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Green's Function for Second Order Elliptic Equations in Non-divergence Form

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Abstract
We construct the Green function for second order elliptic equations in non-divergence form when the mean oscillations of the coefficients satisfy the Dini condition and the domain has $C^{1,1}$ boundary. We also obtain pointwise bounds for the Green functions and its derivatives.

Keywords Green's function · Non-divergent elliptic equation · Dini mean oscillation

Mathematics Subject Classification (2010) Primary 35J08, 35B45 · Secondary 35J47

1 Introduction and Main Results

Let Ω be a bounded $C^{1,1}$ domain (open connected set) in $\mathbb{R}^n$ with $n \geq 3$. We consider a second-order elliptic operator L in non-divergence form

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u. \tag{1.1}$$

We assume that the coefficient $\mathbf{A} := (a^{ij})$ is an $n \times n$ real symmetric matrix-valued function defined on $\mathbb{R}^n$, which satisfies the uniform ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x) \xi^i \xi^j \leq \Lambda |\xi|^2, \quad \forall \xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n \tag{1.2}$$

for some constants $0 < \lambda \leq \Lambda$.

In this article, we are concerned with construction and pointwise estimates for the Green's function $G(x, y)$ of the non-divergent operator L (1.1) in Ω. Unlike the Green's

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function for uniformly elliptic operators in divergence form, the Green's function for non-divergent elliptic operators does not necessarily enjoy the usual pointwise bound

$$G(x, y) \leq c|x - y|^{2-n} \tag{1.3}$$

even in the case when the coefficient **A** is uniformly continuous; see [1]. On the other hand, in the case when the coefficient **A** is Hölder continuous, then it is well known that the Green's function satisfies the pointwise bound (1.3); see e.g., [10] for the construction of fundamental solutions of parabolic operators by the parametrix method. In this perspective, it is an interesting question to ask what is the minimal regularity condition to ensure the Green's function to have the pointwise bound (1.3). We shall show that if the coefficient **A** is of *Dini mean oscillation*, then the Green's function exists and satisfies the pointwise bound (1.3). We shall say that a function is of Dini mean oscillation if its mean oscillation satisfies the Dini condition. Here, we briefly describe the role of this Dini mean oscillation condition because it will be used somewhat implicitly in the paper. First, it will imply that the coefficient **A** is uniformly continuous so that the Calderón-Zygmund L^p theory can be applied. Also, it will provide us a local L^∞ estimate for the solutions of the adjoint equation $L^*u = 0$ as appears in Eq. 2.18, which is one of the main results of the very recent papers by the second author and collaborators [4, 5]. This L^∞ estimate is crucial for the pointwise bound (1.3) and the uniform continuity of the coefficient **A** alone is not enough to produce such an estimate. Below is a more precise formulation of Dini mean oscillation condition.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the Euclidean ball with radius r centered at x, and denote

$$\Omega(x, r) := \Omega \cap B(x, r).$$

We shall say that a function $g : \Omega \to \mathbb{R}$ is of Dini mean oscillation in Ω if the mean oscillation function $\omega_g : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\omega_g(r) := \sup_{x \in \overline{\Omega}} \fint_{\Omega(x,r)} |g(y) - \bar{g}_{\Omega(x,r)}|\, dy \quad \text{where } \bar{g}_{\Omega(x,r)} := \fint_{\Omega(x,r)} g$$

satisfies the Dini condition; i.e.,

$$\int_0^1 \frac{\omega_g(t)}{t}\, dt < +\infty.$$

It is clear that if g is Dini continuous, then g is of Dini mean oscillation. However, the Dini mean oscillation condition is strictly weaker than the Dini continuity; see [4] for an example. Also if g is of Dini mean oscillation, then g is uniformly continuous in Ω with its modulus of continuity controlled by ω_g; see Appendix.

The formal adjoint operator L^* is given by

$$L^*u = \sum_{i,j=1}^{n} D_{ij}(a^{ij}(x)u). \tag{1.4}$$

We need to consider the boundary value problem of the form

$$L^*u = \operatorname{div}^2 \mathbf{g} + f \ \text{ in } \ \Omega, \quad u = \frac{\mathbf{g}\nu \cdot \nu}{\mathbf{A}\nu \cdot \nu} \ \text{ on } \ \partial\Omega, \tag{1.5}$$

where $\mathbf{g} = (g^{ij})$ is an $n \times n$ matrix-valued function,

$$\operatorname{div}^2 \mathbf{g} = \textstyle\sum_{i,j=1}^{n} D_{ij} g^{ij},$$

and ν is the unit exterior normal vector of $\partial\Omega$. For $\mathbf{g} \in L^p(\Omega)$ and $f \in L^p(\Omega)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we say that u in $L^p(\Omega)$ is an adjoint solution of Eq. 1.5 if u satisfies

$$\int_\Omega uLv = \int_\Omega \operatorname{tr}(\mathbf{g}D^2 v) + \int_\Omega fv \tag{1.6}$$

for any v in $W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$. The following lemma is quoted from [7, Lemma 2].

Lemma 1.7 *Let $1 < p < \infty$ and assume that $\mathbf{g} \in L^p(\Omega)$ and $f \in L^p(\Omega)$. Then there exists a unique adjoint solution u in $L^p(\Omega)$. Moreover, the following estimates holds*

$$\|u\|_{L^p(\Omega)} \le C\left[\|\mathbf{g}\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right],$$

where a constant C depends on Ω, p, n, λ, Λ, and the continuity of **A**.

We clarify that "the continuity" of **A** in Lemma 1.7 specifically means "the modulus of continuity" of **A**, which is clear from the context in [7]. By the modulus of continuity of **A**, we mean the function $\varrho_{\mathbf{A}}$ defined by

$$\varrho_{\mathbf{A}}(t) :=\supset , |\mathbf{A}(x) - \mathbf{A}(y)| : x, y \in \Omega, , |x - y| \le t, , \quad \forall t \ge 0.$$

Therefore, in the case when coefficient **A** is of Dini mean oscillation, the constant C in Lemma 1.7 depends only on Ω, p, n, λ, Λ, and $\omega_{\mathbf{A}}$.

It is also known that if $f \in L^p(\Omega)$ with $p > \frac{n}{2}$, then the adjoint solution of the problem

$$L^* u = f \;\text{ in }\; \Omega, \quad u = 0 \;\text{ on }\; \Omega, \tag{1.8}$$

is uniformly continuous in Ω; see Theorem 1.8 in [5].

We say $\partial\Omega$ is $C^{k,\mathrm{Dini}}$ if for each point $x_0 \in \partial\Omega$, there exist a constant $r > 0$ independent of x_0 and a $C^{k,\mathrm{Dini}}$ function (i.e., C^k function whose kth derivatives are uniformly Dini continuous) $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that (upon relabeling and reorienting the coordinates axes if necessary) in a new coordinate system $(x', x^n) = (x^1, \ldots, x^{n-1}, x^n)$, x_0 becomes the origin and

$$\Omega \cap B(0, r) = \{x \in B(0, r) : x^n > \gamma(x^1, \ldots, x^{n-1})\}, \quad \gamma(0') = 0.$$

A few remarks are in order before we state our main theorem. There are many papers in the literature dealing with the existence and estimates of Green's functions or fundamental solutions of non-divergence form elliptic operators with measurable or continuous coefficients. To our best knowledge, the first author who considered Green's function for non-divergence form elliptic operators with measurable coefficients is Bauman [2, 3], who introduced the concept of normalized adjoint solutions; see also Fabes et al. [9]. Fabes and Stroock [8] established L^p-integrablity of Green's functions for non-divergence form elliptic operators with measurable coefficients. Krylov [14] showed that the weak uniqueness property holds for solutions of non-divergence form elliptic equations in Ω if and only if there is a unique Green's function in Ω. Escauriaza [6] established bounds for fundamental solution for non-divergence form elliptic operators in terms of nonnegative adjoint solution. We would like to thank Luis Escauriaza for bringing our attention to these results in the literature.

Now, we state our main theorem.

Theorem 1.9 *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ with $n \ge 3$. Assume the coefficient $\mathbf{A} = (a^{ij})$ of the non-divergent operator L in* Eq. 1.1 *satisfies the uniform ellipticity condition* (1.2) *and is of Dini mean oscillation in Ω. Then, there exists a Green's function $G(x, y)$ (for any $x, y \in \Omega$, $x \ne y$) of the operator L in Ω and it is unique in the following* *sense: if u is the unique adjoint solution of the problem* (1.8)*, where* $f \in L^p(\Omega)$ *with* $p > \frac{n}{2}$*, then u is represented by*

$$u(y) = \int_\Omega G(x, y) f(x)\, dx. \tag{1.10}$$

The Green function $G(x, y)$ *satisfies the following pointwise estimates:*

$$|G(x, y)| \le C|x - y|^{2-n}, \tag{1.11}$$

$$|D_x G(x, y)| \le C|x - y|^{1-n}, \tag{1.12}$$

where $C = C(n, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$*. Moreover, if the boundary* $\partial\Omega$ *is* $C^{2,Dini}$*, then we have*

$$|D_x^2 G(x, y)| \le C|x - y|^{-n}, \tag{1.13}$$

where $C = C(n, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$.

Remark 1.14 In the proof of Theorem 1.9, we will construct the Green's function $G^*(x, y)$ for the adjoint operator L^* as a by-product. It is characterized as follows: for $q > \frac{n}{2}$ and $f \in L^q(\Omega)$, if $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ is the strong solution of

$$Lv = f \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \partial\Omega,$$

then, we have the representation formula

$$v(y) = \int_\Omega G^*(x, y) f(x)\, dx.$$

Also, in the proof of Theorem 1.9, we shall show that

$$G(x, y) = G^*(y, x), \quad \forall x, y \in \Omega, \quad x \neq y. \tag{1.15}$$

Finally, by the maximum principle, it is clear that $G(x, y) \ge 0$.

2 Proof of Theorem 1.9

2.1 Construction of Green's Function

To construct Green's function, we follow the scheme of [13], which in turn is based on [12]. For $y \in \Omega$ and $\epsilon > 0$, let $v_\epsilon = v_{\epsilon;y} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ be a unique strong solution of the problem

$$Lv = \frac{1}{|\Omega(y, \epsilon)|} \chi_{\Omega(y,\epsilon)} \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \partial\Omega. \tag{2.1}$$

Note that $\mathbf{A}$ is uniformly continuous in Ω with its modulus of continuity controlled by $\omega_{\mathbf{A}}$. Therefore, the unique solvability of the problem (2.1) is a consequence of standard L^p theory; see e.g., Chapter 9 of [11]. Also, by the same theory, we see that v_ϵ belong to $W^{2,p}(\Omega)$ for any $p \in (1, \infty)$ and we have an estimate

$$\|v_\epsilon\|_{W^{2,p}(\Omega)} \le C\,\epsilon^{-n+\frac{n}{p}}, \quad \forall \epsilon \in (0, \operatorname{diam}\Omega), \tag{2.2}$$

where $C = C(n, \lambda, \Lambda, p, \Omega, \omega_{\mathbf{A}})$. In particular, we see that v_ϵ is continuous in Ω. Next, for $f \in C_c^\infty(\Omega)$, consider the adjoint problem

$$L^* u = f \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega.$$

By Lemma 1.7, there exists a unique adjoint solution u in $L^2(\Omega)$, and by Eq. 1.6, we have

$$\fint_{\Omega(y,\epsilon)} u = \int_\Omega f v_\epsilon. \tag{2.3}$$

Let w be the solution of the Dirichlet problem

$$\Delta w = f \ \text{ in } \ \Omega, \quad w = 0 \ \text{ on } \ \partial\Omega. \tag{2.4}$$

By the standard L^p theory and Sobolev's inequality, for $1 < p < \frac{n}{2}$, we have

$$\|w\|_{L^q(\Omega)} \le C(n, \Omega, p)\|f\|_{L^p(\Omega)}, \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{2}{n}, \tag{2.5}$$

and for $1 < p < n$, we have

$$\|\nabla w\|_{L^q(\Omega)} \le C(n, \Omega, p)\|f\|_{L^p(\Omega)}, \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

In particular, for $\frac{n}{2} < p < n$, we have by Morrey's theorem that

$$[w]_{C^{0,\mu}(\Omega)} \le C(n, \Omega, p)\|f\|_{L^p(\Omega)}, \quad \text{where } \mu = 1 - \frac{n}{q} = 2 - \frac{n}{p}. \tag{2.6}$$

Hereafter, we set

$$\mathbf{g} := w\mathbf{I}.$$

Note that by Eqs. 1.8 and 2.4, $u \in L^2(\Omega)$ is an adjoint solution of

$$L^* u = \operatorname{div}^2 \mathbf{g} \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{2.7}$$

By Lemma 1.7 and Eq. 2.5, we see that $u \in L^q(\Omega)$ for $q \in (\frac{n}{n-2}, \infty)$ and that it satisfies

$$\|u\|_{L^q(\Omega)} \le C\|f\|_{L^{nq/(n+2q)}(\Omega)}, \quad \text{where } C = C(n, q, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}}). \tag{2.8}$$

Also, by Eq. 2.6, we see that $\mathbf{g}$ is of Dini mean oscillation in Ω with

$$\omega_{\mathbf{g}}(t) \le C\|f\|_{L^p(\Omega)}\, t^{2-\frac{n}{p}}.$$

Therefore, by Theorem 1.8 of [5], we see that $u \in C(\overline{\Omega})$. As a matter of fact, Lemma 2.27 of [5] and Theorem 1.10 of [4] with a scaling argument ($x \mapsto rx$) reveals that for any $x_0 \in \Omega$ and $0 < r < \operatorname{diam}\Omega$, we have

$$\sup_{\Omega(x_0, \frac{1}{2}r)} |u| \le C\left(r^{-n}\|u\|_{L^1(\Omega(x_0,r))} + r^{2-\frac{n}{p}}\|f\|_{L^p(\Omega)}\right),$$

where $C = C(n, p, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$. In particular, if f is supported in $\Omega(y, r)$, then by Eq. 2.8 and Hölder's inequality, we have

$$\sup_{\Omega(y, \frac{1}{2}r)} |u| \le C\left(r^{-\frac{n}{q}}\|f\|_{L^{nq/(n+2q)}(\Omega(y,r))} + r^{2-\frac{n}{p}}\|f\|_{L^p(\Omega(y,r))}\right) \le Cr^2\|f\|_{L^\infty(\Omega(y,r))}. \tag{2.9}$$

Therefore, if f is supported in $\Omega(y, r)$, then it follows from Eqs. 2.3 and 2.9 that

$$\left|\int_{\Omega(y,r)} f v_\epsilon\right| \le Cr^2\|f\|_{L^\infty(\Omega(y,r))}, \quad \forall \epsilon \in \left(0, \tfrac{1}{2}r\right).$$

By duality, we obtain

$$\|v_\epsilon\|_{L^1(\Omega(y,r))} \le Cr^2, \quad \forall \epsilon \in \left(0, \tfrac{1}{2}r\right), \quad \forall r \in (0, \operatorname{diam}\Omega), \tag{2.10}$$

where $C = C(n, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$. We define the approximate Green's function

$$G_\epsilon(x, y) = v_{\epsilon,y}(x) = v_\epsilon(x).$$

Lemma 2.11 *Let $x, y \in \Omega$ with $x \neq y$. Then*

$$|G_\epsilon(x,y)| \le C|x-y|^{2-n}, \quad \forall \epsilon \in \left(0, \tfrac{1}{3}|x-y|\right), \tag{2.12}$$

where $C = C(n, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$.

Proof Let $r = \frac{2}{3}|x-y|$. If $\epsilon < \frac{1}{2}r$, then v_ϵ satisfies $Lv_\epsilon = 0$ in $\Omega(x,r)$. Therefore, by the standard elliptic estimate (see [11, Theorem 9.26]) and Eq. 2.10, we have

$$|v_\epsilon(x)| \le Cr^{-n}\|v_\epsilon\|_{L^1(\Omega(x,r))} \le Cr^{-n}\|v_\epsilon\|_{L^1(\Omega(y,3r))} \le Cr^{2-n}. \qquad \square$$

Lemma 2.13 *For any $y \in \Omega$ and $0 < \epsilon < \operatorname{diam}\Omega$, we have*

$$\int_{\Omega\setminus\overline{B}(y,r)} |G_\epsilon(x,y)|^{\frac{2n}{n-2}}\,dx \le Cr^{-n}, \quad \forall r > 0, \tag{2.14}$$

$$\int_{\Omega\setminus\overline{B}(y,r)} |D_x^2 G_\epsilon(x,y)|^2\,dx \le Cr^{-n}, \quad \forall r > 0, \tag{2.15}$$

where $C = C(n, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$.

Proof We first establish Eq. 2.14. In the case when $r > 3\epsilon$, we get from Eq. 2.12 that

$$\int_{\Omega\setminus\overline{B}(y,r)} |G_\varepsilon(x,y)|^{\frac{2n}{n-2}}\,dx \le C\int_{\Omega\setminus\overline{B}(y,r)} |x-y|^{-2n}\,dx \le Cr^{-n}.$$

In the case when $r \le 3\epsilon$, by Eq. 2.2 with $p = \frac{2n}{n+2}$ and the Sobolev's inequality, we have

$$\|v_\epsilon\|_{L^{\frac{2n}{n-2}}(\Omega)} \le C\|v_\epsilon\|_{W^{2,\frac{2n}{n+2}}(\Omega)} \le C\epsilon^{1-\frac{n}{2}} \le Cr^{1-\frac{n}{2}}, \tag{2.16}$$

and thus we still get Eq. 2.14.

Next, we turn to the proof of Eq. 2.15. It is enough to consider the case when $r > 2\epsilon$. Indeed, by Eq. 2.2, we have

$$\int_{\Omega\setminus\overline{B}(y,r)} |D^2 v_\epsilon|^2 \le \int_\Omega |D^2 v_\epsilon|^2 \le C\epsilon^{-n} \le Cr^{-n}.$$

For $\mathbf{g} \in C_c^\infty(\Omega \setminus \overline{B}(y,r))$, let $u \in L^2(\Omega)$ be an adjoint solution of Eq. 2.7 so that we have

$$\fint_{\Omega(y,\epsilon)} u = \int_\Omega \operatorname{tr}(\mathbf{g}D^2 v_\epsilon). \tag{2.17}$$

Since $\mathbf{g} = 0$ in $\Omega(y,r)$, we see that u is continuous on $\overline{\Omega}(y, \frac{1}{2}r)$ by [5, Theorem 1.8]. In fact, it follows from [5, Lemma 2.27] that

$$\sup_{\Omega(y,\frac{1}{2}r)} |u| \le Cr^{-n}\|u\|_{L^1(\Omega(y,r))}. \tag{2.18}$$

Therefore, by Hölder's inequality and Lemma 1.7, we have

$$\sup_{\Omega\left(y,\frac{1}{2}r\right)} |u| \le Cr^{-\frac{n}{2}}\|u\|_{L^2(\Omega(y,r))} \le r^{-\frac{n}{2}}\|\mathbf{g}\|_{L^2(\Omega)}.$$

Since $\mathbf{g}$ is supported in $\Omega \setminus \overline{B}_r(y)$, by Eq. 2.17 and the above estimate, we have

$$\left| \int_{\Omega \setminus \overline{B}(y,r)} \operatorname{tr}(\mathbf{g} D^2 v_\epsilon) \right| \leq C r^{-\frac{n}{2}} \|\mathbf{g}\|_{L^2(\Omega \setminus \overline{B}(y,r))}.$$

Therefore, Eq. 2.15 follows by duality. □

Lemma 2.19 *For any* $y \in \Omega$ *and* $0 < \epsilon < \operatorname{diam} \Omega$*, we have*

$$|\{x \in \Omega : |G_\epsilon(x,y)| > t\}| \leq C t^{-\frac{n}{n-2}}, \quad \forall t > 0, \tag{2.20}$$

$$|\{x \in \Omega : |D_x^2 G_\epsilon(x,y)| > t\}| \leq C t^{-1}, \quad \forall t > 0, \tag{2.21}$$

where $C = C(n, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$.

Proof We first establish Eq. 2.20. Let

$$A_t = \{x \in \Omega : |G_\epsilon(x,y)| > t\}$$

and take $r = t^{-\frac{1}{n-2}}$. Then, by Eq. 2.14, we get

$$|A_t \setminus \overline{B}(y,r)| \leq t^{-\frac{2n}{n-2}} \int_{A_t \setminus \overline{B}(y,r)} |G_\epsilon(x,y)|^{\frac{2n}{n-2}}\, dx \leq C t^{-\frac{2n}{n-2}} t^{\frac{n}{n-2}} = C t^{-\frac{n}{n-2}}.$$

Since $|A_t \cap \overline{B}(y,r)| \leq C r^n = C t^{-\frac{n}{n-2}}$, obviously we thus obtain (2.20).

Next, we prove Eq. 2.21. Let

$$A_t = \{x \in \Omega : |D_x^2 G_\epsilon(x,y)| > t\}$$

and take $r = t^{-\frac{1}{n}}$. Then, by Eq. 2.15, we have

$$|A_t \setminus \overline{B}(y,r)| \leq t^{-2} \int_{A_t \setminus \overline{B}(y,r)} |D_x^2 G_\epsilon(x,y)|^2\, dx \leq C t^{-2} t = C t^{-1}.$$

Since $|A_t \cap \overline{B}(y,r)| \leq C r^n = C t^{-1}$, we get Eq. 2.21. □

We are now ready to construct a Green's function. By Lemma 2.13, for any $r > 0$, we have

$$\sup_{0<\epsilon<\operatorname{diam}\Omega} \|G_\epsilon(\cdot, y)\|_{W^{2,2}(\Omega \setminus \overline{B}(y,r))} < +\infty.$$

Therefore, by applying a diagonalization process, we see that there exists a sequence of positive numbers $\{\epsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} \epsilon_i = 0$ and a function $G(\cdot, y)$, which belongs to $W^{2,2}(\Omega \setminus \overline{B}(y,r))$ for any $r > 0$, such that

$$G_{\epsilon_i}(\cdot, y) \rightharpoonup G(\cdot, y) \text{ weakly in } W^{2,2}(\Omega \setminus \overline{B}(y,r)), \quad \forall r > 0. \tag{2.22}$$

Note that by Eq. 2.15, we have

$$\int_{\Omega \setminus \overline{B}(y,r)} |D_x^2 G(x,y)|^2\, dy \leq C r^{-n}, \quad \forall r > 0. \tag{2.23}$$

By compactness of the embedding $W^{2,2} \hookrightarrow L^{\frac{2n}{n-2}}$, we also get from Eq. 2.14 that

$$\int_{\Omega \setminus \overline{B}(y,r)} |G(x,y)|^{\frac{2n}{n-2}}\, dx \leq C r^{-n}, \quad \forall r > 0,$$

On the other hand, Eq. 2.20 implies that for $1 < p < \frac{n}{n-2}$, we have (see e.g., Section 3.5 in [13])

$$\sup_{0<\epsilon<\operatorname{diam}\Omega} \|G_\epsilon(\cdot, y)\|_{L^p(\Omega)} < +\infty.$$

Therefore, by passing to a subsequence if necessary, we see that

$$G_{\epsilon_i}(\cdot, y) \rightharpoonup G(\cdot, y) \ \text{ weakly in } \ L^p(\Omega), \quad \forall\, p \in \left(1, \tfrac{n}{n-2}\right).$$

For $f \in L^q(\Omega)$ with $q > \frac{n}{2}$, let u be the unique adjoint solution in $L^q(\Omega)$ of the problem (1.8). Then by Eq. 2.3, we have

$$\fint_{\Omega(y,\epsilon_i)} u(x)\, dx = \int_\Omega G_{\epsilon_i}(x, y) f(x)\, dx.$$

By taking the limit, we get the representation formula (1.10), which yields the uniqueness of the Green's function.

Finally, from Eqs. 2.22 and 2.1, we find that $G(\cdot, y)$ belongs to $W^{2,2}(\Omega \setminus \overline{B}(y,r))$ and satisfies $LG(\cdot, y) = 0$ in $\Omega \setminus \overline{B}(y,r)$ for all $r > 0$. Since $\mathbf{A}$ is uniformly continuous in Ω, by the standard L^p theory (see e.g., [11]), we then see that $G(\cdot, y)$ is continuous in $\Omega \setminus \{y\}$. Moreover, by the same reasoning, we see from Lemma 2.13 that $G_\epsilon(\cdot, y)$ is equicontinuous on $\Omega \setminus \overline{B}(y,r)$ for any $r > 0$. Therefore, we may assume, by passing if necessary to another subsequence, that

$$G_{\epsilon_i}(\cdot, y) \to G(\cdot, y) \ \text{ uniformly on } \ \Omega \setminus \overline{B}(y,r), \quad \forall\, r > 0. \tag{2.24}$$

In particular, from Lemma 2.11, we see that

$$|G(x, y)| \le C|x - y|^{2-n}.$$

Therefore, we have shown Eq. 1.11.

2.2 Construction of Green's Function for the Adjoint Operator

To construct Green's function for the adjoint operator L^* given in Eq. 1.4, we follow the same scheme in Section 2.1. For $y \in \Omega$ and $\epsilon > 0$, consider the following adjoint problem:

$$L^* u = \frac{1}{|\Omega(y,\epsilon)|}\chi_{\Omega(y,\epsilon)} \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{2.25}$$

By Lemma 1.7, there exists a unique adjoint solution $u_\epsilon = u_{\epsilon;y}$ in $L^2(\Omega)$ such that

$$\|u_\epsilon\|_{L^2(\Omega)} \le C\epsilon^{-\frac{n}{2}}, \quad \forall\, \epsilon \in (0, \operatorname{diam}\Omega),$$

where $C = C(n, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$. Next, for $f \in C_c^\infty(\Omega)$, consider the problem

$$Lv = f \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \partial\Omega.$$

By the standard L^p theory (see e.g., [11]) and the Sobolev's inequality, we have

$$\|v\|_{L^q(\Omega)} \le C\|f\|_{L^{nq/(n+2q)}(\Omega)},$$

where $C = C(n, \lambda, \Lambda, \Omega, q, \omega_{\mathbf{A}})$. Also, we have

$$\sup_{\Omega\left(x_0, \frac{1}{2}r\right)} |v| \le C\left(r^{-n}\|v\|_{L^1(\Omega(x_0,r))} + r^2\|f\|_{L^\infty(\Omega(x_0,r))}\right), \quad \forall\, x_0 \in \Omega, \quad \forall\, r \in (0, \operatorname{diam}\Omega).$$

Therefore, if f is supported in $\Omega(y, r)$, then by the above estimates and Hölder's inequality, we get

$$\sup_{\Omega\left(y,\frac{1}{2}r\right)} |v| \leq Cr^2 \|f\|_{L^\infty(\Omega(y,r))}.$$

From the identity

$$\fint_{\Omega(y,\epsilon)} v = \int_\Omega f u_\epsilon, \tag{2.26}$$

we then see that (c.f. Eq. 2.10 above)

$$\|u_\epsilon\|_{L^1(\Omega(y,r))} \leq Cr^2, \quad \forall \epsilon \in \left(0, \tfrac{1}{2}r\right), \quad \forall r \in (0, \operatorname{diam}\Omega),$$

where $C = C(n, \lambda, \Lambda, \Omega, \omega_{\mathbf{A}})$. We define

$$G^*_\epsilon(x, y) = u_{\epsilon,y}(x) = u_\epsilon(x).$$

Then, similar to Lemma 2.11, for any $x, y \in \Omega$ with $x \neq y$, we have

$$|G^*_\epsilon(x, y)| \leq C|x - y|^{2-n}, \quad \forall \epsilon \in \left(0, \tfrac{1}{3}|x - y|\right). \tag{2.27}$$

Indeed, if we set $r = \frac{2}{3}|x - y|$, then for $\epsilon < \frac{1}{2}r$, we have $L^* u_\epsilon = 0$ in $\Omega(x, r)$. Since $\mathbf{A}$ is of Dini mean oscillation, by [5, Lemma 2.27], we have

$$|u_\epsilon(x)| \leq Cr^{-n}\|u_\epsilon\|_{L^1(\Omega(x,r))} \leq Cr^{-n}\|u_\epsilon\|_{L^1(\Omega(y,3r))} \leq Cr^{2-n},$$

which yields Eq. 2.27.

By using Eq. 2.27 and following the proof of Lemma 2.13, we get the following estimate, which is a counterpart of Eq. 2.14. For any $y \in \Omega$ and $0 < \epsilon < \operatorname{diam}\Omega$, we have

$$\int_{\Omega\setminus\overline{B}(y,r)} |G^*_\epsilon(x, y)|^{\frac{2n}{n-2}}\, dx \leq Cr^{-n}, \quad \forall r > 0. \tag{2.28}$$

Indeed, by taking $f = \frac{1}{|\Omega(y,\epsilon)|}\chi_{\Omega(y,\epsilon)}$ and $q = \frac{2n}{n-2}$ in Eq. 2.8, we have

$$\|u_\epsilon\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C\|f\|_{L^{\frac{2n}{n+2}}(\Omega)} \leq C\epsilon^{1-\frac{n}{2}},$$

which corresponds to Eq. 2.16. Then, by the same proof of Lemma 2.13, we get Eq. 2.28. By using Eq. 2.28 and proceeding as in the proof of Eq. 2.20, we obtain

$$|\{x \in \Omega : |G^*_\epsilon(x, y)| > t\}| \leq Ct^{-\frac{n}{n-2}}, \quad \forall t > 0,$$

which in turn implies that for $0 < p < \frac{n}{n-2}$, there exists a constant C_p such that

$$\int_\Omega |G^*_\epsilon(x, y)|^p\, dx \leq C_p, \quad \forall y \in \Omega, \quad \forall \epsilon \in (0, \operatorname{diam}\Omega).$$

Therefore, for any $1 < p < \frac{n}{n-2}$, we obtain

$$\sup_{0<\epsilon<\operatorname{diam}\Omega} \|G^*_\epsilon(\cdot, y)\|_{L^p(\Omega)} < +\infty,$$

and thus, there exists a sequence of positive numbers $\{\epsilon_j\}_{j=1}^\infty$ with $\lim_{j\to\infty} \epsilon_j = 0$ and a function $G^*(\cdot, y) \in L^p(\Omega)$ such that

$$G^*_{\epsilon_j}(\cdot, y) \rightharpoonup G^*(\cdot, y) \;\text{ weakly in }\; L^p(\Omega). \tag{2.29}$$

For $f \in L^q(\Omega)$ with $q > \frac{n}{2}$, let $v \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ be the strong solution of

$$Lv = f \;\text{ in }\; \Omega, \quad v = 0 \;\text{ on }\; \partial\Omega.$$

Then, we have (c.f. Eq. 2.26 above)

$$\fint_{\Omega(y,\epsilon_j)} v(x)\,dx = \int_\Omega G^*_{\epsilon_j}(x,y) f(x)\,dx,$$

and thus, by taking the limit, we also get the representation formula

$$v(y) = \int_\Omega G^*(x,y) f(x)\,dx.$$

Finally, by Eqs. 2.25, 2.29, and Theorem 1.10 of [4], we see that $G^*(\cdot,y)$ is continuous away from its singularity at y.

2.3 Proof of Symmetry (1.15)

For any $x \neq y$ in Ω, choose two sequences $\{\epsilon_i\}_{i=1}^\infty$ and $\{\delta_j\}_{j=1}^\infty$ such that $0 < \epsilon_i, \delta_j < \frac{1}{3}|x-y|$ for all i, j and $\epsilon_i, \delta_j \to 0$ as $i, j \to \infty$. From the construction of $G_\epsilon(\cdot,y)$ and $G^*_\epsilon(\cdot,x)$, we observe the following:

$$\fint_{\Omega(x,\delta_j)} G_{\epsilon_i}(\cdot,y) = \int_\Omega G_{\epsilon_i}(\cdot,y)\, L^* G^*_{\delta_j}(\cdot,x) = \int_\Omega L G_{\epsilon_i}(\cdot,y)\, G^*_{\delta_j}(\cdot,x) = \fint_{\Omega(y,\epsilon_i)} G^*_{\delta_j}(\cdot,x).$$

By Eq. 2.29 and the continuity of $G^*(\cdot,x)$ away from its singularity, we get

$$\lim_{i\to\infty}\lim_{j\to\infty} \fint_{\Omega(y,\epsilon_i)} G^*_{\delta_j}(\cdot,x) = \lim_{i\to\infty} \fint_{\Omega(y,\epsilon_i)} G^*(\cdot,x) = G^*(y,x).$$

On the other hand, by the continuity of $G_{\epsilon_i}(\cdot,y)$ and Eq. 2.24, we obtain

$$\lim_{i\to\infty}\lim_{j\to\infty} \fint_{\Omega(x,\delta_j)} G_{\epsilon_i}(\cdot,y) = \lim_{i\to\infty} G_{\epsilon_i}(x,y) = G(x,y).$$

We have thus shown that

$$G(x,y) = G^*(y,x), \quad \forall x \neq y.$$

So far, we have seen that there is a subsequence of $\{\epsilon_i\}$ tending to zero such that $G_{\epsilon_{k_i}}(\cdot,y) \to G(\cdot,y)$. However, for any $x \neq y$, we have

$$G_\epsilon(x,y) = \lim_{j\to\infty} \fint_{\Omega(x,\delta_j)} G_\epsilon(\cdot,y) = \lim_{j\to\infty} \fint_{\Omega(y,\epsilon)} G^*_{\delta_j}(\cdot,x) = \fint_{\Omega(y,\epsilon)} G^*(\cdot,x) = \fint_{\Omega(y,\epsilon)} G(\cdot,y).$$

That is, we have

$$G_\epsilon(x,y) = \fint_{\Omega(y,\epsilon)} G(x,z)\,dz.$$

Therefore, we find that

$$\lim_{\epsilon\to 0} G_\epsilon(x,y) = G(x,y), \quad \forall x \neq y.$$

2.4 Proof of Estimates Eqs. 1.12 and 1.13

We now show Eqs. 1.12 and 1.13. Let $v = G(\cdot,y)$ and $r = \frac{1}{3}|x-y|$. Note that we have

$$Lv = 0 \ \text{ in } \ \Omega(x,2r), \qquad v = 0 \ \text{ on } \ \partial\Omega(x,2r).$$

By the standard elliptic theory, we have

$$\|Dv\|_{L^\infty(\Omega(x,r))} \le Cr^{-1-n}\|v\|_{L^1(\Omega(x,2r))}.$$

Therefore, by Eq. 1.11, we get

$$|Dv(x)| \le Cr^{-1-n}\|v\|_{L^1(\Omega(y,4r))} \le Cr^{1-n},$$

from which Eq. 1.12 follows.

In the case when Ω has $C^{2,\mathrm{Dini}}$ boundary, by Theorem 1.5 of [5], we have

$$\sup_{\Omega(x,r)} |D^2 v| \le Cr^{-n}\|D^2 v\|_{L^1(\Omega(x,2r))}.$$

Therefore, by Hölder's inequality and Eq. 2.23, we have

$$|D^2 v(x)| \le Cr^{-\frac{n}{2}}\|D^2 v\|_{L^2(\Omega(x,2r))} \le Cr^{-\frac{n}{2}}\|D^2 v\|_{L^2(\Omega\setminus B(y,r))} \le Cr^{-n},$$

from which Eq. 1.13 follows.

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Appendix

The following lemma is well known to experts and essentially due to Campanato. We provide its proof for the reader's convenience.

Lemma A.1 *Let $\Omega \in \mathbb{R}^n$ be a domain satisfying the following condition: there exists a constant $A_0 \in (0,1]$ such that for every $x \in \Omega$ and $0 < r < \operatorname{diam}\Omega$, we have*

$$|\Omega(x,r)| \ge A_0 |B(x,r)|, \quad \text{where } \Omega(x,r) := \Omega \cap B(x,r).$$

Suppose that a function $u \in L^1_{loc}(\overline{\Omega})$ is of Dini mean oscillation in Ω, then there exists a uniformly continuous function u^* on Ω such that $u^* = u$ a.e. in Ω.

Proof In the proof we shall denote

$$\bar{u}_{x,r} = \bar{u}_{\Omega(x,r)} = \fint_{\Omega(x,r)} u \quad \text{and} \quad \omega(r) = \omega_u(r) = \sup_{x\in\overline{\Omega}} \fint_{\Omega(x,r)} |u(y) - \bar{u}_{x,r}|\,dy.$$

By taking the average over $\Omega(x, \frac{1}{2}r)$ to the triangle inequality

$$|\bar{u}_{x,r} - \bar{u}_{x,\frac{1}{2}r}| \le |u - \bar{u}_{x,r}| + |u - \bar{u}_{x,\frac{1}{2}r}|$$

and using $|\Omega(x,r)|/|\Omega(x,\frac{1}{2}r)| \le 2^n/A_0$, we get

$$|\bar{u}_{x,r} - \bar{u}_{x,r/2}| \le (2^n/A_0)\omega(r) + \omega\left(\tfrac{1}{2}r\right) \le (2^n/A_0)\left(\omega(r) + \omega\left(\tfrac{1}{2}r\right)\right).$$

By telescoping, we get

$$\begin{aligned}|\bar{u}_{x,r}-\bar{u}_{x,2^{-k}r}| &\le \sum_{j=0}^{k-1}|\bar{u}_{x,2^{-j}r}-\bar{u}_{x,2^{-(j+1)}r}|\\ &\le (2^n/A_0)\left(\omega(r)+2\omega\left(\tfrac{1}{2}r\right)+\cdots+2\omega\left(\tfrac{1}{2^{k-1}}r\right)+\omega\left(\tfrac{1}{2^k}r\right)\right)\\ &\le (2^{n+1}/A_0)\sum_{j=0}^{\infty}\omega\left(\tfrac{1}{2^j}r\right)\lesssim\int_0^r\frac{\omega(t)}{t}\,dt, \end{aligned}\tag{A.2}$$

where in the last step we used the fact that $\omega(t)\simeq\omega\left(\frac{1}{2^j}r\right)$ when $t\in\left(\frac{1}{2^{j+1}}r,\,\frac{1}{2^j}r\right]$; see [4]. Note that the last inequality also implies that

$$\omega(r)\lesssim\int_0^r\frac{\omega(t)}{t}\,dt.\tag{A.3}$$

Now, we define the function u^* on Ω by setting $u^*(x)=\lim_{r\to 0}\bar{u}_{x,r}$. By the Lebesgue differentiation theorem, we have $u=u^*$ a.e. By letting $k\to\infty$ in (A.2), we obtain

$$|u^*(x)-\bar{u}_{x,r}|\lesssim\int_0^r\frac{\omega(t)}{t}\,dt\quad\text{for a.e. } x\in\Omega.\tag{A.4}$$

For any x, y in Ω, let $r=|x-y|$, $z=\frac{1}{2}(x+y)$, and use Eq. A.4 to get

$$|u^*(x)-u^*(y)|\le|u^*(x)-\bar{u}_{x,r}|+|u^*(y)-\bar{u}_{y,r}|+|\bar{u}_{x,r}-\bar{u}_{y,r}|\lesssim\int_0^{|x-y|}\frac{\omega(t)}{t}\,dt+|\bar{u}_{x,r}-\bar{u}_{y,r}|.$$

By taking the average over $\Omega(z,\frac{1}{2}r)$ to the triangle inequality

$$|\bar{u}_{x,r}-\bar{u}_{y,r}|\le|u-\bar{u}_{x,r}|+|u-\bar{u}_{y,r}|$$

and noting that $\Omega(z,\frac{1}{2}r)\subset\Omega(x,r)\cap\Omega(y,r)$, we get

$$|\bar{u}_{x,r}-\bar{u}_{y,r}|\le(2^{n+1}/A_0)\omega(|x-y|).$$

Combining together and using Eq. A.3, we conclude that

$$|u^*(x)-u^*(y)|\lesssim\int_0^{|x-y|}\frac{\omega(t)}{t}\,dt+\omega(|x-y|)\lesssim\int_0^{|x-y|}\frac{\omega(t)}{t}\,dt.$$

Therefore, we see that u^* is uniformly continuous with it the modulus of continuity dominated by the function $\rho(r):=\displaystyle\int_0^r\frac{\omega(t)}{t},dt.$ □

References

1. Bauman, P.: Equivalence of the Green's functions for diffusion operators in $\mathbb{R}^n$: a counterexample. Proc. Amer. Math. Soc. **91**(1), 64–68 (1984)
2. Bauman, P.: Positive solutions of elliptic equations in nondivergence form and their adjoints. Ark. Mat. **22**(2), 153–173 (1984)
3. Bauman, P.: A Wiener test for nondivergence structure, second-order elliptic equations. Indiana Univ. Math. J. **34**(4), 825–844 (1985)
4. Dong, H., Kim, S.: On C^1, C^2, and weak type-(1, 1) estimates for linear elliptic operators. Commun. Partial Differ. Equ. **42**(3), 417–435 (2017)

5. Dong, H., Escauriaza, L., Kim, S.: On C^1, C^2, and weak type-$(1, 1)$ estimates for linear elliptic operators: part II. Math. Ann. **370**(1–2), 447–489 (2018)
6. Escauriaza, L.: Bounds for the fundamental solution of elliptic and parabolic equations in nondivergence form. Commun. Partial Differ. Equ. **25**(5–6), 821–845 (2000)
7. Escauriaza, L., Montaner, S.: Some remarks on the L^p regularity of second derivatives of solutions to non-divergence elliptic equations and the Dini condition. Rend. Lincei Mat. Appl. **28**, 49–63 (2017)
8. Fabes, E.B., Stroock, D.W.: The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations. Duke Math. J. **51**(4), 997–1016 (1984)
9. Fabes, E., Garofalo, N., Marín-Malave, S., Salsa, S.: Fatou theorems for some nonlinear elliptic equations. Rev. Mat. Iberoamericana **4**(2), 227–251 (1988)
10. Friedman, A.: Partial Differential Equations of Parabolic Type. Prentice-Hall, Inc., Englewood Cliffs (1964)
11. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Reprint of the 1998 edition. Classics in Mathematics. Springer, Berlin (2001)
12. Grüter, M., Widman, K.-O.: The Green function for uniformly elliptic equations. Manuscr. Math. **37**(3), 303–342 (1982)
13. Hofmann, S., Kim, S.: The Green function estimates for strongly elliptic systems of second order. Manuscr. Math. **124**(2), 139–172 (2007)
14. Krylov, N.V.: On one-point weak uniqueness for elliptic equations. Commun. Partial Differ. Equ. **17**(11–12), 1759–1784 (1992)