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Stochastic Completeness and Gradient Representations for Sub-Riemannian Manifolds

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Abstract Given a second order partial differential operator L satisfying the strong Hörmander condition with corresponding heat semigroup P_t, we give two different stochastic representations of $dP_t f$ for a bounded smooth function f. We show that the first identity can be used to prove infinite lifetime of a diffusion of $\frac{1}{2}L$, while the second one is used to find an explicit pointwise bound for the horizontal gradient on a Carnot group. In both cases, the underlying idea is to consider the interplay between sub-Riemannian geometry and connections compatible with this geometry.

Keywords Diffusion process · Stochastic completeness · Hypoelliptic operators · Gradient bound · Sub-Riemannian geometry

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1 Introduction

A Brownian motion on a Riemannian manifold (M, g) is a diffusion process with infinitesimal generator equal to one-half of the Laplace-Beltrami operator Δ_g on M. If (M, g) is a complete Riemannian manifold, a lower bound for the Ricci curvature is a sufficient condition for Brownian motion to have infinite lifetime [47]. Stated in terms of the minimal heat kernel $p_t(x, y)$ to $\frac{1}{2}\Delta_g$, this means that

$$\int_M p_t(x, y)\, d\mu(y) = 1$$

for any $(t, x) \in (0, \infty) \times M$, where $\mu = \mu_g$ is the Riemannian volume measure. Infinite lifetime of the Brownian motion is equivalent to uniqueness of solutions to the heat equation in L^∞, see e.g. [23, 27 Section 5]. Furthermore, let P_t denote the minimal heat semigroup of $\frac{1}{2}\Delta_g$ and let ∇f denote the gradient of a smooth function with respect to g. Then a lower Ricci bound also guarantees that $t \mapsto \|\nabla P_t f\|_{L^\infty(g)}$ is bounded on any finite interval whenever ∇f is bounded. This fact allows one to use the Γ_2-calculus of Bakry-Émery, see e.g. [5, 6].

For a second order partial differential operator L on M, let $\sigma(L) \in \Gamma(\mathrm{Sym}^2\, TM)$ denote its symbol, i.e. the symmetric, bilinear tensor on the cotangent bundle T^*M uniquely determined by the relation

$$\sigma(L)(df, d\phi) = \frac{1}{2}\left(L(f\phi) - fL\phi - \phi Lf\right), \quad f, \phi \in C^\infty(M). \tag{1.1}$$

If L is elliptic, then $\sigma(L)$ coincides with the cometric g^* of some Riemannian metric g and L can be written as $L = \Delta_g + Z$ for some vector field Z. Hence, we can use the geometry of g along with the vector field Z to study the properties of the heat flow of L, see e.g. [46]. If $\sigma(L)$ is only positive semi-definite we can still associate a geometric structure known as a sub-Riemannian structure. Recently, several results have appeared linking sub-Riemannian geometric invariants to properties of diffusions of corresponding second order operators and their heat semigroup, see [8, 10, 12, 24, 25]. These results are based on a generalization of the Γ_2-calculus for sub-Riemannian manifolds, first introduced in [11]. As in the Riemannian case, the preliminary requirements for using this Γ_2-calculus is that the diffusion of L has infinite lifetime and that the gradient of a function does not become unbounded under the application of the heat semigroup.

Consider the following example of an operator L with positive semi-definite symbol. Let (M, g) be a complete Riemannian manifold with a foliation $\mathcal{F}$ corresponding to an integrable distribution V. Let H be the orthogonal complement of V with corresponding orthogonal projection pr_H and define a second order operator L on M by

$$Lf = \operatorname{div}(\mathrm{pr}_H \nabla f), \quad f \in C^\infty(M). \tag{1.2}$$

If H satisfies the bracket-generating condition, meaning that the sections of H along with their iterated brackets span the entire tangent bundle, then L is a hypoelliptic operator by Hörmander's classical theorem [30]. The operator L corresponds to the sub-Riemannian metric $g_H = g|H$. Let us make the additional assumption that leaves of the foliation are totally geodesic submanifolds of M and that the foliation is Riemannian. If only the first order brackets are needed to span the entire tangent bundle, it is known that any $\frac{1}{2}L$-diffusion X_t has infinite lifetime given certain curvature bounds [25, Theorem 3.4]. Furthermore, if H satisfies the Yang-Mills condition, then no assumption on the number of brackets is needed to span the tangent bundle is necessary [12, Section 4], see Remark 3.16

for the definition of the Yang-Mills condition. Under the same restrictions, for any smooth function f with bounded gradient, $t \mapsto \|\nabla P_t f\|_{L^\infty(g)}$ remains bounded on a finite interval.

We will show how to modify the argument in [12] to go beyond the requirement of the Yang-Mills condition and even beyond foliations. We will start with some preliminaries on sub-Riemannian manifolds and sub-Laplacians in Section 2. In Section 3.1 we will show that existence of a Weitzenböck type formula for a connection sub-Laplacian always corresponds to the adjoint of a connection compatible with a sub-Riemannian structure. Our results on infinite lifetime are presented in Section 3.3 based on a Feynman-Kac representation of $dP_t f$ using a particular adjoint of a compatible connection. Using recent results of [18], we also show that our curvature requirement in the case of totally geodesic foliations implies that the Brownian motion of the full Riemannian metric g has infinite lifetime as well, see Section 3.7.

Our Feynman-Kac representation in Section 3.3 uses parallel transport with respect to a connection that does not preserve the horizontal bundle. In Section 4.1 we give an alternative stochastic representation of $dP_t f$ using parallel transport along a connection that preserves the sub-Riemannian structure. This rewritten representation allows us to derive an explicit pointwise bound for the horizontal gradient in Carnot groups. For a smooth function f on M, the horizontal gradient $\nabla^H f$ is defined by the condition that $\alpha(\nabla^H f) = \sigma(L)(df, \alpha)$ for any $\alpha \in T^*M$. Carnot groups are the 'flat model spaces' in sub-Riemannian geometry in the sense that their role is similar to that of Euclidean spaces in Riemannian geometry. See Section 4.3 for the definition. It is known that there exists pointwise bounds for the horizontal gradient on Carnot groups. From [34], there exist constants C_p such that

$$|\nabla^H P_t f|_{g_H} \leq C_p \left(P_t |\nabla^H f|^p_{g_H} \right)^{1/p}, \quad p \in (1, \infty), \tag{1.3}$$

holds pointwise for any $t > 0$. This can even be extended to $p = 1$ in the case of the Heisenberg group [32]. According to [16], the constant C_p has to be strictly larger than 1. We give explicit constants for the gradient estimates on Carnot groups. Our results improve on the constant found in [4] for the special case of the Heisenberg group. Also, for $p > 2$ we find a constant that does not depend on the heat kernel.

Appendix A deals with Feynman-Kac representations of semigroups whose generators are not necessarily self-adjoint, which is needed for the result in Section 3.3.

2 Sub-Riemannian Manifolds and Sub-Laplacians

2.1 Sub-Riemannian Manifolds

We define *a sub-Riemannian manifold* as a triple (M, H, g_H) where M is a connected manifold, $H \subseteq TM$ is a subbundle of the tangent bundle and g_H is a metric tensor defined only on H. Such a structure induces a map $\sharp^H : T^*M \to H \subseteq TM$ by the formula

$$\alpha(v) = \langle \sharp^H \alpha, v \rangle_{g_H} := g_H(\sharp^H \alpha, v), \quad \alpha \in T_x^* M, \; v \in H_x, \; x \in M. \tag{2.1}$$

The kernel of this map is the subbundle $\operatorname{Ann}(H) \subseteq T^*M$ of covectors vanishing on H. This map $\sharp^H$ induces a cometric g_H^* on T^*M by the formula

$$\langle \alpha, \beta \rangle_{g_H^*} = \langle \sharp^H \alpha, \sharp^H \beta \rangle_{g_H}, \tag{2.2}$$

which is degenerate unless $H = TM$. Conversely, given a cometric g_H^* degenerating along a subbundle of T^*M, we can define $\sharp^H \alpha = g_H^*(\alpha, \cdot)$ and use (2.2) to obtain g_H. Going forward, we will refer to g_H^* and (H, g_H) interchangeably as a *sub-Riemannian structure* on M. We will call H *the horizontal bundle*. For the rest of the paper, n is the rank of H while $n + \nu$ denotes the dimension of M.

Let μ be a chosen smooth volume density with corresponding divergence div_μ. Relative to μ, we can define a second order operator

$$\Delta_H f := \Delta_{g_H} f = \operatorname{div}_\mu \sharp^H df. \tag{2.3}$$

By means of definition (1.1), the symbol of Δ_H satisfies $\sigma(\Delta_H) = g_H^*$. Locally the operator Δ_H can be written as

$$\Delta_H f = \sum_{i=1}^n A_i^2 f + A_0 f, \quad n = \operatorname{rank} H,$$

where $A_0, A_1, \dots, A_n$ are vector fields taking values in H such that $A_1, \dots, A_n$ form a local orthonormal basis of H.

The horizontal bundle H is called *bracket-generating* if the sections of H along with its iterated brackets span the entire tangent bundle. The horizontal bundle is said to have *step* k at x if $k - 1$ is the minimal order of iterated brackets needed to span $T_x M$. From the local expression of Δ_H, it follows that H is bracket-generating if and only if Δ_H satisfies *the strong Hörmander condition* [30]. We shall assume that this condition indeed holds, giving us that both Δ_H and $\frac{1}{2}\Delta_H - \partial_t$ are hypoelliptic and that

$$\mathsf{d}_{g_H}(x, y) := \sup\left\{|f(x) - f(y)| : \ f \in C_c^\infty(M), \ \sigma(\Delta_H)(df, df) \le 1\right\}, \tag{2.4}$$

is a well defined distance on M. Here, and in the rest of the paper, $C_c^\infty(M)$ denotes the smooth, compactly supported functions on M. Alternatively, the distance $\mathsf{d}_{g_H}(x, y)$ can be realized as the infimum of the lengths of all absolutely continuous curves tangent to H and connecting x and y. The bracket-generating condition ensures that such curves always exist between any pair of points. For more information on sub-Riemannian manifolds, we refer to [36].

In what follows, we will always assume that H is bracket-generating, unless otherwise stated explicitly. We note that if Δ_H satisfies the strong Hörmander condition and if d_{g_H} is a complete metric, then $\Delta_H | C_c^\infty(M)$ is essentially self-adjoint by [41, Chapter 12].

For the remainder of the paper, we make the following notational conventions. If $p : E \to M$ is a vector bundle, we denote by $\Gamma(E)$ the space of smooth sections of E. If E is equipped with a connection ∇ or a (possibly degenerate) metric tensor g, we denote the induced connections on E^*, $\bigwedge^2 E$, etc. by the same symbol, while the induced metric tensors are denoted by g^*, $\wedge^2 g$, etc. For elements e_1, e_2, we write $g(e_1, e_2) = \langle e_1, e_2 \rangle_g$ and $|e_1|_g = \langle e_1, e_1 \rangle_g^{1/2}$ even in the cases when g is only positive semi-definite. If μ is a chosen volume density on M and f is a function on M, we write $\|f\|_{L^p}$ for the corresponding L^p-norm with the volume density being implicit. If $Z \in \Gamma(E)$ then $\|Z\|_{L^p(g)} := \||Z|_g\|_{L^p}$.

For $x \in M$, if $\mathscr{A} \in \operatorname{End} T_x M$ is an endomorphism, we let $\mathscr{A}^\mathsf{T} \in \operatorname{End} T_x^* M$ denote its transpose. If M is equipped with a Riemannian metric g, then $\mathscr{A}^* \in \operatorname{End} T_x^* M$ denotes its dual. In other words,

$$\langle \mathscr{A} v, w \rangle_g = \langle v, \mathscr{A}^* w \rangle_g, \quad (\mathscr{A}^\mathsf{T} \alpha)(v) = \alpha(\mathscr{A} v), \quad \alpha \in T_x^* M, \quad v, w \in T_x M.$$

The same conventions apply for endomorphisms of $T^* M$. If $\mathscr{A}$ is a differential operator, then $\mathscr{A}^*$ is defined with respect to the L^2-inner product of g.

2.2 Taming Metrics

Given a sub-Riemannian manifold (M, H, g_H), a Riemannian metric g on M is said to *tame* g_H if $g|H = g_H$. If d_g is the corresponding Riemannian distance, then $\mathsf{d}_g(x, y) \leq \mathsf{d}_{g_H}(x, y)$ for any $x, y \in M$, since curves tangent to H have equal length with respect to both metrics, while d_g considers the infimum of the lengths over curves that are not tangent to H as well. It follows that if d_g is complete, then d_{g_H} is a complete metric as well, as observed in [41, Theorem 7]. By [40, Theorem 2.4], if g is a complete Riemannian metric taming g_H, then the sub-Laplacian Δ_H with respect to the volume density of g and the Laplace-Beltrami operator Δ_g are both essentially self adjoint on $C_c^\infty(M)$.

Given g, we denote the corresponding orthogonal projection to H by pr_H. Let $\flat : TM \to T^*M$ be the vector bundle isomorphism $v \mapsto \langle v, \cdot \rangle_g$ with inverse $\sharp$. The fact that g tames g_H is equivalent to the statement that $\sharp^H = \mathrm{pr}_H \sharp$. Let V denote the orthogonal complement of H with corresponding projection. *The curvature* $\mathcal{R}$ and *the cocurvature* $\bar{\mathcal{R}}$ of H with respect to the complement V are defined as

$$\mathcal{R}(A, Z) = \mathrm{pr}_V[\mathrm{pr}_H A, \mathrm{pr}_H Z], \qquad \bar{\mathcal{R}}(A, Z) = \mathrm{pr}_H[\mathrm{pr}_V A, \mathrm{pr}_V Z], \tag{2.5}$$

for $A, Z \in \Gamma(TM)$. By definition, $\mathcal{R}$ and $\bar{\mathcal{R}}$ are vector-valued two-forms, and $\bar{\mathcal{R}}$ vanishes if and only if V is integrable. The curvature and the cocurvature only depend on the direct sum $TM = H \oplus V$ and not the metrics g_H or g.

2.3 Connections Compatible with the Metric

Let ∇ be an affine connection on TM. We say that ∇ is compatible with the sub-Riemannian structure (H, g_H) or g_H^* if $\nabla g_H^* = 0$. This condition is equivalent to requiring that ∇ preserves the horizontal bundle H under parallel transport and that $Z\langle A_1, A_2\rangle_{g_H} = \langle \nabla_Z A_1, A_2\rangle_{g_H} + \langle A_1, \nabla_Z A_2\rangle_{g_H}$ for any $Z \in \Gamma(TM)$, $A_1, A_2 \in \Gamma(H)$. For any sub-Riemannian manifold (M, H, g_H), the set of compatible connections is non-empty. Let $\tilde{g}$ be any Riemannian metric on M and define V as the orthogonal complement to H. Let pr_H and pr_V be the corresponding orthonormal projections. Define

$$g = \mathrm{pr}_H^* g_H + \mathrm{pr}_V^* \tilde{g}|V.$$

Then g is a metric taming g_H. Let ∇^g be the Levi-Civita connection of g and define finally

$$\nabla^0 := \mathrm{pr}_H \nabla^g \mathrm{pr}_H + \mathrm{pr}_V \nabla^g \mathrm{pr}_V\, . \tag{2.6}$$

The connection ∇^0 will be compatible with g_H^* and also with g.

2.4 Rough Sub-Laplacians

In this section we introduce rough sub-Laplacians and compare them to the sub-Laplacian as defined in (2.3). Let $g_H^* \in \Gamma(\mathrm{Sym}^2 TM)$ be a sub-Riemannian structure on M with horizontal bundle H. For any two-tensor $\xi \in \Gamma(T^*M^{\otimes 2})$ we write $\mathrm{tr}_H \xi(\times, \times) := \xi(g_H^*)$. We use this notation since for any $x \in M$ and any orthonormal basis $v_1, \dots, v_n$ of H_x

$$\mathrm{tr}_H \xi(x)(\times, \times) = \sum_{i=1}^{n} \xi(x)(v_i, v_i).$$

For any affine connection ∇ on TM, define the Hessian ∇^2 by

$$\nabla^2_{A,B} = \nabla_A \nabla_B - \nabla_{\nabla_A B}.$$

We define *the rough sub-Laplacian* $L(\nabla)$ as $L(\nabla) = \operatorname{tr}_H \nabla^2_{\times,\times}$. Since ∇ induces a connection on all tensor bundles, $L(\nabla)$ defines as an operator on tensors in general. We have the following result.

Lemma 2.1 (a) *Let μ be a volume density on M with corresponding sub-Laplacian Δ_H. Assume that H is a proper subbundle in TM. Then there exists some connection ∇ compatible with g_H^* and satisfying $L(\nabla)f = \Delta_H f$.*

(b) *Let g be a Riemannian metric taming g_H and with volume form μ. Let ∇ be a connection compatible with both g_H^* and g. Let T^∇ be the torsion of ∇ and define the one-form β by*

$$\beta(v) = \operatorname{tr} T^\nabla(v, \cdot).$$

Then the dual of $L = L(\nabla)$ on tensors is given by

$$L^* = L - 2\nabla_{\sharp^H\beta} - \operatorname{div}_\mu \sharp^H\beta = L + (\nabla_{\sharp^H\beta})^* - \nabla_{\sharp^H\beta}.$$

In particular, $Lf = \Delta_H f + \langle \beta, df\rangle_{g_H^*}$ for any $f \in C^\infty(M)$.

Proof (a) If H is properly contained in TM, then there is some Riemannian metric g such that $g|H = g_H$ and such that μ is the volume form g. Define ∇^0 as in (2.6) and for any endomorphism valued one-form $\kappa \in \Gamma(T^*M \otimes \operatorname{End} T^*M)$, define a connection $\nabla^\kappa_v = \nabla^0_v + \kappa(v)$. The connection ∇^κ is compatible with g_H^* if and only if

$$\langle \kappa(v)\alpha, \alpha\rangle_{g_H^*} = 0, \quad v \in TM, \alpha \in T^*M. \tag{2.7}$$

Furthermore, $L(\nabla^\kappa)f = L(\nabla^0)f + (\operatorname{tr}_H \kappa(\times)^\top \times)f$.

Define $Z = \Delta_H - L(\nabla^0)$. We want to show that there is an endomorphism-valued one-form κ such that $\operatorname{tr}_H \kappa(\times)^\top \times = Z$ and such that (2.7) holds. By a partition of unity argument, it is sufficient to consider Z as defined on a small enough neighborhood U such that both TM and H are trivial. Let η be any one-form on U such that

$$|\eta|_{g_H^*} = 1, \qquad \eta(Z) = 0.$$

Let ζ be a one-form such that $\sharp^H\zeta = Z$. Define κ by

$$\kappa(v)\alpha = \eta(v)\big(\alpha(Z)\eta - \alpha(\sharp^H\eta)\zeta\big).$$

We observe that $\langle \kappa(v)\alpha, \alpha\rangle_{g_H^*} = \eta(v)(\alpha(Z)\alpha(\sharp^H\eta) - \alpha(\sharp^H\eta)\alpha(Z)) = 0$. Furthermore, if we choose a local orthonormal basis $A_1, \dots, A_n$ of H such that $A_1 = \sharp^H\eta$, then $\eta(A_j) = \delta_{1,j}$ while $\zeta(A_1) = 0$. Hence

$$\alpha(\operatorname{tr}_H \kappa(\times)^\top \times) = \sum_{j=1}^n \eta(A_j)(\alpha(Z)\eta(A_j) - \alpha(\sharp^H\eta)\zeta(A_j)) = \alpha(Z),$$

and so the one-form κ has the desired properties.

(b) For any connection ∇ preserving the Riemannian metric g, we have

$$\operatorname{div}_\mu Z = \sum_{i=1}^n \langle \nabla_{A_i} Z, A_i\rangle_g + \sum_{s=1}^\nu \langle \nabla_{Z_s} Z, Z_s\rangle_g - \beta(Z), \tag{2.8}$$

with respect to local orthonormal bases $A_1, \dots, A_n$ and $Z_1, \dots, Z_\nu$ of respectively H and V.

For any pair of vector fields A and B consider an operator $F(A \otimes B) = \flat A \otimes \nabla_B$ on tensors with dual

$$F(A \otimes B)^* = -\iota_{(\operatorname{div} B)A} - \iota_{\nabla_B A} - \iota_A \nabla_B.$$

Extend F to arbitrary sections of $TM^{\otimes 2}$ by $C^\infty(M)$-linearity and consider the operator $F(g_H^*)$. Since ∇ preserves H, its orthogonal complement V and their respectice metrics, around any point x we can find local orthonormal bases $A_1, \dots, A_n$ and $Z_1, \dots, Z_\nu$ of respectively H and V that are parallel at any arbitrary point x. Hence, in any local orthonormal basis

$$F(g_H^*)^* = \iota_{\sharp^H\beta} - \sum_{i=1}^n \iota_{A_i} \nabla_{A_i},$$

and so

$$F(g_H^*)^* F(g_H^*) = -L + \nabla_{\sharp^H\beta} = -L^* + \left(\nabla_{\sharp^H\beta}\right)^*. \qquad \square$$

Remark 2.2 As a result of the proof of Lemma 2.1, we actually know that all second order operators on the form $L(\nabla^0) + Z$ for some $Z \in \Gamma(H)$ is given as the rough sub-Laplacian of some connection compatible with the metric g_H.

3 Adjoint Connections and Infinite Lifetime

3.1 A Weitzenböck Formula for Sub-Laplacians

In the case of Riemannian geometry $g_H = g$, one of the central identities involving the rough Laplacian of the Levi-Civita connection $L(\nabla^g)$ is the Weitzenböck formula $L(\nabla^g)df = \mathrm{Ric}_g(\sharp df, \cdot) + dL(\nabla^g)f = \mathrm{Ric}_g(\sharp df, \cdot) + d\Delta_g f$. A similar formula can be introduced in sub-Riemannian geometry, as was observed in [20] using the concept of adjoint connections. Adjoint connections were first considered in [15].

If ∇ is a connection on TM with torsion T^∇, then its adjoint $\hat{\nabla}$ is defined by

$$\hat{\nabla}_A B = \nabla_A B - T^\nabla(A, B).$$

for any $A, B \in \Gamma(TM)$. We remark that $-T^\nabla$ is the torsion of $\hat{\nabla}$, so ∇ is the adjoint of $\hat{\nabla}$.

Proposition 3.1 (Sub-Riemannian Weitzenböck formula) *Let L be any rough sub-Laplacian of an affine connection. Then there exists a vector bundle endomorphism $\mathscr{A} : T^*M \to T^*M$ such that for any $f \in C^\infty(M)$,*

$$(L - \mathscr{A})df = dLf \tag{3.1}$$

if and only if $L = L(\hat{\nabla})$ for some adjoint $\hat{\nabla}$ of a connection ∇ that is compatible with g_H^*. In this case, $\mathscr{A} = \mathrm{Ric}(\nabla)$, where

$$\mathrm{Ric}(\nabla)(\alpha)(v) := \mathrm{tr}_H\, R^\nabla(\times, v)\alpha(\times). \tag{3.2}$$

We note that the bracket-generating assumption is not necessary for this result.

Remark 3.2

(i) Let ∇ be a connection satisfying $\nabla g_H^* = 0$ and let $\hat{\nabla}$ be its adjoint. By [22, Proposition 2.1] any smooth curve γ in M is a normal sub-Riemannian geodesic if and only if there is a one-form $\lambda(t)$ along $\gamma(t)$ such that

$$\sharp^H \lambda(t) = \dot{\gamma}(t), \quad \text{and} \quad \hat{\nabla}_{\dot{\gamma}} \lambda(t) = 0.$$

See the reference for the definition of normal geodesic. In this sense, adjoints of compatible connections occur naturally in sub-Riemannian geometry.

(ii) A Weitzenböck formula in the sub-Riemannian case first appeared in [20, Chapter 2.4], see also [19]. This formulation assumes that the connection ∇ can be represented as a Le Jan-Watanabe connection. For definition and the proof of the fact that all connections on a vector bundle compatible with some metric there are of this type, see [20, Chapter 1]. We will give the proof of Proposition 3.1 without this assumption, in order to obtain an equivalence between existence of a Weitzenböck formula and being an adjoint of a compatible connection.

Before continuing with the proof, we will need the next lemma.

Lemma 3.3 *Let ∇ be an affine connection with adjoint $\hat{\nabla}$. Assume that ∇ is compatible with g_H^* and denote $L = L(\nabla)$, $\mathrm{Ric} = \mathrm{Ric}(\nabla)$ and $\hat{L} = L(\hat{\nabla})$. For any endomorphism-valued one-form $\kappa \in \Gamma(T^*M \otimes \operatorname{End} T^*M)$ let ∇^κ be the connection*

$$\nabla_v^\kappa := \nabla_v + \kappa(v), \quad v \in TM. \tag{3.3}$$

(a) *If the horizontal bundle H is a proper subbundle of TM and bracket-generating then the connection $\hat{\nabla}$ does not preserve H under parallel transport.*

(b) *Define $L^\kappa = L(\nabla^\kappa)$. Then*

$$L^\kappa = L + \nabla_{Z^\kappa} + 2D^\kappa + \kappa(Z^\kappa) + \mathrm{tr}_H(\nabla_\times \kappa)(\times) + \mathrm{tr}_H\, \kappa(\times)\kappa(\times)$$

where $Z^\kappa = \mathrm{tr}_H\, \kappa(\times)^\top \times$ and $D^\kappa = \mathrm{tr}_H\, \kappa(\times)\nabla_\times$. In particular, for any function $f \in C^\infty(M)$,

$$L^\kappa f = Lf + Z^\kappa f \quad \text{and} \quad \hat{L} f = Lf.$$

(c) *The adjoint $\hat{\nabla}^\kappa$ of ∇^κ is given by $\hat{\nabla}_v^\kappa = \hat{\nabla}_v + \hat{\kappa}(v)$ where*

$$(\hat{\kappa}(v)\alpha)(w) := (\kappa(w)\alpha)(v), \quad \text{for } v, w \in TM, \ \alpha \in T^*M.$$

In particular, if ∇^κ is compatible with g_H^* then $\hat{\kappa}(\sharp^H \alpha)\alpha = 0$ for any $\alpha \in T^*M$.

Proof (a) Let $A, B \in \Gamma(H)$ be any two vector fields such that $[A, B]$ is not contained in H. Observe that $\hat{\nabla}_A B = \nabla_B A + [A, B]$ then cannot be contained in H either.

(b) This follows by direct computation: for any local orthonormal basis $A_1, \ldots, A_n$ of H, we have

$$\begin{aligned} L^\kappa &= \sum_{i=1}^n \big(\nabla_{A_i} + \kappa(A_i)\big)\big(\nabla_{A_i} + \kappa(A_i)\big) \\ &\quad - \sum_{i=1}^n \Big(\nabla_{\nabla_{A_i} A_i - \kappa(A_i)^\top A_i} + \kappa(\nabla_{A_i} A_i - \kappa(A_i)^\top A_i)\Big) \\ &= \sum_{i=1}^n \nabla_{A_i}\nabla_{A_i} + \sum_{i=1}^n \nabla_{A_i}\kappa(A_i) + \sum_{i=1}^n \kappa(A_i)\nabla_{A_i} + \sum_{i=1}^n \kappa(A_i)\kappa(A_i) \\ &\quad + \nabla_{Z^\kappa} + \kappa(Z^\kappa) - \sum_{i=1}^n \Big(\nabla_{\nabla_{A_i} A_i} + \kappa(\nabla_{A_i} A_i)\Big) \\ &= L + 2\,\mathrm{tr}_H\, \kappa(\times)\nabla_\times + \mathrm{tr}_H(\nabla_\times \kappa)(\times) + \mathrm{tr}_H\, \kappa(\times)\kappa(\times) + \nabla_{Z^\kappa} + \kappa(Z^\kappa). \end{aligned}$$

For the special case of $\nabla^\kappa = \hat{\nabla}$, we have $\kappa(v)^\top w = -T^\nabla(v, w)$ and hence $Z^\kappa = 0$ as a consequence.

(c) Follows from the definition and (2.7). □

Proof of Proposition 3.1 Notice that $\iota_A \nabla_B df = \iota_B \hat{\nabla}_A df$. Since ∇ is compatible with g_H^*, for any $x \in M$ there is a local orthonormal basis $A_1, \dots, A_n$ of H such that $\nabla A_j(x) = 0$. Hence, for an arbitrary vector field $Z \in \Gamma(TM)$, with the terms below evaluated at $x \in M$ implicitly,

$$\begin{aligned}\iota_Z dL(\hat{\nabla})f &= \iota_Z dL(\nabla)f = Z\sum_{i=1}^n \nabla_{A_i} df(A_i) = \sum_{i=1}^n \nabla_Z \nabla_{A_i} df(A_i) \\ &= \sum_{i=1}^n \iota_{A_i} R^\nabla(Z, A_i)df + \sum_{i=1}^n \nabla_{A_i}\nabla_Z df(A_i) + \nabla_{[Z,A_i]}df(A_i) \\ &= -\operatorname{Ric}(df)(Z) + \sum_{i=1}^n A_i \nabla_Z df(A_i) - \nabla_{\hat{\nabla}_{A_i} Z} df(A_i) \\ &= -\operatorname{Ric}(df)(Z) + \sum_{i=1}^n A_i \hat{\nabla}_{A_i} df(Z) - \hat{\nabla}_{A_i} df(\hat{\nabla}_{A_i} Z) \\ &= \iota_Z(-\operatorname{Ric}(df) + L(\hat{\nabla})df).\end{aligned}$$

Since x was arbitrary, it follows that $L(\hat{\nabla})$ satisfies (3.1).

Conversely, suppose that $L = L(\nabla')$ is an arbitrary rough Laplacian of ∇'. Let ∇ be an arbitrary connection compatible with g_H^* and define κ such that $\nabla'_v = \hat{\nabla}^\kappa_v = \hat{\nabla}_v + \hat{\kappa}(v)$, where ∇^κ is defined as in (3.3). We introduce the vector field $Z = \operatorname{tr}_H \hat{\kappa}(\times)^\top \times$ and the first order operator $D = \operatorname{tr}_H \hat{\kappa}(\times)\nabla_\times$. Using item (3.3) of Lemma 3.3, modulo zero order operators applied to df, $Ldf - dLf$ equals $-dZf + \nabla_Z df + 2Ddf$. Furthermore, $-dZf + \nabla_Z df = (\nabla_Z - \mathcal{L}_Z)df$ and $(\nabla_Z - \mathcal{L}_Z)$ is a zero order operator. Hence, it follows that (3.1) holds if and only if $Ddf = \mathscr{C}df$ for some zero order operator $\mathscr{C}$ and any $f \in C^\infty(M)$.

Let $A_1, \dots, A_n$ be a local orthonormal basis of H and complete this basis to a full basis of TM with vector fields $Z_1, \dots, Z_\nu$. Let $A_1^*, \dots, A_n^*, Z_1^*, \dots, Z_\nu^*$ be the corresponding coframe. Observe that $Z_1^*, \dots, Z_\nu^*$ is a basis for $\operatorname{Ann}(H)$. For any $B \in \Gamma(TM)$ and $f \in C^\infty(M)$,

$$(Ddf)(B) = \sum_{i,k=1}^n \left(\hat{\kappa}(A_i)A_k^*(B)\right)\hat{\nabla}_{A_i} df(A_k) + \sum_{i=1}^n\sum_{s=1}^\nu \left(\hat{\kappa}(A_i)Z_s^*(B)\right)\hat{\nabla}_{A_i} df(Z_s).$$

In order for this to correspond to a zero order operator, we must have $\hat{\kappa}(A_i)Z_s^* = 0$ and $\hat{\kappa}(A_i)(A_k^*) = -\hat{\kappa}(A_k)(A_i^*)$ which is equivalent to $\hat{\kappa}(\sharp^H\alpha)\alpha = 0$ for any $\alpha \in T^*M$. Hence, $\hat{\nabla}^\kappa$ is the adjoint of a connection compatible with g_H^*. □

3.2 Connections with Skew-symmetric Torsion

For a sub-Riemannian manifold (M, H, g_H) with H strictly contained in TM, there exists no torsion-free connection compatible with the metric. Indeed, if ∇ is a connection preserving H, then the equality $\nabla_A B - \nabla_B A = [A, B]$ would imply that H could be bracket-generating only if $H = TM$. For this reason, it has been difficult to find a direct analogue of the Levi-Civita connection in sub-Riemannian geometry.

For a Riemannian metric g, the only compatible connections with the same geodesics as the Levi-Civita connection ∇^g, are the compatible connections with *skew-symmetric torsion*, see e.g. [3, Section 2]. These are the connections ∇ compatible with g such that

$$\zeta(v_1, v_2, v_3) := -\langle T^{\nabla}(v_1, v_2), v_3\rangle_g, \quad v_1, v_2, v_3 \in TM,$$

is a well defined three-form. The connection ∇ is then given by formula $\nabla_A B = \nabla^g_A B + \frac{1}{2}T^{\nabla}(A, B) = \nabla^g_A B - \frac{1}{2}\sharp\iota_{A\wedge B}\zeta$. Equivalently, the connection ∇ is compatible with g and of skew-symmetric torsion if and only if we have both $\nabla g = 0$ and $\hat{\nabla} g = 0$. One can not have a direct analogue for proper sub-Riemannian structures g^*_H, since by Lemma 3.3 (a) it is not possible for both ∇ and $\hat{\nabla}$ to be compatible with g^*_H. In some cases, however, we have the following generalization.

Let (M, H, g_H) be a sub-Riemannian manifold with taming Riemannian metric g and $V = H^{\perp}$. Let $\mathcal{L}_A$ denote the Lie derivative with respect to the vector field A. Introduce a vector-valued symmetric bilinear tensor $I\!I$ by the formula

$$\langle I\!I(A, A), Z\rangle_g = -\frac{1}{2}(\mathcal{L}_{\mathrm{pr}_V Z}g)(\mathrm{pr}_H A, \mathrm{pr}_H A) - \frac{1}{2}(\mathcal{L}_{\mathrm{pr}_H Z}g)(\mathrm{pr}_V A, \mathrm{pr}_V A) \tag{3.4}$$

for any $A, Z \in \Gamma(TM)$. Observe that $I\!I = 0$ is equivalent to the assumption

$$(\mathcal{L}_A g)(Z, Z) = 0, \qquad (\mathcal{L}_Z g)(A, A) = 0, \tag{3.5}$$

for any $A \in \Gamma(H)$ and $Z \in \Gamma(V)$.

Proposition 3.4 *Let ∇ be a connection compatible with g^*_H and with adjoint $\hat{\nabla}$. Assume that there exists a Riemannian metric g taming g_H such that $\hat{\nabla} g = 0$. Then $I\!I = 0$. Furthermore, if Δ_H is defined relative to the volume density of g, then*

$$\Big(L(\hat{\nabla}) - \mathrm{Ric}(\nabla)\Big)\, df = dL(\hat{\nabla})f = dL(\nabla)f = d\Delta_H f, \quad f \in C^{\infty}(M).$$

Conversely, suppose that g is a Riemannian metric taming g_H and satisfying $I\!I = 0$. Define $\mathcal{R}$ and $\bar{\mathcal{R}}$ as in (2.5) and introduce a three-form ζ by

$$\zeta(v_1, v_2, v_3) = \circlearrowleft\langle \mathcal{R}(v_1, v_2), v_3\rangle_g + \circlearrowleft\langle \bar{\mathcal{R}}(v_1, v_2), v_3\rangle_g, \tag{3.6}$$

with $\circlearrowleft$ denoting the cyclic sum. Then the connection

$$\nabla_A B = \nabla^g_A B - \frac{1}{2}\sharp\iota_{A\wedge B}\zeta \tag{3.7}$$

is compatible with g^*_H, and both it and its adjoint $\hat{\nabla}_A B = \nabla^g_A B + \frac{1}{2}\sharp\iota_{A\wedge B}\zeta$ are compatible with $\hat{\nabla} g = 0$.

Furthermore, among all such possible choices of connections, ∇ gives the maximal value with regard to the lower bound of $\alpha \mapsto \langle \mathrm{Ric}(\nabla)\alpha, \alpha\rangle_{g^*_H}$.

Remark 3.5 (i) *Analogy to the Levi-Civita connection:* Applying Proportion 3.4 to the case when $g_H = g$ is a Riemannian metric, the Levi-Civita connection can be described as the connection such that both ∇ and $\hat{\nabla}$ are compatible with g and that also maximizes the lower bound $\alpha \mapsto \langle \mathrm{Ric}(\nabla)\alpha, \alpha\rangle_{g^*}$ which was observed in [20, Corollary C.7]. In this sense, the connection in (3.7) is analogous to the Levi-Civita connection.

(ii) *Existence and uniqueness for a Riemannian metrics g taming g_H and satisfying* (3.5): Every taming Riemannian metric g with $I\!I = 0$ is uniquely determined by the orthogonal complement V of H and its value at one point [24, Remark 3.10]. Conversely, suppose that (M, H, g_H) is a sub-Riemannian manifold and let V be a subbundle such that $TM = H \oplus V$. Then one can use horizontal holonomy to determine if there exists a Riemannian metric g taming g_H, satisfying (3.5) and making H and V orthogonal. See [14] for more details and examples where no such metric can be found. Two Riemannian metrics g_1 and g_2 may tame g_H, satisfy (3.5) and have the same volume density but their orthogonal complements of H may be different, see [24, Example 4.6] and [14, Example 4.2].

(iii) *Geometric interpretation of* (3.5): From [22], the condition (3.5) holds if and only if the Riemannian and the sub-Riemannian geodesic flow commute. See also Section 3.7 for more relations to geometry and explanation of the notation $I\!I$ for the tensor in (3.4).

(iv) If we define ∇ as in (3.7) and assume $\bar{\mathcal{R}} = 0$, then its adjoint $\hat{\nabla}$ equals the connection ∇^ε in [7] with $\varepsilon = \frac{1}{2}$.

Proof Let ∇^g be the Levi-Civita connection of g. Define the connection ∇^0 as in (2.6) which is compatible with both g_H^* and the Riemannian metric g. Let T be the torsion of ∇^0. Define $\mathcal{R}$ and $\bar{\mathcal{R}}$ as in (2.5). We write T_Z for the vector valued form $T_Z(A) = T(Z, A)$ and use similar notation for $\mathcal{R}$, $\bar{\mathcal{R}}$ and $I\!I$. By the definition of the Levi-Civita connection, we have

$$T_Z = -\mathcal{R}_Z + \frac{1}{2}\mathcal{R}_Z^* - \bar{\mathcal{R}}_Z + \frac{1}{2}\bar{\mathcal{R}}_Z^* + I\!I_Z^* - I\!I_\cdot^* Z - \frac{1}{2}\mathcal{R}_\cdot^* Z - \frac{1}{2}\bar{\mathcal{R}}_\cdot^* Z,$$

with dual

$$T_Z^* = -\mathcal{R}_Z^* + \frac{1}{2}\mathcal{R}_Z - \bar{\mathcal{R}}_Z^* + \frac{1}{2}\bar{\mathcal{R}}_Z + I\!I_Z - I\!I_\cdot^* Z + \frac{1}{2}\mathcal{R}_\cdot^* Z + \frac{1}{2}\bar{\mathcal{R}}_\cdot^* Z,$$

Hence, if we introduce $T_Z^s := \frac{1}{2}(T_Z + T_Z^*)$ then

$$2T_Z^s = -\frac{1}{2}(\mathcal{R}_Z + \mathcal{R}_Z^*) - \frac{1}{2}(\bar{\mathcal{R}}_Z + \bar{\mathcal{R}}_Z^*) + (I\!I_Z^* + I\!I_Z) - 2\,I\!I_\cdot^* Z.$$

Let ∇' be a connection compatible with g_H. Define an End T^*M-valued one-form κ such that $\nabla'_v = \nabla^\kappa_v = \nabla^0_v + \kappa(v)$, and let $\hat{\nabla}'_v = \hat{\nabla}^0_v + \hat{\kappa}(v)$ be its adjoint. Define

$$\hat{\kappa}^s(Z) = \frac{1}{2}\left(\hat{\kappa}(Z) + \hat{\kappa}(Z)^*\right), \quad \hat{\kappa}^a(Z) = \frac{1}{2}\left(\hat{\kappa}(Z) - \hat{\kappa}(Z)^*\right).$$

In order for the adjoint to be compatible with g, we must have

$$(\hat{\nabla}^\kappa_Z g)(A, A) = 2\langle (T_Z + \hat{\kappa}(Z)^\mathsf{T})A, A\rangle_g = 0,$$

giving us the requirement $\hat{\kappa}^s(Z)^\mathsf{T} = -T_Z^s$. However, since ∇^κ is compatible with g_H, we also have $\hat{\kappa}(\sharp^H \alpha)\alpha = 0$ by Lemma 3.3. The latter condition is equivalent to $\hat{\kappa}(A)^{\mathsf{T}*}(A + B) = 0$ for any $A \in \Gamma(H)$ and $B \in \Gamma(V)$. This means that

$$\begin{aligned} 0 &= \langle \hat{\kappa}(A)^{\mathsf{T}*}(A + B), A + B\rangle_g = \langle \hat{\kappa}^s(A)^\mathsf{T}(A + B), A + B\rangle_g \\ &= -\langle T_A^s(A + B), A + B\rangle_g = -\langle I\!I(A, A), B\rangle_g + \langle A, I\!I(B, B)\rangle_g. \end{aligned}$$

The condition holds for any $A \in \Gamma(H)$ and $B \in \Gamma(V)$ if and only if $I\!I = 0$. It follows that $4\hat{\kappa}^s(Z)^\mathsf{T} = \mathcal{R}_Z + \mathcal{R}_Z^* + \bar{\mathcal{R}}_Z + \bar{\mathcal{R}}_Z^*$.

For the anti-symmetric part, we observe that

$$\begin{aligned} 0 &= -4\hat{\kappa}(A)^{\mathsf{T}*}(A + B) = 4\hat{\kappa}^a(A)^\mathsf{T}(A + B) - 4\hat{\kappa}^s(A)^\mathsf{T}(A + B) \\ &= 4\hat{\kappa}^a(A)^\mathsf{T}(A + B) - \mathcal{R}_A^* B \end{aligned}$$

for any $A \in \Gamma(H)$, $B \in \Gamma(V)$. This relation and anti-symmetry give us

$$\hat{\kappa}^a(Z)^{\mathsf{T}}(A+B) = \hat{\kappa}^a(\mathrm{pr}_V Z)(A+B) - \frac{1}{4}(\mathcal{R}_Z - \mathcal{R}_Z^*)(A+B) + \sharp\iota_{Z\wedge A}\beta,$$

where β is a three-form vanishing on V.

In conclusion, for any $Z_1, Z_2 \in \Gamma(TM)$,

$$\begin{aligned}\nabla^\kappa_{Z_1} Z_2 &= \nabla^0_{Z_1} Z_2 - \hat{\kappa}(Z_2)^{\mathsf{T}}(Z_1) \\ &= \nabla^0_{Z_1} Z_2 - \frac{1}{4}(2\mathcal{R}^*_{Z_2} + \bar{\mathcal{R}}_{Z_2} + \bar{\mathcal{R}}^*_{Z_2})Z_1 + \hat{\kappa}^a(\mathrm{pr}_V Z_2)(Z_1) + \sharp\iota_{Z_1\wedge Z_2}\beta.\end{aligned}$$

Furthermore, since

$$\begin{aligned}\nabla^0_Z &= \nabla^g_Z + \frac{1}{2}T_Z - \frac{1}{2}T^*_Z - \frac{1}{2}T^*Z \\ &= \nabla^g_Z + \frac{1}{2}\left(-\mathcal{R}_Z + \frac{1}{2}\mathcal{R}^*_Z - \bar{\mathcal{R}}_Z + \frac{1}{2}\bar{\mathcal{R}}^*_Z - \frac{1}{2}\mathcal{R}^*_\cdot Z - \frac{1}{2}\bar{\mathcal{R}}^*_\cdot Z\right) \\ &\quad -\frac{1}{2}\left(-\mathcal{R}^*_Z + \frac{1}{2}\mathcal{R}_Z - \bar{\mathcal{R}}^*_Z + \frac{1}{2}\bar{\mathcal{R}}_Z + \frac{1}{2}\mathcal{R}^*_\cdot Z + \frac{1}{2}\bar{\mathcal{R}}^*_\cdot Z\right) \\ &\quad -\frac{1}{2}\left(-\mathcal{R}^*_\cdot Z - \frac{1}{2}\mathcal{R}_Z - \bar{\mathcal{R}}^*_\cdot Z - \frac{1}{2}\bar{\mathcal{R}}_Z + \frac{1}{2}\mathcal{R}^*_Z + \frac{1}{2}\bar{\mathcal{R}}^*_Z\right) \\ &= \nabla^g_Z + \frac{1}{2}\left(-\mathcal{R}_Z + \mathcal{R}^*_Z - \bar{\mathcal{R}}_Z + \bar{\mathcal{R}}^*_Z\right),\end{aligned}$$

we get

$$\nabla^\kappa_Z = \nabla^g_Z + \frac{1}{2}\left(-\mathcal{R}_Z + \mathcal{R}^*_Z - \bar{\mathcal{R}}_Z + \bar{\mathcal{R}}^*_Z - \mathcal{R}^*Z_1 - \bar{\mathcal{R}}^*Z_1\right)Z_2 + \lambda(Z_2)Z_1 + \sharp^H\iota_{Z_1\wedge Z_2}\beta$$

where $\lambda(Z)A = \frac{1}{4}(\bar{\mathcal{R}}_Z - \bar{\mathcal{R}}^*_Z)A - \hat{\kappa}^a(\mathrm{pr}_V Z)A$. It follows that if ∇' and $\hat{\nabla}'$ are compatible with g^*_H and g respectively, and ∇ is defined as in (3.7), then $I\!I = 0$ and

$$\nabla'_{Z_1} Z_2 = \nabla^{\lambda,\beta}_{Z_1} Z_2 := \nabla_{Z_1} Z_2 + \lambda(Z_2)Z_1 + \sharp^H\iota_{Z_1\wedge Z_2}\beta, \tag{3.8}$$

for some three-form β vanishing on V and some $\mathrm{End}\,TM$-valued one-form λ vanishing on H and satisfying $\lambda(v)^* = -\lambda(v)$, $v \in TM$. It is straightforward to verify that $\mathrm{tr}\, T^{\nabla^{\lambda,\beta}}(v, \cdot) = 0$ for any $v \in H$, and hence $L(\nabla')f = L(\hat{\nabla}')f = \Delta_H f$ by Lemma 2.1.

All that remains to be proven is that

$$\langle \alpha, \mathrm{Ric}(\nabla^{\lambda,\beta})\alpha\rangle_{g^*_H} \leq \langle \alpha, \mathrm{Ric}(\nabla)\alpha\rangle_{g^*_H}.$$

If $\nabla^\beta = \nabla^{0,\beta}$ then $\hat{L}^\beta := L(\hat{\nabla}^\beta) = L(\hat{\nabla}^{\lambda,\beta})$ since λ vanishes on H. If we define $\hat{L} = L(\hat{\nabla})$, then for any smooth function f and local orthonormal basis $A_1, \ldots, A_n$ of H,

$$\begin{aligned}\hat{L}^\beta df(Z) &= \hat{L}df(Z) + 2\sum_{i=1}^n \hat{\nabla}_{A_i} df(\sharp\iota_{A_i\wedge Z}\beta) \\ &\quad + \sum_{i=1}^n df(\sharp\iota_{A_i\wedge Z}(\hat{\nabla}_{A_i}\beta)) + \sum_{i=1}^n df(\sharp_{A_i\wedge\sharp\iota_{A_i\wedge Z}\beta}\beta) \\ &= \hat{L}df(Z) + \sum_{i=1}^n df(T^\nabla(A_i, \sharp\iota_{A_i\wedge Z}\beta)) + \sum_{i=1}^n (\hat{\nabla}_{A_i}\beta)(\sharp df, A_i, Z) - 2\langle\iota_{\sharp df}\beta, \iota_Z\beta\rangle_{\wedge^2 g^*_H} \\ &= \hat{L}df(Z) + 2\langle\iota_{\mathcal{R}} df, \iota_Z\beta\rangle_{\wedge^2 g^*_H} - \mathrm{tr}_H(\hat{\nabla}_\times\beta)(\times, \sharp df, Z) - 2\langle\iota_{\sharp df}\beta, \iota_Z\beta\rangle_{\wedge^2 g^*_H}.\end{aligned}$$

We use that

$$\langle (\hat{L}^{\beta} - \hat{L})df, \alpha\rangle_g = \langle (\mathrm{Ric}(\nabla^{\beta}) - \mathrm{Ric}(\nabla))df, \alpha\rangle_g = \langle (\mathrm{Ric}(\nabla^{\lambda,\beta}) - \mathrm{Ric}(\nabla))df, \alpha\rangle_g.$$

As a consequence, for any $\alpha \in T^*M$,

$$\langle \alpha, \mathrm{Ric}(\nabla^{\lambda,\beta})\alpha\rangle_{g^*} = \langle \alpha, \mathrm{Ric}(\nabla)\alpha\rangle_{g^*} + 2\langle \iota_{\mathcal{R}}\alpha, \iota_{\sharp\alpha}\beta\rangle_{\wedge^2 g_H^*} - 2\langle \iota_{\sharp\alpha}\beta, \iota_{\sharp\alpha}\beta\rangle_{\wedge^2 g_H^*}.$$

Denoting $\alpha_H = \mathrm{pr}_H^* \alpha$, we get

$$\langle \alpha, \mathrm{Ric}(\nabla^{\lambda,\beta})\alpha\rangle_{g_H^*} = \langle \alpha_H, \mathrm{Ric}(\nabla)\alpha_H\rangle_{g^*} - 2|\iota_{\sharp\alpha_H}\beta|^2_{\wedge^2 g_H^*}.$$

The result follows. □

3.3 Infinite Lifetime of the Diffusion to the Sub-Laplacian

Assume now that the taming metric g is a complete Riemannian metric. Then both the sub-Laplacian Δ_H of $\mu = \mu_g$ and the Laplacian Δ_g are essentially self-adjoint on compactly supported functions. We denote their unique self-adjoint extension by the same symbol.

Let ∇ be a connection compatible with g_H^* and let $X_t(\cdot)$ be the stochastic flow of $\frac{1}{2}L(\nabla)$ with explosion time $\tau(\cdot)$. For any $x \in M$, let $/\!/_t = /\!/_t(x)\colon T_xM \to T_{X_t(x)}M$ be parallel transport along $X_t(x)$ with respect to ∇. Using arguments similar to [24, Section 2.5], we know that the anti-development $W_t(x)$ at x determined by

$$dW_t(x) = /\!/_t^{-1} \circ dX_t(x), \quad W_t(0) = 0 \in T_xM,$$

is a Brownian motion in the inner product space $(H_x, \langle\cdot,\cdot\rangle_{g_H(x)})$ with lifetime $\tau(x)$. Consider the semigroup P_t on bounded Borel measurable functions corresponding to $X_t(\cdot)$

$$P_t f(x) = \mathbb{E}[1_{t<\tau(x)} f(X_t(x))].$$

We search for statements about the explosion time $\tau(\cdot)$ using connections that are compatible with g_H^*. Let $C_b^\infty(M)$ denote the space of smooth bounded functions. For a vector bundle endomorphism $\mathscr{A}$ of T^*M write $\mathscr{A}_{/\!/_t}(x) = /\!/_t^{-1}\mathscr{A}(X_t(x))/\!/_t$ and let $\hat{/\!/}_t$ denote the parallel transport along X_t with respect to $\hat{\nabla}$.

We make the following three assumptions:

(A) If $I\!I$ is defined as in (3.4), then $I\!I = 0$.

(B) Consider the two-form $\mathcal{C} \in \Gamma(\bigwedge^2 T^*M)$ defined by

$$\mathcal{C}(v, w) = \mathrm{tr}\,\bar{\mathcal{R}}(v, \mathcal{R}(w,\cdot)) - \mathrm{tr}\,\bar{\mathcal{R}}(w, \mathcal{R}(v,\cdot)), \quad v, w \in TM. \tag{3.9}$$

We suppose that $\delta\mathcal{C} = 0$ where δ is the codifferential with respect to g.

(C) Let ∇ be defined as in (3.7). We assume that there exists a constant $K \geq 0$ such that for $\mathrm{Ric} = \mathrm{Ric}(\nabla)$,

$$\langle \mathrm{Ric}\,\alpha, \alpha\rangle_{g^*} \geq -K|\alpha|^2_{g^*}.$$

Theorem 3.6 *Assuming that* (3.3), (3.3) *and* (3.3) *hold, we have the following results.*

(a) Δ_g *and* Δ_H *spectrally commute.*

(b) $\tau(x) = \infty$ *a.s. for any* $x \in M$.

(c) *Define* $\hat{Q}_t = \hat{Q}_t(x) \in \mathrm{End}\, T_x^*M$ *as solution to the ordinary differential equation*

$$\frac{d}{dt}\hat{Q}_t = -\frac{1}{2}\hat{Q}_t \,\mathrm{Ric}_{\hat{/\!/}_t}, \quad \hat{Q}_0 = \mathrm{id}.$$

Then, for any $f \in C_b^\infty(M)$ *with* $\|df\|_{L^\infty(g^*)} < \infty$*, we have*

$$dP_t f(x) = \mathbb{E}[\hat{Q}_t \hat{/\!/}_t^{-1} df(X_t(x))]$$

and

$$\|dP_t f\|_{L^\infty(g^*)} \leq e^{Kt} \|df\|_{L^\infty(g^*)}.$$

In particular,

$$\sup_{t\in[0,t_1]} \|dP_t f\|_{L^\infty(g^*)} \leq e^{Kt_1} \|df\|_{L^\infty(g^*)} < \infty$$

whenever $\|df\|_{L^\infty(g^*)} < \infty$.

Remark that since ∇ preserves H under parallel transport, and hence also $\mathrm{Ann}(H)$, we have $\mathrm{Ric}\,\alpha = 0$ for any $\alpha \in \mathrm{Ann}(H)$. For this reason it is not possible to have a positive lower bound of $\langle \mathrm{Ric}\,\alpha, \alpha\rangle_{g^*}$ unless $H = TM$. The results of Theorem 3.6 appear as necessary conditions for the Γ_2-calculus on sub-Riemannian manifolds, see e.g. [11, 12, 25] . We will use the remainder of this section to prove this statement.

3.4 Anti-symmetric Part of Ricci Curvature

Let ζ and ∇ be as in (3.6) and (3.7), respectively. The operator $\mathrm{Ric}(\nabla)$ is not symmetric in general. We consider its anti-symmetric part. Letting $\mathrm{Ric} = \mathrm{Ric}(\nabla)$ we define

$$\mathrm{Ric}^s = \frac{1}{2}\left(\mathrm{Ric} + \mathrm{Ric}^*\right), \qquad \mathrm{Ric}^a = \frac{1}{2}\left(\mathrm{Ric} - \mathrm{Ric}^*\right). \tag{3.10}$$

Lemma 3.7 *For any* $\alpha, \beta \in T^*M$,

$$2\langle \mathrm{Ric}^a \alpha, \beta\rangle_{g^*} = \mathrm{tr}_H (\nabla_\times \zeta)(\times, \sharp\alpha, \sharp\beta) = \mathrm{tr}_H (\nabla_\times \zeta_H)(\times, \sharp\alpha, \sharp\beta),$$

where $\zeta_H(v_1, v_2, v_3) = \circlearrowright \langle \mathcal{R}(v_1, v_2), v_3\rangle_g$ *and* $\circlearrowright$ *denotes the cyclic sum. In particular,*

$$\langle \beta, \mathrm{Ric}^a \alpha\rangle_{g^*} = \langle \mathrm{pr}_V^* \beta, \mathrm{Ric}^s \alpha\rangle_{g^*} - \langle \mathrm{pr}_V^* \alpha, \mathrm{Ric}^s \beta\rangle_{g^*},$$

so if Ric^s *has a lower bound then* Ric^a *is a bounded operator. Furthermore, if we define* $\mathcal{C}$ *by* (3.9)*, then whenever the* L^2 *inner product is finite,*

$$2\langle \mathrm{Ric}^a df, d\phi\rangle_{L^2(g^*)} = \langle \mathcal{C}, df \wedge d\phi\rangle_{L^2(\wedge^2 g^*)} \quad \text{for any } f, \phi \in C^\infty(M).$$

The first part of this result is also found in [20, Proposition C.6]. When $\bar{\mathcal{R}} = 0$, the condition $\mathrm{Ric}^a = 0$ is called *the Yang-Mills condition*. For more details, see Remark 3.16.

Proof of Lemma 3.7 For the proof, we will use the first Bianchi identity

$$\circlearrowright R^\nabla(B_1, B_2)B_3 = \circlearrowright(\nabla_{B_1} T)(B_2, B_3) + \circlearrowright T(T(B_1, B_2), B_3) \tag{3.11}$$

and the identity $\langle R(B_1, B_2)A, A\rangle_g = 0$ which follows from the compatibility of ∇ with g. We first compute,

$$2\langle \mathrm{Ric}^a \alpha, \beta\rangle_{g^*} = \sum_{i=1}^n \langle A_i, R^\nabla(A_i, \sharp\beta)\sharp\alpha - R^\nabla(A_i, \sharp\alpha)\sharp\beta\rangle_g$$

$$= -\sum_{i=1}^n \langle A_i, \circlearrowright R^\nabla(A_i, \sharp\alpha)\sharp\beta\rangle_g = -\sum_{i=1}^n \langle A_i, \circlearrowright(\nabla_{A_i} T)(\sharp\alpha, \sharp\beta) + \circlearrowright T(T(A_i, \sharp\alpha), \sharp\beta)\rangle_g$$

$$= -\sum_{i=1}^{n}\langle A_i, (\nabla_{A_i} T)(\sharp\alpha, \sharp\beta) + T(T(A_i, \sharp\alpha), \sharp\beta) + T(T(\sharp\beta, \sharp A_i), \sharp\alpha)\rangle_g$$
$$= \sum_{i=1}^{n}(\nabla_{A_i}\zeta)(A_i, \sharp\alpha, \sharp\beta) - \sum_{i=1}^{n}\langle T(A_i, \sharp\alpha), T(\sharp\beta, A_i)\rangle_g - \sum_{i=1}^{n}\langle T(\sharp\beta, A_i), T(\sharp\alpha, A_i)\rangle_g$$
$$= \mathrm{tr}_H(\nabla_\times\zeta)(\times, \sharp\alpha, \sharp\beta).$$

Write $\zeta = \zeta_H + \zeta_V$ where $\zeta_H(v_1, v_2, v_3) = \circlearrowright\langle v_1, \mathcal{R}(v_2, v_3)\rangle_g$ and $\zeta_V(v_1, v_2, v_3) = \circlearrowright\langle v_1, \bar{\mathcal{R}}(v_2, v_3)\rangle_g$. Recall that $\mathrm{Ric}\,\alpha = 0$ whenever α vanishes on H. Hence, for $\alpha, \beta \in \mathrm{Ann}(H)$,

$$2\langle \mathrm{Ric}^a\,\alpha, \beta\rangle_{g^*} = 0 = \mathrm{tr}_H(\nabla_\times\zeta)(\times, \sharp\alpha, \sharp\beta) = \mathrm{tr}_H(\nabla_\times\zeta_V)(\times, \sharp\alpha, \sharp\beta),$$

and so we can write $2\langle \mathrm{Ric}_a\,\alpha, \beta\rangle = \mathrm{tr}_H(\nabla_\times\zeta_H)(\times, \sharp\alpha, \sharp\beta)$. We remark for later purposes that by reversing the place of V and H and writing $g_V = g|V$, we have also $\mathrm{tr}_{g_V}(\nabla_\times\zeta_H)(\times, \sharp\alpha, \sharp\beta) = 0$ by the same argument.

We note that

$$\begin{aligned} 2\langle \mathrm{Ric}^a\,\alpha, \beta\rangle_{g^*} &= \mathrm{tr}_H(\nabla_\times\zeta_H)(\times, \sharp\alpha, \sharp\beta) \\ &= \mathrm{tr}_H(\nabla_\times\zeta_H)(\times, \mathrm{pr}_H\,\sharp\alpha, \mathrm{pr}_V\,\sharp\beta) + \mathrm{tr}_H(\nabla_\times\zeta_H)(\times, \mathrm{pr}_V\,\sharp\alpha, \mathrm{pr}_H\,\sharp\beta). \end{aligned}$$

We again use that Ric vanishes on $\mathrm{Ann}(H)$ to get

$$\begin{aligned} 2\langle \mathrm{Ric}^a\,\alpha, \beta\rangle_{g^*} &= 2\langle \mathrm{Ric}^a\,\mathrm{pr}_H^*\,\alpha, \mathrm{pr}_V^*\,\beta\rangle_{g^*} + 2\langle \mathrm{Ric}^a\,\mathrm{pr}_V^*\,\alpha, \mathrm{pr}_H^*\,\beta\rangle_{g^*} \\ &= \langle \mathrm{Ric}\,\alpha, \mathrm{pr}_V^*\,\beta\rangle_{g^*} - \langle \mathrm{pr}_V^*\,\alpha, \mathrm{Ric}\,\beta\rangle_{g^*} \\ &= 2\langle \mathrm{Ric}^s\,\alpha, \mathrm{pr}_V^*\,\beta\rangle_{g^*} - 2\langle \mathrm{pr}_V^*\,\alpha, \mathrm{Ric}^s\,\beta\rangle_{g^*}. \end{aligned}$$

Continuing, if $A_1, \dots, A_n$ and $Z_1, \dots, Z_\nu$ are local orthonormal bases of H and V, respectively, observe that since ∇ preserves the metric g, for any one-form η, we have

$$d\eta = \sum_{i=1}^{n} \flat A_i \wedge \nabla_{A_i}\eta + \sum_{i=1}^{\nu} \flat Z_\nu \wedge \nabla_{Z_\nu}\eta + \iota_T\eta,$$

where $\iota_T\eta = \eta(T(\cdot, \cdot))$. The formula above becomes valid for arbitrary forms η if we extend ι_T by the rule that $\iota_T(\alpha \wedge \beta) = (\iota_T\alpha) \wedge \beta + (-1)^k\alpha \wedge \iota_T\beta$ for any k-form α and form β. Observe that $\mathrm{tr}\,T(v, \cdot) = 0$ for any $v \in TM$. Hence, by arguments similar to the proof of Lemma 2.1 (b), we obtain a local formula for the codifferential

$$\delta\eta = -\sum_{i=1}^{n} \iota_{A_i}\nabla_{A_i}\eta - \sum_{i=1}^{\nu} \iota_{Z_\nu}\nabla_{Z_\nu}\eta + \iota_T^*\eta. \tag{3.12}$$

By the relation $\mathrm{tr}_{g_V}(\nabla_\times\zeta_H)(\times, \sharp\alpha, \sharp\beta) = 0$, we finally have

$$\mathrm{tr}_H(\nabla_\times\zeta_H)(\times, \sharp\alpha, \sharp\beta) = (\iota_T^*\zeta_H)(\sharp\alpha, \sharp\beta) - (\delta\zeta_H)(\sharp\alpha, \sharp\beta) = \langle \mathcal{C} - \delta\zeta_H, \alpha \wedge \beta\rangle_{g^*}.$$

Inserting $\alpha \wedge \beta = df \wedge d\phi = d(f\,d\phi)$ and integrating over the manifold, we obtain the result. □

3.5 Commutation Relations Between the Laplacian and the Sub-Laplacian

Let (M, H, g_H) be a sub-Riemannian manifold and let g be a taming Riemannian metric with $I\!I = 0$. Define Δ_g as the Laplacian of g and let Δ_H be defined relative to the volume density of g.

Proposition 3.8 *We keep the definition of* $\mathcal{C}$ *as in* (3.9).

(a) *We have* $\Delta_g \Delta_H f = \Delta_H \Delta_g f$ *for all* $f \in C^\infty(M)$ *if and only if* $\delta\mathcal{C} = 0$.
(b) *Assume* $\delta\mathcal{C} = 0$ *and that* $\mathrm{Ric}(\nabla)$ *is bounded from below by some constant* $-K$. *Then* Δ_g *and* Δ_H *spectrally commute.*

See Example 3.12 for a concrete example where $\mathcal{C} \neq 0$ while $\delta\mathcal{C} = 0$. Before starting the proof, we shall need the following lemmas.

Lemma 3.9 ([33, Proposition], [11, Proposition 4.1]) *Let* $\mathcal{A}$ *be equal to the Laplacian* Δ_g *or sub-Laplacian* Δ_H *defined relative to a complete Riemannian or sub-Riemannian metric, respectively. Let* $M \times [0, \infty)$, $(x, t) \mapsto u_t(x)$ *be a function in* L^2 *of the solving the heat equation*

$$(\partial_t - \mathcal{A})u_t = 0, \quad u_0 = f,$$

for an L^2*-function* f. *Then* $u_t(x)$ *is the unique solution to this equation in* L^2.

Lemma 3.10 *Let* (M, H, g_H) *be a sub-Riemannian manifold and define* Δ_H *as the sub-Laplacian with respect to a volume form* μ. *Let* g *be a taming metric of* g_H *with volume form* μ. *Assume that* ∇ *and its adjoint* $\hat{\nabla}$ *are compatible with* g_H^* *and* g*, respectively. If* $\hat{L} = L(\hat{\nabla})$*, then with respect to* g*,*

$$\hat{L}^* = \hat{L} = -(\hat{\nabla}_{\mathrm{pr}_H})^* \hat{\nabla}_{\mathrm{pr}_H}.$$

In particular, $\hat{L} f = \Delta_H f$ *for any* $f \in C^\infty(M)$.

Proof Define $\hat{F}(A \otimes B) = \flat A \otimes \hat{\nabla}_B$ and extend it by linearity to all sections of $TM^{\otimes 2}$. Again we know that for any point x, there exists a basis $A_1, \ldots, A_n$ such that $\nabla A_i(x) = 0$. This means that $\hat{\nabla}_Z A_i(x) = T^\nabla(A_i, Z)(x)$ for the same basis, and hence locally

$$\hat{F}(g_H^*)^* = -\iota_{\sharp^H \hat{\beta}} - \sum_{i=1}^n \iota_{A_i} \hat{\nabla}_{A_i}, \quad \hat{\beta}(v) = \mathrm{tr}\, T^{\hat{\nabla}}(v, \cdot).$$

However, since $\hat{\nabla}$ is the adjoint of a connection compatible with g_H^* we have $\hat{\beta} = 0$ since $\hat{\nabla}$ has to be on the form (3.8). Hence $\hat{F}(g_H^*)^* \hat{F}(g_H^*) = -\hat{L}$ and the result follows. □

Proof of the Proposition 3.8

(a) It is sufficient to prove the statement for compactly supported functions. Note that for $f, \phi \in C_c^\infty(M)$, $\langle \Delta_H \Delta_g f, \phi \rangle_{L^2} = \langle f, \Delta_g \Delta_H \phi \rangle_{L^2}$. Hence, we need to show that $\Delta_g \Delta_H$ is its own dual on compact supported forms.

Let ∇ be as in (3.7) with adjoint $\hat{\nabla}$. Define $L = L(\nabla)$, $\hat{L} = L(\hat{\nabla})$, $\mathrm{Ric} = \mathrm{Ric}(\nabla)$ and introduce $\mathrm{Ric}^a = \frac{1}{2}\left(\mathrm{Ric} - \mathrm{Ric}^*\right)$. By Lemma 3.10 we have $\hat{L}^* = \hat{L}$. In addition,

$$\begin{aligned} \langle \Delta_g \Delta_H f, \phi \rangle_{L^2} &= -\langle dLf, d\phi \rangle_{L^2(g^*)} \\ &= -\langle (\hat{L} - \mathrm{Ric}) df, d\phi \rangle_{L^2(g^*)} \\ &= -\langle df, (\hat{L} - \mathrm{Ric}) d\phi \rangle_{L^2(g^*)} + 2\langle \mathrm{Ric}^a\, df, d\phi \rangle_{L^2(g^*)} \\ &= \langle f, \Delta_g \Delta_H \phi \rangle_{L^2} + 2\langle \mathrm{Ric}^a\, df, d\phi \rangle_{L^2(g^*)}. \end{aligned}$$

Furthermore, $2\langle \mathrm{Ric}^a\, df, d\phi \rangle_{L^2(g^*)} = \langle \mathcal{C}, df \wedge d\phi \rangle_{L^2(\wedge^2 g^*)} = \langle \delta\mathcal{C}, f d\phi \rangle_{L^2(g^*)}$. Since all one-forms can we written as sums of one-forms of the type $f d\phi$, it follows that $(\Delta_g \Delta_H)^* f = \Delta_g \Delta_H f$ for $f \in C_c^\infty(M)$ if and only if $\delta\mathcal{C} = 0$.

(b) Write $\Delta_g = \Delta_H + \Delta_V$ and $df = d_H f + d_V f$, with $d_H f = \mathrm{pr}_H^* df$ and $d_V f = \mathrm{pr}_V^* df$. Then $\langle \Delta_H f, \phi \rangle_{L^2} = -\langle d_H f, d_H \phi \rangle_{L^2(g^*)}$ and similarly for Δ_V.

Observe that for any compactly supported f,

$$\begin{aligned} &\|\Delta_g f\|_{L^2} \|\Delta_H f\|_{L^2} \geq \langle \Delta_g f, \Delta_H f \rangle_{L^2} \\ &= -\langle df, (\hat{L} - \mathrm{Ric}) df \rangle_{L^2(g^*)} \\ &= \|\hat{\nabla} df\|^2_{L^2(g^{*\otimes 2})} + \langle df, \mathrm{Ric}\, d_H f \rangle_{L^2(g^*)} \\ &\geq \frac{1}{n} \|\Delta_H f\|^2_{L^2} - K \|df\|_{L^2(g^*)} \|d_H f\|_{L^2(g^*)}. \end{aligned}$$

and ultimately

$$\|\Delta_H f\|^2_{L^2} \leq n \sqrt{\|\Delta_g f\|_{L^2} \|\Delta_H f\|_{L^2}} \left(\sqrt{\|\Delta_g f\|_{L^2} \|\Delta_H f\|_{L^2}} + K \|f\|_{L^2} \right). \quad (3.13)$$

By approaching any $f \in \mathrm{Dom}(\Delta_g)$ by compactly supported functions, we conclude from (3.13) that any such function must satisfy $\|\Delta_H f\|_{L^2} < \infty$. As a consequence, $\mathrm{Dom}(\Delta_g) \subseteq \mathrm{Dom}(\Delta_H)$.

Let $Q_t = e^{t\Delta_g/2}$ and $P_t = e^{t\Delta_H/2}$ be the semigroups of Δ_g and Δ_H, which exists by the spectral theorem. For any $f \in \mathrm{Dom}(\Delta_H)$, $u_t = \Delta_H Q_t f$ is an L^2 solution of

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \Delta_g \right) u_t = 0, \quad u_0 = \Delta_H f.$$

By Lemma 3.9 we obtain $\Delta_H Q_t f = Q_t \Delta_H f$. Furthermore, for any $s > 0$ and $f \in L^2$, we know that $Q_s f \in \mathrm{Dom}(\Delta_g) \subseteq \mathrm{Dom}(\Delta_H)$, and since

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \Delta_H \right) Q_s P_t f = 0,$$

it again follows from Lemma 3.9 that $P_t Q_s f = Q_s P_t f$ for any $s, t \geq 0$ and $f \in L^2$. The operators consequently spectrally commute, see [38, Chapter VIII.5].

□

Remark 3.11 The results of Lemma 2.1 and Lemma 3.10 do not require the bracket generating assumptions. The result of $\hat{L}$ being symmetric is also found in [20, Theorem 2.5.1] for the case when ∇ and $\hat{\nabla}$ preserves the metric.

Example 3.12 ($\mathcal{C}$ nonzero and coclosed) For $j = 1, 2$, define $\mathfrak{g}_j = \mathfrak{su}(2)$ with basis A^j, B^j, C^j satisfying

$$[A^j, B^j] = C^j, \qquad [B^j, C^j] = A^j, \qquad [C^j, A^j] = B^j.$$

Let $\mathfrak{g}$ denote the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as Lie algebras and give it a bi-invariant inner product such that $A^1, A^2, B^1, B^2, C^1, C^2$ form an orthonormal basis. Consider the elements $A^\pm \in \mathfrak{g}$ where $A^\pm = A^1 \pm A^2$ and define $B^\pm$ and $C^\pm$ analogously. As vector spaces, write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v} = \mathrm{span}\{A^+, B^+, C^1\} \oplus \mathrm{span}\{A^-, B^-, C^2\},$$

Consider the Lie group $M = \mathrm{SU}(2) \times \mathrm{SU}(2)$ with a Riemannian metric g defined by left translation of the inner product on its Lie algebra $\mathfrak{g}$. Furthermore, define H and V as the left translation of respectively $\mathfrak{h}$ and $\mathfrak{v}$. Then the condition $I\!I = 0$ follows from bi-invariance. Furthermore, observe that if we use the same symbol for elements in $\mathfrak{g}$ and their corresponding left invariant vector fields, then

$\mathcal{R}$	A^+	B^+	C^1
A^+	0	C^2	$-\frac{1}{2}B^-$
B^+	$-C^2$	0	$\frac{1}{2}A^-$
C^1	$\frac{1}{2}B^-$	$-\frac{1}{2}A^-$	0

$\bar{\mathcal{R}}$	A^-	B^-	C^2
A^-	0	C^1	$\frac{1}{2}B^+$
B^-	$-C^1$	0	$-\frac{1}{2}A^+$
C^2	$-\frac{1}{2}B^+$	$\frac{1}{2}A^+$	0

We then have

$$\begin{aligned} 2\,\mathrm{Ric}^a: \quad & A^+ \mapsto A^-, \quad B^+ \mapsto B^-, \quad C^1 \mapsto 2C^2 \\ & A^- \mapsto -A^+, \quad B^- \mapsto -B^+, \quad C^2 \mapsto -2C^1 \end{aligned}$$

and $\mathcal{C} = \frac{1}{2}\flat C_2 \wedge \flat C_1$. The form $\mathcal{C}$ is in fact coclosed. To see this, let ∇^l denote the connection defined such that all left invariant vector fields are parallel and let T^l denote its torsion. If A and B are left invariant, then $T^l(A, B) = -[A, B]$. Bi-invariance of the inner product gives us $\operatorname{tr} T^l(v, \cdot) = 0$, so formula (3.12) is still valid when using the connection ∇^l. Hence $\delta\mathcal{C} = \frac{1}{2}\iota_{T^*}\flat C_2 \wedge \flat C_1 = -\frac{1}{2}\flat[C_2, C_1] = 0$.

3.6 Proof of Theorem 3.6

We consider the assumptions that $\delta\mathcal{C} = 0$ and that the symmetric part Ric^s of the Ric is bounded from below. By Lemma 3.7, the anti-symmetric part Ric^a is a bounded operator. Furthermore, the operators Δ_g and Δ_H spectrally commute by Proposition 3.8.

Let $X_t(x)$, $\hat{/\!/}_t$ and $\hat{Q}_t$ be as in the statement of the theorem. If

$$N_t = \hat{Q}_t \hat{/\!/}_t^{-1} \alpha(X_t(x))$$

for an arbitrary $\alpha \in \Gamma(T^*M)$, then by Itô's formula

$$dN_t \overset{\text{loc. m.}}{=} \frac{1}{2}\hat{Q}_t \hat{/\!/}_t^{-1}(\hat{L} - \mathrm{Ric})\alpha(X_t(x))dt$$

where $\overset{\text{loc. m.}}{=}$ denotes equivalence modulo differential of local martingales. Consider $L^2(T^*M)$ as the space of L^2-one-forms on M with respect to g. Since g is complete and Ric^s bounded from below, the operator $\hat{L} - \mathrm{Ric}^s$ is essentially self-adjoint by Lemma 3.10 and Lemma A.1. Hence, by Lemma A.4, there is a strongly continuous semigroup $P_t^{(1)}$ on $L^2(T^*M)$ with generator $(\hat{L} - \mathrm{Ric}, \mathrm{Dom}(\hat{L} - \mathrm{Ric}^s))$ such that

$$P_t^{(1)}\alpha(x) = \mathbb{E}[1_{t<\tau(x)} N_t] = \mathbb{E}[1_{t<\tau(x)}\hat{Q}_t \hat{/\!/}_t^{-1}\alpha(X_t(x))].$$

We want to show that for any compactly supported function f, $P_t^{(1)} df = dP_t f$ where $P_t f(x) = \mathbb{E}[f(X_t(x))1_{t<\tau(x)}]$. Following the arguments in [17, Appendix B.1], we have $P_t f = e^{t\Delta_H/2} f$ where the latter semigroup is the L^2-semigroup defined by the spectral theorem and the fact that Δ_H is essentially self-adjoint on compactly supported functions. To this end, we want to show that $dP_t f$ is contained in the domain of the generator of $P_t^{(1)}$. This observation will then imply $P_t^{(1)} df = dP_t f$, since $P_t^{(1)} df$ is the unique solution to

$$\frac{\partial}{\partial t}\alpha_t = \frac{1}{2}L\alpha_t, \quad \alpha_0 = df,$$

with values in $\mathrm{Dom}(\hat{L} - \mathrm{Ric}^s)$ by strong continuity, [21, Chapter II.6].

We will first need to show that $dP_t f$ is indeed in L^2. Let Δ_g denote the Laplace-Beltrami operator of g, which will also be essentially self-adjoint on compactly supported functions since g is complete. Denote its unique self-adjoint extension by the same symbol. Since the operators spectrally commute, $e^{s\Delta_g}e^{t\Delta_H} = e^{t\Delta_H}e^{s\Delta_g}$ for any $s, t \geq 0$ which implies $\Delta_g e^{t\Delta_H} f = e^{t\Delta_H}\Delta_g f$ for any f in the domain of Δ_g. In particular,

$$\langle dP_t f, dP_t f\rangle_{L^2(g^*)} = -\langle \Delta_g P_t f, P_t f\rangle_{L^2(g^*)} = -\langle P_t\Delta_g f, P_t f\rangle_{L^2(g^*)} < \infty.$$

Next, since $\langle(\hat{L} - \mathrm{Ric}^s)\alpha, \alpha\rangle_{L^2(g^*)} \geq -K\|\alpha\|^2_{L^2(g^*)}$, the domain $\mathrm{Dom}(\hat{L} - \mathrm{Ric}^s)$ coincides with the completion of compactly supported one-forms $\Gamma_c(T^*M)$ with respect to the quadratic form

$$\begin{aligned} q(\alpha, \alpha) &= (K+1)\langle\alpha, \alpha\rangle_{L^2(g^*)} - \langle(\hat{L} - \mathrm{Ric}^s)\alpha, \alpha\rangle_{L^2(g^*)} \\ &= (K+1)\langle\alpha, \alpha\rangle_{L^2(g)} - \langle(\hat{L} - \mathrm{Ric})\alpha, \alpha\rangle_{L^2(g^*)}. \end{aligned}$$

Since $P_t f$ is in the domain of both Δ_g and Δ_H for any compactly supported f, we have that for any fixed t, there is a sequence of compactly supported functions h_n such that $h_n \to P_t f$, $\Delta_H h_n \to \Delta_H P_t f$ and $\Delta_g h_n \to \Delta_g P_t f$ in L^2. From the latter fact, it follows that dh_n converges to $dP_t f$ in L^2 as well. Furthermore,

$$\begin{aligned} q(dh_n, dh_n) &= (K+1)\langle dh_n, dh_n\rangle_{L^2(g)} - \langle(\hat{L} - \mathrm{Ric})dh_n, dh_n\rangle_{L^2(g)} \\ &= -(K+1)\langle h_n, \Delta_g h_n\rangle_{L^2(g)} - \langle d\Delta_H h_n, dh_n\rangle_{L^2(g)} \\ &= -(K+1)\langle h_n, \Delta_g h_n\rangle_{L^2(g)} + \langle \Delta_H h_n, \Delta_g h_n\rangle_{L^2(g)}, \end{aligned}$$

which has a finite limit as $n \to \infty$. Hence, $dP_t f \in \mathrm{Dom}(\hat{L} - \mathrm{Ric}^s)$ and $P_t^{(1)} df = dP_t f$.

Using that $\langle \mathrm{Ric}\,\alpha, \alpha\rangle_{g^*} \geq -K|\alpha|^2_{g^*}$, Gronwall's lemma and the fact that $\hat{\nabla}$ preserves the metric means that

$$|1_{t<\tau(x)}\hat{Q}_t /\!/_t^{-1}\alpha(X_t(x))|_{g^*} \leq e^{Kt/2}1_{t<\tau(x)}|\alpha|_{g^*}(X_t(x)).$$

Hence,

$$|P_t^{(1)}\alpha(x)|_{g^*} \leq e^{Kt/2}P_t|\alpha|_{g^*}(x). \tag{3.14}$$

We assumed that g was complete, so we know that there exists a sequence of compactly supported functions g_n such that $g_n \uparrow 1$ and such that $\|dg_n\|^2_{L^\infty(g^*)} \to 0$. Since $|dP_t g_n|_{g^*} \to 0$ uniformly by (3.14) and we know that $P_t g_n \to P_t 1$, we obtain $dP_t 1 = 0$. Hence, we know that $P_t 1 = 1$, which is equivalent to $\tau(x) = \infty$ almost surely.

It is a standard argument to extend the formulas from functions of compact support to bounded functions with $\|df\|_{L^\infty(g^*)} < \infty$.

3.7 Foliations and a Counter-example

Let (M, H, g_H) be a sub-Riemannian manifold and let g be a Riemannian metric taming g_H and satisfying $I\!I = 0$ with $I\!I$ as in (3.4). Write V for the orthogonal complement of H. Define *the Bott connection*, by

$$\begin{aligned} \mathring{\nabla}_{Z_1} Z_2 &= \mathrm{pr}_H \nabla^g_{\mathrm{pr}_H Z_1} \mathrm{pr}_H Z_2 + \mathrm{pr}_V \nabla^g_{\mathrm{pr}_V Z_1} \mathrm{pr}_V Z_2 \\ &\quad + \mathrm{pr}_H[\mathrm{pr}_V Z_1, \mathrm{pr}_H Z_2] + \mathrm{pr}_V[\mathrm{pr}_H Z_1, \mathrm{pr}_V Z_2] \end{aligned} \tag{3.15}$$

where ∇^g denote the Levi-Civita connection. Its torsion $\mathring{T} := T^{\mathring{\nabla}}$ equals $\mathring{T} = -\mathcal{R} - \bar{\mathcal{R}}$ and $\mathring{\nabla} g = 0$ is equivalent to requiring $I\!I = 0$. Since $\mathring{\nabla}$ is compatible with the metric, we have

$$\mathring{\nabla}_Z = \nabla^g_Z + \frac{1}{2}\mathring{T}_Z - \frac{1}{2}\mathring{T}^*_Z - \frac{1}{2}\mathring{T}^*_{\cdot} Z, \quad T_Z(A) = T(Z, A).$$

If ζ and ∇ are as in (3.6) and (3.7), respectively, then

$$\zeta(v_1, v_2, v_2) = -\circlearrowleft\langle \mathring{T}(v_1, v_2), v_3\rangle_g, \quad \text{and} \quad \nabla_Z = \mathring{\nabla}_Z + \mathring{T}^*_{\cdot} Z.$$

The connection $\mathring{\nabla}$ does not have skew-symmetric torsion, however, it does have the advantage that $\mathring{\nabla}_A B$ is independent of $g|V$ if either A or B takes its values in H, see [24, Section 3.1].

3.7.1 Totally Geodesic, Riemannian Foliations

Assume that $\bar{\mathcal{R}} = 0$, i.e. assume that the orthogonal complement V of H is integrable. Let $\mathcal{F}$ be the corresponding foliation of V that exists from the Frobenius theorem. We have the following way of interpreting the condition $I\!I = 0$. The tensor $I\!I(\mathrm{pr}_V \cdot, \mathrm{pr}_V \cdot)$ equals the second fundamental form of the leaves, i.e. $\mathrm{pr}_H \nabla^g_Z W = I\!I(Z, W)$ for any $Z, W \in \Gamma(V)$. Hence, $I\!I(\mathrm{pr}_V \cdot, \mathrm{pr}_V \cdot) = 0$ is equivalent to the leaves of $\mathcal{F}$ being totally geodesic immersed submanifolds. On the other hand, the condition $0 = -2\langle I\!I(A, A), Z\rangle = (\mathcal{L}_Z g)(A, A)$ for any $A \in \Gamma(H)$, $Z \in \Gamma(V)$ is the definition of $\mathcal{F}$ being a Riemannian foliation. Locally, such a foliation $\mathcal{F}$ consists of the fibers of a Riemannian submersion. In other words, every $x_0 \in M$ has a neighborhood U such that there exists a surjective submersion between two Riemannian manifolds,

$$\pi : (U, g|_U) \to (\check{M}_U, \check{g}_U), \tag{3.16}$$

satisfying

$$TU = H|U \oplus_\perp \ker \pi_*, \quad \mathcal{F}|U = \{\pi^{-1}(\check{x}) : \check{x} \in \check{M}_U\}$$

and that $\pi_* \colon H_x \to T_{\pi(x)}\check{M}_U$ is an isometry for every $x \in U$.

Let $X_t(\cdot)$ be a stochastic flow with generator $\frac{1}{2}\Delta_H$ where the latter is defined relative to the volume density of g. The following result is found in [18] for totally geodesic Riemannian foliations.

Theorem 3.13 *If (M, g) is a stochastically complete Riemannian manifold, then $X_t(x)$ has infinite lifetime.*

In particular, if the Riemannian Ricci curvature Ric_g is bounded from below, $X_t(x)$ has infinite lifetime. We want to compare this result using the entire Riemannian geometry with our result using $\mathrm{Ric}(\nabla)$, an operator only defined by taking the trace over horizontal vectors. For this special case, it turns out that Ric_g being bounded from below is actually a weaker condition than $\mathrm{Ric}(\nabla)$ being bounded from below.

Proposition 3.14 *Let (M, H, g_H) be a sub-Riemannian manifold with H bracket-generating. Let $\mathcal{F}$ be a foliation of M corresponding to an integrable subbundle V such that $TM = H \oplus V$. Let g be any Riemannian metric taming g_H such that $I\!I = 0$, making $\mathcal{F}$ a totally geodesic Riemannnian foliation. Assume finally that g is complete. For $x \in M$, let F_x denote the leaf of the foliation $\mathcal{F}$ containing x. Write Ric_{F_x} for the Ricci curvature tensor of F_x.*

(a) *For any $x, y \in M$, there exist neighborhoods $x \in U_x \subseteq F_x$ and $y \in U_y \subseteq F_y$, and an isometry*

$$\Phi \colon U_x \to U_y, \quad \Phi(x) = y.$$

As a consequence, if we define $\mathrm{Ric}_{\mathcal{F}}$ such that

$$\mathrm{Ric}_{\mathcal{F}}(v, w) = \mathrm{Ric}_{F_x}(\mathrm{pr}_V v, \mathrm{pr}_V w), \quad \textit{for any } v, w \in T_x M,$$

then $\mathrm{Ric}_{\mathcal{F}}$ *is bounded.*

(b) *Let* Ric_g *be the Ricci curvature of the Riemannian metric* g. *Let* ∇ *be defined as in* (3.7). *Then for any* $v \in T_x M$, $x \in M$ *and for any local orthonormal basis* $A_1, \dots, A_n$ *of* H *about* x,

$$\mathrm{Ric}_g(v, v) = \mathrm{Ric}(\nabla)(\flat v)(v) + \frac{1}{2}\sum_{i=1}^{n} |\mathcal{R}(A_i, v)|_g^2 + \frac{1}{4}\sum_{i=1}^{n} |\mathcal{R}^*_{A_i} v|^2 + \mathrm{Ric}_{\mathcal{F}}(v, v). \tag{3.17}$$

In particular, Ric_g *has a lower bound if* $\mathrm{Ric}(\nabla)$ *has a lower bound.*

Before presenting the proof we need the next lemma. Let (M, g) be a complete Riemannian manifold and let $\mathcal{F}$ be a Riemannian foliation with totally geodesic leaves. Let V be the integrable subbundle of TM corresponding to $\mathcal{F}$ and define H as its orthogonal complement. Write n for the rank of H and ν for the rank of V. Define

$$\mathrm{O}(n) \to \mathrm{O}(H) \xrightarrow{p} M$$

as the orthonormal frame bundle of H. Introduce the principal connection E on p corresponding to the restriction of $\mathring{\nabla}$ to H. In other words, E is the subbundle of $T\,\mathrm{O}(H)$ satisfying $T\,\mathrm{O}(H) = E \oplus \ker p_*$, $E_\phi \cdot a = E_{\phi \cdot a}$, $\phi \in \mathrm{O}(H)$, $a \in \mathrm{O}(n)$ and defined such that a curve $\phi(t)$ in $\mathrm{O}(H)$ is tangent to E if and only if the frame is $\mathring{\nabla}$-parallel along $p(\phi(t))$. For any $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, define $\hat{A}_u$ as the vector field on $\mathrm{O}(H)$ taking values in E uniquely determined by the property

$$p_* \hat{A}_u(\phi) = \sum_{j=1}^{n} u_j \phi_j, \qquad \text{for any } \phi = (\phi_1, \dots, \phi_n) \in \mathrm{O}(H).$$

For any $\phi \in \mathrm{O}(H)_x$, define $\hat{F}_\phi$ as all points that can be reached from ϕ by an E-horizontal lift of a curve in F_x starting in x. We then have the following result, found in [18], see also [43, Chapter 10] and [35].

Lemma 3.15 *The collection* $\hat{\mathcal{F}} = \{\hat{F}_\phi : \phi \in \mathrm{O}(H)\}$ *gives a foliation of* $\mathrm{O}(H)$ *with* ν-*dimensional leaves such that for each* $\phi \in O(H)$ *the map*

$$p|\hat{F}_\phi : \hat{F}_\phi \to F_{p(\phi)}$$

is a cover map. Furthermore, giving each leaf of $\hat{\mathcal{F}}$ *a Riemannian structure by pulling back the metric from the leaves of* $\mathcal{F}$, *then for any* $u \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$, *the flow* $\Psi_u(t) = e^{t\hat{A}_u}$ *maps* $\hat{F}_\phi$ *onto* $\hat{F}_{\Psi_u(t)(\phi)}$ *isometrically for each* $\phi \in \mathrm{O}(H)$.

Note that the reason for using the connection $\mathring{\nabla}$ in the definition of $\hat{\mathcal{F}}$, is that $R^{\mathring{\nabla}}(Z, W)A = 0$ for any $Z, W \in \Gamma(V)$ and $A \in \Gamma(H)$.

Proof of Proposition 3.14

(a) For any $x \in M$, choose a fixed element ϕ_0 in $\mathrm{O}(H)_x$. With the notation of Lemma 3.15, define

$$\mathcal{O}_{\phi_0} = \left\{ \Psi_{u_k}(t_k) \circ \cdots \circ \Psi_{u_1}(t_1)(\phi) : t_j \in \mathbb{R}, u_j \in \mathbb{R}^n,\ k \in \mathbb{N} \right\}.$$

Clearly, by definition of the set, for any $\phi \in \mathcal{O}_{\phi_0}$, there is an isometry $\hat{\Phi} : \hat{F}_{\phi_0} \to \hat{F}_\phi$ such that $\hat{\Phi}(\phi_0) = \phi$. Consider the vector bundle $\hat{H} = \operatorname{span}\{\hat{A}_u : u \in \mathbb{R}^n\}$ and define

$$\begin{aligned}\operatorname{Lie}_\phi \hat{H} &:= \operatorname{span}\left\{[B_1, [B_2, \cdots, [B_{k-1}, B_k]] \cdots]\big|_\phi : B_j \in \Gamma(\hat{H}),\ k \in \mathbb{R}\right\} \\ &= \operatorname{span}\left\{[\hat{A}_{u_1}, [\hat{A}_{u_2}, \cdots, [\hat{A}_{u_{k-1}}, \hat{A}_{u_k}]] \cdots]\big|_\phi : u_j \in \mathbb{R}^n,\ k \in \mathbb{R}\right\},\end{aligned}$$

for any $\phi \in \mathrm{O}(H)$. By the Orbit Theorem, see e.g. [2, Chapter 5], $\mathcal{O}_{\phi_0}$ is an immersed submanifold of $\mathrm{O}(H)$, and furthermore,

$$\operatorname{Lie}_\phi \hat{H} \subseteq T_\phi \mathcal{O}_{\phi_0}, \qquad \text{for any } \phi \in \mathcal{O}_{\phi_0}.$$

Since $p_* \hat{H} = H$ and since H is bracket-generating, we have that $p_* \operatorname{Lie}_\phi \hat{H} = T_{p(\phi)} M$. It follows that $p(\mathcal{O}_{\phi_0}) = M$. Hence, for any $y \in M$, there is an isometry $\hat{\Phi} : \hat{F}_{\phi_0} \to \hat{F}_\phi$ with $\hat{\Phi}(\phi_0) = \phi$ for some $\phi \in \mathrm{O}(H)_y$. As a consequence, there is a local isometry Φ taking x to y.

(b) Recall that $\nabla_A B = \nabla^g_A B + \frac{1}{2} T(A, B) = \nabla^g_A B - \frac{1}{2} \sharp \iota_{A \wedge B} \zeta$. Hence, if R^g is the curvature of the Levi-Civita connection, then

$$\begin{aligned}R^g(Z_1, Z_2) B_1 = R^\nabla(Z_1, Z_2) B_1 - \frac{1}{2}(\nabla_{Z_1} T)(Z_2, B_1) + \frac{1}{2}(\nabla_{Z_2} T)(Z_1, B_1) \\ - \frac{1}{2} T(T(Z_1, Z_2), B_1) + \frac{1}{4} T(Z_1, T(Z_2, B_1)) - \frac{1}{4} T(Z_2, T(Z_1, B_1))\end{aligned}$$

and we can write

$$\begin{aligned}\langle R^g(Z_1, Z_2) B_1, B_2\rangle_g &= \langle R^\nabla(Z_1, Z_2) B_1, B_2\rangle_g + \frac{1}{2}(\nabla_{Z_1}\zeta)(Z_2, B_1, B_2) \\ &\quad - \frac{1}{2}(\nabla_{Z_2}\zeta)(Z_1, B_1, B_2) - \frac{1}{2}\langle T(Z_1, Z_2), T(B_1, B_2)\rangle_g \\ &\quad - \frac{1}{4}\langle T(Z_1, B_2), T(Z_2, B_1)\rangle + \frac{1}{4}\langle T(Z_1, B_1), T(Z_2, B_2)\rangle\end{aligned}$$

for $Z_j, B_j \in \Gamma(TM)$. Since all the leaves of the foliation are totally geodesic, we have $\langle R^g(Z_1, Z_2) B_1, B_2\rangle = \langle R^{\mathcal{F}}(Z_1, Z_2) B_1, B_2\rangle$ whenever all vector fields take values in V. Using any local orthonormal bases $A_1, \dots, A_n$ and $Z_1, \dots, Z_\nu$ of H and V, respectively, then

$$\begin{aligned}\langle R^g(A_i, v) v, A_i\rangle_g &= \langle R^\nabla(A_i, v) v, A_i\rangle_g + \frac{1}{4}|T(A_i, v)|^2_g \\ &= \langle R^\nabla(A_i, v) v, A_i\rangle_g + \frac{1}{4}|\mathcal{R}(A_i, v)|^2_g + \frac{1}{4}|\mathcal{R}^*_{A_i} v|^2_g\end{aligned}$$

and

$$\begin{aligned}\langle R^g(Z_s, v) v, Z_s\rangle_g &= \langle R^\nabla(Z_s, v) v, Z_s\rangle_g + \frac{1}{4}|T(Z_s, v)|^2_g \\ &= \langle R^\nabla(Z_s, \operatorname{pr}_H v) \operatorname{pr}_H v, Z_s\rangle_g + \frac{1}{4}|\mathcal{R}^*_v Z_s|^2_g.\end{aligned}$$

Here we have used the first Bianchi identity (3.11) to obtain

$$\begin{aligned}&\langle R^\nabla(Z_s, v) v, Z_s\rangle_g = \langle R^\nabla(Z_s, \operatorname{pr}_H v) \operatorname{pr}_V v, Z_s\rangle + \langle R^\nabla(Z_s, \operatorname{pr}_V v) \operatorname{pr}_V v, Z_s\rangle \\ &= \langle \circlearrowright R^\nabla(Z_s, \operatorname{pr}_H v) \operatorname{pr}_V v, Z_s\rangle + \langle R^\nabla(Z_s, \operatorname{pr}_V v) \operatorname{pr}_V v, Z_s\rangle \\ &= \langle R^\nabla(Z_s, \operatorname{pr}_V v) \operatorname{pr}_V v, Z_s\rangle.\end{aligned}$$

In summary

$$\begin{aligned}\operatorname{Ric}_g(v,v) &= \sum_{i=1}^{n}\langle R^g(A_i,v)v,A_i\rangle_g + \sum_{s=1}^{\nu}\langle R^g(Z_s,v)v,Z_s\rangle_g.\\ &= \operatorname{Ric}(\nabla)(\flat v)(v) + \frac{1}{2}\sum_{i=1}^{n}|\mathcal{R}(A_i,v)|_g^2 + \frac{1}{4}\sum_{i=1}^{n}|\mathcal{R}^*_{A_i}v|^2 + \operatorname{Ric}_{\mathcal{F}}(v,v).\end{aligned}$$

The result now follows from (a). □

Remark 3.16

(a) Let g be any metric taming g_H such that $I\!I = 0$. Let $\mathring{\nabla}$ be the Bott connection defined in (3.15). Write V for the orthogonal complement of H. Then for any $\varepsilon > 0$, the scaled Riemannian metric

$$g_\varepsilon(v,w) = g(\operatorname{pr}_H v, \operatorname{pr}_H w) + \frac{1}{\varepsilon}g(\operatorname{pr}_V v, \operatorname{pr}_V w),$$

also tames g_H and satisfies $I\!I = 0$. Since $\mathring{\nabla}_A B$ is independent of $g|V$ whenever at least one of the vector fields takes values only in H, it behaves better with respect to the scaled metric. Such scalings of the extended metric are important for sub-Riemannian curvature-dimension inequalities, see [8, 10–12, 24, 25].

(b) If $\bar{\mathcal{R}} = 0$ then we have that $\operatorname{tr}_H(\nabla_\times \mathcal{R})(\times,\cdot) = \operatorname{tr}_H(\mathring{\nabla}_\times \mathcal{R})(\times,\cdot)$. If this map vanishes, i.e. if $\operatorname{Ric}(\nabla)$ is a symmetric operator, then H is said to satisfy *the Yang-Mills condition*. One may consider subbundles H satisfying this condition as locally minimizing the curvature $\mathcal{R}$. See [25, Appendix A.4] for details.

3.7.2 Regular Foliations

We give a short remark on the case in Section 3.7.1 when the foliation is also regular, i.e. when there is a global Riemannian submersion $\pi: (M,g) \to (\check{M},\check{g})$ with foliation $\mathcal{F} = \{F_y = \pi^{-1}(y):\ y \in \check{M}\}$. We can rewrite (3.17) as

$$\begin{aligned}\operatorname{Ric}_g(v,v) &= \operatorname{Ric}(\mathring{\nabla})(\flat v)v - \frac{1}{2}|\mathcal{R}(v,\cdot)|^2_{g^*\otimes g} + \frac{1}{4}|\mathcal{R}^*_\cdot v|^2_{g^*\otimes g}\\ &\quad + \langle v, \operatorname{tr}_H(\mathring{\nabla}_\times\mathcal{R})(\times,v)\rangle_g + \operatorname{Ric}_{\mathcal{F}}(\operatorname{pr}_V v, \operatorname{pr}_V v).\end{aligned}$$

Furthermore, as $\operatorname{Ric}(\flat v)v = \operatorname{Ric}(\flat \operatorname{pr}_H v)\operatorname{pr}_H v$, requiring that $\operatorname{Ric}(\mathring{\nabla})$ is bounded from below is even weaker than requiring this for Ric_g. This weaker condition is a sufficient requirement for infinite lifetime for the case of regular foliations.

To prove this, we need a result in [29]. Fix a point $y_0 \in \check{M}$ and let $\sigma : [0,1] \to \check{M}$ be a smooth curve with $\sigma(0) = y_0$. Define $F = F_{y_0}$ and write σ^x for the H-horizontal lift of σ starting at $x \in F$. Then the map

$$\Psi_{\sigma(t)}: F \to F_{\sigma(t)}, \quad \Psi_{\sigma(t)}(x) := \sigma^x(t),$$

is an isometry, so all leaves of $\mathcal{F}$ are isometric. Write G for the isometry group of F and Q_y for the space of isometries $q: F \to F_y$. Then $Q = \coprod_{y\in\check{M}} Q_y$ can be given a structure of a principal bundle, such that

$$p: Q \times F \to M \cong (Q\times F)/G, \quad (q,z) \mapsto q(z).$$

In the above formula, $\phi \in G$ acts on F on the right by $z \cdot \phi = \phi^{-1}(z)$. Finally, if we define

$$E = \left\{ \frac{d}{dt} \Psi_{\sigma(t)} \circ \phi : \begin{array}{l} \sigma \in C^\infty([0,1], \check{M}) \\ \sigma(0) = y_0, \ \phi \in G, \ t \in [0,1] \end{array} \right\} \subseteq TQ,$$

then E is a principal connection on Q and $p_* E = H$.

One can verify that if $Y_t(y)$ is the Brownian motion in $\check{M}$ starting at $y \in \check{M}$ with horizontal lift $\tilde{Y}_t(q)$ to $q \in Q_y$ with respect to E, then $X_t(x) = p(\tilde{Y}_t(q), z)$ is a diffusion in M with infinitesimal generator $\frac{1}{2}\Delta_H$ starting at $x = p(q, z)$. Hence, if $Y_t(y)$ has infinite lifetime so does $X_t(x)$, as a process and its horizontal lifts to principal bundles have the same lifetime [39]. Since a lower bound of $\mathrm{Ric}(\mathring{\nabla})$ is equivalent to a lower bound of the Ricci curvature of $\check{M}$ by [24, Section 2], this is a sufficient condition for infinite lifetime of $X_t(x)$.

The above argument does not depend on H being bracket-generating. However, in the case of H bracket-generating, F is a homogeneous space by a similar argument to that of the proof of Proposition 3.14.

3.7.3 A Counter-example

We will give an example showing that the assumption $\bar{\mathcal{R}} = 0$ is essential for the conclusion of Proposition 3.14.

Example 3.17 Consider $M = \mathrm{SU}(2) \times \mathrm{SU}(2)$ with vector fields $A^\pm, B^\pm, C^\pm$ as in defined in Example 3.12. Consider $\mathbb{R}$ with coordinate c and introduce $\tilde{M} = M \times \mathbb{R}$. Let f be an arbitrary smooth function on M that factors through the projection to $\mathbb{R}$, i.e. $f(x, y, c) = f(c)$ for $(x, y, c) \in \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}$. We write $\partial_c f$ simply as f'. Let Z_j, $j = 1, 2, 3$ be the vector fields on M given by

$$Z_1 = e^f A^+, \quad Z_2 = e^f B^+, \quad Z_3 = e^f A^-,$$

and define a Riemannian metric g on $\tilde{M}$ such that $Z_1, Z_2, Z_3, C^+, B^-, C^-, \partial_c$ form an orthonormal basis. Define a sub-Riemannian manifold $(\tilde{M}, H, g_H)$ such that H is the span of Z_1, Z_2, Z_3 and ∂_c with g_H the restriction of g to this bundle. Defining $\mathit{I\!I}$ and $\mathcal{C}$ as in respectively (3.4) and (3.9), we have $\mathit{I\!I} = 0$ and $\mathcal{C} = 0$, even though $\bar{\mathcal{R}} \neq 0$. If ∇ is as in (3.7), then $\mathrm{Ric}(\nabla)$ is given by

$$\mathrm{Ric}(\nabla) : \begin{cases} \flat Z_1 \mapsto \left(f'' - e^{2f}(e^{2f} - 1) - 3(f')^2\right) \flat Z_1, \\ \flat Z_2 \mapsto \left(f'' - 2e^{2f}(e^{2f} - 1) - 3(f')^2\right) \flat Z_2, \\ \flat Z_3 \mapsto \left(f'' - e^{2f}(e^{2f} - 1) - 3(f')^2\right) \flat Z_3, \\ \flat \partial_c \mapsto 3\left(f'' - (f')^2\right) \flat \partial_c. \end{cases}$$

However, one can also verify that if Ric_g is the Ricci curvature of g, then

$$\mathrm{Ric}_g(B^-, B^-) = 2 - e^{-f}.$$

Hence, if f' and f'' are bounded and f is bounded from above but not from below, then $\mathrm{Ric}(\nabla)$ has a lower bound, but not Ric_g. For example, one may take $f(c) = -c \tan^{-1} c$.

4 Torsion, Integration by Parts and a Bound for the Horizontal Gradient on Carnot Groups

4.1 Torsion and Integration by Parts

For a function $f \in C^\infty(M)$ on a sub-Riemannian manifold define the horizontal gradient $\nabla^H f = \sharp^H df$. The fact that the parallel transport $\hat{//}_t$ in Theorem 3.6 does not preserve the horizontal bundle, makes it difficult to bound $\nabla^H P_t f$ by terms only involving the horizontal part of the gradient of f and not the full gradient. We therefore give the following alternative stochastic representation of the gradient.

Let (M, g_H^*) be a sub-Riemannian manifold and let ∇ be compatible with g_H^*. Let g be a Riemannian metric taming g_H and assume that ∇ is compatible with g as well. Introduce a zero order operator

$$\begin{aligned}\mathscr{A}(\alpha) &:= \mathrm{Ric}(\nabla)\alpha - \alpha(\mathrm{tr}_H(\nabla_\times T^\nabla)(\times, \cdot)) - \alpha(\mathrm{tr}_H T^\nabla(\times, T^\nabla(\times, \cdot))) \\ &= \mathrm{Ric}(\hat{\nabla})\alpha + \alpha(\mathrm{tr}_H T^\nabla(\times, T^\nabla(\times, \cdot))).\end{aligned} \tag{4.1}$$

Let $X_t(\cdot)$ be the stochastic flow of $\frac{1}{2}L(\nabla)$ with explosion time $\tau(\cdot)$. Write $//_t = //_t(x) : T_xM \to T_{X_t(x)}M$ for parallel transport with respect to ∇ along $X_t(x)$. Observe that this parallel transport along ∇ preserves H and its orthogonal complement. Let $W_t = W_t(x)$ denote the anti-development of $X_t(x)$ with respect to ∇ which is a Brownian motion in $(H_x, \langle \cdot, \cdot \rangle_{g_H(x)})$.

Theorem 4.1 *Assume that $\tau(x) = \infty$ a.s. for any $x \in M$ and that for any $t_1 > 0$ and any $f \in C_b^\infty(M)$ with bounded gradient, we have $\sup_{t\in[0,t_1]} \|dP_t f\|_{L^\infty(g^*)} < \infty$. Furthermore, assume that $|T^\nabla|_{\wedge^2 g^* \otimes g} < \infty$ and that $\mathscr{A}$ is bounded from below. Define stochastic processes $Q_t = Q_t(x)$ and $U_t = U_t(x)$ taking values in $\mathrm{End}\, T_x^* M$ as follows:*

$$\frac{d}{dt} Q_t = -\frac{1}{2} Q_t \mathscr{A}_{//_t} \quad Q_0 = \mathrm{id},$$

resp.

$$U_t\alpha(v) = \int_0^t \alpha T^\nabla_{//_s}(dW_s, Q_s^\intercal v), \qquad T^\nabla_{//_t}(v, w) = //_t^{-1} T(//_t v, //_t w).$$

Then for any $f \in C_b^\infty(M)$,

$$dP_t f(x) = \mathbb{E}\Big[(Q_t + U_t)//_t^{-1} df(X_t(x))\Big]. \tag{4.2}$$

For a geometric interpretation of $\mathscr{A}$ for different choices of ∇, see Section 4.2. Equality (4.2) allows us to choose the connection ∇ convenient for our purposes and gives us a bound for the horizontal gradient on Carnot groups in Section 4.3.

For the proof of this result, we rely on ideas from [17]. *A multiplication m* of T^*M is a map $m : T^*M \otimes T^*M \to T^*M$. Corresponding to a multiplication and a connection ∇, we have a corresponding first order operator

$$D^m \alpha = m(\nabla_\cdot \alpha).$$

Lemma 4.2 *Let ∇ be a connection compatible with g_H^* and with torsion T. Define $L = L(\nabla)$, $\mathrm{Ric} = \mathrm{Ric}(\nabla)$ and $T = T^\nabla$. Then for any $f \in C^\infty(M)$,*

$$Ldf - dLf = -2D^m df + \mathscr{A}(df),$$

where $m(\beta \otimes \alpha) = \alpha(T(\sharp^H \beta, \cdot))$ and $\mathscr{A}$ as in (4.1).

Proof Recall that if $\hat{\nabla}$ is the adjoint of ∇ and $\hat{L} = L(\hat{\nabla})$, then

$$(\hat{L}df - dLf) = \operatorname{Ric} df.$$

The result now follows from Lemma 3.3 and the fact that for any $A \in \Gamma(H)$,

$$\hat{\nabla}_A = \nabla_A + \kappa(A),$$

where $\kappa(A)\alpha = \alpha(T(A, \cdot)) = m(\flat A \otimes \alpha)$. □

Proof of Theorem 4.1 Let $x \in M$ be fixed. To simplify notation, we shall write $X_t(x)$ simply as X_t. Define $//_t$ as parallel transport with respect to ∇ along X_t. Define Q_t as in Theorem 4.1. For any $t_1 > 0$, consider the stochastic process on $[0, t_1]$ with values in $T_x^* M$,

$$N_t = //_t^{-1} dP_{t_1 - t} f(X_t).$$

By Lemma 4.2 and Itô's formula

$$dN_t = //_t^{-1} \nabla_{//_t dW_t} dP_{t_1-t} f(X_t) - //_t^{-1} D^m dP_{t_1-t} f(X_t) dt + \frac{1}{2} //_t^{-1} \mathscr{A}(dP_{t_1-t} f(X_t)) dt,$$

and so

$$dQ_t N_t = Q_t //_t^{-1} \nabla_{//_t dW_t} dP_{t_1-t} f(X_t) - Q_t //_t^{-1} D^m dP_{t_1-t}(X_t)\, dt.$$

Since W_t is a Brownian motion in H_x and $//_t$ preserves H and its inner product, the differential of the quadratic covariation equals

$$d[U_t, N_t] = Q_t //_t^{-1} D^m dP_{t_1-t} f(X_t)\, dt.$$

Hence, $(Q_t + U_t)N_t$ is a local martingale which is a true martingale from our assumptions. The result follows. □

4.2 Geometric Interpretation

We will look at some specific examples to interpret Theorem 4.1 and the zero order operator $\mathscr{A}$ in (4.1).

4.2.1 Totally Geodesic Riemannian Foliation and its Generalizations

Assume that condition (3.5) holds, so that we are in the case of Section 3.2. Define ∇ as in (3.7) and let $\mathring{\nabla}$ be the Bott connection defined as in (3.15). Recall that its torsion $\mathring{T}$ equals $\mathring{T} = -\mathcal{R} - \bar{\mathcal{R}}$ and that $\nabla_Z = \mathring{\nabla}_Z + \mathring{T}_\cdot^* Z$. It can then be computed that $\mathscr{A}$ is given by

$$\begin{aligned}
\langle \mathscr{A} \operatorname{pr}_H^* \alpha, \operatorname{pr}_H^* \beta \rangle_{g^*} &= \langle \operatorname{Ric}(\mathring{\nabla})\alpha, \beta \rangle_{g^*}, \\
\langle \mathscr{A} \operatorname{pr}_H^* \alpha, \operatorname{pr}_V^* \beta \rangle_{g^*} &= \mathcal{C}(\sharp^V \beta, \sharp^H \alpha) \\
\langle \mathscr{A} \operatorname{pr}_V^* \alpha, \operatorname{pr}_H^* \beta \rangle_{g^*} &= \mathcal{C}(\sharp^V \alpha, \sharp^H \beta) + \alpha(\operatorname{tr}_H \mathring{\nabla}_\times \mathcal{R})(\times, \sharp\beta) \\
\langle \mathscr{A} \operatorname{pr}_V^* \alpha, \operatorname{pr}_V^* \beta \rangle_{g^*} &= \langle \mathcal{R}_\cdot^* \sharp\alpha, \mathcal{R}_\cdot^* \sharp\alpha \rangle_{g^* \otimes g} + \langle \bar{\mathcal{R}}(\sharp\alpha, \cdot), \bar{\mathcal{R}}(\sharp\beta, \cdot) \rangle_{g^* \otimes g}.
\end{aligned}$$

4.2.2 Lie Groups of Polynomial Growth

Let G be a connected Lie group with unit $\mathbf{1}$ of polynomal growth. Consider a subspace $\mathfrak{h}$ that generates all of $\mathfrak{g}$. Equip $\mathfrak{h}$ with an inner product and define a sub-Riemannian structure (H, g_H) by left translation of $\mathfrak{h}$ and its inner product. Let g be any left invariant metric taming g_H. Let ∇ be the connection defined such that any left invariant vector field on G is ∇-parallel. Then ∇ is compatible with g_H^* and g. Let $X_t(\cdot)$ be the stochastic flow of $\frac{1}{2} L(\nabla)$, which has infinite lifetime by [26]. Furthermore, $\|dP_t f\|_{L^\infty(g^*)} < \infty$ for any bounded $f \in C_b^\infty(G)$ by [42]. Hence we can use Theorem 4.1.

Let $l_x : G \to G$ denote left multiplication on G and write $x \cdot v := dl_x v$. Notice that since we have a left invariant system, $X_t(x) = x \cdot X_t(\mathbf{1}) =: x \cdot X_t$. Furthermore, parallel transport with respect to ∇ is simply left translation so

$$//_t(x)v = (x \cdot X_t \cdot x^{-1}) \cdot v.$$

If $W_t(x)$ is the anti-development of $X_t(x)$ with respect to ∇ then

$$W_t(x) = x \cdot W_t(\mathbf{1}) =: x \cdot W_t.$$

As ∇ is a flat connection and since

$$T^\nabla(A_1, A_2) = -[A_1, A_2],$$

for any pair of left invariant vector fields A_1 and A_2, we have that $\mathscr{A}$ in (4.1) equals

$$\mathscr{A} = -\alpha(\mathrm{tr}_H\, T(\times, T(\times, \cdot))).$$

In other words, if we define a map $\psi : \mathfrak{g} \to \mathfrak{g}$, by

$$\psi = \mathrm{tr}_{H_1}\, \mathrm{ad}(\times)\, \mathrm{ad}(\times), \tag{4.3}$$

then

$$\mathscr{A}\alpha = -l_{x^{-1}}^* \psi^* l_x^* \alpha, \quad \alpha \in T_x^* G.$$

Both $\mathscr{A}$ and T^∇ are bounded in g. Hence, we can conclude that for any $v \in \mathfrak{g}$ and $x \in G$,

$$dP_t f(x \cdot v) = \mathbb{E}\left[df\left((x \cdot X_t) \cdot \left(Q_t^\intercal v + \int_0^t \mathrm{ad}(Q_s^\intercal v) dW_s\right)\right)\right]$$

where

$$Q_t = \exp\left(-t\psi^*/2\right).$$

Note that Q_t is deterministic in this case.

4.3 Carnot Groups and a Gradient Bound

Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and identity $\mathbf{1}$. Assume that there exists a stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ into subspaces, each of strictly positive dimension, such that $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{1+j}$ for any $1 \le j \le k$ with convention $\mathfrak{g}_{k+1} = 0$. Write $\mathfrak{h} = \mathfrak{g}_1$ and choose an inner product on this vector space. Define the sub-Riemannian structure (H, g_H) on G by left translation of $\mathfrak{h}$ and its inner product. Then (G, H, g_H) is called *a Carnot group of step k*. Carnot groups are important as they are the analogue of Euclidean space in Riemannian geometry in the sense that any sub-Riemannian manifold has a Carnot group as its metric tangent cone at points where the horizontal bundle is equiregular. See [13] for details and the definition of equiregular.

Let (G, H, g_H) be a Carnot group with $n = \mathrm{rank}\, H$. Let Δ_H be defined with respect to left Haar measure on G, which equals the right Haar measure since nilpotent groups are unimodular. Consider the commutator ideal $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ with corresponding normal subgroup K. Define the corresponding quotient map

$$\pi : G \to G/K \cong \mathfrak{h},$$

and write $|\pi| \colon x \mapsto |\pi(x)|_{g_H(\mathbf{1})}$.

It is known from [16] and [34] that for each $p \in (1, \infty)$, there exists a constant C_p such that $|\nabla^H P_t f|_{g_H} \le C_p (P_t |\nabla^H f|_{g_H})^{1/p}$ pointwise for any $f \in C^\infty(G)$. We want to give a more explicit description of constants satisfying this inequality.

Theorem 4.3 *Let ψ be defined as in (4.3) and assume that $\psi|\mathfrak{h} = 0$. Let $p_t(x, y)$ denote the heat kernel of Δ_H and define $\varrho(x) = p_1(\mathbf{1}, x)$. Define a probability measure $\mathbb{P}$ on G by $d\mathbb{P} = \varrho d\mu$. Let Q be the homogeneous dimension of G,*

$$Q := \sum_{j=1}^{k} j\,(\operatorname{rank} \mathfrak{g}_j). \tag{4.4}$$

(a) *Consider the function $\vartheta(x) = n + |\pi|(x) \cdot |\nabla^H \log \varrho|_{g_H}(x)$ and for any $p \in (1, \infty]$, the constant*

$$C_p = \left(\int_G \varrho(y) \cdot \vartheta^q(y)\, d\mu(y) \right)^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{4.5}$$

Then the constants C_p are finite and for any $x \in G$ and $t \geq 0$, we have

$$|\nabla^H P_t f|_{g_H}(x) \leq C_p (P_t |\nabla^H f|^p_{g_H}(x))^{1/p}, \qquad f \in C^\infty(G).$$

Furthermore, $C_2 < n + (nQ - 2\operatorname{Cov}_{\mathbb{P}}[|\pi|^2, \log \varrho])^{1/2}$ where $\operatorname{Cov}_{\mathbb{P}}$ is the covariance with respect to $\mathbb{P}$.

(b) *For any n and $q \in [2, \infty)$, define*

$$c_{n,q} = \left(\frac{2^{(q+n+1)/2} \pi^{(n-1)/2}}{\sqrt{n}} \frac{\Gamma(\frac{n+q}{2})}{\Gamma(\frac{n}{2})} \right)^{1/q}.$$

Then for $p \in (2, \infty)$, we have

$$|\nabla^H P_t f| \leq (n + c_{n,q} \sqrt{Q}) \left(P_t |df|^p \right)^{1/p}, \qquad \frac{1}{q} + \frac{1}{p} = \frac{1}{2}.$$

The condition $\psi|\mathfrak{h} = 0$ is actually equal to the Yang-Mills condition in the case of Carnot groups, see Remark 4.6. In the definition of ϱ, the choices of $t = 1$ and $x = \mathbf{1}$ are arbitrary. For any fixed t and x, if we replace ϱ by $\varrho_{t,x}(y) := p_t(x, y)$ in (4.5), we would still obtain the same bounds. Taking into account [34, Cor 3.17], we get the following immediate corollary.

Corollary 4.4 *For any smooth function $f \in C^\infty(G)$ and $t \geq 0$, we have*

$$P_t f^2 - (P_t f)^2 \leq t\, C_2^2 P_t |\nabla^H f|^2_{g_H}$$

with C_2 as in Eq. (4.5).

We introduce the theory necessary for the proof of Theorem 4.3. Let g be a left invariant metric on G taming g_H. Let ∇ be the connection on M defined such that all left invariant vector fields are parallel. As

$$\beta(v) = \operatorname{tr} T^\nabla(v, \cdot) = 0, \qquad v \in TG$$

we have that $L(\nabla)^* = L(\nabla)$ by Lemma 2.1. Furthermore, if $A_1, \ldots, A_n$ is a basis of $\mathfrak{g}$, then $L(\nabla) f = \sum_{i=1}^n A_i^2 f$ by [1]. Let $X_t := X_t(\mathbf{1})$ be a $\frac{1}{2}\Delta_H$-diffusion starting at the identity $\mathbf{1}$ and let $//_t$ denote the corresponding parallel transport along X_t with respect to ∇. Let $\pi : G \to \mathfrak{h}$ denote the quotient map.

(i) For any $v, w \in H$ we have $\langle v, w \rangle_{g_H} = \langle \pi_* v, \pi_* w \rangle_{g_H(\mathbf{1})}$. Hence we can consider our sub-Riemannian structure as obtained by choosing a principal Ehresmann connection H on π and lifting the metric on $\mathfrak{h}$. It follows by [24, Section 2] that Δ_H is the horizontal lift of the Laplacian of $(\mathfrak{h}, \langle\cdot,\cdot\rangle_{g_H(\mathbf{1})})$ and so we have that $W_t = \pi(X_t)$ is a Brownian motion in the inner product space $\mathfrak{h}$. Since

$$\pi_* v = \mathrm{pr}_{\mathfrak{h}}\, x^{-1} \cdot v, \quad v \in T_x G,$$

we may identify W_t with the anti-development of X_t.

(ii) Since Δ_H is left invariant, $X_t(x) := x \cdot X_t$ is a $\frac{1}{2}\Delta_H$-diffusion starting at x, and $P_t f(x) = P_t(f \circ l_x)(\mathbf{1})$ where l_x denotes left translation. In particular, if $\varrho_t(x) := p_t(\mathbf{1}, x)$ then

$$p_t(x, y) = \varrho_t(x^{-1}y).$$

(iii) Since the Lie algebra $\mathfrak{g}$ has a stratification, for any $s > 0$, the map $(\mathrm{Dil}_s)_* \colon \mathfrak{g} \mapsto \mathfrak{g}$ given by

$$(\mathrm{Dil}_s)_* A \in \mathfrak{g}_j \mapsto s^j A \tag{4.6}$$

is a Lie algebra automorphism. It corresponds to a Lie group automorphism Dil_s of G since G is simply connected. These automorphisms are called *dilations*. It can be verified that if $A \in \mathfrak{g}_j$ and we use the same symbol for the corresponding left invariant vector field then

$$A(f \circ \mathrm{Dil}_s) = s^j (Af) \circ \mathrm{Dil}_s\,.$$

(iv) As a consequence of item (4.3) we have

$$\Delta_H(f \circ \mathrm{Dil}_s) = s^2(\Delta_H f) \circ \mathrm{Dil}_s,$$

and hence

$$P_t(f \circ \mathrm{Dil}_s) = (P_{s^2 t} f) \circ \mathrm{Dil}_s\,.$$

Also, for any function f, we have $|df|_{g_H^*} \circ \mathrm{Dil}_s = s^{-1}|d(f \circ \mathrm{Dil}_s)|_{g_H^*}$.

(v) Let Q be the homogeneous dimension of G as in (4.4). By definition $\mathrm{Dil}_s^* \mu = s^Q \mu$, and considering (4.3), the heat kernel has the behavior

$$\varrho_{s^2 t}(\mathrm{Dil}_s(x)) = s^{-Q} \varrho_t(x).$$

(vi) Clearly $R^\nabla = 0$ and $\nabla T = 0$ since the torsion takes left invariant vector fields to left invariant vector fields. Hence, for any left invariant vector field A, we have $\mathscr{A}^\top A = \psi A$ with ψ as in (4.3). If $\psi|\mathfrak{h} = 0$, we can apply Theorem 4.1. We obtain that for any $v \in \mathfrak{h}$,

$$dP_t f(v) = \mathbb{E}\left[/\!/_t^{-1} df(X_t)\big(v + \mathrm{ad}(W_t)v\big)\right].$$

Theorem 4.3 now follows as a result of the next Lemma. Note that for any function $f \in C^\infty(M)$, we have $|\nabla^H f|_{g_H} = |df|_{g_H^*}$.

Lemma 4.5 *Assume that $\psi|\mathfrak{h} = 0$. For every $t > 0$, define*

$$\vartheta_t = n + |\pi||d \log \varrho_t|_{g_H^*}$$

where $|\pi|(x) = |\pi(x)|_{g_H(\mathbf{1})}$. For any $p \in (1, \infty]$, let $q \in [1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and consider

$$C_{t,p} := \mathbb{E}\left[\vartheta_t(X_t)^q\right]^{1/q}. \tag{4.7}$$

Then

(a) $C_{t,p} = C_{1,p} = C_p$ *for any* $t > 0$.

(b) *The constants C_p are finite. Furthermore, we have the inequality*

$$C_2 \le n + \left(nQ + 2\int_G (n - |\pi|^2)\varrho \log \varrho \, d\mu\right)^{1/2} = n + (nQ - 2\,\mathrm{Cov}_{\mathbb{P}}[|\pi|^2, \log \varrho])^{1/2}.$$

Proof To keep the notation simple, we write $\langle \cdot, \cdot \rangle_{L^2(\wedge^j \mathfrak{g}^*)}$ as $\langle \cdot, \cdot \rangle$ and let $r = |\pi|^2$.

(a) We use dilations to prove the statement. Observe that $r \circ \mathrm{Dil}_s = s^2 r$ and that $|d \log \varrho_t|_{g_H^*} \circ \mathrm{Dil}_s = s^{-1} |d \log \varrho_{t/s^2}|_{g_H^*}$, and so $\vartheta_t \circ \mathrm{Dil}_s = \vartheta_{t/s^2}$. It follows that

$$\begin{aligned}(C_{t,p})^q &= \int_G \varrho_t \vartheta_t^q \, d\mu \overset{\mathrm{Dil}^*_{\sqrt{t}}}{=} \int_G (\varrho_t \circ \mathrm{Dil}_{\sqrt{t}}) \left(\vartheta_t \circ \mathrm{Dil}_{\sqrt{t}}\right)^q t^{Q/2} \, d\mu \\ &= \int_G \varrho_1 \vartheta_1^q \, d\mu = (C_p)^q.\end{aligned}$$

(b) We only need to show that for any $1 < q < \infty$,

$$\int_G \varrho (r^{1/2} |d \log \varrho|_{g_H^*})^q d\mu = \int_G r^{q/2} \varrho^{1-q} |d\varrho|^q_{g_H^*} \, d\mu < \infty.$$

Define $\mathsf{d}(x) = \mathsf{d}_{g_H}(\mathbf{1}, x)$. Then π is distance decreasing, so $r(x) \leq \mathsf{d}(x)^2$. By [44, Theorem 1], for any $0 < \varepsilon < \frac{1}{2}$ there is a constant k_ε such that

$$\frac{1}{\varrho(x)} \leq k_\varepsilon \exp\left(\frac{\mathsf{d}^2(x)}{2 - \varepsilon}\right).$$

Furthermore, by [45, Theorem IV.4.2], for every $\varepsilon' > 0$ there are constants $k_{\varepsilon'}$ such that

$$|d\varrho|_{g_H^*}(x) \leq k_{\varepsilon'} \exp\left(-\frac{\mathsf{d}^2(x)}{2 + \varepsilon'}\right).$$

Since we can always find appropriate values of ε and ε' such that

$$\frac{q-1}{q} \leq \frac{2 - \varepsilon}{2 + \varepsilon'},$$

it follows that $\int_G r^{q/2} \varrho^{1-q} |d\varrho|^q_{g_H^*} \, d\mu < \infty$.

Next, define the vector field D by $Df = \frac{d}{ds}(f \circ \mathrm{Dil}_{1+s})|_{s=0}$ for any function f. If f satisfies $f \circ \mathrm{Dil}_\varepsilon = \varepsilon^k f$, then by definition $Df = kf$. By item (v), we have $\operatorname{div} D = Q$ since

$$\mathcal{L}_D \mu = \frac{d}{ds} \mathrm{Dil}^*_{1+s} \mu|_{s=0} = \frac{d}{ds} (1+s)^Q \mu|_{s=0} = Q\mu.$$

Furthermore, again by item (v),

$$\begin{aligned}-Q\varrho_t &= \frac{d}{ds}(1+s)^{-Q} \varrho_t|_{s=0} \\ &= \frac{d}{ds} \varrho_{(1+s)^2 t} \circ \mathrm{Dil}_{1+s}|_{s=0} = 2t \cdot \frac{1}{2} \Delta_H \varrho_t + D\varrho_t,\end{aligned}$$

so

$$(t\Delta_H + D + Q) p_t = (t\Delta_H - D^*) p_t = 0.$$

This equality along with the observation that

$$\Delta_H (\varrho_t \log \varrho_t) = (\log \varrho_t + 1) \Delta_H \varrho_t + \varrho_t |d \log \varrho_t|^2_{g_H^*}$$

allows us to compute

$$\begin{aligned}(C_2 - n)^2 &\leq \langle r, \varrho |d \log \varrho|^2_{g_H^*} \rangle = \langle r, \Delta_H (\varrho \log \varrho) - (\log \varrho + 1) \Delta_H \varrho \rangle \\ &= \langle \Delta_H r, \varrho \log \varrho \rangle + \langle r, (\log \varrho + 1) D\varrho \rangle + Q \langle r, (\log \varrho + 1) \varrho \rangle \\ &= 2n \langle \varrho, \log \varrho \rangle + \langle r, (D + Q) \varrho \log \varrho \rangle + Q \langle r, \varrho \rangle\end{aligned}$$

$$= 2n\langle \varrho, \log \varrho\rangle - \langle Dr, \varrho \log \varrho\rangle + Qn$$
$$= 2\langle (n-r), \varrho \log \varrho\rangle + Qn$$

which equals to the covariance since $\int_G r\varrho d\mu = n$. □

Proof of Theorem 4.3 Again, for simplicity, we write $\langle\cdot,\cdot\rangle_{L^2(\wedge^j g^*)}$ as $\langle\cdot,\cdot\rangle$ and $r = |\pi|^2$.

(a) By left invariance, it is sufficient to prove the inequality at the point $x = \mathbf{1}$. Let $v \in H_{\mathbf{1}} = \mathfrak{h}$ be arbitrary. We will use Theorem 4.1 and item (4.3). For every $x \in G$ we have $\sharp dr(x) = 2x \cdot \pi(x)$. Let us consider the form α^v defined by $\alpha^v(x) = \flat(x \cdot v)$. Then

$$\begin{aligned} dP_t f(v) &= \mathbb{E}\left[//_t^{-1} df(X_t)\,(v - \mathcal{R}(W_t, v))\right] \\ &= \mathbb{E}[//_t^{-1} df(X_t)(v)] - \mathbb{E}\left[df(X_t)\mathcal{R}(//_t(\pi(X_t) \wedge v))\right] \\ &= \mathbb{E}[//_t^{-1} df(X_t)(v)] - \frac{1}{2}\mathbb{E}\left[df\mathcal{R}(\sharp dr, \sharp\alpha^v)(X_t)\right]. \end{aligned}$$

Define $F(A, B) = \flat A \wedge \nabla_B$ and extend F to general sections of $TG^{\otimes 2}$ by $C^\infty(G)$-linearity. Consider $F_H = F(g_H^*)$ and notice that

$$F_H f = d_H f = \mathrm{pr}_H^* df, \quad F_H^2 f = df\mathcal{R}(\cdot,\cdot).$$

Hence

$$\begin{aligned} &\mathbb{E}\big[\langle df\mathcal{R}(\sharp dr, \sharp\alpha^v)(X_t)\big] = \langle F_H^2 f, \varrho_t dr \wedge \alpha^v\rangle \\ &= \langle F_H f, F_H^*(\varrho_t dr \wedge \alpha^v)\rangle \\ &= -\langle d_H f, \iota_{\sharp^H d\varrho_t} dr \wedge \alpha^v\rangle - \langle d_H f, \varrho_t(\Delta_{g_H^*} r)\alpha^v\rangle + \langle d_H f, \varrho_t \nabla_{\sharp^H \alpha} dr\rangle \end{aligned}$$

since $\nabla\alpha^v = 0$. Using the identities $\Delta_H r = 2n$ and $\nabla_A dr = 2\flat\,\mathrm{pr}_H A$, we obtain

$$\mathbb{E}\left[\left\langle F_H^2 f, dr \wedge \alpha^v\right\rangle_{g^*}(X_t)\right] = -\langle d_H f, \iota_{\sharp^H d\varrho_t} dr \wedge \alpha^v\rangle - 2(n-1)\langle d_H f, \varrho_t \alpha^v\rangle$$
$$= -\mathbb{E}\left[\left\langle d_H f, \iota_{\sharp^H d\log\varrho_t} dr \wedge \alpha^v\right\rangle_{g^*}(X_t)\right] - 2(n-1)\mathbb{E}\left[//_t^{-1} d_H f(X_t)(v)\right].$$

Hence, if we define $\mathscr{N}_t\colon T_{\mathbf{1}}^*G \to T_{\mathbf{1}}^*G$ by

$$\mathscr{N}_t\beta = n\beta + \frac{1}{2}//_t^{-1}\iota_{\sharp dr(X_t)}(d\log\varrho_t(X_t) \wedge //_t\beta),$$

then $dP_t f(v) = \mathbb{E}[\mathscr{N}_t //_t^{-1} df(v)]$ for any $v \in H$.

Observe that $|\mathscr{N}_t\beta|_{g_H^*} \le \vartheta_t|\beta|_{g_H^*}$. Using Hölder's inequality, this leads us to the conclusion

$$\begin{aligned} |dP_t f|_{g_H^*}(\mathbf{1}) &= \sup_{v\in\mathfrak{h}, |v|_{g_H}=1} dP_t f(v) \\ &= \sup_{v\in\mathfrak{h}, |v|_{g_H}=1} \mathbb{E}[\mathscr{N}_t //_t^{-1} df(X_t)(v)] \\ &\le \mathbb{E}[\vartheta_t^q \circ X_t]^{1/q}\mathbb{E}[|df|_{g_H^*}^p \circ X_t]^{1/p} \\ &\le C_{t,p}(P_t|df|_{g_H^*}^p(\mathbf{1}))^{1/p}. \end{aligned}$$

(b) Using $dP_t f(v) = \mathbb{E}[\mathscr{N}_t //_t^{-1} df(v)]$, for $p \in (2,\infty]$, $q \in [2,\infty)$ satisfying

$$\frac{1}{q} + \frac{1}{p} + \frac{1}{2} = 1,$$

we have

$$
\begin{aligned}
|dP_1 f|_{g_H^*}(1) &\leq n\mathbb{E}[|df|_{g_H}(X_1)] + \mathbb{E}\left[(|\pi||\log\varrho|_{g_H^*}|df|_{g_H^*})(X_1)\right] \\
&\leq nP_1|df|_{g_H^*} + \mathbb{E}\left[|\pi|^q(X_1)\right]^{1/q}\mathbb{E}\left[|\log\varrho|^2_{g_H^*}(X_1)\right]^{1/2}\mathbb{E}\left[|df|^p_{g_H^*}(X_1)\right]^{1/p}.
\end{aligned}
$$

As observed in [9, page 9], we have

$$
\begin{aligned}
\mathbb{E}\left[|d\log\varrho|^2_{g_H^*}(X_1)\right] &= \int_G \varrho|d\log\varrho|^2_{g_H^*}\,d\mu \\
&= \int_G (\Delta_H(\varrho\log\varrho) - (\log\varrho+1)\Delta_H\varrho)\,d\mu \\
&= \int_G (\log\varrho+1)(D+Q)\varrho\,d\mu \\
&= \int_G D(\varrho\log\varrho)d\mu + Q\int_G(\log\varrho+1)\varrho\,d\mu \\
&= \int_G (D+Q)(\varrho\log\varrho)d\mu + Q\int_G \varrho\,d\mu = Q
\end{aligned}
$$

while

$$
\mathbb{E}[|\pi|^q(X_1)] = \mathbb{E}[|W_1|^q] = \frac{2^{(q+n+1)/2}\pi^{(n-1)/2}}{\sqrt{n}}\frac{\Gamma(\frac{n+q}{2})}{\Gamma(\frac{n}{2})}.
$$

The result follows. □

Remark 4.6 Consider a Carnot group (G, H, g_H) and let V be the complement of V defined by left translation of $\mathfrak{g}_2\oplus\cdots\oplus\mathfrak{g}_k$. Since this is an ideal, we obtain the same subbundle using right translation. We extend the g_H to a Riemannian metric g by defining a right invariant metric on V. Then condition (3.5) holds, but if ∇ is defined as in (3.7), then $\mathrm{Ric}(\nabla)$ does not have a lower bound for $k \geq 3$. However, the Yang-Mills condition $\mathrm{tr}_H(\nabla_\times\mathcal{R})(\times,\cdot) = 0$ of Remark 3.16 equals exactly the condition $\psi|\mathfrak{h} = 0$.

Appendix A: Feynman-Kac Formula for Perturbations of Self-Adjoint Operators

A.1 Essentially Self-Adjoint Operator on Forms

Let M be a manifold with a sub-Riemannian structure (H, g_H) with H bracket-generating. Consider the rough sub-Laplacian $L = L(\nabla)$ relative to some affine connection ∇ on TM. Let g be a complete sub-Riemannian metric taming g_H such that $\nabla g = 0$. Assume that

$$
L^* = L = -(\nabla_{\mathrm{pr}_H})^*(\nabla_{\mathrm{pr}_H}).
$$

We then have the following statement for operators of the type $L - \mathscr{C}$ where $\mathscr{C} \in \Gamma(End(T^*M))$. To simplify notation, we denote $\langle\cdot,\cdot\rangle_{L^2(\wedge^j g^*)}$ as simply $\langle\cdot,\cdot\rangle$ for the rest of this section.

Lemma A.1 *Assume that $\mathscr{C}^* = \mathscr{C}$. If $\mathcal{A} = L - \mathscr{C}$ is bounded from above on compactly supported forms, i.e. if*

$$\lambda_0 = \lambda_0(\mathcal{A}) = \sup \left\{ \frac{\langle \mathcal{A}\alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} : \alpha \in \Gamma_c(T^*M) \right\} < \infty,$$

then $\mathcal{A}$ is essentially self-adjoint on compactly supported one-forms.

We follow the argument of [40, Section 2]. We begin by introducing the following lemma.

Lemma A.2 [37, Section X.1] *Let $\mathcal{A}$ be any closed, symmetric, densely defined operator on a Hilbert space with domain $Dom(\mathcal{A})$. Assume that $\mathcal{A}$ is bounded from above by $\lambda_0(\mathcal{A})$ on its domain. Then $\mathcal{A} = \mathcal{A}^*$ if and only if there are no eigenvectors in the domain of $\mathcal{A}^*$ with eigenvalue $\lambda > \lambda_0(\mathcal{A})$.*

Proof of Lemma A.1 Let pr_H be the orthogonal projection to H. Since $L = -(\nabla_{\mathrm{pr}_H})^*(\nabla_{\mathrm{pr}_H})$, we have $-\langle \mathscr{C}\alpha, \alpha \rangle \leq \lambda_0 \langle \alpha, \alpha \rangle$. Denote the closure of $\mathcal{A}|\Gamma_c(T^*M)$ by $\mathcal{A}$ as well. Assume that there exists a one-form α in L^2 satisfying $\mathcal{A}^*\alpha = \lambda\alpha$ with $\lambda > \lambda_0$. We know that α is smooth, since L is hypoelliptic. To see the latter, consider any point $x \in M$, and let U be a neighborhood of x such that we can trivialize T^*M. Recalling the definition of step from Section 2.1, let r denote the step of H at x. Relative to the trivialization, we have that L equals Δ_H along with terms of lower order derivatives in horizontal directions in each component, so by possibly shrinking U, we have that L is maximal hypoelliptic of degree $1/r$ and hence hypoelliptic on this neighborhood, see [28, Chapter 1] for details. As it is a local property, L is hypoelliptic globally. Let f be an arbitrary function of compact support and write $d_H f = \mathrm{pr}_H^* df$. Then

$$\begin{aligned} \lambda \langle f^2\alpha, \alpha \rangle &= \langle f^2\alpha, \mathcal{A}^*\alpha \rangle = \langle \mathcal{A}(f^2\alpha), \alpha \rangle \\ &= -\langle f^2 \nabla_{\mathrm{pr}_H \cdot}\, \alpha, \nabla_{\mathrm{pr}_H \cdot}\, \alpha \rangle - \langle f^2 \mathscr{C}\alpha, \alpha \rangle - 2\langle f d_H f \otimes \alpha, \nabla_{\mathrm{pr}_H \cdot}\, \alpha \rangle \\ &\leq -\| f \nabla_{\mathrm{pr}_H \cdot}\, \alpha \|^2_{L^2(g^*)} + \lambda_0 \langle f^2\alpha, \alpha \rangle - 2\langle d_H f \otimes \alpha, f \nabla_{\mathrm{pr}_H \cdot}\, \alpha \rangle. \end{aligned}$$

Since $(\lambda - \lambda_0)\langle f^2\alpha, \alpha \rangle \geq 0$, we have

$$\left\| f \nabla_{\mathrm{pr}_H \cdot}\, \alpha \right\|^2_{L^2(g^*)} \leq -2\langle d_H f \otimes \alpha, f \nabla_{\mathrm{pr}_H \cdot}\, \alpha \rangle,$$

and hence

$$\left\| f \nabla_{\mathrm{pr}_H \cdot}\, \alpha \right\|^2_{L^2(g^*)} \leq 2\| d_H f \|_{L^\infty(g^*)} \|\alpha\|_{L^2(g^*)} \| f \nabla_{\mathrm{pr}_H \cdot}\, \alpha \|_{L^2(g^*)}. \tag{A.1}$$

Since we assumed that g was complete, there exists a sequence of smooth functions $f_j \uparrow 1$ of compact support satisfying $\|df_j\|_{L^\infty(g^*)} \to 0$. By inserting f_j in (A.1) and taking the limit we obtain $\|\nabla_{\mathrm{pr}_H \cdot}\, \alpha\|^2_{L^2(g^*)} = -\langle L\alpha, \alpha \rangle = 0$. However, this contradicts our initial hypothesis $\mathcal{A}^*\alpha = \lambda\alpha$ for $\lambda > \lambda_0$. Hence, we obtain our result. □

Remark A.3 By replacing the sequence f_j in the proof of Lemma A.1 with (an appropriately smooth approximation of) the sequence found in [41, Theorem 7.3], we can deduce essential self-adjointness of $L - \mathscr{C}$ just by assuming completeness of d_{g_H}.

A.2 Stochastic Representation of a Semigroup

Let (M, H, g_H) be a sub-Riemannian manifold and let g be a complete Riemannian metric taming g_H. Define $L^2(T^*M)$ as the space of all one-forms in L^2 relative to g. Let ∇ be a connection satisfying $\nabla g = 0$ and $L^* = L$. Relative to $L(\nabla)$, consider the stochastic flow $X_t(\cdot)$ with explosion time $\tau(\cdot)$. Define $//_t(x)$ as parallel transport along $X_t(x)$ with respect to ∇.

Let $\mathscr{C}$ be a zero order operator on M, with

$$\mathscr{C}^s = \frac{1}{2}(\mathscr{C} + \mathscr{C}^*), \quad \mathscr{C}^a = \frac{1}{2}(\mathscr{C} - \mathscr{C}^*).$$

Lemma A.4 *Assume that $L - \mathscr{C}^s$ is bounded from above and assume that $\mathscr{C}^a$ is bounded. For each x, let $Q_t(x) \in \operatorname{End} T_x^*M$ be a continuous process adapted to the filtration of $X_t(x)$ such that for any $\alpha \in \Gamma_c(T_x^*M)$, we have*

$$d\left(Q_t(x)//_t^{-1}\alpha(X_t(x))\right) \overset{loc.\,m.}{=} Q_t(x)//_t^{-1}(L - \mathscr{C})\alpha(X_t(x))dt,$$

where $\overset{loc.\,m.}{=}$ denotes equality modulo differentials of local martingales.

Then there exists a strongly continuous semigroup $P_t^{(1)}$ on $L^2(T^*M)$ such that for any $\alpha \in L^2(T^*M)$,

$$P_t^{(1)}\alpha(x) = \mathbb{E}\left[1_{t<\tau(x)} Q_t(x)//_t^{-1}\alpha(X_t)(x)\right],$$

and such that $\lim_{t\downarrow 0} \frac{d}{dt} P_t^{(1)}\alpha = (L - \mathscr{C})\alpha$ for any $\alpha \in \Gamma_c(TM)$.

For the proof, we need to consider a special class of Volterra operators. To this end, we follow the arguments of [21, Section III.1]. Let $\mathfrak{B}$ be a Banach space and let $\mathscr{L}(\mathfrak{B})$ be the space of all bounded operators on $\mathfrak{B}$ with the strong operator topology. Consider any strongly continuous semigroup $\mathbb{R}_{\geq 0} \to \mathscr{L}(\mathfrak{B})$, $t \mapsto S_t$ and let $\mathscr{A} : \mathfrak{B} \to \mathfrak{B}$ be a bounded operator. We define the corresponding Volterra operator $\mathsf{V}(S; \mathscr{A})$ on continuous functions $\mathbb{R}_{\geq 0} \to \mathscr{L}(\mathfrak{B})$, $(t, \alpha) \mapsto F_t\alpha$ by

$$(\mathsf{V}(S; \mathscr{A})F)_t\alpha = \int_0^t S_{t-r}\mathscr{A} F_r\alpha \, dr,$$

and introduce the operator $\mathsf{T}(S; \mathscr{A})$ by

$$\mathsf{T}(S; \mathscr{A})F = \sum_{n=0}^{\infty} \mathsf{V}(S; \mathscr{A})^n F.$$

The operator $\mathsf{T}(S; \mathscr{A})$ is well defined, and if S_t has generator $(L, \operatorname{Dom}(L))$ then $\tilde{S}_t := (\mathsf{T}(S; \mathscr{A})S)_t$ defines a strongly continuous semigroup with generator $(L + \mathscr{A}, \operatorname{Dom}(L))$.

Proof By Lemma A.1 the operator $L - \mathscr{C}^s$ is essentially self-adjoint. Let P_t^s be the corresponding semigroup on $L^2(T^*M)$ with domain $\operatorname{Dom}^s = \operatorname{Dom}(L - \mathscr{C}^s)$.

Let D^n be an exhausting sequence of M of relative compact domains, see e.g. [17, Appendix B.1] for construction. Consider the Friedrichs extension $(\Lambda^n, \operatorname{Dom}(\Lambda^n))$ of $L - \mathscr{C}^s$ restricted to compactly supported forms on D^n and let $\tilde{P}_t^n$ be the corresponding semigroup defined by the spectral theorem. Since the operators Λ^n are bounded from above by assumption, the semigroups $\tilde{P}^n$ are strongly continuous by [21, Chapter II.3 c]. Define P_t^s similarly with respect to the unique self-adjoint extension of $L - \mathscr{C}^s$ restricted to compactly supported forms. Let $(\Lambda, \operatorname{Dom}(\Lambda))$ denote the generator of P_t^s

and note that for any compactly supported forms α, we have that $\tilde{P}_t^n\alpha$ converge to $P_t^s\alpha$ in $L^2(T^*M)$, by e.g. [31, Chapter VIII.3.3]. Define $P_t^n = (\mathsf{T}(\tilde{P}^n; \mathscr{A})\tilde{P}^n)_t$ and finally $P_t^{(1)} = (\mathsf{T}(P^s; \mathscr{C}^a)P^s)_t$. These semigroups are strongly continuous with respective generators $(\Lambda^n + \mathscr{C}^a, \mathrm{Dom}(\Lambda^n))$ and $(\Lambda + \mathscr{C}^a, \mathrm{Dom}(\Lambda))$. Furthermore, $P_t^n\alpha$ converge to $P_t^{(1)}\alpha$ in $L^2(TM)$ by [31, Theorem IV.2.23 (c)].

For $x \in M$, let $\tau_n(x)$ denote the first exist time for $X_t(x)$ of the domain D^n. For any form α with support in D^k, we have that for $S > 0$ and $n \geq k$,

$$N_t^n = Q_t(x)//_t^{-1}(P_{S-t}^n\alpha)|_{X_t(x)}$$

is a bounded local martingale, giving us

$$P_t^n\alpha(x) = \mathbb{E}\left[1_{t<\tau(x)}Q_t(x)//_t^{-1}\alpha(X_t(x))\right].$$

Taking the limit, and using that P_t^n converges to $P_t^{(1)}$, we obtain

$$P_t^{(1)}\alpha(x) = \mathbb{E}\left[1_{t<\tau(x)}Q_t(x)//_t^{-1}\alpha(X_t(x))\right].$$

□

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