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Existence of Solution for an Asymptotically Linear Schrödinger-Kirchhoff Equation

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Abstract We consider the Kirchhoff equation

$$-\Big(1+\lambda\int|\nabla u|^2\Big)\Delta u+V(x)u=f(u) \quad \text{in} \quad \mathbb{R}^N,$$

where $N\in\{3,4\}$, $\lambda\geq 0$, the potential V is radial and f can be superlinear or aysmptotically linear at infinity. By using variational methods we obtain, for $N=4$, the existence of a ground state radial solution when λ is small. The same holds for $N=3$ with no restriction on λ. We also prove that, when $\lambda\to 0^+$, the solutions strongly converge to a solution of $-\Delta u+V(x)u=f(u)$.

Keywords Kirchhoff equation · Nonlocal problems · Radial problems · Ground state solution

Mathematics Subject Classification (2010) Primary 35J20 · Secondary 35J25, 35J60

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1 Introduction

Consider the equation

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u+V(x)u=f(u),\quad x\in\mathbb{R}^N,$$

with $a,\ b\in\mathbb{R}$, V and f satisfying some suitable conditions. The presence of the term $\int_{\mathbb{R}^N}|\nabla u|^2dx$ shows that this equation is not a pointwise identity and therefore the problem is called nonlocal. Although these feature provides many mathematical difficulties, the main interest in this problem is due to the fact that it arises in the following physical context: if we set $V\equiv 0$ and replace the entire space by $\Omega\subset\mathbb{R}^N$, then we get the problem

$$\begin{cases}-\Big(a+b\int_{\Omega}|\nabla u|^2dx\Big)\Delta u=f(x,u), & x\in\Omega\\ u=0, & x\in\partial\Omega,\end{cases}\tag{1.1}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2u}{\partial x^2}=0.$$

presented by Kirchhoff in [10]. The above equation is an extension of the classical d'Alembert wave equation by considering the effects of the changes in the length of the string during vibrations. Actually, in the physical model, the parameters have the following meaning: L is the length of the string, h is the area of cross-section, E is the young modulus of the material, ρ is the mass density and P_0 is the initial tension. After J.L.Lions [12] presented an abstract functional analysis framework to the evolution equation related with Eq 1.1, this kind of problem has been extensively studied (see [1–4, 11] and references there in).

In this paper we assume, with no loss of generality, that $a=1$ and we consider $b=\lambda$ as a parameter. We deal with the equation

$$-\Big(1+\lambda\int|\nabla u|^2\Big)\Delta u+V(x)u=f(u)\quad\text{in}\quad\mathbb{R}^N,\tag{K_λ}$$

where $N\in\{3,4\}$, $\lambda\geq 0$ and the potential V satisfies the following assumptions

(V_0) $V\in C^2(\mathbb{R}^N)$ and the map $x\mapsto(V(x),\nabla V(x)\cdot x)$ is radially symmetric;
(V_1) $V_\infty:=\lim_{|x|\to+\infty}V(x)>0$;
(V_2) $\nabla V(x)\cdot x\leq 0$, for any $x\in\mathbb{R}^N$;
(V_3) if we define $H(x):=\Big(V(x)+\frac{\nabla V(x)\cdot x}{N}\Big)$ then, for any $x\in\mathbb{R}^{\mathrm{N}}$, there hold

$$H(x)\geq V_\infty,\qquad \nabla H(x)\cdot x\leq 0.$$

For the subcritical nonlinearity f, we shall suppose that

(f_0) $f\in C(\mathbb{R},\mathbb{R})$;
(f_1) there exist $a_1,a_2>0$ and $1<p<(N+2)/(N-2)$ such that, for any $s\in\mathbb{R}$,

$$|f(s)|\leq a_1|s|+a_2|s|^p;$$

(f_2) $\lim_{s\to 0}f(s)/s=0$;
(f_3) there exists $\zeta>0$ such that

$$\int_0^\zeta\Big(f(s)-V_\infty s\Big)ds>0.$$

In the main result of this paper we prove the following.

Theorem 1.1 *Suppose that* $N = 4$*, the potential* V *satisfy* $(V_0) - (V_3)$ *and* f *satisfy* $(f_0) - (f_3)$*. Then there exists* $\lambda^* > 0$ *such that, for any* $\lambda \in (0, \lambda^*)$*, the problem* (K_λ) *has a ground state solution* $u_\lambda \in H^1_{rad}(\mathbb{R}^4)$*. Moreover, as* $\lambda \to 0^+$*, we have that* $u_\lambda \to u_0$ *strongly in* $H^1_{rad}(\mathbb{R}^4)$ *and* u_0 *is a weak solution of*

$$-\Delta u_0 + V(x)u_0 = f(u_0) \quad \text{in } \mathbb{R}^4.$$

If $N = 3$*, the same result holds with* $\lambda^* = +\infty$*.*

Since our potential V is radial, the notion of ground state solution stated above is related with the space of radial funcions $H^1_{rad}(\mathbb{R}^N)$. The conditions $(V_2) - (V_3)$ have already appeared in [15] for a Schödinger equation with asymptotically linear nonlinearity f. They also appeared, in the Kirchhoff context, in the paper [19], where the authors considered only the superlinear case. As a model case for the potential, we can take $V(x) = V_\infty + (1 + |x|^a)^{-1}$, with $0 < a \leq N$.

Concernig the nonlinearity f, we notice that condition (f_3) was introduced in the celebrated paper of Berestycki and Lions [5]. It permits to deal with superlinear or aysmptotically linear nonlinearities f. Indeed, a straightforward computation shows that (f_3) is a consequence of each one of the conditions below

(f_4) $\lim_{s\to+\infty} f(s)/s = +\infty$;

(f_5) $\lim_{s\to+\infty} f(s)/s = l > V_\infty$, for some $l \in \mathbb{R}$.

In a recent paper [2], the author considered the condition (f_3) and obtained existence of solution for an autonomous version of the problem (K_λ). Except for [2], we do not know any paper which deals with superlinear and asymptotically linear functions in a unified way.

In what follow we quote some results for the superlinear case. They are in some sense related with ours. We start with [19], where the authors considered a possible non-randial potential $V \in C^2(\mathbb{R}^N)$ verifying $(V_1) - (V_3)$. They also assumed that V is bounded from above by V_∞ plus a quantity related with the ground-state solution of the limit problem associated with (K_λ). Under $(f_0) - (f_2)$, a superlinear condition slightly weaker than (f_4) and

(M) $s \mapsto f(s)/s$ is non decreasing,

they obtained the existence of a positive solution (with high level energy) for $N = 3$. With the same monotonicity condition, in [11] (see also [9]), the authors considered the homogeneous case $f(u) = |u|^{p-1}u$ but with different conditions on V. In particular, they assumed

(R) $V(x) \leq \liminf_{|y|\to+\infty} V(y) = V_\infty$, for any $x \in \mathbb{R}^N$.

This same condition was used in [8], in the nonhomogeneous case, but also considering (M). Is is worth to mention that the above hypothesis has first appeared in the paper of Rabinowitz [18] and it is a sufficient condition to recover compactness for problems in unbounded domains. Nnotice that here the conditions $(V_2) - (V_3)$ imply that $V(x) \geq V_\infty$, and therefore we need a different approach. We would also like cite the paper [16], where the authors do not impose monotonicity conditions but considered the autonomous case.

The literature for the asymptotically linear case is not vast. We first quote the paper [13], where the authors considered a nonautonomous nonlinearity f satisfying a sort of condition (f_5), but with $0 < l < V_\infty$. Under some technical assumptions they obtained a positive solution. We also cite the papers [6, 17], where some multiplicity results were proved in a different (and not comparable) setting of hypotheses on V.

In the proof of our main theorem we apply variational methods. Actually, the weak solutions of problem (K_λ) are the critical points of the energy functional $I_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla u|^2 + V(x)u^2\Big)\mathrm{d}x - \frac{\lambda}{2}\Big(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)^2 - \int_{\mathbb{R}^N} F(u)\mathrm{d}x,$$

where $F(s) := \int_0^s f(t)\mathrm{d}t$. In some of the aforementioned works the authors take advantage of the condition (M) for minimizing I_λ constrained to its Nehari manifold $\{u \in H^1(\mathbb{R}^N) \setminus \{0\} : I'_\lambda(u)u = 0\}$. Since in our case the ratio $f(s)/s$ is not supposed to be monotonic we need a different approach. Thus, we follow [2] and notice that the solutions of (K_λ) verify

$$\frac{N-2}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)}\Big(1 + \lambda\|\nabla u\|_2^2\Big) = N\int \Big(F(u) - H(x)\frac{u^2}{2}\Big)\mathrm{d}x.$$

Hence, we can define the Pohozaev manifold $\mathcal{P}_\lambda$ as being the collection of the nonzero functions satisfying above equality. After a carefull analysis of the fibration maps $\theta \mapsto I_\lambda(u(\cdot/\theta))$, we prove that this set is a natural constraint for I_λ and that it is possible to minimize I_λ constrained to $\mathcal{P}_\lambda$.

The paper contains two more sections. In the next one, we present some auxiliar results and present a detailed study of the fibration maps. In the final section we prove our main theorem.

2 Some Preliminary Results

In this section we state and prove some technical results. For any $2 \le q \le \infty$, we denote by $\|u\|_q$ the L^q-norm of a function $u \in L^q(\mathbb{R}^N)$. To simplify notation, we write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. We denote by X the Sobolev space $H^1(\mathbb{R}^N)$ endowed with the norm

$$\|u\|^2 := \int \Big(|\nabla u|^2 + V(x)u^2\Big), \qquad u \in X.$$

It is easy to use $(V_1) - (V_3)$ to prove that this norm is well defined.

The proof of the following Pohozaev identity can be found in [7] (see also [14, Proposition 2.1]).

Lemma 2.1 *Let $g \in C(\mathbb{R}^N \times \mathbb{R})$, $G(x,t) := \int_0^t g(x,s)ds$ and $u \in H^1(\mathbb{R}^N) \cap H^2_{loc}(\mathbb{R}^N)$ be a weak solution of the problem*

$$-\Delta u = g(x,u) \quad \text{in } \mathbb{R}^N.$$

If $G(\cdot, u(\cdot))$ and $x_i \frac{\partial G}{\partial x_i}(\cdot, u(\cdot))$ are in $L^1(\mathbb{R}^N)$, then

$$\frac{(N-2)}{2}\int |\nabla u|^2 = N\int G(x,u) + \sum_{i=1}^N \int x_i \frac{\partial G}{\partial x_i}(x,u).$$

By using this result we conclude that, if $u \in H^1(\mathbb{R}^N) \cap H^2_{loc}(\mathbb{R}^N)$ weakly solves

$$-\Big(1 + \lambda\|\nabla u\|_2^2\Big)\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

then

$$\frac{N-2}{2}\|\nabla u\|_2^2\Big(1 + \lambda\|\nabla u\|_2^2\Big) = N\int \Big(F(u) - H(x)\frac{u^2}{2}\Big).$$

Hence, all the nonzero solutions of the problem (K_λ) belong to the set

$$\mathcal{P}_\lambda := \Big\{u \in X\backslash\{0\} \,:\, J_\lambda(u) = 0\Big\},$$

where $J_\lambda \in C^1(X, \mathbb{R})$ is given by

$$J_\lambda(u) := \frac{N-2}{2}\|\nabla u\|_2^2\Big(1 + \lambda\|\nabla u\|_2^2\Big) - N\int\Big(F(u) - H(x)\frac{u^2}{2}\Big). \tag{2.1}$$

We first present some conditions on the parameter λ which guarantee that set $\mathcal{P}_\lambda$ is non empty.

Lemma 2.2 *If $N = 3$, then $P_\lambda \neq \emptyset$ for any $\lambda > 0$. If $N = 4$, then there exists $\lambda^* > 0$ such that the same conclusion holds for $\lambda \in [0, \lambda^*)$.*

Proof If $N = 3$, it follows from [5] that the problem

$$-\Delta u + V_\infty u = f(u) \quad \text{in } \mathbb{R}^3$$

has a solution $\omega_1 \in X$ such that

$$3\int\Big(F(\omega_1) - V_\infty\frac{\omega_1^2}{2}\Big) = \frac{1}{2}\|\nabla\omega_1\|_2^2 > 0. \tag{2.2}$$

Let $g_\lambda : \mathbb{R}^+ \to \mathbb{R}$ be given by

$$g_\lambda(\theta) := I(\omega_1(\cdot/\theta)) = \frac{\theta}{2}\|\nabla\omega_1\|_2^2 + \frac{\lambda\theta^2}{4}\|\nabla\omega_1\|_2^4 - \theta^3\int\Big(F(\omega_1) - V(x\theta)\frac{\omega_1^2}{2}\Big).$$

By (V_1), Lebesgue Dominate Convergence Theorem and EqE 2.2 we get

$$\lim_{\theta\to+\infty}\int\Big(F(\omega_1) - V(x\theta)\frac{\omega_1^2}{2}\Big) = \int\Big(F(\omega_1) - V_\infty\frac{\omega_1^2}{2}\Big) > 0,$$

and therefore $\lim_{\theta\to+\infty} g_\lambda(\theta) = -\infty$. By using (V_1) again, we obtain

$$g_\lambda(\theta) \geq \frac{\theta}{2}\|\nabla\omega_1\|_2^2 + \frac{\lambda\theta^2}{4}\|\nabla\omega_1\|_2^4 - \theta^3\int\Big(F(\omega_1) - c_V\frac{\omega_1^2}{2}\Big),$$

and therefore $g_\lambda(\theta) > 0$, for $\theta > 0$ small. It follows that g_λ attains its maximum at some $\theta_0 > 0$. Since $g_\lambda'(\theta_0)\theta_0 = 0$, we can use (2.1) and a straightforward calculation to conclude that $\omega_1(\cdot/\theta_0) \in \mathcal{P}_\lambda$.

If $N = 4$ we consider $v_0 \in X \setminus \{0\}$ such that $-\Delta v_0 + V_\infty v_0 = f(v_0)$ in $\mathbb{R}^4$. As proved in [4], for any $\lambda \in [0, \|\nabla v_0\|_2^{-2})$, the problem

$$-\Big(1 + \lambda\|\nabla u\|_2^2\Big)\Delta u + V_\infty u = f(u) \quad \text{in } \mathbb{R}^4,$$

has a solution $\omega_2 \in X$ such that

$$4\int\Big(F(\omega_2) - V_\infty\frac{\omega_2^2}{2}\Big)dx - \lambda\|\nabla\omega_2\|_2^4 = \|\nabla\omega_2\|_2^2 > 0.$$

The result follows as in the case $N = 3$ by considering the function $\theta \mapsto I(\omega_2(\cdot/\theta)$. We omit the details. □

From now on we assume that $N = 3$ or $N = 4$ and the number λ belongs to the interval $(0, \lambda^*)$ of the above lemma. With this assumption, the set $\mathcal{P}_\lambda$ is non empty.

Lemma 2.3 *There exists $c_0 > 0$, independent of $\lambda \geq 0$, such that*

$$\|\nabla u\|_2^2 \geq c_0, \qquad \forall\, u \in \mathcal{P}_\lambda.$$

Proof Let $2^* := 2N/(N-2)$ and $\alpha \in (0,1)$ be such that

$$\frac{1}{p+1} = \frac{\alpha}{2} + \frac{(1-\alpha)}{2^*}.$$

For any $u \in \mathcal{P}_\lambda$, the interpolation inequality provides

$$\|u\|_{p+1}^{p+1} \leq \|u\|_2^{(p+1)\alpha} \|u\|_{2^*}^{(p+1)(1-\alpha)}.$$

Recall that, if $a, b \geq 0$ and $s > 1$ then, for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that $ab \leq \varepsilon a^s + c_\varepsilon b^{s'}$, where $(1/s) + (1/s') = 1$. Since

$$\frac{(p+1)\alpha}{2} + \frac{\big(2-(p+1)\alpha\big)}{2} = 1,$$

we can use the last inequality with $a = \|u\|_2^{(p+1)\alpha}$, $b = \|u\|_{2^*}^{(p+1)(1-\alpha)}$ and $s = 2/(p+1)\alpha$, to obtain

$$\|u\|_{p+1}^{p+1} \leq \varepsilon \|u\|_2^2 + c_\varepsilon \|u\|_{2^*}^{k(p,\alpha)}, \tag{2.3}$$

where

$$k(p,\alpha) := \frac{2(p+1)(1-\alpha)}{2-(p+1)\alpha}. \tag{2.4}$$

It follows from EqE 2.1 and (V_3) that

$$\frac{N-2}{2}\|\nabla u\|_2^2 \leq N \int \left(F(u) - H(x)\frac{u^2}{2}\right) \leq N \int \left(F(u) - V_\infty \frac{u^2}{2}\right). \tag{2.5}$$

Given $\delta > 0$, the hypotheses $(f_0) - (f_2)$ provide $C_\delta > 0$ such that

$$|F(s)| \leq \frac{\delta}{2}s^2 + \frac{C_\delta}{p+1}|s|^{p+1}, \quad \forall\, s \in \mathbb{R}.$$

Thus, by EqE 2.3 and the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, we obtain $c > 0$ satisfying

$$\frac{N-2}{2}\|\nabla u\|_2^2 \leq c\|\nabla u\|_2^{k(p,\alpha)} + N \int \left(\frac{(\delta - V_\infty)}{2} + \frac{C_\delta \varepsilon}{p+1}\right) u^2.$$

By choosing $0 < \delta < V_\infty$ and $\varepsilon > 0$ small, we can discard the last term on the right-hand side above and obtain

$$\frac{N-2}{2}\|\nabla u\|_2^2 \leq c_1 \|\nabla u\|_2^{k(p,\alpha)}.$$

Since $k(p,\alpha) > 2$, the lemma is proved. □

We now recall that the weak solutions of (K_λ) are the critical points of the energy functional $I_\lambda \in C^1(X, \mathbb{R})$ given by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\|\nabla u\|^4 - \int F(u).$$

Given $u \in \mathcal{P}_\lambda$, we can use (2.1) to obtain

$$I_\lambda(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{4}\|\nabla u\|_2^4 - \int \left(\frac{\nabla V(x)\cdot x}{N}\right)\frac{u^2}{2} \\ - \frac{(N-2)}{2N}\|\nabla u\|_2^2 - \frac{\lambda(N-2)}{2N}\|\nabla u\|_2^4.$$

This, (V_2) and straightforward calculations provide

$$I_\lambda(u) \geq \frac{1}{N}\|\nabla u\|_2^2 + \frac{\lambda(4-N)}{4N}\|\nabla u\|_2^4 \geq \frac{1}{N}c_0^2 > 0. \tag{2.6}$$

Hence, as consequence of the previous lemma, we have that

$$p_\lambda := \inf_{u\in\mathcal{P}_\lambda} I_\lambda(u) > 0.$$

We shall obtain a solution for the problem (K_λ) by showing that the above infimum is attained. We first prove that the set $\mathcal{P}_\lambda$ is a regular manifold, in such way that we can use the Lagrange Multiplier Theorem.

Lemma 2.4 *The set $\mathcal{P}_\lambda$ is a C^1-manifold.*

Proof Given $u \in \mathcal{P}_\lambda$, we claim that $J_\lambda'(u)\varphi \neq 0$, for some $\varphi \in X$. Indeed, if this is not true, then

$$(N-2)\int(\nabla u\cdot\nabla)\varphi + (N-2)2\lambda\|\nabla u\|_2^2\int(\nabla u\cdot\nabla)\varphi = N\int\Big(f(u)-H(x)u\Big)\varphi,$$

for all $\varphi \in X$. Hence, we can use (V_0) and the Principle of Symmetric Criticality to conclude that $u \in H^1(\mathbb{R}^N)$ weakly solves

$$-(N-2)\Big(1+2\lambda\|\nabla u\|_2^2\Big)\Delta u = N\Big(f(u)-H(x)u\Big) \quad \text{in } \mathbb{R}^N.$$

It follows from Lemma 2.1 that

$$\frac{(N-2)^2}{2}\|\nabla u\|_2^2\Big(1+2\lambda\|\nabla u\|_2^2\Big) = N^2\int\Big(F(u)-H(x)\frac{u^2}{2}\Big) - N\int\Big(\nabla H(x)\cdot x\Big)\frac{u^2}{2}.$$

Since $u \in \mathcal{P}_\lambda$, we can use EqE 2.1 and (V_3) to get

$$\frac{(N-2)^2}{2}\|\nabla u\|_2^2\Big(1+2\lambda\|\nabla u\|_2^2\Big) \geq N\Big\{\frac{N-2}{2}\|\nabla u\|_2^2\Big(1+\lambda\|\nabla u\|_2^2\Big)\Big\},$$

that is

$$-2\|\nabla u\|_2^2 + \lambda(N-4)\|\nabla u\|_2^4 \geq 0.$$

This and $N \in \{3,4\}$ provide $\|\nabla u\|_2^2 = 0$, which contradicts Lemma 2.3. Therefore, for any $u \in \mathcal{P}_\lambda$, we have that $J_\lambda'(u) \neq 0$ and the conclusion follows from the Implicit Function Theorem. □

Lemma 2.5 *If $u \in \mathcal{P}_\lambda$, then*

$$I_\lambda(u) = \max_{\theta>0} I_\lambda(u(\cdot/\theta)).$$

Proof If $N = 3$ and $u \in \mathcal{P}_\lambda$, it follows from EqE 2.5 and Lemma 2.3 that

$$\int\left(F(u) - V_\infty\frac{u^2}{2}\right) > 0.$$

Hence, we can argue as in the proof of Lemma 2.2 to conclude that the function $g_\lambda(\theta) := I_\lambda(u(\cdot/\theta))$, defined for $\theta > 0$, attains its maximum value at $\theta_0 > 0$ such that

$$\frac{1}{2}\theta_0\|\nabla u\|_2^2\Big(1+\lambda\theta_0\|\nabla u\|_2^2\Big) = 3\theta_0^3\int\Big(F(u)-H(x\theta_0)\frac{u^2}{2}\Big). \tag{2.7}$$

It follows from the second inequality of (V_3) that $\theta \frac{d}{d\theta} H(x\theta) \le 0$, and therefore the map $\theta \mapsto H(x\theta)$ is nonincreasing in $(0, +\infty)$. Thus, if $\theta_0 > 1$, since

$$\frac{1}{2}\theta_0 \|\nabla u\|_2^2 \Big(\theta_0 + \lambda\theta_0 \|\nabla u\|_2^2\Big) > \frac{1}{2}\theta_0 \|\nabla u\|_2^2 \Big(1 + \lambda\theta_0 \|\nabla u\|_2^2\Big),$$

we can use Eq 2.7 and $u \in \mathcal{P}_\lambda$ to get

$$\frac{1}{2}\|\nabla u\|_2^2\Big(1 + \lambda\|\nabla u\|_2^2\Big) > 3\theta_0 \int \Big(F(u) - H(x\theta_0)\frac{u^2}{2}\Big) > 3 \int \Big(F(u) - H(x)\frac{u^2}{2}\Big),$$

which contradicts $u \in \mathcal{P}_\lambda$. Analogously, we cannot have $\theta_0 < 1$. Thus, $\theta_0 = 1$ and the lemma is proved.

If $N = 4$ and $u \in \mathcal{P}_\lambda$ we can use (V_3) to obtain

$$\|\nabla u\|_2^2 = 4 \int \Big(F(u) - H(x)\frac{u^2}{2}\Big) - \lambda\|\nabla u\|_2^4 \le 4 \int \Big(F(u) - V_\infty \frac{u^2}{2}\Big) - \lambda\|\nabla u\|_2^4.$$

As in the case $N = 3$, the function $\theta \mapsto I_\lambda(u(x/\theta))$ attains its maximum value at $\theta_0 = 1$ and the conclusion follows. □

Lemma 2.6 *If $u \in \mathcal{P}_\lambda$ satisfies*

$$I(u) = \inf_{v \in \mathcal{P}_\lambda} I_\lambda(v),$$

then $I_\lambda'(u)\varphi = 0$, for any $\varphi \in H^1(\mathbb{R}^N)$.

Proof If I constrained to $\mathcal{P}_\lambda$ attains its maximum value at $u \in \mathcal{P}_\lambda$ then, by Lemma 2.4, there is a multiplier $\mu \in \mathbb{R}$ such that $I_\lambda'(u) + \mu J_\lambda'(u) = 0$. Using this equality and arguing as in the proof of Lemma 2.4, we conclude that $u \in H^1(\mathbb{R}^N)$ weakly solves

$$-c\Delta u = (1 + \mu N)(f(u) - V(x)u) - \mu(\nabla V(x) \cdot x)u, \quad \text{in } \mathbb{R}^N,$$

with

$$c := \Big\{1 + \mu(N-2) + \Big(\lambda + \mu\lambda 2(N-2)\Big)\|\nabla u\|_2^2\Big\}.$$

Lemma 2.1 provides

$$\frac{(N-2)}{2} c\|\nabla u\|_2^2 = N(1 + \mu N) \int \Big(F(u) - H(x)\frac{u^2}{2}\Big) - N\mu \int \Big(\nabla H(x) \cdot x\Big)\frac{u^2}{2}.$$

If $\mu > 0$, the above equality, Eq 2.1, the second inequality of (V_3) and the definition of c imply that

$$\mu \frac{(N-2)}{2}\|\nabla u\|_2^2\Big(2 + \lambda(4 - N)\|\nabla u\|_2^2\Big) \le 0.$$

Recalling that $N \in \{3, 4\}$, we conclude that $\|\nabla u\|_2^2 = 0$, which contradicts Lemma 2.3. Since an analogous argument discards the inequality $\mu < 0$, we conclude that $\mu = 0$ and this implies that $I_\lambda'(u) = 0$. □

In our last lemma we take advantage of the radiality of the functions in X to obtain the following convergence result.

Lemma 2.7 *If f satisfis $(f_0) - (f_2)$ and $(u_n) \subset X$ is such that $u_n \rightharpoonup u$ weakly in X, then*

$$\lim_{n \to +\infty} \int F(u_n) = \int F(u).$$

Proof Given $\varepsilon > 0$, we can use $(f_0) - (f_2)$ to obtain $c_\varepsilon > 0$ such that

$$\left|F(u_n) - F(u)\right| \leq \varepsilon\left(|u_n|^2 + |u|^2\right) + c_\varepsilon\left(|u_n|^p + |u|^p\right). \tag{2.8}$$

Since the embedding $X \hookrightarrow L^p(\mathbb{R}^N)$ is compact we may suppose that, for a.e. $x \in \mathbb{R}^N$, we have that $u_n(x) \to u(x)$ and $|u_n(x)| \leq \psi(x)$, for some $\psi \in L^p(\mathbb{R}^N)$. Hence, if we set

$$g_n := \max\left\{\left|F(u_n) - F(u)\right| - \varepsilon\left(|u_n|^2 + |u|^2\right), 0\right\},$$

it follows from the Lebesgue Dominate Convergence Theorem that $\int g_n \to 0$. The definition of g_n and Eq 2.8 provide $n_0 \in \mathbb{N}$ such that

$$\int \left|F(u_n) - F(u)\right| < \varepsilon \int \left(|u_n|^2 + |u|^2\right) + \varepsilon,$$

for $n \geq n_0$. The result follows from the boundedness of (u_n) in $L^2(\mathbb{R}^N)$. □

3 Proof of Theorem 1.1

In this section we prove our main theorem. Let $(u_n) \subset \mathcal{P}_\lambda$ be such that

$$I_\lambda(u_n) \to p_\lambda.$$

We claim that (u_n) is bounded. Indeed, since $(I_\lambda(u_n))$ is bounded, it follows from Eq 2.6 that the sequence $(\|\nabla u_n\|_2)$ is bounded, the same holding for $(\|u_n\|_{2^*})$ due to the Sobolev embedding. Given $\varepsilon, \delta > 0$, we can use $(f_0) - (f_2)$, (V_1), Eq 2.3 and the argument of the proof of Lemma 2.3 to otbain

$$\int \frac{1}{2}\Big(V_\infty - \delta - \frac{2\varepsilon C_\delta}{p+1}\Big)u_n^2 \leq I_\lambda(u_n) - \frac{1}{2}\|\nabla u_n\|_2^2 + c\|u_n\|_{2^*}^{k(p,\alpha)},$$

with $k(p, \alpha) > 2$ given in Eq 2.4. By choosing ε, δ small we infer from the above inequality that $(\|u_n\|_2)$ is bounded. Hence, (u_n) is bounded in X.

Up to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in X. Hence,

$$\|u\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2, \quad \|\nabla u\|^2 \leq \liminf_{n\to\infty} \|\nabla u_n\|^2 \quad \text{and} \quad \|u\|_2^2 \leq \liminf_{n\to\infty} \|u_n\|_2^2. \tag{3.1}$$

By using Eq 2.5 and Lemma 2.3, we obtain $c_1 > 0$ such that

$$0 < c_1 < \frac{N-2}{2N}\|\nabla u_n\|_2^2 \leq \int \left(F(u_n) - V_\infty \frac{u_n^2}{2}\right).$$

By taking the limit, using (3.1) and Lemma 2.7 we conclude that $\int (F(u) - V_\infty u^2/2) > 0$. Hence, as in the proof of Lemma 2.2, we obtain $\theta > 0$ such that $u(\cdot/\theta) \in \mathcal{P}_\lambda$. Thus, using (3.1), Fatou's lemma and Lemma 2.7, we get

$$I_\lambda(u(\cdot/\theta)) \leq \liminf_{n\to\infty} I_\lambda(u_n(\cdot/\theta)).$$

Since $(u_n) \subset \mathcal{P}_\lambda$, Lemma 2.5 gives $I_\lambda(u_n(\cdot/\theta)) \leq I_\lambda(u_n(\cdot))$. Hence,

$$p_\lambda \leq I_\lambda(u(\cdot/\theta)) \leq \liminf_{n\to\infty} I_\lambda(u_n(\cdot/\theta)) \leq \liminf_{n\to\infty} I_\lambda(u_n(\cdot)) = p_\lambda,$$

and we conclude that the function $u_\lambda := u(\cdot/\theta) \in \mathcal{P}_\lambda$ satisfies

$$I(u_\lambda) = \inf_{v \in \mathcal{P}_\lambda} I_\lambda(v).$$

It follows from Lemma 2.6 that u_λ is a solution of problem (K_λ).

In what follows we prove the concentration result. For each $\lambda \in (0, \lambda^*)$ (suppose $\lambda^* = +\infty$ if $N = 3$), let $u_\lambda \in \mathcal{P}_\lambda$ be a solution such that $p_\lambda = I_\lambda(u_\lambda)$. By using Eq 2.1, (V_2) and $N \in \{3, 4\}$ we obtain

$$Np_\lambda = \|\nabla u_\lambda\|_2^2 + \frac{\lambda(4-N)}{4}\|\nabla u_\lambda\|_2^4 - \int (\nabla V(x)\cdot x)\frac{u_\lambda^2}{2} \geq \|\nabla u_\lambda\|_2^2. \tag{3.2}$$

For any fixed $u \in P_\lambda$, it follows from Lemma 2.5 that

$$\begin{aligned} p_\lambda = I_\lambda(u_\lambda) = \min_{v\in\mathcal{P}_\lambda} I_\lambda(v) &= \min_{v\in\mathcal{P}_\lambda}\max_{\theta>0} I_\lambda(v(\cdot/\theta)) \\ &\leq \max_{\theta>0} I_\lambda(u(\cdot/\theta)) \\ &\leq \max_{\theta>0} I_{\lambda^*}(u(\cdot/\theta)). \end{aligned}$$

Since $u \in \mathcal{P}_\lambda$, we infer from Eq 2.5 that $\int (F(u) - V_\infty u^2/2) > 0$. Thus, arguing as in the proof of Lemma 2.2, we obtain

$$\lim_{\theta\to\infty} I_{\lambda^*}(u(\cdot/\theta)) = -\infty.$$

It follows that $(p_\lambda)_{\lambda\in(0,\lambda^*)}$ is bounded. By Eq 3.2, the sequence $(\|\nabla u_\lambda\|_2)_{\lambda\in(0,\lambda^*)}$ is also bounded.

Given $0 < \delta < 1$, we can use $(f_0) - (f_2)$ to obtain $c_\delta > 0$ satisfying

$$\|u_\lambda\|^2 \leq \|u_\lambda\|^2 + \lambda\|\nabla u_\lambda\|_2^4 = \int f(u_\lambda)u_\lambda \leq \delta\int u_\lambda^2 + C_\delta\int |u_\lambda|^{p+1}.$$

By using Eq 2.3 and the Sobolev embedding we obtain

$$\|u_\lambda\|^2 \leq (\delta + \varepsilon C_\delta)\|u_\lambda\|^2 + c\|\nabla u_\lambda\|_2^{k(p,\alpha)}.$$

By picking ε small, we can use the above expression and the boundedness of $(\|\nabla u_\lambda\|_2)$ to conclude that $(u_\lambda)_{\lambda\in(0,\lambda^*)}$ is bounded in X. Hence, up to a subsequence, $u_\lambda \rightharpoonup u_0$ weakly in X.

Since $I'(u_\lambda)(u_\lambda - u_0) = 0$, we have that

$$\int f(u_\lambda)(u_\lambda - u_0) = \langle u_\lambda, u_\lambda - u_0\rangle_{H^1} + \lambda\langle u_\lambda, u_\lambda - u_0\rangle_{\mathcal{D}^{1,2}(\mathbb{R}^N)}\|\nabla u_\lambda\|_2^2.$$

Arguing as in the proof of Lemma 2.7, we can prove that $\int f(u_\lambda)(u_\lambda - u_0) \to 0$, as $\lambda \to 0^+$. Hence, taking the limit in the above expression and recalling the weak convergence of (u_n), we get

$$\lim_{\lambda\to 0^+}\langle u_\lambda, u_\lambda - u_0\rangle_{H^1} = 0.$$

This and the weak convergence imply that $u_\lambda \to u_0$ strongly in X.

The strong convergence and Lemma 2.3 imply that $\|\nabla u_0\|_2^2 \geq c > 0$, and therefore $u_0 \neq 0$. Finally, if $\varphi \in X$, then

$$\left(1 + \lambda\|\nabla u_\lambda\|_2^2\right)\int (\nabla u_\lambda \cdot \nabla\varphi) + \int V(x)u_\lambda\varphi = \int f(u_\lambda)\varphi.$$

By taking the limit as $\lambda \to 0^+$ and arguing as above we conclude that

$$\int (\nabla u_0 \cdot \nabla\varphi) + \int V(x)u_0\varphi = \int f(u_0)\varphi,$$

that is, u_0 is a weak solution of

$$-\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N.$$

The theorem is proved. □

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