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Li-Yau Harnack Estimates for a Heat-Type Equation Under the Geometric Flow

Yi Li[1] **· Xiaorui Zhu**[2]

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Abstract

In this paper, we consider the gradient estimates for a postive solution of the nonlinear parabolic equation $\partial_t u = \Delta_t u + hu^p$ on a Riemannian manifold whose metrics evolve under the geometric flow $\partial_t g(t) = -2S_{g(t)}$. To obtain these estimate, we introduce a quantity $\underline{\boldsymbol{S}}$ along the flow which measures whether the tensor S_{ij} satisfies the second contracted Bianchi identity. Under conditions on $\text{Ric}_{g(t)}$, $S_{g(t)}$, and $\underline{\boldsymbol{S}}$, we obtain the gradient estimates.

Keywords Nonlinear parabolic equation · Harnack estimate · Geometric flow

Mathematics Subject Classification (2010) Primary 53C44

1 Introduction

We are continuous to consider the gradient estimates for nonlinear partial differential equations after our previous works [6, 11–13]. Let $(M, g(t))_{t\in[0,T]}$ be a complete solution to the geomtric flow

$$\partial_t g(t) = -2S_{g(t)}, \quad t \in [0, T]. \tag{1.1}$$

on a complete and noncompact n-dimensional manifold M and consider a positive function $u = u(x, t)$ defined on $M \times [0, T]$ solving the equation

$$\partial_t u = \Delta_t u + hu^p, \quad t \in [0, T], \tag{1.2}$$

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✉ Yi Li
yilicms@gmail.com

Xiaorui Zhu
zju-zxr@163.com

[1] Faculty of Science, Technology and Communication (FSTC), Mathematic Research Unit, Campus Belval, Universite du Luxembourg, Maison du Nombre, 6, Avenue de la Fonte L-4364, Esch-sur-Alzette, Luxembourg

[2] China Maritime Police Academy, Ningbo, 315801 China

where Δ_t stands for the Laplacian of $g(t)$, h is a function defined on $M \times [0, T]$ which is C^2 in x and C^1 in t , and p is a positive constant. When metrics are fixed, the study on the gradient estimates of Eq. 1.2 arose from [4]. If $h = 0$, Sun [9] derived the gradient estimates and the Harnack inequalities for the positive solutions of the linear parabolic equation $\partial_t u = \Delta_t u$ under the geometric flow. In this paper, we consider the general case for the nonlinear parabolic equation. Notice that the Δ_t depends on the parameter t, and we should study the Eq. 1.2 coupled with the geometric flow (1.1). The formula (1.1) provides us with additional information about the coefficients of the operator Δ_t appearing in Eq. 1.2 but is itself fully independent of Eq. 1.2.

We introduce notions used throughout this paper. Let $B_{\rho,T} = \{(x, t) \in M \times [0, T] : \mathrm{dist}_{g(t)}(x, x_0) < \rho\}$, where $\mathrm{dist}_{g(t)}(x, x_0)$ denotes the distance between x to a fixed point x_0 with respect to $g(t)$. $\nabla_{g(t)}$ and $|\cdot|_{g(t)}$ stand for the Levi-Civita connection and norm with respect to $g(t)$ respectively. Set

$$\underline{\boldsymbol{S}}_g(t) := \mathrm{div}_{g(t)} S_{g(t)} - \frac{1}{2}\nabla_{g(t)}\left(\mathrm{tr}_{g(t)} S_{g(t)}\right).$$

Locally, one has

$$\underline{\boldsymbol{S}}_i = \nabla^j S_{ij} - \frac{1}{2}\nabla_i\left(\mathrm{tr}_{g(t)} S_{g(t)}\right).$$

For example, if $S_{ij} = R_{ij}$, that is, Eq. 1.1 is the Ricci flow, we arrive at

$$\underline{\boldsymbol{S}}_i = \nabla^j R_{ij} - \frac{1}{2}\nabla_i R_{g(t)} = \frac{1}{2}\nabla_i R_{g(t)} - \frac{1}{2}\nabla_i R_{g(t)} = 0$$

by the second contracted Bianchi identity. Thus, the quantity $\underline{\boldsymbol{S}}_{g(t)}$ measures whether S_{ij} satisfies the second contracted Bianchi identity.

Theorem 1.1 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.1) on M with $-K_1 g(t) \le S_{g(t)} \le K_2 g(t)$, $-K_3 g(t) \le Ric_{g(t)} - S_{g(t)} \le K_3 g(t)$, and $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \le K_4$ for some $K_1, K_2, K_3, K_4 > 0$ on $B_{2R,T}$, with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x, t)$ be a function defined on $M \times [0, T]$ which is C^2 in x and C^1 in t, satisfying $\Delta_{g(t)} h \ge -\theta$ and $|\nabla_{g(t)} h|_{g(t)} \le \gamma$ on $B_{2R,T} \times [0, T]$ for some nonnegative constants θ and γ. If $u(x, t)$ is a positive smooth solution of* Eq. 1.2 *on $M \times [0, T]$, then*

(i) *for $0 < p < 1$, we have*

$$\begin{aligned}\frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le \frac{C_1}{p^2 t} + \frac{n(1-p)}{p^2} M_1 M_2 + \frac{n[3K_1 + 2(K_3 + K_4)p]}{2p^2(1-p)} \\ + \frac{C_1}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1 + K_3}}{R} + \overline{K} + \frac{n}{p(1-p)}\right) \quad (1.3) \\ + \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{n/K_1}}{p}\gamma M_2 + \frac{n}{p^2}\sqrt{\frac{K_4}{2n}},\end{aligned}$$

where C_1 is a positive constant depending only on n and

$$M_1 := \max_{B_{2R,T}} h_-, \quad M_2 := \max_{B_{2R,T}} u^{p-1}, \quad h_- := \max(-h, 0).$$

(ii) *for $p \geq 1$, we have*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{t} \leq \frac{k^2C_2}{p^2t}+\frac{nk^2(p-1)}{p^2}M_4M_5+\frac{k^3n}{k-p}M_3M_4 + \frac{k^2C_2}{p^2}\left(\frac{1}{R^2}+\frac{\sqrt{K_1+K_3}}{R}+\overline{K}+\frac{k^2n}{p(k-p)}\right) + \frac{2k^3n}{(k-p)p^2}\left[K_1+\frac{p}{k}(K_3+K_4)\right]+\frac{k^2\sqrt{n}\gamma}{p}M_4 + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4}+\frac{k^2n}{p^2}\left(\overline{K}+\sqrt{\frac{K_4}{2n}}\right), \quad (1.4)$$

where $k > p$, C_2 is a positive constant depending only on n and

$$M_3 := \max_{B_{2R,T}} h_-, \quad M_4 := \max_{B_{2R,T}} u^{p-1}, \quad M_5 := \max_{B_{2R,T}} h.$$

As an immediate consequence of the above theorem we have

Theorem 1.2 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.1) on M. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t.*

(i) For $0 < p < 1$, assume that $h \geq 0$, $|\nabla_{g(t)}h|_{g(t)} \leq \gamma$, $\Delta_{g(t)}h \geq 0$ along the geometric flow with $-K_1g(t) \leq S_{g(t)} \leq K_2g(t)$, $-K_3g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ for some positive constants $\gamma, K_1, K_2, K_3, K_4$ with $\overline{K} := \max\{K_1, K_2\}$, along the geometric flow. If u is a smooth positive function satisfying the nonlinear parabolic (1.2), then

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{u} \leq \frac{C_1}{p^2t}+\frac{C_1}{p^3(1-p)}+\frac{C_1}{p^2}\overline{K}+\frac{2nK_1}{p^2(1-p)} +\frac{\sqrt{n/K_1}}{p}\gamma M+\frac{n}{p^2}\sqrt{\frac{K_4}{2n}}+\frac{n(K_3+K_4)}{p(1-p)} \quad (1.5)$$

for some positive constant C_1 depending only on n, where $M := \max_{M\times[0,T]} u^{p-1}$.

(ii) For $p = 1$, assume that $-K_1g(t) \leq S_{g(t)} \leq K_2g(t)$, $-K_3g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ for some positive constants K_1, K_2, K_3, K_4 with $\overline{K} := \max\{K_1, K_2\}$, $h \geq 0$, $\Delta_{g(t)}h \geq -\theta$ (θ is nonnegative), and $|\nabla_{g(t)}h|_{g(t)} \leq \gamma$ (γ is nonnegative), along the geometric flow. If u is a smooth positive function satisfying the nonlinear parabolic (1.2), then

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+h-\frac{u_t}{u} \leq \frac{C_2}{t}+C_2\left(1+K_1+K_2+K_3+K_4+\overline{K}+\gamma+\sqrt{\theta}\right) \quad (1.6)$$

for some positive constant C_2 depending only on n.

(iii) For $p > 1$, assume that $-K_1g(t) \leq S_{g(t)} \leq K_2g(t)$, $-K_3g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ for some positive constants $\gamma, K_1, K_2, K_3, K_4$ with $\overline{K} := \max\{K_1, K_2\}$. $\Delta_{g(t)}h \geq -\theta$, $|\nabla_{g(t)}h|_{g(t)} \leq \gamma$, and $-k_1 \leq h \leq k_2$, where

$\theta, \gamma, k_1, k_2 > 0$ *, along the geometric flow. If u is a bounded smooth positive function satisfying the nonlinear parabolic* (1.2)*, then*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le \left(\frac{k}{p}\right)^2 \frac{C_3}{t} + \left(\frac{k}{p}\right)^3 \frac{k}{k-p}C_3 + \left(\frac{k}{p}\right)^2 C_3\Bigg(\overline{K} + \\ + \frac{k}{k-p}(K_1+K_3+K_4) + \sqrt{\frac{K_4}{2n}}\Bigg) \\ + \left(\frac{k}{p}\right)^2 n(p-1)k_2 M + \frac{k^3 n}{k-p}k_1 M \\ + \frac{k^2\sqrt{n}}{p}\gamma M + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M}, \tag{1.7}$$

for some positive constant C_3 *depending only on n, where* $M := \max_{M\times[0,T]} u^{p-1}$ *and* $k > p$*. In particular, taking* $k = 2p$*, we get*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le \frac{C_4}{t} + C_5\left(1 + K_1 + K_2 + K_3 + K_4 + \overline{K}\right) \\ + C_4 p^2\left[(k_1+k_2)M + \gamma M + \sqrt{\theta M}\right], \tag{1.8}$$

for some positive constant C_4 *depending only on n.*

Another type of Harnack inequality is the following

Theorem 1.3 *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a solution to the geometric flow* (1.1) *on M, satisfying* $-K_1 g(t) \le S_{g(t)} \le K_2 g(t)$, $-K_3 g(t) \le Ric_{g(t)} - S_{g(t)} \le K_3 g(t)$, $|\nabla_{g(t)}\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \le K_4$*,for some* $K_1, K_2, K_3, K_4 > 0$*, with* $\overline{K} := \max\{K_1, K_2\}$*. Let* $h(x,t)$ *be a nonnegative function defined on* $M \times [0,T]$ *which is* C^2 *in x and* C^1 *in t,* $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \ge 0$ *on* $M \times [0,T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p > 1$ *and* $C_{n,p} = n$ *if* $p \le 1$*), and* $0 < p \le \frac{2n}{2n-1}$ *(*$n \ge 3$*). If u is a positive solution of* Eq. 1.2*, then*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{u} \le \frac{C}{p^2 t} + \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} \\ + \frac{4n}{p(2-p)}(K_1+K_3+K_4) + \frac{1}{p^2}\sqrt{8nK_4}, \tag{1.9}$$

for some positive constant C depending only on n.

This theorem has three important consequences.

Corollary 1.4 *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a solution to the geometric flow* (1.1) *on M, satisfying* $0 \le S_{g(t)} \le K_2 g(t)$, $-K_3 \le Ric_{g(t)} - S_{g(t)} \le K_3 g(t)$, $|\nabla_{g(t)}\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \le K_4$*, for some positive constants* K_2, K_3, K_4*. Let* $h(x,t)$ *be a nonnegative function defined on* $M \times [0,T]$ *which is* C^2 *in x and* C^1 *in t,* $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \ge 0$ *on* $M \times [0,T]$

(where $C_{n,p} = \frac{p}{p-1}$ *if* $p > 1$ *and* $C_{n,p} = n$ *if* $p \leq 1$*), and* $0 < p \leq \frac{2n}{2n-1}$ *(*$n \geq 3$*). If* u *is a positive solution of* Eq. 1.2*, then*

$$\frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p} u^{p-1} - \frac{2}{p} \frac{u_t}{t} \leq \frac{C}{p^2 t} + \frac{8n}{p^2} K_2 + \frac{4n}{p(2-p)} (K_3 + K_4) + \frac{1}{p^2} \sqrt{8n K_4} \quad (1.10)$$

for some positive constant C *depending only on* n*.*

Corollary 1.5 *Suppose that* $(M, g(t))_{t \in [0,T]}$ *is a solution to the geometric flow* (1.1) *on* M*, satisfying* $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$*,* $-K_3 g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$*,* $|\nabla_{g(t)} \underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$*, for some* $K_1, K_2, K_3, K_4 > 0$*, with* $\overline{K} := \max\{K_1, K_2\}$*. Let* $h(x,t)$ *be a nonnegative function defined on* $M \times [0,T]$ *which is* C^2 *in* x *and* C^1 *in* t*,* $\Delta_{g(t)} h + h_t - 2C_{n,p} p \frac{|\nabla_{g(t)} h|^2_{g(t)}}{h} \geq 0$ *on* $M \times [0,T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p > 1$ *and* $C_{n,p} = n$ *if* $p \leq 1$*), and* $0 < p \leq \frac{2n}{2n-1}$ *(*$n \geq 3$*). If* u *is a positive solution of* Eq. 1.2*, then*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\left[-\frac{1}{2p} \min_{\gamma \in \Theta(x_1,t_1,x_2,t_2)} \int_{t_1}^{t_2} |\dot{\gamma}(t)|^2_{g(t)} dt - 2n(t_2 - t_1) \right.$$
$$\left. \left(\frac{1}{p} \overline{K} + \frac{2}{p} \sqrt{\frac{2n}{p(2-p)} K_1} + \frac{1}{2-p} (K_1 + K_3 + K_4) + \frac{1}{p} \sqrt{2n K_4} \right) \right] \quad (1.11)$$

for some positive constant C *depending only on* n*, where* $(x_1, t_1), (x_2, t_2) \in M \times [0, T]$ *with* $t_1 < t_2$*.*

When $K_1 = 0$, we have the following

Corollary 1.6 *Suppose that* $(M, g(t))_{t \in [0,T]}$ *is a solution to the geometric flow* (1.1) *on* M*, satisfying* $0 \leq S_{g(t)} \leq K_2 g(t)$*,* $-K_3 g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$*,* $|\nabla_{g(t)} \underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$*, for some* $K_2, K_3, K_4 > 0$*. Let* $h(x,t)$ *be a nonnegative function defined on* $M \times [0,T]$ *which is* C^2 *in* x *and* C^1 *in* t*,* $\Delta_{g(t)} h + h_t - 2C_{n,p} p \frac{|\nabla_{g(t)} h|^2_{g(t)}}{h} \geq 0$ *on* $M \times [0,T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p > 1$ *and* $C_{n,p} = n$ *if* $p \leq 1$*), and* $0 < p \leq \frac{2n}{2n-1}$ *(*$n \geq 3$*). If* u *is a positive solution* Eq. 1.2*, then*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\left[-\frac{1}{2p} \min_{\gamma \in \Theta(x_1,t_1,x_2,t_2)} \int_{t_1}^{t_2} |\dot{\gamma}(t)|^2_{g(t)} dt \right.$$
$$\left. - 2n(t_2 - t_1) \left(\frac{K_2}{p} + \frac{K_3 + K_4}{2-p} + \frac{\sqrt{2n K_4}}{p} \right) \right]$$

for some positive constant C *depending only on* n*, where* $(x_1, t_1), (x_2, t_2) \in M \times [0, T]$ *with* $t_1 < t_2$*.*

2 Auxiliary Lemmas

Suppose u is a positive solution of Eq. 1.1, and as in [4], we introduce a function

$$W = u^{-q}, \quad (2.1)$$

where q is a positive constant to be determined later. For convenience, we always omit time variable t and write $\mathcal{Q}_t$ for the partial derivative of $\mathcal{Q}$ relative to t. For example, throughout this paper, $\Delta, \nabla, |\cdot|$ mean the correspondence quantities with respect to $g(t)$. Write

$$\Box := \Delta - \partial_t.$$

A simple computation shows that

$$\begin{aligned} \nabla W &= -qu^{-q-1}\nabla u, \quad |\nabla W|^2 = q^2 u^{-2q-2}|\nabla u|^2, \\ W_t &= -qu^{-q-1}u_t, \quad \Delta W = q(q+1)u^{-q-2}|\nabla u|^2 - qu^{-q-1}\Delta u. \end{aligned}$$

The relation (2.1) yields (see [4, 6])

$$|\nabla u|^2 = \frac{|\nabla W|^2}{q^2 W^{2+2/q}}, \quad u_t = -\frac{W_t}{qW^{1+1/q}}, \tag{2.2}$$

and hence

$$\Box W = \frac{q+1}{q}\frac{|\nabla W|^2}{W} + qhW^{1+\frac{1-p}{q}}. \tag{2.3}$$

Since $|\nabla W|^2/W^2 = q^2|\nabla u|^2/u^2$ and $hW^{(1-p)/q} = hu^{p-1}$, we consider again the same quantities as in [4, 6],

$$F_0 := \frac{|\nabla W|^2}{W^2} + \alpha h W^{(1-p)/q} = |\nabla \ln W|^2 + \alpha h W^{(1-p)/q}, \tag{2.4}$$

$$F_1 := \frac{W_t}{W} = \partial_t \ln W, \tag{2.5}$$

$$F := F_0 + \beta F_1. \tag{2.6}$$

Here α, β are two positive constants to be fixed later.

Introduce a 1-form $\underline{\boldsymbol{S}}_{g(t)}$ defined by

$$\underline{\boldsymbol{S}}_{g(t)} := \mathrm{div}_{g(t)} S_{g(t)} - \frac{1}{2}\nabla_{g(t)}\left(\mathrm{tr}_{g(t)} S_{g(t)}\right). \tag{2.7}$$

Locally, one has

$$\underline{\boldsymbol{S}}_i = \nabla^j S_{ij} - \frac{1}{2}\nabla_i\left(\mathrm{tr}_{g(t)} S_{g(t)}\right).$$

For example, if $S_{ij} = R_{ij}$, that is, Eq. 1.1 is the Ricci flow, we arrive at

$$\underline{\boldsymbol{S}}_i = \nabla^j R_{ij} - \frac{1}{2}\nabla_i R_{g(t)} = \frac{1}{2}\nabla_i R_{g(t)} - \frac{1}{2}\nabla_i R_{g(t)} = 0$$

by the second contracted Bianchi identity. Thus, the quantity $\underline{\boldsymbol{S}}_{g(t)}$ measures whether S_{ij} satisfies the second contracted Bianchi identity.

A analogous quantity like (2.7) also naturally appears in the general relativity, see, for example, Proposition 13.3 in [8].

Lemma 2.1 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the geometric flow* (1.1) *on M. If u is a positive solution of* (1.2)*, then*

$$\begin{aligned} \Box F_1 &= \frac{2}{q}\langle \nabla F_1, \nabla \ln W\rangle + (1-p)hW^{\frac{1-p}{q}}\frac{W_t}{W} + qh_t W^{(1-p)/q} \\ &\quad + 2\left(1+\frac{1}{q}\right)S(\nabla \ln W, \nabla \ln W) - 2\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle - \frac{2\left\langle S, \nabla^2 W\right\rangle}{W}. \end{aligned} \tag{2.8}$$

Here div and tr are respectively divergence operator and trace operator of $g(t)$.

Proof As in [6], we have

$$\Box F_1 = \frac{\Delta W_t - W_{tt}}{W} - \frac{2\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{W_t(\Delta W - W_t)}{W^2} + \frac{2|\nabla W|^2 W_t}{W^3}. \tag{2.9}$$

Since $g(t)$ evolves under the geometric flow (1.1), it follows that

$$\begin{aligned}(\Delta W)_t &= \partial_t\left(g^{ij}\nabla_i\nabla_j W\right) = \left(\partial_t g^{ij}\right)\nabla_i\nabla_j W + g^{ij}\partial_t\left(\partial_i\partial_j W - \Gamma^k_{ij}\partial_k W\right)\\ &= 2S_{ij}\nabla^i\nabla^j W + \Delta(W_t) - g^{ij}\partial_k W\partial_t\Gamma^k_{ij}\\ &= \Delta(W_t) + 2\left\langle S, \nabla^2 W\right\rangle + 2\langle \underline{\mathbf{S}}, \nabla W\rangle\end{aligned}$$

using the fact that $g^{ij}\partial_t\Gamma^k_{ij} = -2\nabla^j S_j{}^k + \nabla^k(\mathrm{tr}(S)) = -2\underline{\mathbf{S}}^k$. The term $\Delta W_t - W_{tt} = (\Delta W - W_t)_t - 2\langle S, \nabla^2 W\rangle - 2\langle \underline{\mathbf{S}}, \nabla W\rangle$ can be simplified as [6] into

$$\begin{aligned}\Delta W_t - W_{tt} &= 2\left(1+\frac{1}{q}\right)\frac{\langle\nabla W, \nabla W_t\rangle}{W} - \left(1+\frac{1}{q}\right)\frac{|\nabla W|^2 W_t}{W^2} + q h_t W^{1+\frac{1-p}{q}}\\ &\quad + \left(1+\frac{1}{q}\right)\frac{2S(\nabla W, \nabla W)}{W} - 2\langle \underline{\mathbf{S}}, \nabla W\rangle\\ &\quad + h(q+1-p)W^{\frac{1-p}{q}} W_t - 2\left\langle S, \nabla^2 W\right\rangle.\end{aligned}$$

Plugging it into Eq. 2.9 yields

$$\begin{aligned}\Box F_1 &= \frac{2}{q}\frac{\langle\nabla W, \nabla W_t\rangle}{W^2} - \frac{2}{q}\frac{|\nabla W|^2 W_t}{W^3} + (1-p)hW^{\frac{1-p}{q}-1}W_t + q h_t W^{\frac{1-p}{q}}\\ &\quad + \left(1+\frac{1}{q}\right)\frac{2S(\nabla W, \nabla W)}{W} - \frac{2\left\langle S, \nabla^2 W\right\rangle}{W} - \frac{2\langle \underline{\mathbf{S}}, \nabla W\rangle}{W}.\end{aligned}$$

The desired (2.8) immediately follows. □

Similarly, we can find the evolution equation of Eq. 2.5.

Lemma 2.2 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the geometric flow* (1.1) *on M. If u is a positive solution of* Eq. 1.2*, then*

$$\begin{aligned}\Box F_0 &\geq 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle\nabla F_0, \nabla\ln W\rangle\\ &\quad - \frac{2\alpha p}{q}W^{\frac{1-p}{q}}\langle\nabla\ln W, \nabla h\rangle + \alpha W^{\frac{1-p}{q}}(\Delta h - h_t) + \frac{2(Ric - S)(\nabla W, \nabla W)}{W^2}\\ &\quad + (1-p)\left(2-\frac{\alpha p}{q^2}\right)hW^{\frac{1-p}{q}}|\nabla\ln W|^2 + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}}\end{aligned} \tag{2.10}$$

where $\epsilon \in (0, 1]$ is any given constant.

Proof Recall from [6] that ΔF_0 satisfies

$$\begin{aligned}\Delta F_0 = & \frac{2|\nabla^2 W|^2}{W^2} + \frac{2\langle \nabla W, \Delta \nabla W\rangle}{W^2} - 8\frac{\langle \nabla^2 W, \nabla W \otimes \nabla W\rangle}{W^3} - \frac{2|\nabla W|^2 \Delta W}{W^3} \\ & + \frac{6|\nabla W|^4}{W^4} + \alpha W^{\frac{1-p}{q}} \Delta h + 2\alpha \left(\frac{1-p}{q}\right) W^{\frac{1-p}{q}-1} \langle \nabla W, \nabla h\rangle \\ & + \alpha \left(\frac{1-p}{q}\right)\left(\frac{1-p}{q} - 1\right) h W^{\frac{1-p}{q}} \frac{|\nabla W|^2}{W^2} + \alpha \left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-1} \Delta W.\end{aligned} \tag{2.11}$$

On the other hand, the time derivative of F_0 equals

$$\begin{aligned}\partial_t F_0 = & \frac{2\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{2|\nabla W|^2 W_t}{W^3} + \alpha h_t W^{\frac{1-p}{q}} \\ & + \alpha \left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-1} W_t + \frac{2S(\nabla W, \nabla W)}{W^2}.\end{aligned} \tag{2.12}$$

From Eqs. 2.11, 2.12 and the Ricci identity $\Delta \nabla_i W = \nabla_i \Delta W + R_{ij} \nabla^j W$, we have

$$\begin{aligned}\Box F_0 = & \frac{2\langle \nabla W, \nabla(\Delta W - W_t)\rangle}{W^2} - \frac{2|\nabla W|^2 (\Delta W - W_t)\rangle}{W^3} \\ & + \left(\frac{2|\nabla^2 W|^2}{W^2} - \frac{8\langle \nabla^2 W, \nabla W \otimes \nabla W\rangle}{W^3} + \frac{6|\nabla W|^4}{W^4}\right) \\ & + \alpha W^{\frac{1-p}{q}} (\Delta h - h_t) + \alpha \left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-1} (\Delta W - W_t) \\ & + 2\alpha \left(\frac{1-p}{q}\right) W^{\frac{1-p}{q}-1} \langle \nabla W, \nabla h\rangle + \frac{2(\mathrm{Ric} - S)(\nabla W, \nabla W)}{W^2} \\ & + \alpha \left(\frac{1-p}{q}\right)\left(\frac{1-p}{q} - 1\right) h W^{\frac{1-p}{q}} \frac{|\nabla W|^2}{W^2}.\end{aligned} \tag{2.13}$$

The following argument is the same as Lemma 2.2 in [6]. □

Combing Lemma 2.1 with Lemma 2.2, we get

Proposition 2.3 Suppose that $(M, g(t))_{t \in [0,T]}$ is a complete solution to the geometric flow (1.1) on M. If u is a positive solution of Eq. 1.2, Define

$$W = u^{-q}, \quad F = \frac{|\nabla W|^2}{W^2} + \alpha h W^{\frac{1-p}{q}} + \beta \frac{W_t}{W}.$$

Then for all $\epsilon \in (0, 1]$ we have

$$\begin{aligned}\Box F \geq & 2(1-\epsilon) \left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1 - \frac{1}{\epsilon}\right) |\nabla \ln W|^4 + \frac{2}{q} \langle \nabla F, \nabla \ln W\rangle \\ & + 2\beta \left(1 + \frac{1}{q}\right) S(\nabla \ln W, \nabla \ln W) - \frac{2\alpha p}{q} W^{\frac{1-p}{q}} \langle \nabla \ln W, \nabla h\rangle \\ & + (1-p)\left(2 - \frac{\alpha p}{q^2}\right) h W^{\frac{1-p}{q}} |\nabla \ln W|^2 + W^{\frac{1-p}{q}} [\alpha \Delta h + h_t (q\beta - \alpha)] \\ & + \alpha (1-p) h^2 W^{\frac{2(1-p)}{q}} + \beta (1-p) h W^{\frac{1-p}{q}} \frac{W_t}{W} - 2\beta \left\langle S, \frac{\nabla^2 W}{W}\right\rangle \\ & + 2\,(\mathrm{Ric} - S)\,(\nabla \ln W, \nabla \ln W) - 2\beta \left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle\end{aligned} \tag{2.14}$$

3 Two Special Cases

As in [6], we consider two special cases. The first special case of Eq. 2.14 is to choose

$$\beta := \frac{\alpha}{q}, \quad \alpha = \frac{kq^2}{p}. \tag{3.1}$$

Then $q\beta - \alpha = 0$ so that Eq. 2.14 becomes

$$\begin{aligned}\Box F \geq\ & 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2\alpha(1+q)}{q^2} S(\nabla \ln W, \nabla \ln W) \\ & + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle - \frac{2\alpha p}{q} W^{\frac{1-p}{q}} \langle \nabla \ln W, \nabla h\rangle \\ & + (1-p)\left(2 - \frac{\alpha p}{q^2}\right) h W^{\frac{1-p}{q}} |\nabla \ln W|^2 + \alpha W^{\frac{1-p}{q}} \Delta h \\ & + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}} + \frac{\alpha(1-p)}{q} h W^{\frac{1-p}{q}} \frac{W_t}{W} - \frac{2\alpha}{q}\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \\ & + 2(\mathrm{Ric} - S)(\nabla \ln W, \nabla \ln W) - \frac{2\alpha}{q}\left\langle \underline{S}, \nabla \ln W\right\rangle.\end{aligned} \tag{3.2}$$

Recall the inequality in [6] (cf. (3.4))

$$2\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{2\alpha}{q}\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \geq 2\left[\frac{a\alpha}{q}\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{\alpha}{4bq}|S|^2\right], \tag{3.3}$$

for any positive real numbers a, b satisfying $a+b = \frac{q}{\alpha}$, with the equality if $S = 2b\nabla^2 W/W$. Using the inequality $|\nabla^2 W|^2 \geq (\Delta W)^2/n$, we conclude from Eqs. 3.2 and 3.3 that

$$\begin{aligned}\Box F \geq\ & \frac{2}{n}\left(\frac{a\alpha}{q} - \epsilon\right)\left|\frac{\Delta W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\ & + \frac{2\alpha(1+q)}{q^2} S(\nabla \ln W, \nabla \ln W) - \frac{2\alpha p}{q} W^{\frac{1-p}{q}} \langle \nabla \ln W, \nabla h\rangle \\ & + \alpha W^{\frac{1-p}{q}} \Delta h + (1-p)\left(2 - \frac{\alpha p}{q^2}\right) h W^{\frac{1-p}{q}} |\nabla \ln W|^2 \\ & + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}} + \frac{\alpha(1-p)}{q} h W^{\frac{1-p}{q}} \frac{W_t}{W} - \frac{\alpha}{2bq}|S|^2 \\ & + 2(\mathrm{Ric} - S)(\nabla \ln W, \nabla \ln W) - \frac{2\alpha}{q}|\underline{S}||\nabla \ln W|.\end{aligned} \tag{3.4}$$

By Eq. 2.3, we get

$$\frac{\Delta W}{W} = \frac{q+1}{q}\frac{|\nabla W|^2}{W^2} + \frac{W_t}{W} + qhW^{\frac{1-p}{q}} = \frac{q}{\alpha}F + \left(\frac{1+q}{q} - \frac{q}{\alpha}\right)|\nabla \ln W|^2.$$

Because of the assumption $\alpha = kq^2/p$, we arrive at

$$\frac{\Delta W}{W} = \frac{p}{kq}F + \left(\frac{1+q-p/k}{q}\right)|\nabla \ln W|^2 \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain

Lemma 3.1 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the geometric flow* (1.1) *on M. If u is a positive solution of* Eq. 1.2*, then*

$$\begin{aligned}\Box F \geq\ & \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle + \frac{2}{n}\left(\frac{akq}{p} - \epsilon\right)\frac{p^2}{k^2q^2}F^2 + (1-p)hW^{\frac{1-p}{q}}F \\ & + \frac{4p}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)F|\nabla \ln W|^2 - \frac{kq}{2bp}|S|^2 \\ & + 2\left[\frac{1}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 \\ & + \frac{2k(1+q)}{p}S(\nabla \ln W, \nabla \ln W) - 2qkW^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\ & + \frac{kq^2}{p}W^{\frac{1-p}{q}}\Delta h + (1-p)(1-k)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 \\ & + 2(Ric - S)(\nabla \ln W, \nabla \ln W) - \frac{2kq}{p}|\underline{\boldsymbol{S}}||\nabla \ln W|,\end{aligned}$$

where ϵ is a positive real number satisfying $\epsilon \in (0, 1]$, p, q, k, a, b are positive real numbers such that $a + b = p/kq$, and

$$W = u^{-q}, \quad F = \frac{|\nabla W|^2}{W^2} + \frac{kq^2}{p}hW^{\frac{1-p}{q}} + \frac{kq}{p}\frac{W_t}{W}.$$

The second special case is to choose

$$\beta := \frac{2\alpha}{q}, \quad \alpha = \frac{q^2}{p} \tag{3.6}$$

in (2.14). Then the inequality (2.14) becomes

$$\begin{aligned}\Box F \geq\ & 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1 - \frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\ & + \frac{4(1+q)}{p}S(\nabla \ln W, \nabla \ln W) + (1-p)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 \\ & + \frac{q^2}{p}W^{\frac{1-p}{q}}(\Delta h + h_t) + q^2\left(\frac{1}{p} - 1\right)h^2W^{\frac{2(1-p)}{q}} - 2qW^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\ & + 2q\left(\frac{1}{p} - 1\right)hW^{\frac{1-p}{q}}\frac{W_t}{W} - \frac{4q}{p}\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \\ & + 2\,(\mathrm{Ric} - S)\,(\nabla \ln W, \nabla \ln W) - \frac{4q}{p}\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle.\end{aligned} \tag{3.7}$$

For any positive real numbers a, b with $a + b = q/2\alpha = p/2q$, we have (cf. [6])

$$2\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{4q}{p}\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \geq \frac{4aq}{p}\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{q}{bp}|S|^2. \tag{3.8}$$

Together (3.7), (3.8) with $|\nabla^2 W|^2 \geq (\Delta W)^2/n$ implies

$$\begin{aligned}\Box F \geq\ & \frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left|\frac{\Delta W}{W}\right|^2 + 2\left(1 - \frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\ & + \frac{4(1+q)}{p} S(\nabla \ln W, \nabla \ln W) + (1-p) h W^{\frac{1-p}{q}} |\nabla \ln W|^2 - \frac{q}{bp}|S|^2 \\ & + \frac{q^2}{p} W^{\frac{1-p}{q}}(\Delta h + h_t) + q^2\left(\frac{1}{p} - 1\right) h^2 W^{\frac{2(1-p)}{q}} + 2q\left(\frac{1}{p} - 1\right) h W^{\frac{1-p}{q}} \frac{W_t}{W} \\ & - 2q W^{\frac{1-p}{q}} \langle \nabla \ln W, \nabla h\rangle + 2(\mathrm{Ric} - S)(\nabla \ln W, \nabla \ln W) - \frac{4q}{p}\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle.\end{aligned} \tag{3.9}$$

Substituting the identity (by Eq. 2.3)

$$\frac{\Delta W}{W} = \frac{p}{2q} F + \frac{q}{2} h W^{\frac{1-p}{q}} + \left(\frac{1 + q - p/2}{q}\right)|\nabla \ln W|^2.$$

into Eq. 3.9 yields

$$\begin{aligned}\Box F \geq\ & \frac{1}{2n}\left(\frac{2aq}{p} - \epsilon\right)\frac{p^2}{q^2} F^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right) F |\nabla \ln W|^2 \\ & + \left[\frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2 + 2\left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{q}{bp}|S|^2 \\ & + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle + \frac{4(1+q)}{p} S(\nabla \ln W, \nabla \ln W) + \frac{q^2}{p} W^{\frac{1-p}{q}}(\Delta h + h_t) \\ & + \frac{q^2}{2n}\left(\frac{2aq}{p} - \epsilon\right) h^2 W^{\frac{2(1-p)}{q}} + \left[\frac{p}{n}\left(\frac{2aq}{p} - \epsilon\right) + (1-p)\right] h W^{\frac{1-p}{q}} F \\ & + \frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right) h W^{\frac{1-p}{q}} |\nabla \ln W|^2 - 2q W^{\frac{1-p}{q}} \langle \nabla \ln W, \nabla h\rangle \\ & - \frac{4q}{p}\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle + 2(\mathrm{Ric} - S)(\nabla \ln W, \nabla \ln W).\end{aligned}$$

The term $2qW^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle$ is bounded from above by (where we assume that h is nonnegative)

$$\eta h W^{\frac{1-p}{q}} |\nabla \ln W|^2 + \frac{q^2}{\eta} W^{\frac{1-p}{q}} \frac{|\nabla h|^2}{h}$$

for any given $\eta > 0$. Therefore

$$\begin{aligned}\Box F \geq & \frac{1}{2n}\left(\frac{2aq}{p}-\epsilon\right)\frac{p^2}{q^2}F^2+\frac{2p}{nq^2}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)F|\nabla \ln W|^2 \\ & +\left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2+2\left(1-\frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4-\frac{q}{bp}|S|^2 \\ & +\frac{2}{q}\langle \nabla F, \nabla \ln W\rangle+\frac{4(1+q)}{p}S(\nabla \ln W, \nabla \ln W) \\ & +\frac{q^2}{p}W^{\frac{1-p}{q}}\left(\Delta h+h_t-\frac{p}{\eta}\frac{|\nabla h|^2}{h}\right) \\ & +\frac{q^2}{2n}\left(\frac{2aq}{p}-\epsilon\right)h^2W^{\frac{2(1-p)}{q}}+\left[\frac{p}{n}\left(\frac{2aq}{p}-\epsilon\right)+(1-p)\right]hW^{\frac{1-p}{q}}F \\ & +\left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)-\eta\right]hW^{\frac{1-p}{q}}|\nabla \ln W|^2 \\ & +2\left(\mathrm{Ric}-S\right)(\nabla \ln W, \nabla \ln W)-\frac{4q}{p}\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle .\end{aligned} \tag{3.10}$$

By choosing the same conditions on positive real numbers p, q, a, b, ϵ as in [6], Lemma 3.2, we obtain

Lemma 3.2 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the geometric flow (1.1) on an n-dimensional manifold M. Let $h(x,t)$ be a nonnegative function defined on $M\times[0,T]$ which is C^2 in x and C^1 in t, and $\Delta_{g(t)}h+h_t-\frac{p}{\eta}\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h}\geq 0$ on $M\times[0,T]$ for some $p, \eta>0$. Let p, q, a, b, ϵ be positive real numbers satisfying*

- (i) *q is a priori given positive real number;*
- (ii) *$0<\epsilon\leq 1$;*
- (iii) *$a+b=p/2q$;*
- (iv) *either $0<\epsilon\leq\frac{2aq-n(p-1)}{p}$ and $1<p<1+\frac{2aq}{n}$ (then we choose $0<\eta\leq\frac{p-1}{2p}$), or $0<p\leq 1$ and $\frac{2aq}{p}-\epsilon>0$ (then we choose $0<\eta\leq\frac{1}{n}(\frac{2aq}{p}-\epsilon)$).*

If u is a positive solution of Eq. 1.2*, $F(x_0,t_0)>0$ for some point $(x_0,t_0)\in M\times[0,T]$, where*

$$F=\frac{|\nabla W|^2}{W^2}+\frac{q^2}{p}hW^{\frac{1-p}{q}}+\frac{2q}{p}\frac{W_t}{W},$$

then at the point (x_0,t_0) we have

$$\begin{aligned}\Box F \geq & \frac{1}{2n}\left(\frac{2aq}{p}-\epsilon\right)\frac{p^2}{q^2}F^2+\frac{2p}{nq^2}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)F|\nabla \ln W|^2 \\ & +\left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2+2\left(1-\frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4-\frac{q}{bp}|S|^2 \\ & +\frac{2}{q}\langle \nabla F, \nabla \ln W\rangle+\frac{4(1+q)}{p}S(\nabla \ln W, \nabla \ln W) \\ & +2\left(Ric-S\right)(\nabla \ln W, \nabla \ln W)-\frac{4q}{p}\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle .\end{aligned} \tag{3.11}$$

4 Gradient Estimates and Some Relative Results

In this section, we will use previous lemmas to get the gradient estimates for the positive solution of the Eq. 1.2 under the geometric flow.

Theorem 4.1 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.1) on M with $-K_1 g(t) \le S_{g(t)} \le K_2 g(t)$, $-K_3 g(t) \le Ric_{g(t)} - S_{g(t)} \le K_3 g(t)$, and $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \le K_4$ for some $K_1, K_2, K_3, K_4 > 0$ on $B_{2R,T}$, with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, satisfying $\Delta_t h \ge -\theta$ and $|\nabla_t h|_t \le \gamma$ on $B_{2R,T} \times [0,T]$ for some nonnegative constants θ and γ. If $u(x,t)$ is a positive smooth solution* Eq. 1.2 *on $M \times [0,T]$, then*

(i) *for $0 < p < 1$, we have*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le \frac{C_1}{p^2 t} + \frac{n(1-p)}{p^2}M_1M_2 + \frac{n[3K_1 + 2(K_3+K_4)p]}{2p^2(1-p)} + \frac{C_1}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{n}{p(1-p)}\right) + \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{n/K_1}}{p}\gamma M_2 + \frac{n}{p^2}\sqrt{\frac{K_4}{2n}}, \tag{4.1}$$

where C_1 is a positive constant depending only on n and

$$M_1 := \max_{B_{2R,T}} h_-, \quad M_2 := \max_{B_{2R,T}} u^{p-1}, \quad h_- := \max(-h, 0).$$

(ii) *for $p \ge 1$, we have*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{t} \le \frac{k^2C_2}{p^2 t} + \frac{nk^2(p-1)}{p^2}M_4M_5 + \frac{k^3 n}{k-p}M_3M_4 + \frac{k^2C_2}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{k^2 n}{p(k-p)}\right) + \frac{2k^3 n}{(k-p)p^2}\left[K_1 + \frac{p}{k}(K_3+K_4)\right] + \frac{k^2\sqrt{n}\gamma}{p}M_4 + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4} + \frac{k^2 n}{p^2}\left(\overline{K} + \sqrt{\frac{K_4}{2n}}\right), \tag{4.2}$$

where $k > p$, C_2 is a positive constant depending only on n and

$$M_3 := \max_{B_{2R,T}} h_-, \quad M_4 := \max_{B_{2R,T}} u^{p-1}, \quad M_5 := \max_{B_{2R,T}} h.$$

Proof The proof is along the outline in [1, 4, 5] and is identically as it in [6]. For completeness, we give a proof here. Firstly, we introduce a cut-off function (see [1, 3, 5–7, 10]) on $B_{\rho,T} := \{(x,t) \in M \times [0,T] : \mathrm{dist}_{g(t)}(x, x_0) < \rho\}$, where $\mathrm{dist}_{g(t)}(x, x_0)$ stands for the distance between x and x_0 with respect to the metric $g(t)$, which satisfies a basic analytical result stated in the following lemma.

Lemma 4.2 *Given $\tau \in (0,T]$, there exists a smooth function $\overline{\Psi} : [0,\infty) \times [0,T] \to R$ satisfying the following requirements:*

(1) *The support of* $\overline{\Psi}(r,t)$ *is a subset of* $[0,\rho]\times[0,T]$, $0\le\overline{\Psi}(r,t)\le 1$ *in* $[0,\rho]\times[0,T]$, *and* $\overline{\Psi}(r,t)=1$ *holds in* $[0,\frac{\rho}{2}]\times[\tau,T]$.

(2) $\overline{\Psi}$ *is decreasing as a radial function in the spatial variables.*

(3) *The estimate* $|\partial_t\overline{\Psi}|\le\frac{\overline{C}}{\tau}\overline{\Psi}^{1/2}$ *is satisfied on* $[0,\infty)\times[0,T]$ *for some* $\overline{C}>0$.

(4) *The inequalities* $-\frac{C_\alpha}{\rho}\overline{\Psi}^\alpha\le\partial_r\overline{\Psi}\le 0$ *and* $|\partial_r^2\overline{\Psi}|\le\frac{C_\alpha}{\rho^2}\overline{\Psi}^\alpha$ *hold on* $[0,\infty)\times[0,T]$ *for every* $\alpha\in(0,1)$ *with some constant* C_α *dependent on* α.

Proof See [1]. □

For the fixed $\tau\in(0,T]$, choose the above cut-off function $\overline{\Psi}$. Define $\Psi: M\times[0,T]\to\mathbf{R}$ by setting

$$\Psi(x,t):=\overline{\Psi}\left(\mathrm{dist}_{g(t)}(x,x_0),t\right)$$

with $\rho:=2R$ in Lemma 4.2. Consider the function $\varphi(x,t)=tF(x,t)$. Using the argument of Calabi [2], we may assume that the function $G(x,t):=\varphi(x,t)\Psi(x,t)$ with support in $B_{2R,T}$ is smooth. Let (x_0,t_0) be the point where G achieves its maximum in the set $\{(x,t):0\le t\le\tau,\ d_t(x,x_0)\le\rho\}$. Without loss of generality, assuming $G(x_0,t_0)>0$, we have

$$\nabla G=0,\quad \partial_t G\ge 0,\quad \Delta G\le 0$$

at (x_0,t_0). Now apply Lemma 4.2 and the Laplacian comparison theorem (observe that the hypothesis implies that $-(K_1+K_3)g(t)\le\mathrm{Ric}_{g(t)}\le(K_2+K_3)g(t)$), we have

$$\begin{aligned}
\frac{|\nabla\Psi|^2}{\Psi}&\le\frac{C_{1/2}^2}{\rho^2},\\
\Delta\Psi&\ge-\frac{C_{1/2}\Psi^{1/2}}{\rho^2}-\frac{C_{1/2}\Psi^{1/2}}{\rho}(n-1)\sqrt{K_1+K_3}\coth(\sqrt{K_1+K_3}\rho)\\
&\ge-\frac{d_1}{\rho^2}-\frac{d_1\Psi^{1/2}}{\rho}\sqrt{K_1+K_3},\\
-\partial_t\Psi&\ge-\frac{\overline{C}\Psi^{1/2}}{\tau}-C_{1/2}\overline{K}\Psi^{1/2}
\end{aligned}$$

where $C_{1/2}$, $\overline{C}$ and d_1 are positive constants depending only on n. It is easy to show that

$$0\ge\Box G=\varphi\Box\Psi+2\langle\nabla\varphi,\nabla\Psi\rangle+\Psi\Box\varphi \tag{4.3}$$

at (x_0,t_0). Setting $p\in(0,1)$ and $k=1$ in Lemma 3.1, we obtain from $\Box\varphi=t\Box F-\varphi/t$ that

$$\begin{aligned}
\Box\varphi\ge\ &\frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla\ln W|^2-\frac{q^2\theta t}{p}W^{\frac{1-p}{q}}\\
&+2t\left[\frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q}\right)^2+\left(1-\frac{1}{\epsilon}\right)\right]|\nabla\ln W|^4-\frac{nq\overline{K}^2}{2bp}t\\
&-\frac{2(1+q)K_1t+2pK_3t}{p}|\nabla\ln W|^2+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle+(1-p)hW^{\frac{1-p}{q}}\varphi\\
&-2q\gamma tW^{\frac{1-p}{q}}|\nabla\ln W|-\frac{2qt}{p}K_4|\nabla\ln W|-\frac{\varphi}{t}.
\end{aligned}\tag{4.4}$$

According to Hölder's inequality,

$$2q\gamma tW^{\frac{1-p}{q}}|\nabla \ln W| \le \frac{(1+q)K_1 t}{p}|\nabla \ln W|^2 + \frac{pq^2\gamma^2}{(1+q)K_1}tW^{2\frac{1-p}{q}}$$
$$\frac{2qt}{p}K_4|\nabla \ln W| \le 2K_4 t|\nabla \ln W|^2 + \frac{q^2K_4}{2p^2}t$$

we have

$$\begin{aligned}\Box\varphi \ge\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2 + \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla \ln W|^2 - \frac{q^2\theta t}{p}W^{\frac{1-p}{q}}\\ &+ 2t\left[\frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q}\right)^2 + \left(1-\frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{nq\overline{K}^2}{2bp}t\\ &- \frac{3(1+q)K_1 t + 2p(K_3+K_4)t}{p}|\nabla \ln W|^2 + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle - \frac{\varphi}{t}\\ &+ (1-p)hW^{\frac{1-p}{q}}\varphi - \frac{pq^2\gamma^2}{(1+q)K_1}tW^{2\frac{1-p}{q}} - \frac{q^2K_4}{2p^2}t.\end{aligned}$$

Using Hölder's inequality again we have

$$\begin{aligned}&\frac{3(1+q)K_1 t + 2p(K_3+K_4)t}{p}|\nabla \ln W|^2\\ &\le \frac{n[3(1+q)K_1 + 2p(K_3+K_4)]^2 t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2\\ &\quad + \frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1-p}{q}\right)^2 2t|\nabla \ln W|^4.\end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4) yields

$$\begin{aligned}\Box\varphi \ge\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2 + \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla \ln W|^2\\ &+ 2t\left[\frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{2-2p+q}{q}\right) + \left(1-\frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{nq\overline{K}^2}{2bp}t\\ &- \frac{n[3(1+q)K_1 + 2p(K_3+K_4)p]^2 t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2 + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle - \frac{\varphi}{t}\\ &+ (1-p)hW^{\frac{1-p}{q}}\varphi - \frac{pq^2\gamma^2}{(1+q)K_1}tW^{2\frac{1-p}{q}} - \frac{q^2\theta t}{p}W^{\frac{1-p}{q}} - \frac{q^2K_4}{2p^2}t.\end{aligned}$$

Take $\epsilon \in (0, 1/4)$ and choose q so that $1/q \ge n(1-\epsilon)/2\epsilon^2(1-p)$. For such a pair (p, q), we may choose a positive real number a such that $aq/p \ge 2\epsilon$ and then the condition $a + b = p/q$ holds for some $b > 0$ (because in this case $0 < aq/p < 1$). Under the above assumption, we have (as in [6])

$$\frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{2-2p+q}{q}\right) + \left(1-\frac{1}{\epsilon}\right) \ge 0.$$

and hence

$$\begin{aligned}\Box\varphi \geq\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla \ln W|^2\\ & -\frac{nq\overline{K}^2}{2bp}t-\frac{n[3(1+q)K_1+2p(K_3+K_4)p]^2t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2-\frac{\varphi}{t}-\frac{q^2K_4}{2p^2}t \quad (4.6)\\ & -\frac{pq^2\gamma^2}{(1+q)K_1}tW^{2\frac{1-p}{q}}-\frac{q^2\theta t}{p}W^{\frac{1-p}{q}}+\frac{2}{q}\langle\nabla\varphi,\nabla \ln W\rangle+(1-p)hW^{\frac{1-p}{q}}\varphi.\end{aligned}$$

Plugging (4.6) into (4.3) and using the estimate for $\Box\Psi$, we arrive at, where $\rho := 2R$,

$$\begin{aligned}0 \geq\ & \varphi\Box\Psi-\frac{2\varphi}{\Psi}|\nabla\Psi|^2+\Psi\Box\varphi\\ \geq\ & \varphi d_1\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right)-\frac{2d_1}{\rho^2}\varphi+\Psi\Box\varphi\\ \geq\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\Psi\varphi^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi\Psi|\nabla \ln W|^2\\ & -\frac{n[3(1+q)K_1+2(K_3+K_4)p]^2t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2-\left(\frac{\theta M_2}{p}+\frac{pM_2^2\gamma^2}{(1+q)K_1}\right)q^2t\Psi\\ & -(1-p)M_1M_2\varphi\Psi-\frac{nq\overline{K}^2}{2bp}t\Psi-\frac{\varphi\Psi}{t}+\varphi d_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right)\\ & -\frac{K_4q^2}{2p^2}t\Psi-\frac{2}{q}\langle\nabla\Psi,\nabla \ln W\rangle\varphi,\end{aligned}$$

where d_1, d_2 are positive constants depending only on n, and

$$M_1 := \sup_{B_{2R,T}} h_-,\quad M_2 := \sup_{B_{2R,T}} u^{p-1}.$$

Multiplying the above inequality by Ψ on both sides, we get, where $G = \varphi\Psi$

$$\begin{aligned}0 \geq\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)G^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)G\Psi|\nabla \ln W|^2\\ & -\frac{n[3(1+q)K_1+2(K_3+K_4)p]^2t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2-\left(\frac{\theta M_2}{p}+\frac{pM_2^2\gamma^2}{(1+q)K_1}\right)tq^2 \quad (4.7)\\ & -\frac{nq\overline{K}^2}{2bp}t-\frac{G}{t}+Gd_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right)-\frac{2}{q}\langle\nabla\Psi,\nabla \ln W\rangle G\\ & -(1-p)M_1M_2G-\frac{K_4q^2}{2p^2}t.\end{aligned}$$

Using Hölder's inequality

$$\begin{aligned}\frac{2}{q}\langle\nabla\Psi,\nabla \ln W\rangle G \leq\ & \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)G\Psi|\nabla \ln W|^2\\ & +\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G,\end{aligned}$$

the inequality (4.7) gives us the estimate (because $t \le \tau$)

$$
\begin{aligned}
0 \ge\ & \frac{2p^2}{nq^2}\left(\frac{aq}{p}-\epsilon\right)G^2-(1-p)M_1M_2Gt-d_3G \\
& -t\left[\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\right]d_3G \qquad (4.8)\\
& -t^2\left[\frac{n[3(1+q)K_1+2(K_3+K_4)p]^2}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2+\frac{q^2}{p}M_2\theta\right.\\
& \left.+\frac{q^2p}{(1+q)K_1}(M_2\gamma)^2+\frac{nq}{2bp}\overline{K}^2+\frac{K_4q^2}{2p^2}\right].
\end{aligned}
$$

for some positive constant d_3 depending only on n. The following inequality

$$
aG^2-bG-c\le 0 \quad (a,b,c>0)\Longrightarrow G\le\frac{b}{a}+\sqrt{\frac{c}{a}},
$$

implies

$$
\begin{aligned}
G\ \le\ & \frac{d_3+(1-p)M_1M_2t+td_3\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\right)}{\frac{2p^2}{nq^2}\left(\frac{aq}{p}-\epsilon\right)}\\
& +t\sqrt{\frac{\frac{n[3(1+q)K_1+2(K_3+K_4)p]^2}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2+\frac{q^2M_2\theta}{p}+\frac{q^2p(M_2\gamma)^2}{(1+q)K_1}+\frac{nq\overline{K}^2}{2bp}+\frac{K_4q^2}{2p^2}}{\frac{2p^2}{nq^2}\left(\frac{aq}{p}-\epsilon\right)}}.
\end{aligned}
$$

Recall the conditions on p, q, ϵ, a, b that

$$
0<p<1,\quad 0<\epsilon<\frac{1}{4},\quad \frac{1}{q}\ge\frac{n(1-\epsilon)}{2\epsilon^2(1-p)},\quad a+b=\frac{p}{q},\quad a\ge 2\epsilon\frac{p}{q}.
$$

Choose p, ϵ, q as above and

$$
a=\left(\frac{1}{2}+2\epsilon\right)\frac{p}{q},\quad b=\left(\frac{1}{2}-2\epsilon\right)\frac{p}{q}. \tag{4.9}
$$

The additional condition (4.9), plugging into the inequality for G, yields

$$
\begin{aligned}
G\ \le\ & \frac{tnq^2\left[\frac{d_3}{t}+(1-p)M_1M_2+d_3\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{2(1+2\epsilon)p(1-p)}\right)\right]}{p^2(1+2\epsilon)}\\
& +t\sqrt{\frac{nq^4}{p^2(1+2\epsilon)}\left(\begin{array}{c}\frac{n[3(1+q)K_1+2(K_3+K_4)p]^2}{4(1+2\epsilon)p^2(1-p)^2}+\frac{M_2\theta}{p}\\ +\frac{p(M_2\gamma)^2}{(1+q)K_1}+\frac{n\overline{K}^2}{p^2(1-4\epsilon)}+\frac{K_4}{2p^2}\end{array}\right)}
\end{aligned}
$$

at (x_0, t_0). Since

$$
G=tq^2\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{u}\right)\Psi
$$

and $q \le 2\epsilon^2(1-p)/n(1-\epsilon)$, it follows that, by letting $\epsilon \to 0$,

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le\ & \frac{d_4}{p^2 t} + \frac{n(1-p)}{p^2}M_1M_2 \\ & + \frac{d_4}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{n}{2p(1-p)}\right) \\ & + \frac{n}{p^2}\sqrt{\begin{array}{c}\frac{[3(1+q)K_1+2(K_3+K_4)p]^2}{4(1-p)^2} + \frac{p\theta}{n}M_2 \\ + \frac{(p\gamma)^2}{nK_1}M_2^2 + \overline{K}^2 + \frac{K_4}{2n}\end{array}}\end{aligned}$$

on $B_{R,\tau}$, for some positive constant d_4 depending only on n. Because $\tau \in (0,T]$ was arbitrary, we arrive at

$$\begin{aligned}\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le\ & \frac{d_4}{p^2 t} + \frac{n(1-p)}{p^2}M_1M_2 + \frac{n[3K_1 + 2(K_3+K_4)p]}{2p^2(1-p)} \\ & + \frac{d_4}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{n}{2p(1-p)}\right) \\ & + \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{\frac{n}{K_1}}}{p}\gamma M_2 + \frac{n}{p^2}\overline{K} + \frac{n}{p^2}\sqrt{\frac{K_4}{2n}},\end{aligned}$$

on $B_{R,T}$. Arranging terms yields (4.1).

When $p \ge 1$,applying Lemma 3.1, we have

$$\begin{aligned}\Box\varphi \ge\ & \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p} - \epsilon\right)\varphi^2 + \frac{4p}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi|\nabla \ln W|^2 - \frac{\varphi}{t} \\ & - \frac{kq^2\theta t}{p}W^{\frac{1-p}{q}} + 2t\left[\frac{1}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 \\ & - \frac{knq\overline{K}^2}{2bp}t + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle + (1-p)hW^{\frac{1-p}{q}}\varphi \\ & - 2qk\gamma tW^{\frac{1-p}{q}}|\nabla \ln W| + (1-p)(1-k)thW^{\frac{1-p}{q}}|\nabla \ln W|^2 \\ & - \frac{2k(1+q)K_1t + 2p(K_3+K_4)t}{p}|\nabla \ln W|^2 - \frac{k^2q^2K_4t}{2p^2},\end{aligned}$$

where $\epsilon \in (0,1]$ and p, q, k, a, b are positive real numbers such that $a + b = p/kq$ and $k \ge 1$. Define

$$M_3 := \max_{B_{2R,T}} h_-, \quad M_4 := \max_{B_{2R,T}} u^{p-1}, \quad M_5 := \max_{B_{2R,T}} h,$$

and

$$\begin{aligned}M_6 :=\ & \min_{q\ge 0}\min_{y\ge 0}\frac{1}{q^2}\Bigg\{2\left[\frac{1}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]y^2 \\ & - (p-1)(k-1)M_3M_4y - \frac{2k(1+q)K_1 + 2p(K_3+K_4)}{p}y - 2qkM_4\gamma y^{\frac{1}{2}}\Bigg\}.\end{aligned}$$

Observe that $M_6 \leq 0$. Therefore, we arrive at the following inequality

$$\begin{aligned}\Box\varphi \geq\ & \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi|\nabla\ln W|^2-\frac{\varphi}{t}\\ &-\frac{kq^2\theta t}{p}M_4+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle-(p-1)M_4M_5\varphi+M_6q^2t\\ &-\frac{k^2q^2K_4}{2p^2}t-\frac{knq\overline{K}^2}{2bp}t.\end{aligned}$$

As before, using $0=\nabla G=\Psi\nabla\varphi+\varphi\nabla\Psi$ at (x_0,t_0), we arrive at, where $\rho := 2R$,

$$\begin{aligned}0 &\geq \varphi\Box\Psi-2\varphi\frac{|\nabla\Psi|^2}{\Psi}+\Psi\Box\varphi\\ &\geq \varphi d_1\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right)-\frac{2d_1}{\rho^1}\varphi+\Psi\Box\varphi\\ &\geq \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)\Psi\varphi^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi\Psi|\nabla\ln W|^2\\ &\quad+M_6q^2\Psi t-\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle\varphi-\frac{kq^2\theta M_4}{p}\Psi t-(p-1)M_4M_5\Psi\varphi\\ &\quad-\frac{\Psi\varphi}{t}+\varphi d_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{t}-\overline{K}\right)-\left(\frac{k^2q^2K_4}{2p^2}+\frac{knq\overline{K}^2}{2bp}\right)\Psi t\end{aligned}$$

for some positive constants d_1, d_2. Multiplying the above inequality by Ψ on both sides, we get, where $G=\varphi\Psi$,

$$\begin{aligned}0 \geq\ & \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)G^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)G\Psi|\nabla\ln W|^2\\ &+M_6q^2t-\frac{kq^2\theta t}{p}M_4-(p-1)M_4M_5G-\left(\frac{k^2q^2K_4}{2p^2}+\frac{knq\overline{K}^2}{2bp}\right)t-\frac{G}{t} \qquad (4.10)\\ &+Gd_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right)-\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle G.\end{aligned}$$

Using Hölder's inequality, where we choose $akq > \epsilon p$ and $k+kq > p$,

$$\begin{aligned}\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle G \leq\ & \frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)G\Psi|\nabla\ln W|^2\\ &+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G,\end{aligned}$$

the inequality (4.10) gives the following estimate

$$
\begin{aligned}
0 \geq\ & \frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)G^2-(p-1)M_4M_5Gt-d_3G \\
& -t\left[\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\right]d_3G \qquad (4.11)\\
& -t^2\left(\frac{knq\overline{K}^2}{2bp}+\frac{k^2q^2K_4}{2p^2}+\frac{kq^2}{p}M_4\theta-M_6q^2\right)
\end{aligned}
$$

that is similar to Eq. 4.8 at (x_0, t_0), where d_3 is a positive constant. Hence

$$
\begin{aligned}
G \leq\ & \frac{d_3+(p-1)M_4M_5t+td_3\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\right)}{\frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)} \\
& +t\sqrt{\frac{\frac{knq\overline{K}^2}{2bp}+\frac{k^2q^2K_4}{2p^2}+\frac{kq^2}{p}M_4\theta-M_6q^2}{\frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)}}
\end{aligned}
$$

at (x_0, t_0). Finally, we obtain

$$
\begin{aligned}
G \leq\ & \frac{tnk^2q^2}{p^2}\left[\frac{d_3}{t}+(p-1)M_4M_5+d_3\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{k^2n}{2p(k+kq-p)}\right)\right] \\
& +tq^2\left[\frac{nk^2}{p^2}\left(\frac{k^2n\overline{K}^2}{p^2(1-2\epsilon)}+\frac{k^2K_4}{2p^2}+\frac{k}{p}M_4\theta-M_6\right)\right]^{1/2}
\end{aligned}
$$

by taking $a = (\epsilon + \frac{1}{2})\frac{p}{kq}$, $b = (\frac{1}{2} - \epsilon)\frac{p}{kq}$ with $\epsilon \in (0, 1/2)$ and $k \geq p$. As before, we conclude that

$$
\begin{aligned}
\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{t} \leq\ & \frac{k^2d_4}{p^2t}+\frac{nk^2(p-1)}{p^2}M_4M_5 \\
& +\frac{k^2d_4}{p^2}\left(\frac{1}{R^2}+\frac{\sqrt{K_1+K_3}}{R}+\overline{K}+\frac{k^2n}{2p(k-p)}\right) \\
& +\frac{k^2n}{p^2}\sqrt{-M_6\frac{p^2}{k^2n}+\frac{p\theta}{kn}M_4+\overline{K}^2+\frac{K_4}{2n}}
\end{aligned}
$$

on $B_{R,\tau}$, for some positive constant d_4 depending only on n. Because $\tau \in (0, T]$ was arbitrary, we arrive at

$$
\begin{aligned}
\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{t} \leq\ & \frac{k^2d_4}{p^2t}+\frac{nk^2(p-1)}{p^2}M_4M_5+\frac{k^2n}{p^2}\left(\overline{K}+\sqrt{\frac{K_4}{2n}}\right) \\
& +\frac{k^2d_4}{p^2}\left(\frac{1}{R^2}+\frac{\sqrt{K_1+K_3}}{R}+\overline{K}+\frac{k^2n}{2p(k-p)}\right) \\
& +\frac{k\sqrt{n}}{p}\sqrt{-M_6}+\left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4}
\end{aligned}
$$

on $B_{R,\tau}$. In the following we shall show that $-M_6 > 0$ is bounded from above by some constant. For any $q, y \geq 0$ we have

$$q^2 M_6 \geq \left[\frac{1}{n}\left(1+\frac{k-p}{kq}\right)^2 + 2\left(1-\frac{1}{\epsilon}\right)\right] y^2 - Ay - By^{1/2}$$

where

$$A := (p-1)(k-1)M_3M_4 + \frac{2k(1+q)K_1 + 2p(K_3+K_4)}{p}, \quad B := 2qkM_4\gamma.$$

Since $Ay \leq \eta_1 y^2 + A^2/4\eta_1$ and $By^{1/2} \leq \eta_2 y + B^2/4\eta_2$ for any $\eta_1, \eta_2 > 0$, it follows that (as in [6]), where we choose $\eta_1 = [(k-p)/kq]^2/2n$,

$$\begin{aligned} -M_6 \leq \; & \frac{nk^2}{2(k-p)^2}\eta_2^2 + \frac{nk^2}{2(k-p)^2}\bigg[(p-1)(k-1)M_3M_4 \\ & + \frac{2k(1+q)K_1 + 2p(K_3+K_4)}{p}\bigg]^2 + \frac{k^2M_4^2\gamma^2}{\eta_2} \end{aligned} \tag{4.12}$$

holds for any $q > 0$. Because the right-hand side of Eq. 4.12 as a function of q is increasing, letting $q \to 0$ yields

$$\begin{aligned} -M_6 \leq \; & \frac{nk^2}{2(k-p)^2}\eta^2 + \frac{k^2\gamma^2M_4^2}{\eta} \\ & + \frac{nk^2}{2(k-p)^2}\left[(p-1)(k-1)M_3M_4 + \frac{2kK_1}{p} + 2(K_3+K_4)\right]^2 \end{aligned} \tag{4.13}$$

where $\eta > 0$. Using Eq. 4.13, we prove (4.2). □

As an immediate consequence of the above theorem we have

Theorem 4.3 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.1) on M. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t.*

(i) For $0 < p < 1$, assume that $h \geq 0$, $|\nabla_t h|_t \leq \gamma$, $\Delta_t h \geq 0$ along the geometric flow with $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\mathbf{S}}_{g(t)}|_{g(t)} \leq K_4$ for some positive constants $\gamma, K_1, K_2, K_3, K_4$ with $\overline{K} := \max\{K_1, K_2\}$, along the geometric flow. If u is a smooth positive function satisfying the nonlinear parabolic (1.2), then

$$\begin{aligned} \frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \leq \; & \frac{C_1}{p^2 t} + \frac{C_1}{p^3(1-p)} + \frac{C_1}{p^2}\overline{K} + \frac{2nK_1}{p^2(1-p)} \\ & + \frac{\sqrt{n/K_1}}{p}\gamma M + \frac{n}{p^2}\sqrt{\frac{K_4}{2n}} + \frac{n(K_3+K_4)}{p(1-p)} \end{aligned} \tag{4.14}$$

for some positive constant C_1 depending only on n, where $M := \max_{M\times[0,T]} u^{p-1}$.

(ii) For $p = 1$, assume that $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\mathbf{S}}_{g(t)}|_{g(t)} \leq K_4$ for some positive constants K_1, K_2, K_3, K_4 with $\overline{K} := \max\{K_1, K_2\}$, $h \geq 0$, $\Delta_{g(t)} h \geq -\theta$ (θ is nonnegative), and $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$ (γ is nonnegative), along the geometric flow. If u is a smooth positive function satisfying the nonlinear parabolic (1.2), then

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + h - \frac{u_t}{u} \leq \frac{C_2}{t} + C_2\left(1 + K_1 + K_2 + K_3 + K_4 + \overline{K} + \gamma + \sqrt{\theta}\right) \tag{4.15}$$

for some positive constant C_2 depending only on n.

(iii) For $p > 1$, assume that $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ for some positive constants $\gamma, K_1, K_2, K_3, K_4$ with $\overline{K} := \max\{K_1, K_2\}$. $\Delta_{g(t)} h \geq -\theta$, $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$, and $-k_1 \leq h \leq k_2$, where $\theta, \gamma, k_1, k_2 > 0$, along the geometric flow. If u is a bounded smooth positive function satisfying the nonlinear parabolic (1.2), then

$$\begin{aligned}\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{u} \leq &\left(\frac{k}{p}\right)^2\frac{C_3}{t}+\left(\frac{k}{p}\right)^3\frac{k}{k-p}C_3+\left(\frac{k}{p}\right)^2C_3\Bigg(\overline{K}+\\ &+\frac{k}{k-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}}\Bigg)\\ &+\left(\frac{k}{p}\right)^2 n(p-1)k_2M+\frac{k^3n}{k-p}k_1M \qquad (4.16)\\ &+\frac{k^2\sqrt{n}}{p}\gamma M+\left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M},\end{aligned}$$

for some positive constant C_3 depending only on n, where $M := \max_{M\times[0,T]} u^{p-1}$ and $k > p$. In particular, taking $k = 2p$, we get

$$\begin{aligned}\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{u} \leq &\frac{C_4}{t}+C_5\left(1+K_1+K_2+K_3+K_4+\overline{K}\right)\\ &+C_4p^2\left[(k_1+k_2)M+\gamma M+\sqrt{\theta M}\right], \qquad (4.17)\end{aligned}$$

for some positive constant C_4 depending only on n.

In Lemma 3.2, we required that

$$\Delta_{g(t)}h + h_t - \frac{p}{\eta}\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$$

for some positive constant p, η. In the following proof, we shall see that when $0 < p \leq \frac{2n}{2n-1}$, we need only to assume that

$$\Delta_{g(t)} + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$$

where

$$C_{n,p} = \begin{cases} n, & p \leq 1,\\ \frac{p}{p-1}, & p > 1.\end{cases}$$

Theorem 4.4 *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.1) on M, satisfying $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_t \underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$, for some $K_1, K_2, K_3, K_4 > 0$, with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_{g(t)}h + h_t -$*

$2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ *on* $M \times [0,T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p > 1$ *and* $C_{n,p} = n$ *if* $p \leq 1$*), and* $0 < p \leq \frac{2n}{2n-1}$ *(*$n \geq 3$*). If* u *is a positive solution of* Eq. 1.2*, then*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{u} \leq \frac{C}{p^2t} + \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{4n}{p(2-p)}(K_1 + K_3 + K_4) + \frac{1}{p^2}\sqrt{8nK_4}, \quad (4.18)$$

for some positive constant C *depending only on* n.

Proof As in the proof of Theorem 4.1, we have

$$\begin{aligned}\Box\varphi \geq\ & \frac{1}{2nt}\left(\frac{2aq}{p} - \epsilon\right)\frac{p^2}{q^2}\varphi^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right)\varphi|\nabla \ln W|^2 \\ & + 2t\left[\frac{1}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{qnt}{bp}\overline{K}^2 - \frac{\varphi}{t} \\ & + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle - \frac{4(1+q)K_1 + 2(K_3+K_4)p}{p}t|\nabla \ln W|^2 - \frac{2q^2K_4t}{p^2},\end{aligned}$$

where $\varphi = tF$, from Lemma 3.2. Using Hölder's inequality

$$\frac{4(1+q)K_2 + 2K_1}{p}t|\nabla \ln W|^2 \leq \frac{1}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1-p/2}{q}\right)^2 2t|\nabla \ln W|^4 + \frac{2nt[(1+q)K_1 + \frac{1}{2}(K_3+K_4)p]^2}{p(2aq - p\epsilon)}\left(\frac{q}{1-p/2}\right)^2,$$

we see that

$$\begin{aligned}\Box\varphi \geq\ & \frac{1}{2nt}\left(\frac{2aq}{p} - \epsilon\right)\frac{p^2}{q^2}\varphi^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right)\varphi|\nabla \ln W|^2 \\ & - \frac{nq\overline{K}^2t}{bp} - \frac{2nt[(1+q)K_1 + \frac{1}{2}(K_3+K_4)p]^2}{p(2aq - p\epsilon)}\left(\frac{q}{1-p/2}\right)^2 \\ & + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle - \frac{2q^2K_4t}{p^2} - \frac{\varphi}{t}.\end{aligned}$$

Writing $G := \varphi\Psi$ and using $\Box G = \varphi\Box\Psi - 2\varphi|\nabla\Psi|^2/\Psi + \Psi\Box\varphi$, as before, we arrive at

$$\begin{aligned}0 \geq\ & \frac{p^2}{2ntq^2}\left(\frac{2aq}{p} - \epsilon\right)G^2 + \frac{2p}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)G\Psi|\nabla \ln W|^2 \\ & - \frac{2nt[(1+q)K_1 + \frac{1}{2}(K_3+K_4)p]^2}{p(2aq - p\epsilon)}\left(\frac{q}{1-p/2}\right)^2 - \frac{nq\overline{K}^2}{bp}t - \frac{G}{t} \\ & - \frac{2}{q}\langle\nabla\Psi, \nabla \ln W\rangle G + Gd_1\left(-\frac{1}{\rho^2} - \frac{\sqrt{K_1+K_3}}{\rho} - \frac{1}{\tau} - \overline{K}\right) - \frac{2q^2K_4}{p^2}t,\end{aligned} \quad (4.19)$$

for some positive constant d_1 depending only on n. Plugging the inequality

$$\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle G \le \frac{2p}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)G\Psi|\nabla\ln W|^2 + \frac{\frac{1}{q^2}}{\frac{2p}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G$$

into Eq. 4.19 yields

$$\begin{aligned}0 \ge\ & \frac{p^2}{2nq^2}\left(\frac{2aq}{p}-\epsilon\right)G^2 - d_2G\\ & - t\left[\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{2(2aq-p\epsilon)(1+q-p/2)}\right]d_2G\\ & - t^2\left[\frac{2n[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2+\frac{nq}{bp}\overline{K}^2+\frac{2q^2K_4}{p^2}\right]\end{aligned}$$

for some positive constant d_2 depending only on n. Hence

$$G \le \frac{d_2+t\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{2(2aq-p\epsilon)(1+q-p/2)}\right)}{\frac{p^2}{2nq^2}\left(\frac{2aq}{p}-\epsilon\right)} + t\sqrt{\frac{\frac{2n[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2+\frac{nq}{bp}\overline{K}^2+\frac{2q^2K_4}{p^2}}{\frac{p^2}{2nq^2}\left(\frac{2aq}{p}-\epsilon\right)}}.$$

The above calculation is based on the assumption that

$$\Delta_t h + h_t - \frac{p}{\eta}\frac{|\nabla_t h|_t^2}{h} \ge 0$$

for some positive constant $\eta, p > 0$. We now choose appropriate constants, together with the our assumption that

$$\Delta_t h + h_t - 2C_{n,p}p\frac{|\nabla_t h|_t^2}{h} \ge 0$$

to verify this assumption in Lemma 3.2. Recall the conditions on p, q, ϵ, a, b. First we consider the case,

$$q > 0,\quad 0<\epsilon\le 1,\quad a+b=\frac{p}{2q},\quad 0<p\le 1,\quad 0<\epsilon<\frac{2aq}{p}. \tag{4.20}$$

Choose

$$a=\left(\epsilon+\frac{1}{2}\right)\frac{p}{2q},\quad b=\left(\frac{1}{2}-\epsilon\right)\frac{p}{2q},\quad 0<\epsilon<\frac{1}{2}. \tag{4.21}$$

Then we can choose $\eta=\frac{1}{n}(\frac{2aq}{p}-\epsilon)=\frac{1}{2n}$ so that $p/\eta = 2np > 2p$, and furthermore

$$G \le \frac{4nq^2t}{p^2}\left[\frac{d_2}{t}+\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)}\right)\right] + \frac{4nq^2t}{p^2}\sqrt{\frac{1}{1-2\epsilon}\overline{K}^2+\frac{[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{\left(1-\frac{p}{2}\right)^2}+\frac{K_4}{2n}}.$$

Letting $\epsilon \to 0$ and $R \to \infty$ implies

$$\frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) \le \frac{d_2}{t}+\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)} + \sqrt{\overline{K}^2+\frac{[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{\left(1-\frac{p}{2}\right)^2}+\frac{K_4}{2n}}.$$

Now we minimize the above inequality for any $q > 0$ by the following observation

$$\frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) \le \frac{d_2}{t}+2\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)}+\frac{1+q-\frac{p}{2}}{1-\frac{p}{2}}K_1 + \frac{p}{2-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}}.$$

Hence

$$\begin{aligned}\frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) &\le \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{n}{p\left(1-\frac{p}{2}\right)}K_1} \\ &\quad + \frac{p}{2-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}} \\ &= \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{2n}{p(2-p)}K_1} \\ &\quad + \frac{p}{2-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}}.\end{aligned}$$

Next we consider the second case; that is,

$$q>0,\quad 0<\epsilon\le 1,\quad a+b=\frac{p}{2q},\quad 1<p<1+\frac{2aq}{n},\quad 0<\epsilon\le\frac{2aq-n(p-1)}{p}. \tag{4.22}$$

We have proved that $1 < p < \frac{n}{n-1} \le 2$ and $1+q-\frac{p}{2} > 0$ in this case. Choose

$$a=\left(\epsilon+\frac{1}{2}\right)\frac{p}{2q},\quad b=\left(\frac{1}{2}-\epsilon\right)\frac{p}{2q},\quad 0<\epsilon<\frac{1}{2},\quad 1<p\le\frac{2n}{2n-1} \tag{4.23}$$

and $\eta = \frac{p-1}{2p} \in (0, \frac{1}{4n}]$ so that $p/\eta = 2p\frac{p}{p-1} > 2p$. This choice of positive constants a, b, p, q, ϵ satisfies the mentioned condition (4.22). Then we obtain the same inequality

$$G \le \frac{4nq^2t}{p^2}\left[\frac{d_2}{t}+\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)}\right)\right] + \frac{4nq^2t}{p^2}\sqrt{\frac{1}{1-2\epsilon}\overline{K}^2+\frac{[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{\left(1-\frac{p}{2}\right)^2}+\frac{K_4}{2n}}.$$

Letting $\epsilon \to 0$ and $R \to \infty$, and minimizing over all $q > 0$, we obtain

$$\frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) \le \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{p}{2-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}}.$$

In both cases, we proved Theorem 4.4. □

Corollary 4.5 *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a solution to the geometric flow* (1.1) *on* M, *satisfying* $0 \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_{g(t)}\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$, *for some positive constants* K_2, K_3, K_4. *Let* $h(x,t)$ *be a nonnegative function defined on* $M\times[0,T]$ *which is* C^2 *in* x *and* C^1 *in* t, $\Delta_{g(t)}h + h_t - 2C_{n,p}\,p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ *on* $M\times[0,T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p>1$ *and* $C_{n,p}=n$ *if* $p\leq 1$*), and* $0 < p \leq \frac{2n}{2n-1}$ $(n\geq 3)$. *If* u *is a positive solution* Eq. 1.2, *then*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{t} \leq \frac{C}{p^2 t} + \frac{8n}{p^2}K_2 + \frac{4n}{p(2-p)}(K_3+K_4) + \frac{1}{p^2}\sqrt{8nK_4} \quad (4.24)$$

for some positive constant C *depending only on* n.

Under the hypotheses of Theorem 4.4, we let $f := \ln u$. Then

$$|\nabla_t f|_t^2 - \frac{2}{p}f_t \leq \frac{C}{p^2 t} + \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{4n}{p(2-p)}(K_1+K_3+K_4) + \frac{1}{p^2}\sqrt{8nK_4} \quad (4.25)$$

on $M\times[0,T]$. For any two points $(x_1,t_1), (x_2,t_2) \in M\times[0,T]$ with $t_1 < t_2$, as [1], we let $\Theta(x_1,t_1,x_2,t_2)$ the set of all the smooth paths $\gamma : [t_1,t_2] \to M$ that connect x_1 to x_2. Using the same argument in the proof of Lemma 2.10 in [1] and the inequality (4.25), for any $\gamma \in \Theta(x_1,t_1,x_2,t_2)$ we have

$$\begin{aligned}\frac{d}{dt}f(\gamma(t),t) &= \nabla_t f(\gamma(t),t)\dot\gamma(t) + \frac{\partial}{\partial s}f(\gamma(t),s)\Big|_{s=t}\\ &\geq -|\nabla_t f(\gamma(t),t)|_t|\dot\gamma(t)|_t + \frac{p}{2}\left(|\nabla_t f(\gamma(t),t)|_t^2 - \frac{C}{p^2 t} - A\right)\\ &\geq -\frac{1}{2p}|\dot\gamma(t)|_t^2 - \frac{p}{2}\left(\frac{C}{p^2 t} + A\right),\end{aligned}$$

where

$$A := \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{4n}{p(2-p)}(K_1+K_3+K_4) + \frac{1}{p^2}\sqrt{8nK_4}.$$

Therefore, we arrive at

$$\begin{aligned}f(x_2,t_2) - f(x_1,t_1) &= \int_{t_1}^{t_2}\frac{d}{dt}f(\gamma(t),t)\,dt\\ &\geq -\frac{1}{2p}\int_{t_1}^{t_2}|\dot\gamma(t)|^2_{g(t)}dt - \frac{pA}{2}(t_2-t_1) - \frac{C}{2p}\ln\frac{t_2}{t_1}.\end{aligned}$$

Corollary 4.6 *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a solution to the geometric flow* (1.1) *on* M, *satisfying* $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_{g(t)}\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$, *for some* $K_1, K_2, K_3, K_4 > 0$, *with* $\overline{K} := \max\{K_1, K_2\}$. *Let* $h(x,t)$ *be a nonnegative function defined on* $M\times[0,T]$ *which is* C^2 *in* x *and* C^1 *in* t,

$\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ *on* $M \times [0, T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p > 1$ *and* $C_{n,p} = n$ *if* $p \leq 1$*), and* $0 < p \leq \frac{2n}{2n-1}$ *(*$n \geq 3$*). If* u *is a positive solution of* Eq. 1.2*, then*

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\left[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot{\gamma}(t)|^2_{g(t)}dt - 2n(t_2 - t_1)\right.$$
$$\left.\left(\frac{1}{p}\overline{K} + \frac{2}{p}\sqrt{\frac{2n}{p(2-p)}}K_1 + \frac{1}{2-p}(K_1 + K_3 + K_4) + \frac{1}{p}\sqrt{2nK_4}\right)\right] \tag{4.26}$$

for some positive constant C *depending only on* n*, where* $(x_1, t_1), (x_2, t_2) \in M \times [0, T]$ *with* $t_1 < t_2$*.*

When $K_1 = 0$, we have the following

Corollary 4.7 *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a solution to the geometric flow* (1.1) *on* M*, satisfying* $0 \leq S_{g(t)} \leq K_2 g(t)$*,* $-K_3 g(t) \leq Ric_{g(t)} - S_{g(t)} \leq K_3 g(t)$*,* $|\nabla_{g(t)}\underline{\mathbf{S}}_{g(t)}|_{g(t)} \leq K_4$*, for some* $K_2, K_3, K_4 > 0$*. Let* $h(x, t)$ *be a nonnegative function defined on* $M \times [0, T]$ *which is* C^2 *in* x *and* C^1 *in* t*,* $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ *on* $M \times [0, T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p > 1$ *and* $C_{n,p} = n$ *if* $p \leq 1$*), and* $0 < p \leq \frac{2n}{2n-1}$ *(*$n \geq 3$*). If* u *is a positive solution* Eq. 1.2*, then*

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\left[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot{\gamma}(t)|^2_{g(t)}dt\right.$$
$$\left. - 2n(t_2 - t_1)\left(\frac{K_2}{p} + \frac{K_3 + K_4}{2-p} + \frac{\sqrt{2nK_4}}{p}\right)\right]$$

for some positive constant C *depending only on* n*, where* $(x_1, t_1), (x_2, t_2) \in M \times [0, T]$ *with* $t_1 < t_2$*.*

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