy the *strong Legendre condition*.

Let $V \subset H$ be a continuous inclusion of Hilbert spaces (with non-closed image, in general). Let V^* be the complex conjugate of V with pairing $V \times V^* \to \mathbb{C}$ restricting to the scalar product of H on $V \times H$. Recall that an operator $T : V \to V^*$ is *coercive* on V if it satisfies the Gårding inequality, that is, if there exist $\gamma > 0$ and $R \in \mathbb{R}$ such that, for all $u \in V$,

$$\Re\langle Tu, u\rangle \geq \gamma\|u\|_V^2 - R\|u\|_H^2. \tag{27}$$

Then $T+\lambda$ is strongly coercive for $\Re(\lambda) > R$ (see Definition 6.11), and hence it satisfies the conditions of the Lax-Milgram lemma. Therefore, it satisfies regularity in view of the results of the next section. Coercive operators on *bounded* domains were characterized by Agmon in [1] as strongly elliptic operators satisfying suitable conditions at the boundary (which we shall call the "Agmon condition."). We shall need a *uniform* version of this condition, to account for the non-compactness of the boundary.

Let now $\tilde{P}\tilde{V} \to V^*$ be as in Eq. 8 (so it is associated to the sesquilinear form B and has principal symbol a). Let C be the boundary conditions associated to $\tilde{P}$. That is, $C = (e_0, e_1\partial_\nu^a + Q)$. We let e_{0x} and e_{1x} be the values at x of the endomorphisms e_0 and e_1. Similarly, let $Q_x^{(0)}$ be the principal part of Q with coefficients frozen at 0, where Q_x is regarded as a first order differential operator (so $Q_x^{(0)} = 0$ if Q is of order zero. Let $P_x^{(0)}$ be the principal part of the operator P and $C_x^{(0)} = (e_{0x}, e_{1x}\partial_\nu^a + Q_x^{(0)})$ be the principal part of the boundary conditions C with coefficients frozen at some $x \in \partial M$, as in Notation 6.6. Let $B_x^{(0)}$ be the associated Dirichlet bilinear form to $P_x^{(0)}$ equipped with the above boundary conditions (again with coefficients frozen at x), that is

$$B_x^{(0)}(u, v) := \int_{T_x^+} a_x^{(0)}(du, dv) + \int_{T_x\partial M} (Q_x^{(0)}u, v). \tag{28}$$

This defines a continuous sesquilinear form on

$$V_x := \{u \in H^1(T_x^+M; E_x) \mid u|_{T_x\partial M} \in (F_1)_x\}.$$

The associated operator in divergence form $\tilde{P}_x^{(0)}$ will be called *the principal part of $\tilde{P}$ with coefficients frozen at x.*

Definition 6.12 We say that $\tilde{P}$ satisfies the *uniform Agmon condition (on ∂M)* if it is uniformly strongly elliptic and if there exists $C > 0$ such

$$\langle \tilde{P}_x^{(0)}u, u\rangle = B_x^{(0)}(u, u) \geq C|u|_{H^1}^2,$$

for all $x \in \partial M$ and all $u \in \mathcal{C}_c^\infty(T_x^+M)$ with $e_{0x}u = 0$ on the boundary of T_x^+M.

We have then the following result that is proved, *mutatis mutandis*, as the regularity result of Theorem 6.9.

Theorem 6.13 *Let M be a manifold with boundary and bounded geometry, $E \to M$ be a vector bundle with bounded geometry, and $\tilde{P}$ be a second order differential operator in divergence form, as above. We assume $\tilde{P}$ has coefficients in $W^{1,\infty}$. We have that $\tilde{P}$ is coercive on $V := \{u \in H^1(M; E) \mid e_0u = 0 \text{ on} \partial M\}$ if, and only if, $\tilde{P}$ is uniformly strongly elliptic and it satisfies the uniform Agmon condition on ∂M.*

The proof is essentially the same as that of Theorem 6.9, more precisely, of the equivalence (i) $\Leftrightarrow$ (iii). We need at least $W^{1,\infty}$ to make the partition of unity argument work.

Remark 6.14 As is well-known, coercivity estimates lead to solutions of evolution equations [6, 7, 46, 59]. Let $H^1_0(M; E) \subset V \subset H^1(M; E)$ be the space defining our variational boundary value problem, see 4.1, Eq. 12, and Remark 4.13. Let V^* be the complex conjugate dual of V, as before, $\mathcal{W} := L^2(0, T; V)$, $T > 0$, so that $\mathcal{W}^* = L^2(0, T; V^*)$. Assume $P: V \to V^*$ is coercive (i.e. it satisfies Eq. 27). Then Theorem 4.1 of [46, Section 3.4.4] states that, for any $f \in \mathcal{W}^*$, there exists a unique $w \in \mathcal{W} \cap C([0, T], L^2(M; E))$ such that $\partial_t w(t) - Pw(t) = f(t)$ and $w(0) = 0$. Moreover, we also have $w \in H^1(0, T; V^*)$.

This leads to the following result.

Theorem 6.15 *Let us assume that $\tilde{P}$ is as in Remark 6.14, whose notation we continue to use, and that $\tilde{P}$ satisfies the uniform Agmon condition. Then, for any $f \in \mathcal{W}^*$, there is a unique $w \in \mathcal{W} \cap C([0, T], L^2(M; E))$ such that $\partial_t w(t) - Pw(t) = f(t)$ and $w(0) = 0$. Moreover, we also have $w \in H^1(0, T; V^*)$.*

For manifolds with bounded geometry (no boundary), this result was proved in [49]. The result in [49] was generalized to higher order equations in [7]. For the particular case mixed (Dirichlet/Neumann) boundary conditions and scalar equations, this result was proved in [6].

6.4 Conformal Invariance

Both the uniform Shapiro-Lopatinski regularity condition and the uniform Agmon condition are conformally invariant in an obvious sense that we make explicit in this subsection. Let $\rho > 0$ be a smooth function on M such that $\rho^{-1} d\rho$ is in $W^{\infty,\infty}$ (a function ρ with these properties will be called an *admissible* weight). Let $(P, C) = (P, C_0, C_1)$ be a boundary value problem. (We no longer assume that C has constant order on the boundary). Recall that the metric on M is denoted g, and consider $g' := \rho^{-2} g$, $P' := \rho^2 P$, $C_0' := C_0$, $C_1' := \rho C_1$, and $C' := (C_0', C_1')$. All differential operators will act on the same vector bundle E, whose metric we do *not* change.

Proposition 6.16 *Assume that (P, C) has coefficients in $W^{\ell+1,\infty}$ (i.e. $(P, C) \in \mathcal{D}^{\ell+1,j}(M; E)$ if C is of constant order j). We have that (P, C) satisfies a uniform $H^{\ell+1}$-Shapiro-Lopatinski regularity condition (with respect to the metric g) if, and only if, (P', C') satisfies a uniform $H^{\ell+1}$-Shapiro-Lopatinski regularity condition (with respect to the metric g'). The same statement remains true if we replace "a uniform $H^{\ell+1}$-Shapiro-Lopatinski regularity condition" with "a uniform Agmon condition."*

Proof This follows directly from definitions, by taking into account how the homogeneous Sobolev space norms change under the change of metric. More precisely, the semi-norm $|\cdot|'_{H^s} = |\cdot|'_{H^s(T_xM)}$ associated to the metric $g' := \rho^{-2} g = \rho(x)^{-2} g$ on T_xM is related to the original semi-norm $|\cdot|_{H^s} = |\cdot|_{H^s(T_xM)}$ associated to the metric g on T_xM by the relation

$$|v|'_{H^s} = \rho(x)^{s+m/2} |v|_{H^s}, \tag{29}$$

where m is the dimension of M. Taking into account this equation and, assuming, for simplicity that we have constant order j at the boundary, we have the following (where $\rho = \rho(x)$)

$$|w|_{H^{k+1}(T_x^+ M)} \leq c_{SL} \left(|P_x^{(0)} w|_{H^{k-1}(T_x^+ M)} + |C_x^{(0)} w|_{H^{k-j+1/2}(T_x \partial M)} \right)$$

$$\begin{aligned}
\Leftrightarrow \rho^{k+1+m/2}|w|_{H^{k+1}(T_x^+M)} &\leq \rho^{k+1+m/2}c_{SL}\Big(|P_x^{(0)}w|_{H^{k-1}(T_x^+M)} \\
&\quad +|C_x^{(0)}w|_{H^{k-j+1/2}(T_x\partial M)}\Big) \\
\Leftrightarrow |w|'_{H^{k+1}(T_x^+M)} &\leq c_{SL}\Big(\rho^{k-1+m/2}|\rho^2 P_x^{(0)}w|_{H^{k-1}(T_x^+M)} \\
&\quad +\rho^{k-j+1/2+(m-1)/2}|\rho^j C_x^{(0)}w|_{H^{k-j+1/2}(T_x\partial M)}\Big) \\
\Leftrightarrow |w|'_{H^{k+1}(T_x^+M)} &\leq c_{SL}\Big(|(P')_x^{(0)}w|'_{H^{k-1}(T_x^+M)}+|(C')_x^{(0)}w|'_{H^{k-j+1/2}(T_x\partial M)}\Big).
\end{aligned}$$

This completes the proof for the case of Shapiro-Lopatinski regularity condition, in view of the definition of the uniform $\overset{\bullet}{H}{}^{\ell+1}$-Shapiro-Lopatinski regularity condition, Definition 6.8.

The proof for the uniform Agmon condition is completely similar (only shorter), once one notices that the "full" operator $\tilde{P}'$ associated to (P', C') (and the associated bilinear form) scales in the right way, that is $\tilde{P}' = \rho^2\tilde{P}$. □

Note that for the above proof we did not need that ρ be an admissible weight. We continue to use the notation (P', C') introduced right before Proposition 6.16. We obtain the following consequence. The regularity estimates and the coercivity are stable under conformal changes of metric with bounded, admissible weights. More precisely, we have the following theorem.

Theorem 6.17 *Assume that (P, C) has coefficients in $W^{\ell+1,\infty}$ and that ρ is an admissible weight. Then (P, C) satisfies an $H^{\ell+1}$-regularity estimate on M (with respect to the metric g) if, and only if, (P', C') satisfies an $H^{\ell+1}$-regularity estimate on M (with respect to the metric g'). The same statement remains true for coercivity.*

Proof Let us notice that P is uniformly elliptic (respectively, uniformly strongly elliptic) with respect to the metric g if, and only if, P' satisfies the same property for the metric g'. Also, the fact that the weight is admissible and bounded guarantees that P' is in divergence form with bounded coefficients. Then the result follows by combining Proposition 6.16 with Theorem 6.9 (for the regularity part), respectively with Theorem 6.13 for the coercivity part. □

7 Coercivity, Legendre Condition, and Regularity

In this section we use coercivity (or positivity) to obtain regularity results. As an application, we study mixed Dirichlet/Robin boundary conditions for operators satisfying the strong Legendre condition. We continue to assume that M is a manifold with boundary and bounded geometry and that $E \to M$ is a vector bundle with bounded geometry.

7.1 Well-Posedness in Energy Spaces Implies Regularity

Often the regularity conditions (including the Shapiro-Lopatinski ones discussed below) are obtained from the invertibility of the given operator. This is the case also in the bounded geometry setting, owing to Proposition 3.2. The results of this subsection will be used to deal with the Neumann and Robin boundary conditions. The approach below is based on the so called "Nirenberg trick" (see also [50] and the references therein).

Lemma 7.1 *Let X_j be as in Lemma 3.1 and $Y = X_j$, for some $j > 1$ fixed.*

(i) *There exists a one parameter group of diffeomorphisms $\phi_t : M \to M$, $t \in \mathbb{R}$, that integrates Y, that is $\frac{d}{dt} f(\phi_t(x))_{t=0} = (Yf)(x)$, for any $x \in M$ and any smooth function $f : M \to \mathbb{C}$.*

(ii) *Let $\pi : E \to M$ be a vector bundle. Then there exists a one parameter group of diffeomorphisms $\psi_t : E \to E$, $t \in \mathbb{R}$, with $\pi \circ \psi_t = \phi_t \circ \pi$, that integrates ∇_Y, that is $\frac{d}{dt}(\psi_{-t} \circ \xi)|_{t=0} = \nabla_Y \xi$, for any smooth section $\xi : M \to E$.*

Proof (i) Let $\hat{M}$ be as in Definition (2.5). Let $\hat{Y} \in W^{\infty,\infty}(\hat{M}, T\hat{M})$ be an extension of Y. Then $\hat{Y}$ is a bounded vector field on a complete manifold. Hence, by [5, Sec. 3.9] it admits a global flow, i.e. a smooth solution $\phi : \mathbb{R} \times \hat{M} \to \hat{M}$ of

$$\frac{d}{dt}\phi(t, p) = \hat{Y}(p), \quad \phi(0, p) = p$$

such that $\phi_t := \phi(t, .)$ is a one-parameter family of diffeomorphisms of $\hat{M}$. Since $\hat{Y}$ is tangent to $\partial M \subset \hat{M}$, $\phi(t, \partial M) = \partial M$. Since ∂M divides $\hat{M}$ into two parts, ϕ_t restricts to diffeomorphisms of M.

(ii) We choose a connection ∇^E on E. Let $\psi_t : E \to E$ be defined by $e \mapsto e(t)$ where $e(t)$ is the solution of $\nabla^E_{Y(p)=\partial_t \phi_t(p)} e(t) = 0$ with $e(0) = e$. By the standard properties of parallel transport, respectively of the underlying linear ordinary differential equation, we have the global existence and uniqueness of the solution and the claimed properties.

□

We then obtain the following abstract regularity result. Let X_j be as in Lemmas 3.1 and 7.1. We can assume that X_1 is a unit vector field normal at the boundary. Recall that $V := H^1(M; E) \cap \{u|_{\partial M} \in \Gamma(\partial M; F_1)\}$.

Theorem 7.2 *Let $\tilde{P}$ be a second order differential operator in divergence form with associated form a and $W^{\ell,\infty}$-coefficients. Let $\zeta := a(X_1, X_1)$ and let us assume that ζ is invertible and ζ^{-1} bounded. Also, let us assume that $\tilde{P} : V \to V^*$ is a continuous bijection (i.e., an isomorphism). Then P satisfies an $H^{\ell+1}$-regularity estimate on M.*

Proof The proof is classic, except maybe the fact that we have a slightly weaker assumption on the coefficients; typically one requires C^{k+1}-coefficients in textbooks. We include, nevertheless, a very brief outline of the proof. Recall that the proof is done by induction, with the general step the same as the first step (going from well-posedness to H^2-regularity). Let us assume then that $\ell = 1$.

Let $F \in j_0(L^2(M; E) \oplus H^{1/2}(M; F_1))$ and let $\tilde{P}u = F$, with $u \in V$. For simplicity, let us assume E is one-dimensional. In general, we just replace X_j with ∇_{X_j}. We want to show that $u \in H^2(M)$, with continuous dependence on F. To this end, in view of Proposition 3.2, it is enough to check that $X_i X_j u \in L^2(M)$, with continuous dependence on F, since we already know that $u \in V \subset H^1(M)$, with continuous dependence on F.

In particular, $X_j u \in L^2(M)$. Nirenberg's trick is to give conditions on X_j such that we can formally apply X_j to the equation $\tilde{P}u = F$ to obtain that $X_j u \in V$ is the unique solution of

$$\tilde{P}(X_j u) = [\tilde{P}, X_j](u) + X_j(F) \in V^*. \tag{30}$$

This is possible whenever $[\tilde{P}, X_j]\colon V \to V^*$ continuously and X_j generates a continuous parameter semi-group on V (see [14] for a general approach and more details). These conditions are satisfied since $\tilde{P}$ has coefficients in $W^{1,\infty}$ and if $j > 1$, since then X_j is tangent to the boundary and we can invoke Lemma 7.1. This argument gives that $X_i X_j u \in L^2(M)$ if *at least* one of the i and j is > 1.

It remains to prove that $X_1^2 u \in L^2(M)$. This is proved using the equation

$$\zeta X_1^2 u = Pu - \sum_{i+j>2} c_{ij} X_i X_j u \in L^2(M),$$

since we can choose $c_{ij} \in L^\infty(M; \mathrm{End}(E))$ and $Pu \in L^2(M; E)$, by assumption. (Recall that we assumed that X_1 is a unit vector everywhere on the boundary.) □

Remark 7.3 Except the above theorem, it is very likely that most of the results obtained so far extend to higher order equations, but we have not checked all the details. The above theorem will require, however, some additional ideas in order to extend it to the setting of Remark 4.13.

7.2 Robin vs Shapiro-Lopatinski

Let us discuss, as an example, Robin (and hence also Neumann) boundary conditions from the perspective of the Shapiro-Lopatinski conditions. We do that now in the case of a model problem on the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, \infty)$. The sesquilinear form a of Section 4 (see Assumption 4.1) is now simply a sesquilinear form on $\mathbb{C}^{nN} = \mathbb{R}^n \otimes \mathbb{C}^N$. We allow now F_0 and F_1 to be both non-trivial, which amounts to a decomposition $N = N_0 + N_1$, $N_j \geq 0$. We shall need also the operator Q, which is now a $N_1 \times N_1$ matrix of constant coefficients differential operators on $\mathbb{R}^{n-1}$ acting on the last N_1 components of $\mathbb{C}^N$. The bilinear form that we consider is then

$$B(u, v) := \int_{\mathbb{R}^n_+} \big[a(du, dv) + c(u, v)\big] dx + \int_{\mathbb{R}^{n-1}} (Qu, v) dx', \tag{31}$$

where c is a scalar and $x = (x', x_n)$. This is a particular case of the form considered in Eq. 7. Recall the definition of a strongly coercive form, Definition 6.11. Note that strong coercivity implies uniform strong ellipticity as in Definition 4.9.

Lemma 7.4 *Let us assume that $Q + Q^*$ is of order zero and that a is strongly coercive. Then for c (of* Eq. 31*) large enough, there is a $\gamma > 0$ such that*

$$B(u, u) \geq \gamma \|u\|^2_{H^1(\mathbb{R}^n_+)} \tag{32}$$

for all $u \in H^1(\mathbb{R}^n_+)$.

Proof We identify $\mathbb{R}^{n-1}$ with the boundary of the half-space $\mathbb{R}^n_+$, as usual. Let $c_Q > 0$ be a bound for the norm of the matrix $\frac{1}{2}(Q + Q^*)$. We have, using first the definition

$$\begin{aligned} B(u, u) \; &:= \; \int_{\mathbb{R}^n_+} \big[a(du, du) + c(u, u)\big] dx + \int_{\mathbb{R}^{n-1}} (Qu, u) dx' \\ &\geq \; c_a |u|^2_{H^1(\mathbb{R}^n_+)} + c\|u\|^2_{L^2(\mathbb{R}^n_+)} - c_Q \|u\|_{L^2(\mathbb{R}^{n-1})} \\ &\geq \; \frac{c_a}{2} \|u\|^2_{H^1(\mathbb{R}^n_+)} \end{aligned}$$

for c large. The last statement is proved in the same way one proves the trace inequality $\|u\|_{L^2(\mathbb{R}^{n-1})} \leq c_T \|u\|_{H^1(\mathbb{R}^n_+)}$, but using also Lebesgue's dominated convergence theorem. □

We obtain the following consequence.

Corollary 7.5 *Let a and Q be as in Lemma 7.4 and let $\tilde{P}$ be the differential operator in divergence form associated to the form B of* Eq. 31. *Then $\tilde{P}$ (or, $(P, e_0, e_1\partial_\nu^a + Q)$) satisfies an $H^{\ell+1}$-regularity estimate on $\mathbb{R}^n_+$ for all ℓ. In particular, $(P^{(0)}, e_0, e_1\partial_\nu^a + Q^{(0)})$ satisfies the $H^{\ell+1}$-Shapiro-Lopatinski regularity condition at 0.*

Proof Let $V := \{u \in H^1(\mathbb{R}^n_+) \mid e_0 u = 0 \text{ on} \mathbb{R}^{n-1}\}$. We then have that the restriction of B to $V \times V$ satisfies the assumptions of the Lax-Milgram lemma [33], and hence $\tilde{P} \colon V \to V^*$ is an isomorphism. Theorem 7.2 then gives the first part of the result. The last part follows from Theorem 6.9. □

7.3 Mixed Dirichlet/Robin Boundary Conditions

Let us turn now back to the study of mixed Dirichlet/Robin boundary value problems on M. Let M be a manifold with bounded geometry with a decomposition $E|_{\partial M} = F_0 \oplus F_1$ as the direct sum of two vector bundles with bounded geometry. We consider the same bilinear form B as in Section 4, see Eq. 7, with the data defining it as in Assumptions (4.1). In particular, B is obtained from a form a that can be interpreted as the principal symbol of the associated operators P and $\tilde{P}$. If a is strongly coercive, we say that P (or $\tilde{P}$) satisfies the *strong Legendre condition.* (See [42] for a similar concept for Stokes-type operators.)

The associated boundary conditions are then

$$C_0 u = e_0 u|_{\partial M} \quad \text{and} \quad C_1 u = e_1 \partial_\nu^a u + Q u|_{\partial M} \tag{33}$$

for some first order differential operator Q acting on sections of F_1. We let

$$\|Cv\|_k := \|C_0 v\|_{H^{k+1/2}(\partial M; F_0)} + \|C_1 v\|_{H^{k-1/2}(\partial M; F_1)}.$$

We say that the boundary conditions C are *mixed Dirichlet/Robin boundary conditions.* We shall need the following analog of Corollary 5.9.

Corollary 7.6 *Let $S \subset \mathcal{D}^{\ell+1,0}(B_r^m(0) \times [0, r); E)$ be a bounded family of boundary value problems on $B_r^m(0) \times [0, r) \subset \mathbb{R}^{m+1}$ equipped with the euclidean metric, $r \leq \infty$. We assume that the family S satisfies a uniform strong Legendre condition and all the boundary conditions are are Robin boundary conditions of the form $(e_0, e_1\partial_\nu^a + Q)$, with $Q + Q^*$ of order zero and bounded on S. Then the family S satisfies a uniform $H^{\ell+1}$-regularity estimate on $B_{r'}^m(0) \times [0, r')$, $r' < r$.*

Proof Let $(D_n, C_n) \in S$ converge to $(D, C) \in \mathcal{D}^{\ell,j}(B_{r'}^m(0) \times [0, r'); E)$, with $C_n = (e_0, e_1\partial_\nu^{a_n} + Q_n)$, with $Q_n + Q_n^*$ of order zero. Then D satisfies the strong Legendre condition because the parameter c_a in the definition of the strong Legendre condition (the fact that a is strongly coercive) is assumed to stay away from 0 on S. The limit of Robin boundary conditions is again a Robin boundary condition and the condition that $Q + Q^*$ be scalar is also preserved under limits. Corollary 7.5 then gives that (D, C) satisfies an $H^{\ell+1}$-regularity estimate. This allows us again to use Proposition 5.3 for the relatively compact subset $N := B_{r'}^m(0) \times [0, r')$ of $M := B_r^m(0) \times [0, r)$ to obtain the result. □

We are ready now to prove the result stated in the Introduction, Theorem 1.1.

Proof The proof of Theorem 1.1 is the same as that of Theorem 5.12. Indeed Corollaries 7.6 and 5.10 show that the assumptions of Theorem 5.6 are satisfied. That theorem immediately gives our result. □

Notice that in the statement of the theorem, we have dropped the condition that B be of constant order (it is, nevertheless, of *locally constant* order).

Let us assume now that we have a partition of the boundary $\partial M = \partial_D M \sqcup \partial_R M$ as a disjoint union of two open and closed subsets and that $F_0 = E|_{\partial_D M}$ and $F_1 = E|_{\partial_R M}$. By combining Theorem 7.2 with the Poincaré inequality [9], we can prove following well-posedness result for the mixed Dirichlet/Robin boundary value problem. Let $A \subset \partial M$ (see [9] for details). Recall that we say that (M, A) has finite width [10] if the distance to A is bounded uniformly on M and A intersects all connected components of M.

Theorem 7.7 *We use the same notation as in Theorem 1.1, in particular, M is a manifold with boundary and bounded geometry and $\tilde{P} = (P, \partial_\nu^a u + Q)$ has coefficients in $W^{\ell,\infty}$ and satisfies the strong Legendre condition. Assume that $F_0 = E|_{\partial_D M}$, that $F_1 = E|_{\partial_R M}$, that $Q + Q^* \geq 0$, that there exists $\epsilon > 0$ and an open and closed subset $\partial_{PR} M \subset \partial_R M$ such that $Q + Q^* \geq \epsilon$ on $\partial_{PR} M$, and, finally, that $(M, \partial_D M \cup \partial_{PR} M)$ has finite width. Then the boundary value problem*

$$\begin{cases} Pu = f \in H^{k-1}(M; E) & \text{in } M \\ u = h_D \in H^{k+1/2}(\partial_D M; E) & \text{on } \partial_D M \\ \partial_\nu^a u + Qu = h_R \in H^{k-1/2}(\partial_R M; E) & \text{on } \partial_R M, \end{cases} \tag{34}$$

has a unique solution $u \in H^{k+1}(M; E)$, $k \geq 0$, and this solution depends continuously on the data, $0 \leq k \leq \ell$.

Note that the more general form of the Robin boundary conditions considered in this paper (i.e. corresponding to a splitting of E at the boundary, is useful for treating the Hodge-Laplacian. See also [53, 67]. See [66] also for results on the Robin problem on smooth domains. See [22, 32] for some results on the Robin problem on non-smooth domains. See also [8].

7.4 The Bounded Geometry of the Boundary is Needed

Let us provide now an example of a manifold Ω with smooth metric and smooth boundary $\partial\Omega$ that satisfies the Poincaré inequality, and hence such that $\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, but such that $\Delta\colon H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is not onto. In other words, the operator $D = \Delta$ with Dirichlet boundary condition does not satisfy an H^2-regularity estimate on Ω, in spite of the fact that it has coefficients in $W^{\infty,\infty}$. Our example is based on the loss of regularity for problems on concave polygonal domains.

Let $G \subset \mathbb{R}^n$ be a bounded domain with *Lipschitz* boundary. It will be convenient to consider only *closed* domains. Let $m \in \mathbb{N}$ (we shall need only the case $m = 1$). We denote by $H^m(G)$ the space of functions with m derivatives in L^2. It is the set of restrictions of functions from $H^m(\mathbb{R}^n)$ to G. We let $H_G^m(\mathbb{R}^n)$ be the set of distributions in $H^m(\mathbb{R}^n)$ with support in $G = \overline{G}$ and $H_0^m(G)$ be the closure of the set of test functions with support in G in $H^m(\mathbb{R}^n)$, as usual. It is known that $H_G^1(\mathbb{R}^n) = H_0^1(G) = \ker(H^1(G) \to L^2(G))$.

See [66] for the case G smooth. The Lipschitz case is completely similar and follows from $H_0^1(G) = \ker(H^1(G) \to L^2(G))$, see, for example [52].

Our construction of the manifold with boundary Ω is based on the following lemma.

Lemma 7.8 *Let G and $\omega_1 \supset \omega_2 \supset \ldots \supset \omega_n \supset \ldots$ be closed, bounded domains in $\mathbb{R}^n$ with Lipschitz boundary such that $G := \cap \omega_n$. Let $f \in L^2(G)$, $u_n \in H_0^1(\omega_n)$, and $w \in H_0^1(G)$ satisfy $\Delta u_n = f$ on ω_n and $\Delta w = f$ on G. If the domains ω_n have smooth boundary and $w \notin H^2(G)$, then $\|u_n\|_{H^2(\omega_n)} \to \infty$.*

Proof We have that

$$H_0^1(\omega_1) \supset H_0^1(\omega_2) \supset \ldots \supset H_0^1(\omega_n) \ldots \supset H_0^1(G).$$

Let $\xi \in \cap_{n=1}^{\infty} H_0^1(\omega_n)$. Then ξ has support in $G = \cap_{n=1}^{\infty} \omega_n$, and hence ξ has support in G, which gives $\xi \in H_G^1(\mathbb{R}^n) = H_0^1(G)$, by the discussion preceding this lemma. This shows that $H^1(G) = \cap_{n=1}^{\infty} H_0^1(\omega_n)$. Let $B(u, v) := \int_{\mathbb{R}^n} (\nabla u, \nabla v) d\,\mathrm{vol}$. Then B induces an inner product on $H_0^1(\omega_1)$ equivalent to the initial inner product. Moreover, the relations

$$B(u_n, v) = (f, v) = B(u_{n+1}, v), \quad v \in H_0^1(\omega_{n+1})$$

show that u_{n+1} is the B-orthogonal projection of u_n onto $H_0^1(\omega_{n+1})$. Similarly, w is the B-orthogonal projection of u_n onto $H_0^1(G)$. Since $H^1(G) = \cap_{n=1}^{\infty} H_0^1(\omega_n)$, we obtain that $u_n \to w$ in $H_0^1(\omega_1)$.

To prove that $\|u_n\|_{H^2(\omega_n)} \to \infty$ if $w \notin H^2(G)$, we shall proceed by contradiction. Let us assume then that this is not the case. Then, by passing to a subsequence, we may assume that $\|u_n\|_{H^2(\omega_n)}$ is bounded. Then $\|u_n|_G\|_{H^2(G)} \leq \|u_n\|_{H^2(\omega_n)}$ also forms a bounded sequence. By passing to a subsequence again, we may assume then that $u_n|_G$ converges weakly in $H^2(G)$ to some $\tilde{w}$, by the Alaoglu-Bourbaki theorem. Hence $u_n|_G \to \tilde{w}$ weakly in $H^1(G)$ (even in norm, since $H^2(G) \to H^1(G)$ is compact). We have $\|u_n|_G - w\|_{H^1(G)} \leq \|u_n - w\|_{H^1(\omega_1)} \to 0$. Hence $u_n|_G \to w$ in $H^1(G)$. Consequently, $\tilde{w} = w$, which is a contradiction, since we have assumed that $w \notin H^2(G)$. □

We are ready now to construct our manifold Ω. Let G be a bounded domain whose boundary is smooth, except at one point, where we have an angle $> \pi$ (so G is *not* convex). It is known then that there exists $u \in H_0^1(G)$, $u \notin H^2(G)$ such that $f := \Delta u \in L^2(M)$. See [23, 34, 35, 55]. In fact, the space of functions $\phi \in L^2(W)$ such that $\Delta^{-1}\phi \in H^2(G) \cap H_0^1(G)$ is of codimension one in $L^2(G)$. Let $\ldots \subset \omega_{n+1} \subset \omega_n \subset \ldots \subset \omega_1$ be a sequence of *closed*, smooth, bounded domains whose intersection is G. Then we can take for Ω the (disjoint) union of all the domains $\omega_n \times \{n\}$, $n \in \mathbb{N}$.

Let us check that Ω has the desired properties. We construct $\phi \in L^2(\Omega)$ by taking $\phi := c_n f$ on ω_n, $c_n > 0$. Let $u_n \in H_0^1(\omega_n)$ be the unique solution of $\Delta u_n = f$. We can choose c_n such that $\sum_n c_n^2 < \infty$, and hence $\phi \in L^2(\Omega)$, but $\sum_n c_n^2 \|u_n\|_{H^2(\omega_n)}^2 = \infty$, since the sequence $\|u_n\|_{H^2(\omega_n)}$ is unbounded, by the last lemma. We have that $\Omega \subset \omega_1 \times \mathbb{N}$, and hence it satisfies the Poincaré inequality for all functions in $H_0^1(\Omega)$. Let $u \in H_0^1(\Omega)$ be the unique solution of the equation $\Delta u = \phi$ [10]. Then $u = c_n u_n$ on ω_n, by the uniqueness of u_n, and hence $\|u\|_{H^2(\Omega)}^2 = \sum_{n=1}^{\infty} c_n^2 \|u_n\|_{H^2(\omega_n)}^2 = \infty$. That is, $u \notin H^2(\Omega)$. Note that Ω is *not* of bounded geometry: indeed, the second fundamental form of ω_n cannot be uniformly bounded in n, since the "limit" G of the domains ω_n has a corner.

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