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Weighted Composition Operators Between Different Fock Spaces

Pham Trong Tien[1] · **Le Hai Khoi**[2]

Abstract We study weighted composition operators acting between Fock spaces. The following results are obtained:

  (i) Criteria for the boundedness and compactness.
 (ii) Characterizations of compact differences and essential norm.
(iii) Complete descriptions of path connected components and isolated points of the space of composition operators and the space of nonzero weighted composition operators.

Keywords Fock space · Weighted composition operator · Essential norm · Compact difference · Topological structure

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1 Introduction

Let X be a space of holomorphic functions on a domain G in $\mathbb{C}$. For a holomorphic self-map φ of G and a holomorphic function ψ on G, the *weighted composition operator* $W_{\psi,\varphi}$ is defined by $W_{\psi,\varphi}(f) := \psi(f \circ \varphi)$ for $f \in X$. When the function ψ is identically 1, the operator $W_{\psi,\varphi}$ reduces to the *composition operator* C_φ. A main problem in the investigation of such operators is to relate function theoretic properties of ψ and φ to operator theoretic properties of C_φ and $W_{\psi,\varphi}$.

The study of composition operators on various Banach spaces of holomorphic functions on the unit disk or the unit ball, such as Hardy and Bergman spaces, the space H^∞ of all bounded holomorphic functions, the disk algebra and weighted Banach spaces with sup-norm, etc. received a special attention of many authors during the past several decades (see [9, 24] and references therein for more information). Weighted composition operators on these spaces appeared in some works (see, for instance, [6–8, 12]) with different applications. There is a great number of topics on operators of such a type: boundedness and compactness [5, 10], compact differences [22], topological structure [3, 16, 18, 19], dynamical and ergodic properties [1, 2, 27]. On many spaces, these topics are difficult and not yet solved completely.

Recently, much progress was made in the study of composition operators and weighted composition operators on Fock spaces. One of the main differences between operators C_φ and $W_{\psi,\varphi}$ on Fock spaces and those on the above-mentioned spaces of holomorphic functions on the unit disk or the unit ball is the lack of bounded holomorphic functions in the Fock space setting. In fact, entire functions φ that induce bounded composition operators C_φ and weighted composition operators $W_{\psi,\varphi}$ are quite restrictive, in details, they are only affine functions. We refer the reader to [4, 11] for composition operators on the Hilbert Fock space $\mathcal{F}^2(\mathbb{C}^n)$, to [15, 20, 26] for weighted composition operators on the Hilbert Fock space $\mathcal{F}^2(\mathbb{C})$. It should be noted that in these papers the techniques of adjoint operators in Hilbert spaces played an essential role.

The question to ask is: how about weighted composition operators acting between general Fock spaces $\mathcal{F}^p(\mathbb{C})$ and $\mathcal{F}^q(\mathbb{C})$ ($0 < p, q < \infty$). In this paper, we study several important questions for the operator $W_{\psi,\varphi}$: boundedness, compactness, essential norm, compact differences and topological structure. Roughly speaking, our main result is to give complete answers to all these questions by developing an essentially different approach without adjoint operators.

The paper is organized as follows. Section 2 contains some preliminary results about the Fock spaces and operators defined on them. Section 3 deals with topological properties of weighted composition operators. In details, criteria for the boundedness and compactness of such operators are obtained. Note that in the case when $W_{\psi,\varphi}$ acts from a larger Fock space into a smaller one, these properties are equivalent. In view of this, we provide lower and upper estimates for essential norm of only weighted composition operators acting from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$ with $p \leq q$. In Section 4 we study the topological structure of the space of all composition operators and the space of all nonzero weighted composition operators between different Fock spaces endowed with the operator norm topology. We give complete characterizations of connected path components and isolated points in both these spaces. Necessary and sufficient conditions for the compactness of the difference of two weighted composition operators are also stated.

2 Preliminaries

For a number $p \in (0, \infty)$, the Fock space $\mathcal{F}^p(\mathbb{C})$ is defined as follows

$$\mathcal{F}^p(\mathbb{C}) := \left\{ f \in \mathcal{O}(\mathbb{C}) : \|f\|_p = \left(\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} \, dA(z) \right)^{1/p} < \infty \right\},$$

where $\mathcal{O}(\mathbb{C})$ is the space of entire functions on $\mathbb{C}$ with the usual compact open topology and dA is the Lebesgue measure on $\mathbb{C}$. Furthermore, the space $\mathcal{F}^\infty(\mathbb{C})$ consists of all entire functions $f \in \mathcal{O}(\mathbb{C})$ for which

$$\|f\|_\infty := \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty.$$

It is well known that $\mathcal{F}^p(\mathbb{C})$ with $p \geq 1$ and $\mathcal{F}^\infty(\mathbb{C})$ are Banach spaces. When $0 < p < 1$, $\mathcal{F}^p(\mathbb{C})$ is a complete metric space with the distance $d(f, g) := \|f - g\|_p^p$.

For each $w \in \mathbb{C}$, we define the function

$$k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}, \ z \in \mathbb{C}.$$

These functions play important roles in the study of Fock spaces $\mathcal{F}^p(\mathbb{C})$. Obviously, $\|k_w\|_p = 1$ for every $w \in \mathbb{C}$ and k_w converges to 0 in $\mathcal{O}(\mathbb{C})$ as $|w| \to \infty$.

We refer the reader to the monograph [28] for more details about Fock spaces. Hereby, we give only some auxiliary results which will be needed in the sequel.

Lemma 2.1 *Let* $p \in (0, \infty)$ *be given. For each function* $f \in \mathcal{F}^p(\mathbb{C})$, *the following assertions are valid:*

(i)
$$|f(z)| \leq e^{\frac{|z|^2}{2}} \|f\|_p, \ \forall z \in \mathbb{C}.$$

(ii)
$$|f'(z)| \leq e^2 (1 + |z|) e^{\frac{|z|^2}{2}} \|f\|_p, \ \forall z \in \mathbb{C}.$$

Proof (i) was proved in [28, Corollary 2.8].

(ii). Let $f \in \mathcal{F}^p(\mathbb{C})$. For $|z| \leq 1$, by the classical Cauchy formula and the part (i),

$$|f'(z)| \leq \frac{1}{2\pi} \int_{|\zeta - z| = 1} \frac{|f(\zeta)|}{|\zeta - z|^2} |d\zeta| \leq \max_{|\zeta - z| = 1} |f(\zeta)|$$

$$\leq \|f\|_p \max_{|\zeta - z| = 1} e^{\frac{|\zeta|^2}{2}} \leq e^2 \|f\|_p.$$

On the other hand, for $|z| > 1$, arguing as above, we get

$$|f'(z)| \leq \frac{1}{2\pi} \int_{|\zeta - z| = |z|^{-1}} \frac{|f(\zeta)|}{|\zeta - z|^2} |d\zeta| \leq |z| \max_{|\zeta - z| = |z|^{-1}} |f(\zeta)|$$

$$\leq |z| e^{\frac{(|z| + |z|^{-1})^2}{2}} \|f\|_p \leq e^2 |z| e^{\frac{|z|^2}{2}} \|f\|_p.$$

Combining these estimates yields the desired inequality. $\square$

The following result was proved in [28, Theorem 2.10].

Lemma 2.2 *For $0 < p < q < \infty$, $\mathcal{F}^p(\mathbb{C}) \subset \mathcal{F}^q(\mathbb{C})$, and the inclusion is proper and continuous. Moreover,*

$$\|f\|_q \leq \left(\frac{q}{p}\right)^{\frac{1}{q}} \|f\|_p, \ \forall f \in \mathcal{F}^p(\mathbb{C}).$$

The following two lemmas give necessary and sufficient conditions for compactness of an operator acting from one Fock space into another.

Lemma 2.3 *Let $p, q \in (0, \infty)$ and T be a linear continuous operator from $\mathcal{O}(\mathbb{C})$ into itself and $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ be well-defined. The following two assertions are equivalent:*

(i) *$T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact.*
(ii) *For every bounded sequence $(f_n)_n$ in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$, the sequence $(Tf_n)_n$ also converges to 0 in $\mathcal{F}^q(\mathbb{C})$.*

Proof (i) $\Rightarrow$ (ii). Suppose that $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact and there is a bounded sequence $(f_n)_n$ in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$ such that $(Tf_n)_n$ does not converge to 0 in $\mathcal{F}^q(\mathbb{C})$.

Without loss of generality, we assume that there is a number $c > 0$ such that

$$\|Tf_n\|_q \geq c, \ \forall n \in \mathbb{N}. \tag{2.1}$$

Since $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact, there is a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that Tf_{n_k} converges to some function g in $\mathcal{F}^q(\mathbb{C})$.

On the other hand, since T is continuous on $\mathcal{O}(\mathbb{C})$, then Tf_n, and hence Tf_{n_k} converge to 0 in $\mathcal{O}(\mathbb{C})$.

Consequently, the function g must be identically zero which is a contradiction with Eq. (2.1).

(ii) $\Rightarrow$ (i). Let B be an arbitrary bounded subset of $\mathcal{F}^p(\mathbb{C})$ and $(f_n)_n$ be a sequence in B. By Lemma 2.1[(i)] and Montel's theorem, B is relatively compact in $\mathcal{O}(\mathbb{C})$, and then there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ converging to some function f in $\mathcal{O}(\mathbb{C})$. From this and Fatou's lemma, we have that $f \in \mathcal{F}^p(\mathbb{C})$.

Therefore, the sequence $(f_{n_k} - f)_k$ is bounded in $\mathcal{F}^p(\mathbb{C})$ and converges to 0 in $\mathcal{O}(\mathbb{C})$. By the hypothesis, Tf_{n_k} also converges to Tf in $\mathcal{F}^q(\mathbb{C})$.

Consequently, TB is relatively compact in $\mathcal{F}^q(\mathbb{C})$. □

Note that the assumption that T is a linear continuous operator on $\mathcal{O}(\mathbb{C})$ plays an essential role in the proof of (i) $\Rightarrow$ (ii). Now, for an arbitrary operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ that would be not defined on $\mathcal{O}(\mathbb{C})$, we get the following result.

Lemma 2.4 *Let $p, q \in (1, \infty)$. If the operator $T : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact, then for every sequence $(w_n)_n$ in $\mathbb{C}$ with $\lim_{n \to \infty} |w_n| = \infty$, the sequence $(Tk_{w_n})_n$ converges to 0 in $\mathcal{F}^q(\mathbb{C})$.*

Proof Since $p \in (1, \infty)$, for every sequence $(w_n)_n$ in $\mathbb{C}$ with $\lim_{n \to \infty} |w_n| = \infty$, the sequence $(k_{w_n})_n$ weakly converges to 0 in $\mathcal{F}^p(\mathbb{C})$, and hence, $(Tk_{w_n})_n$ converges to 0 in $\mathcal{F}^q(\mathbb{C})$. □

For entire functions ψ and φ on $\mathbb{C}$, the following quantities play an important role in the present paper:

$$m_z(\psi, \varphi) := |\psi(z)|e^{\frac{|\varphi(z)|^2 - |z|^2}{2}}, \ z \in \mathbb{C},$$

and

$$m(\psi, \varphi) := \sup_{z \in \mathbb{C}} m_z(\psi, \varphi).$$

3 Topological Properties

3.1 Boundedness and Compactness

In this subsection we study the boundedness and compactness for weighted composition operators acting from a Fock space $\mathcal{F}^p(\mathbb{C})$ into an another one $\mathcal{F}^q(\mathbb{C})$.

We obtain the following necessary condition.

Proposition 3.1 *Let* $p, q \in (0, \infty)$. *If the weighted composition operator* $W_{\psi,\varphi}$: $\mathcal{F}^p(\mathbb{C}) \rightarrow \mathcal{F}^q(\mathbb{C})$ *is bounded, then* $\psi \in \mathcal{F}^q(\mathbb{C})$ *and* $m(\psi, \varphi) < \infty$. *In this case,* $\varphi(z) = az + b$ *with* $|a| \leq 1$ *and*

$$m_z(\psi, \varphi) \leq \|W_{\psi,\varphi} k_{\varphi(z)}\|_q \leq \|W_{\psi,\varphi}\|, \ \forall z \in \mathbb{C}. \tag{3.1}$$

Proof Obviously, $\psi = W_{\psi,\varphi}(1) \in \mathcal{F}^q(\mathbb{C})$.

For each $w \in \mathbb{C}$, using $\|k_w\|_p = 1$ and Lemma 2.1[(i)], we have

$$\begin{aligned}\|W_{\psi,\varphi}\| &\geq \|W_{\psi,\varphi} k_w\|_q \geq |W_{\psi,\varphi} k_w(z)| e^{-\frac{|z|^2}{2}} \\ &= |\psi(z)| |e^{\overline{w}\varphi(z) - \frac{|w|^2}{2}}| |e^{-\frac{|z|^2}{2}}, \ \forall z \in \mathbb{C}.\end{aligned}$$

In particular, with $w = \varphi(z)$, the last inequality means that

$$m_z(\psi, \varphi) \leq \|W_{\psi,\varphi} k_{\varphi(z)}\|_q \leq \|W_{\psi,\varphi}\|, \ \forall z \in \mathbb{C}.$$

Then $m(\psi, \varphi) \leq \|W_{\psi,\varphi}\|$. Hence, by [20, Proposition 2.1], $\varphi(z) = az + b$ with $|a| \leq 1$. □

In view of Proposition 3.1, throughout this paper we always assume that ψ is a nonzero function in $\mathcal{F}^q(\mathbb{C})$ and $\varphi(z) = az + b$ with $|a| \leq 1$.

In the case $a = 0$, from Proposition 3.1 we get

Corollary 3.2 *Let* $p, q \in (0, \infty)$ *and* ψ *be a nonzero function in* $\mathcal{F}^q(\mathbb{C})$. *If* $a = 0$, *i.e.,* $\varphi(z) = b$, *then the operator* $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \rightarrow \mathcal{F}^q(\mathbb{C})$ *is compact and*

$$\|W_{\psi,\varphi}\| = e^{\frac{|b|^2}{2}} \|\psi\|_q.$$

Proof By Lemma 2.1[(i)], for each $f \in \mathcal{F}^p(\mathbb{C})$,

$$\|W_{\psi,\varphi} f\|_q = |f(b)| \|\psi\|_q \leq e^{\frac{|b|^2}{2}} \|\psi\|_q \|f\|_p.$$

In particular, with $f = k_b$ we have

$$\|W_{\psi,\varphi} k_b\|_q = |k_b(b)| \|\psi\|_q = e^{\frac{|b|^2}{2}} \|\psi\|_q \|k_b\|_p.$$

Thus, the operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \rightarrow \mathcal{F}^q(\mathbb{C})$ is bounded and

$$\|W_{\psi,\varphi}\| = e^{\frac{|b|^2}{2}} \|\psi\|_q.$$

Moreover, $W_{\psi,\varphi}$ has rank 1, and hence, it is compact. □

The case $0 < |a| \leq 1$ is more complicated. At first, we consider weighted composition operators $W_{\psi,\varphi}$ acting from larger Fock spaces into smaller ones. In this case the boundedness and compactness of $W_{\psi,\varphi}$ are equivalent (see, Theorem 3.3 below).

To show this we will use the Berezin type integral transform

$$B_{\psi,\varphi,q}(w) := \frac{q}{2\pi} \int_{\mathbb{C}} |\psi(z)|^q \left|e^{\overline{w}\varphi(z) - \frac{|w|^2}{2}}\right|^q e^{-\frac{q|z|^2}{2}} dA(z) = \|W_{\psi,\varphi}k_w\|_q^q, \ w \in \mathbb{C}.$$

Since $\varphi(z) = az + b$ with $0 < |a| \leq 1$, we define the following positive pull-back measure $\mu_{\psi,\varphi,q}$ on $\mathbb{C}$ with

$$\mu_{\psi,\varphi,q}(E) := \frac{q}{2\pi} \int_{\varphi^{-1}(E)} |\psi(z)|^q e^{-\frac{q|z|^2}{2}} dA(z)$$

for every Borel subset E of $\mathbb{C}$.

We recall, for the reader's convenience, that for $p, q \in (0, \infty)$ a positive Borel measure μ on $\mathbb{C}$ is called a (p, q)-*Fock Carleson measure*, if the embedding operator $i : \mathcal{F}^p(\mathbb{C}) \to L^q(\mathbb{C}, d\mu)$ is bounded, i.e. there exists a constant $C > 0$ such that for every $f \in \mathcal{F}^p(\mathbb{C})$,

$$\left(\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q|z|^2}{2}} d\mu(z)\right)^{\frac{1}{q}} \leq C\|f\|_p.$$

We will write $\|\mu\|$ for the operator norm of i from $\mathcal{F}^p(\mathbb{C})$ into $L^q(\mathbb{C}, d\mu)$ and refer the reader to [17, Section 3] for more information about (p, q)-Fock Carleson measure.

Theorem 3.3 *Let $0 < q < p < \infty$, ψ a nonzero function in $\mathcal{F}^q(\mathbb{C})$, and $\varphi(z) = az + b$ with $0 < |a| \leq 1$. The following assertions are equivalent:*

(i) *The operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded.*
(ii) *The operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact.*
(iii) $m_z(\psi, \varphi) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.

In this case, for some positive constant C independent of ψ and φ,

$$C^{-1}|a|^{\frac{2(p-q)}{pq}} \|m_z(\psi, \varphi)\|_{L^{\frac{pq}{p-q}}} \leq \|W_{\psi,\varphi}\| \leq C|a|^{-\frac{2}{p}} \|m_z(\psi, \varphi)\|_{L^{\frac{pq}{p-q}}}.$$

Proof (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii). Assume that the operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded. Then for each $f \in \mathcal{F}^p(\mathbb{C})$,

$$\|W_{\psi,\varphi}\|\|f\|_p \geq \|W_{\psi,\varphi}f\|_q = \left(\frac{q}{2\pi} \int_{\mathbb{C}} |\psi(z)|^q |f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z)\right)^{\frac{1}{q}}$$

$$= \left(\int_{\mathbb{C}} |f(z)|^q d\mu_{\psi,\varphi,q}(z)\right)^{\frac{1}{q}} = \left(\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q|z|^2}{2}} d\lambda_{\psi,\varphi,q}(z)\right)^{\frac{1}{q}},$$

where $d\lambda_{\psi,\varphi,q}(z) = e^{\frac{q|z|^2}{2}} d\mu_{\psi,\varphi,q}(z)$. The last inequality means that $\lambda_{\psi,\varphi,q}$ is a (p, q)-Fock Carleson measure. Then by [17, Theorem 3.3], we get

$$\widetilde{\lambda_{\psi,\varphi,q}}(w) := \int_{\mathbb{C}} |k_w(z)|^q e^{-\frac{q|z|^2}{2}} d\lambda_{\psi,\varphi,q}(z) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA).$$

Clearly, for all $w \in \mathbb{C}$,

$$
\begin{aligned}
\widetilde{\lambda_{\psi,\varphi,q}}(w) &= \int_{\mathbb{C}} |k_w(z)|^q e^{-\frac{q|z|^2}{2}} d\lambda_{\psi,\varphi,q}(z) = \int_{\mathbb{C}} |k_w(z)|^q d\mu_{\psi,\varphi,q}(z) \\
&= \frac{q}{2\pi} \int_{\mathbb{C}} |\psi(z)|^q |k_w(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z) = B_{\psi,\varphi,q}(w).
\end{aligned}
$$

Consequently, $B_{\psi,\varphi,q}(w) \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$.

On the other hand, using Lemma 2.1[(i)], we have that, for all $w, z \in \mathbb{C}$,

$$
\begin{aligned}
B_{\psi,\varphi,q}(w) &= \|W_{\psi,\varphi} k_w\|_q^q \geq |W_{\psi,\varphi} k_w(z)|^q e^{-\frac{q|z|^2}{2}} \\
&= |\psi(z)|^q \left| e^{\overline{w}\varphi(z) - \frac{|w|^2}{2}} \right|^q e^{-\frac{q|z|^2}{2}}.
\end{aligned}
$$

In particular, with $w = \varphi(z)$, we have

$$
B_{\psi,\varphi,q}(\varphi(z)) \geq m_z(\psi, \varphi)^q, \ \forall z \in \mathbb{C},
$$

and hence

$$
\begin{aligned}
\int_{\mathbb{C}} m_z(\psi, \varphi)^{\frac{pq}{p-q}} dA(z) &\leq \int_{\mathbb{C}} B_{\psi,\varphi,q}(\varphi(z))^{\frac{p}{p-q}} dA(z) \\
&= |a|^{-2} \int_{\mathbb{C}} B_{\psi,\varphi,q}(w)^{\frac{p}{p-q}} dA(w) < \infty. \quad (3.2)
\end{aligned}
$$

Thus, $m_z(\psi, \varphi) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.

Moreover, by [17, Theorem 3.3], for some constant $C_1 > 0$,

$$
\|W_{\psi,\varphi}\|^q = \|\lambda_{\psi,\varphi,q}\|^q \geq C_1 \|\widetilde{\lambda_{\psi,\varphi,q}}\|_{L^{\frac{p}{p-q}}} = C_1 \|B_{\psi,\varphi,q}\|_{L^{\frac{p}{p-q}}}.
$$

From this and Eq. (3.2) it follows that

$$
\begin{aligned}
\|m_z(\psi, \varphi)\|_{L^{\frac{pq}{p-q}}} &\leq \left(|a|^{-2} \int_{\mathbb{C}} B_{\psi,\varphi,q}(z)^{\frac{p}{p-q}} dA(z) \right)^{\frac{p-q}{pq}} \\
&\leq C_1^{-\frac{1}{q}} |a|^{-\frac{2(p-q)}{pq}} \|W_{\psi,\varphi}\|. \quad (3.3)
\end{aligned}
$$

(iii) $\Rightarrow$ (ii). For each function $f \in \mathcal{F}^p(\mathbb{C})$, using Hölder's inequality, we obtain

$$
\begin{aligned}
\|W_{\psi,\varphi} f\|_q^q &= \frac{q}{2\pi} \int_{\mathbb{C}} m_z(\psi, \varphi)^q |f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \\
&\leq \frac{q}{2\pi} \left(\int_{\mathbb{C}} |f(\varphi(z))|^p e^{-\frac{p|\varphi(z)|^2}{2}} dA(z) \right)^{\frac{q}{p}} \left(\int_{\mathbb{C}} m_z(\psi, \varphi)^{\frac{pq}{p-q}} dA(z) \right)^{\frac{p-q}{p}} \\
&\leq \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}} \|f\|_p^q \left(\int_{\mathbb{C}} m_z(\psi, \varphi)^{\frac{pq}{p-q}} dA(z) \right)^{\frac{p-q}{p}} \\
&\leq \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}} \|f\|_p^q \left(\|m_z(\psi, \varphi)\|_{L^{\frac{pq}{p-q}}} \right)^q.
\end{aligned}
$$

The last inequality means that $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded and

$$
\|W_{\psi,\varphi}\| \leq \left(\frac{q}{2\pi} \right)^{\frac{1}{q}} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{1}{p}} \|m_z(\psi, \varphi)\|_{L^{\frac{pq}{p-q}}}. \quad (3.4)
$$

Next, let $(f_n)_n$ be an arbitrary bounded sequence in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$. For each $n \in \mathbb{N}$ and $R > 0$,

$$
\begin{aligned}
\|W_{\psi,\varphi} f_n\|_q^q &= \frac{q}{2\pi} \int_{\mathbb{C}} |\psi(z)|^q |f_n(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z) \\
&= \frac{q}{2\pi} \left(\int_{|z|\leq R} + \int_{|z|>R} \right) |\psi(z)|^q |f_n(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z) \\
&= \mathcal{I}(n,R) + \mathcal{J}(n,R).
\end{aligned}
$$

Obviously,

$$
\mathcal{I}(n,R) \leq \frac{q}{2\pi} \max_{|z|\leq R} |f_n(\varphi(z))|^q \int_{|z|\leq R} |\psi(z)|^q e^{-\frac{q|z|^2}{2}} dA(z) \leq \|\psi\|_q^q \max_{|z|\leq R} |f_n(\varphi(z))|^q.
$$

For $\mathcal{J}(n,R)$, again using Hölder's inequality, we get

$$
\begin{aligned}
\mathcal{J}(n,R) &= \frac{q}{2\pi} \int_{|z|>R} m_z(\psi,\varphi)^q |f_n(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \\
&\leq \frac{q}{2\pi} \left(\int_{|z|>R} |f_n(\varphi(z))|^p e^{-\frac{p|\varphi(z)|^2}{2}} dA(z) \right)^{\frac{q}{p}} \left(\int_{|z|>R} m_z(\psi,\varphi)^{\frac{pq}{p-q}} dA(z) \right)^{\frac{p-q}{p}} \\
&\leq \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}} \|f_n\|_p^q \left(\int_{|z|>R} m_z(\psi,\varphi)^{\frac{pq}{p-q}} dA(z) \right)^{\frac{p-q}{p}} \\
&\leq M^q \left(\int_{|z|>R} m_z(\psi,\varphi)^{\frac{pq}{p-q}} dA(z) \right)^{\frac{p-q}{p}},
\end{aligned}
$$

where

$$
M^q := \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}} \sup_n \|f_n\|_p^q < \infty.
$$

Consequently, for every $R > 0$, letting $n \to \infty$, we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} \|W_{\psi,\varphi} f_n\|_q^q &\leq \limsup_{n\to\infty} \left(\mathcal{I}(n,R) + \mathcal{J}(n,R) \right) \\
&\leq M^q \left(\int_{|z|>R} m_z(\psi,\varphi)^{\frac{pq}{p-q}} dA(z) \right)^{\frac{p-q}{p}}.
\end{aligned}
$$

Since $m_z(\psi,\varphi) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$, letting $R \to \infty$, we conclude that $W_{\psi,\varphi} f_n$ converges to 0 in $\mathcal{F}^q(\mathbb{C})$ as $n \to \infty$.

Consequently, by Lemma 2.3, the operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(C)$ is compact. Moreover, the desired estimates for $\|W_{\psi,\varphi}\|$ follow from Eqs. (3.3) and (3.4). □

For weighted composition operators $W_{\psi,\varphi}$ acting from smaller Fock spaces into larger ones, we get the following result.

Theorem 3.4 Let $0 < p \leq q < \infty$ and ψ be a nonzero function in $\mathcal{F}^q(\mathbb{C})$ and $\varphi(z) = az + b$ with $0 < |a| \leq 1$.

(a) *The operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if $m(\psi,\varphi) < \infty$. Moreover,*

$$
m(\psi,\varphi) \leq \|W_{\psi,\varphi}\| \leq \left(\frac{q}{p|a|^2} \right)^{\frac{1}{q}} m(\psi,\varphi).
$$

(b) *The operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact if and only if $\lim_{|z|\to\infty} m_z(\psi,\varphi) = 0$.*

Proof For $p = q$, the results were proved in [14]. Hereby we sketch the proof in the case $p \leq q$ for the sake of the completeness.

(a) The necessity follows from Proposition 3.1. Now assume that $m(\psi,\varphi) < \infty$. Then using Lemma 2.2, we have that for every $f \in \mathcal{F}^p(\mathbb{C})$,

$$
\|W_{\psi,\varphi} f\|_q \leq m(\psi,\varphi) \left(\frac{q}{2\pi} \int_{\mathbb{C}} |f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \right)^{\frac{1}{q}}
$$

$$
= m(\psi,\varphi) \frac{1}{|a|^{\frac{2}{q}}} \|f\|_q \leq m(\psi,\varphi) \left(\frac{q}{p|a|^2} \right)^{\frac{1}{q}} \|f\|_p.
$$

Consequently, $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded and

$$
\|W_{\psi,\varphi}\| \leq \left(\frac{q}{p\,|a|^2} \right)^{\frac{1}{q}} m(\psi,\varphi),
$$

which and Eq. (3.1) imply the desired estimates for $\|W_{\psi,\varphi}\|$.

(b) **Necessary.** Suppose that $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact. For every sequence $(z_n)_n$ in $\mathbb{C}$ converging to ∞, we have that $k_{\varphi(z_n)}$ converges to 0 in $\mathcal{O}(\mathbb{C})$. Therefore, by Eq. (3.1) and Lemma 2.3,

$$
m_{z_n}(\psi,\varphi) \leq \|W_{\psi,\varphi} k_{\varphi(z_n)}\|_q \to 0 \text{ as } n \to \infty.
$$

From this, $\lim_{|z|\to\infty} m_z(\psi,\varphi) = 0$.

Sufficiency. By part (a), the operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded.

Let $(f_n)_n$ be an arbitrary bounded sequence in $\mathcal{F}^p(\mathbb{C})$ converging to 0 in $\mathcal{O}(\mathbb{C})$. Then for each $n \in \mathbb{N}$ and $R > 0$, using Lemma 2.2, we have

$$
\|W_{\psi,\varphi} f_n\|_q^q = \frac{q}{2\pi} \int_{\mathbb{C}} |\psi(z)|^q |f_n(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z)
$$

$$
= \frac{q}{2\pi} \left(\int_{|z|\leq R} + \int_{|z|>R} \right) |\psi(z)|^q |f_n(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z)
$$

$$
\leq \frac{q}{2\pi} \max_{|z|\leq R} |f_n(\varphi(z))|^q \int_{|z|\leq R} |\psi(z)|^q e^{-\frac{q|z|^2}{2}} dA(z)
$$

$$
+ \frac{q}{2\pi} \sup_{|z|>R} m_z(\psi,\varphi)^q \int_{|z|>R} |f_n(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} dA(z)
$$

$$
\leq \|\psi\|_q^q \max_{|z|\leq R} |f_n(\varphi(z))|^q + \frac{\|f_n\|_q^q}{|a|^2} \sup_{|z|>R} m_z(\psi,\varphi)^q
$$

$$
\leq \|\psi\|_q^q \max_{|z|\leq R} |f_n(\varphi(z))|^q + \frac{q M^q}{p|a|^2} \sup_{|z|>R} m_z(\psi,\varphi)^q,
$$

where $M := \sup_n \|f_n\|_p < \infty$.

From this, letting $n \to \infty$, and then $R \to \infty$, we get that the sequence $W_{\psi,\varphi} f_n$ converges to 0 in $\mathcal{F}^q(\mathbb{C})$.

Therefore, by Lemma 2.3, $W_{\psi,\varphi}$ is a compact operator from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$. □

From Theorems 3.3 and 3.4 we can get a complete characterization for the boundedness and compactness of composition operators C_φ.

Corollary 3.5 *Let $0 < p \leq q < \infty$.*

(a) *The operator $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if*

$$\varphi(z) = \begin{cases} az + b, & \text{if } |a| < 1, \\ az, & \text{if } |a| = 1. \end{cases} \tag{3.5}$$

(b) *The operator $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact if and only if $\varphi(z) = az + b$ with $|a| < 1$.*

Proof Clearly, $m(1, \varphi) < \infty$ if and only if φ as in Eq. 3.5. Then the assertion immediately follows from Theorem 3.4 and Corollary 3.2. □

Corollary 3.6 *Let $0 < q < p < \infty$. The following assertions are equivalent:*

(i) *The operator $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded.*
(ii) *The operator $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact.*
(iii) *$\varphi(z) = az + b$ with $|a| < 1$.*

Proof We can easily show that for each affine function $\varphi(z) = az + b$ as in Eq. 3.5, $m(1, \varphi) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$ if and only if $|a| < 1$. Then the assertion follows from Theorem 3.3 and Corollary 3.2. □

Before going to study essential norm we summarize all situations of the function $\varphi(z) = az + b$ with $|a| \leq 1$ for both composition operators C_φ and weighted composition operators $W_{\psi,\varphi}$.

Remark 3.7 (1) If $a = 0$, then by Corollary 3.2, both C_φ and $W_{\psi,\varphi}$ are compact from $\mathcal{F}^p(\mathbb{C})$ to $\mathcal{F}^q(\mathbb{C})$ for every $p, q \in (0, \infty)$ and $\psi \in \mathcal{F}^q(\mathbb{C})$.

(2) If $0 < |a| < 1$, then by Corollaries 3.5 and 3.6, $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact for every $p, q \in (0, \infty)$, while there are nonzero functions $\psi \in \mathcal{F}^q(\mathbb{C})$, say $\psi(z) = e^{\frac{az^2}{2}}$ in $\mathcal{F}^q(\mathbb{C})$ with $1 - |a|^2 < \alpha < 1$, which induce unbounded operators $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$, because in this case,

$$m(\psi, \varphi) = \sup_{z \in \mathbb{C}} m_z(\psi, \varphi) = \sup_{z \in \mathbb{C}} e^{\frac{(|a|^2-1)|z|^2 + \alpha \operatorname{Re}(z^2) + 2\operatorname{Re}(az\bar{b}) + |b|^2}{2}} = \infty.$$

Also, there are nonzero functions $\psi \in \mathcal{F}^q(\mathbb{C})$ such that $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded but not compact when $0 < p \leq q < \infty$. Indeed, for $b = 0$ and $\psi(z) = e^{\frac{(1-|a|^2)z^2}{2}} \in \mathcal{F}^q(\mathbb{C})$, we have

$$\limsup_{|z|\to\infty} m_z(\psi, \varphi) = \limsup_{|z|\to\infty} e^{(|a|^2-1)(\operatorname{Im} z)^2} = 1.$$

Then by Theorem 3.4, $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded but not compact.

(3) If $|a| = 1$ and $b \neq 0$, then by Corollaries 3.5 and 3.6, $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is not bounded for every $p, q \in (0, \infty)$.

In this case, by [20, Proposition 2.1], $m(\psi, \varphi) < \infty$ if and only if $\psi(z) = \psi(0)e^{-\bar{b}az}$; and hence, $m_z(\psi, \varphi) = |\psi(0)|e^{\frac{|b|^2}{2}}$ for all $z \in \mathbb{C}$. Then, when $0 < p \leq q < \infty$, by Theorem 3.4, $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded if and only if $\psi(z) = \psi(0)e^{-\bar{b}az}$.

Moreover, by Theorem 3.3, when $0 < q < p < \infty$ there is no bounded operator $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$.

(4) If $|a| = 1$ and $b = 0$, then by Corollaries 3.5 and 3.6, $C_\varphi : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded and not compact when $0 < p \le q < \infty$, and unbounded when $0 < q < p < \infty$.

Also, again by [20, Proposition 2.1], $m(\psi, \varphi) < \infty$ if and only if ψ is a constant function. So, by Theorems 3.3 and 3.4, there is no non-constant function $\psi \in \mathcal{F}^q(\mathbb{C})$ such that $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded, and there is no nonzero function $\psi \in \mathcal{F}^q(\mathbb{C})$ such that $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact for every $p, q \in (0, \infty)$.

3.2 Essential Norm

In a general setting, let X, Y be Banach spaces, and $\mathcal{K}(X, Y)$ be the set of all compact operators from X into Y. The essential norm of a bounded linear operator $L : X \to Y$, denoted as $\|L\|_e$, is defined as

$$\|L\|_e = \inf\{\|L - K\| : K \in \mathcal{K}(X, Y)\}.$$

Clearly, L is compact if and only if $\|L\|_e = 0$.

In view of Corollary 3.2, Theorem 3.3 and Lemma 2.4, we study essential norm of $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ when $1 < p \le q < \infty$ and $\varphi(z) = az + b$ with $0 < |a| \le 1$.

The main result is stated as follows.

Theorem 3.8 *Let $1 < p \le q < \infty$ and $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ be a bounded weighted composition operator induced by a nonzero entire function $\psi \in \mathcal{F}^q(\mathbb{C})$ and an affine function $\varphi(z) = az + b$ with $0 < |a| \le 1$. Then*

$$\limsup_{|z| \to \infty} m_z(\psi, \varphi) \le \|W_{\psi,\varphi}\|_e \le 2 \left(\frac{q}{p|a|^2} \right)^{\frac{1}{q}} \limsup_{|z| \to \infty} m_z(\psi, \varphi).$$

Proof It is clear, by Eq. (3.1), that $\limsup\limits_{|z| \to \infty} m_z(\psi, \varphi)$ is finite.

Lower estimate. We prove the lower estimate for $\|W_{\psi,\varphi}\|_e$ by contradiction. Assume in contrary that

$$\|W_{\psi,\varphi}\|_e < \limsup_{|z| \to \infty} m_z(\psi, \varphi).$$

Then there are positive constants $A < B$ and a compact operator T acting from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$ such that

$$\|W_{\psi,\varphi} - T\| < A < B < \limsup_{|z| \to \infty} m_z(\psi, \varphi).$$

We can find a sequence (z_n) with $|z_n| \uparrow \infty$ so that

$$\lim_{n \to \infty} m_{z_n}(\psi, \varphi) = \limsup_{|z| \to \infty} m_z(\psi, \varphi) > B. \tag{3.6}$$

On the other hand, using Eq. (3.1), for each $n \in N$, we have

$$\|W_{\psi,\varphi} - T\| \ge \|W_{\psi,\varphi} k_{\varphi(z_n)} - T k_{\varphi(z_n)}\|_q \ge \|W_{\psi,\varphi} k_{\varphi(z_n)}\|_q - \|T k_{\varphi(z_n)}\|_q$$
$$\ge m_{z_n}(\psi, \varphi) - \|T k_{\varphi(z_n)}\|_q.$$

Since $a \ne 0$, $\varphi(z_n) \to \infty$ as $n \to \infty$, by Lemma 2.4, $\|T k_{\varphi(z_n)}\|_q \to 0$ as $n \to \infty$.

From this and Eq. (3.6), we obtain

$$A > \|W_{\psi,\varphi} - T\| \ge \lim_{n \to \infty} m_{z_n}(\psi, \varphi) > B,$$

which is a contradiction.

Upper estimate. For each $k \in \mathbb{N}$, we consider the dilation operator $U_{r_k} f(z) := f(r_k z)$ with $r_k := k/(k+1)$. Then by Corollary 3.5 and Theorem 3.4, U_{r_k} is a compact operator on $\mathcal{F}^p(\mathbb{C})$ and

$$\|U_{r_k}\| \le r_k^{-\frac{2}{p}}, \quad \forall k \in \mathbb{N}.$$

Take and fix a number $R > 0$. For every $k \in \mathbb{N}$, we have

$$\|W_{\psi,\varphi}\|_e \le \|W_{\psi,\varphi} - W_{\psi,\varphi} \circ U_{r_k}\| = \sup_{\|f\|_p \le 1} \|W_{\psi,\varphi} \circ (I - U_{r_k}) f\|_q$$

$$= \sup_{\|f\|_p \le 1} \left(\frac{q}{2\pi} \int_{\mathbb{C}} |W_{\psi,\varphi} \circ (I - U_{r_k}) f(z)|^q e^{-\frac{q|z|^2}{2}} dA(z) \right)^{\frac{1}{q}}$$

$$\le \sup_{\|f\|_p \le 1} \left(\frac{q}{2\pi} \int_{|z| \le R} |W_{\psi,\varphi} \circ (I - U_{r_k}) f(z)|^q e^{-\frac{q|z|^2}{2}} dA(z) \right)^{\frac{1}{q}}$$

$$+ \sup_{\|f\|_p \le 1} \left(\frac{q}{2\pi} \int_{|z| > R} |W_{\psi,\varphi} \circ (I - U_{r_k}) f(z)|^q e^{-\frac{q|z|^2}{2}} dA(z) \right)^{\frac{1}{q}}$$

$$= \mathcal{I}(R, k) + \mathcal{J}(R, k),$$

where I is the identity operator on $\mathcal{F}^p(\mathbb{C})$.

On one hand, using Lemma 2.2 we observe that

$$\mathcal{J}(R, k) \le \sup_{|z| > R} m_z(\psi, \varphi) \sup_{\|f\|_p \le 1} \left(\frac{q}{2\pi} \int_{|z| > R} |(I - U_{r_k}) f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \right)^{\frac{1}{q}}$$

$$\le |a|^{-\frac{2}{q}} \sup_{|z| > R} m_z(\psi, \varphi) \sup_{\|f\|_p \le 1} \|(I - U_{r_k}) f\|_q$$

$$\le \left(\frac{q}{p} \right)^{\frac{1}{q}} |a|^{-\frac{2}{q}} \sup_{|z| > R} m_z(\psi, \varphi) \sup_{\|f\|_p \le 1} \|(I - U_{r_k}) f\|_p$$

$$= \left(\frac{q}{p} \right)^{\frac{1}{q}} |a|^{-\frac{2}{q}} \|I - U_{r_k}\| \sup_{|z| > R} m_z(\psi, \varphi)$$

$$\le \left(\frac{q}{p} \right)^{\frac{1}{q}} |a|^{-\frac{2}{q}} \left(1 + r_k^{-\frac{2}{p}} \right) \sup_{|z| > R} m_z(\psi, \varphi).$$

On the other hand, we have

$$\mathcal{I}(R, k) \le \sup_{|z| \le R} m_z(\psi, \varphi) \sup_{\|f\|_p \le 1} \left(\frac{q}{2\pi} \int_{|z| \le R} |(I - U_{r_k}) f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \right)^{\frac{1}{q}}$$

$$\le \left(\frac{q}{2\pi} \int_{|z| \le R} e^{-\frac{q|\varphi(z)|^2}{2}} dA(z) \right)^{\frac{1}{q}} \sup_{|z| \le R} m_z(\psi, \varphi) \sup_{\|f\|_p \le 1} \sup_{|z| \le R} |(I - U_{r_k}) f(\varphi(z))|$$

$$\le |a|^{-\frac{2}{q}} m(\psi, \varphi) \sup_{\|f\|_\infty \le 1} \sup_{|z| \le R} |(I - U_{r_k}) f(\varphi(z))|,$$

where the last inequality is based on the fact that $\|f\|_\infty \le \|f\|_p$ for every $f \in \mathcal{F}^p(\mathbb{C})$.

For each $f(z) = \sum_{j=0}^{\infty} a_j z^j$ with $\|f\|_\infty \leq 1$, we have

$$|a_j| = \frac{|f^{(j)}(0)|}{j!} \leq \frac{1}{\rho^j} \max_{|\zeta|=\rho} |f(\zeta)| \leq \frac{e^{\frac{\rho^2}{2}}}{\rho^j}, \quad \forall j \geq 1, \rho > 0,$$

which gives

$$|a_j| \leq \inf_{\rho>0} \frac{e^{\frac{\rho^2}{2}}}{\rho^j} = \left(\frac{e}{j}\right)^{\frac{j}{2}}, \quad \text{for all } j \geq 1.$$

Putting $R_\varphi := \max_{|z| \leq R} |\varphi(z)|$, we obtain

$$\begin{aligned}
\mathcal{I}(R, k) &\leq |a|^{-\frac{2}{q}} m(\psi, \varphi) \sup_{\|f\|_\infty \leq 1} \sup_{|z| \leq R} \left| (I - U_{r_k}) f(\varphi(z)) \right| \\
&\leq |a|^{-\frac{2}{q}} m(\psi, \varphi) \sup_{\|f\|_\infty \leq 1} \sup_{|z| \leq R_\varphi} \left| (I - U_{r_k}) f(z) \right| \\
&\leq |a|^{-\frac{2}{q}} m(\psi, \varphi) \sup_{\|f\|_\infty \leq 1} \sup_{|z| \leq R_\varphi} \sum_{j=1}^{\infty} |a_j| \left(1 - \frac{k^j}{(k+1)^j} \right) |z|^j \\
&\leq \frac{|a|^{-\frac{2}{q}}}{k+1} m(\psi, \varphi) \sum_{j=1}^{\infty} j R_\varphi^j \left(\frac{e}{j}\right)^{\frac{j}{2}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|W_{\psi,\varphi}\|_e &\leq \limsup_{k\to\infty} \|W_{\psi,\varphi} - W_{\psi,\varphi} \circ U_{r_k}\| \leq \limsup_{k\to\infty} \mathcal{I}(R, k) + \limsup_{k\to\infty} \mathcal{J}(R, k) \\
&\leq 2 \left(\frac{q}{p|a|^2}\right)^{\frac{1}{q}} \sup_{|z|>R} m_z(\psi, \varphi),
\end{aligned}$$

from which the upper estimate of $\|W_{\psi,\varphi}\|_e$ follows by letting $R \to \infty$. $\qquad\square$

In the case $|a| = 1$ we have the following simplified estimates for the essential norm of a bounded weighted composition operator $W_{\psi,\varphi}$.

Corollary 3.9 *Let p, q and $W_{\psi,\varphi}$ be as in* Theorem 3.8. *If $|a| = 1$, then*

$$|\psi(0)|e^{\frac{|b|^2}{2}} \leq \|W_{\psi,\varphi}\|_e \leq \left(\frac{q}{p}\right)^{\frac{1}{q}} |\psi(0)|e^{\frac{|b|^2}{2}}.$$

In particular, $\|W_{\psi,\varphi}\|_e = |\psi(0)|e^{\frac{|b|^2}{2}}$ whenever $p = q$.

Proof The hypothesis and [20, Proposition 2.1] imply that $\psi(z) = \psi(0)e^{-\bar{b}az}$; in this case $m_z(\psi, \varphi) = |\psi(0)|e^{\frac{|b|^2}{2}}$ for all $z \in \mathbb{C}$. Thus, the lower estimate follows directly from Theorem 3.8.

For the upper estimate, using Theorem 3.4 we have

$$\|W_{\psi,\varphi}\|_e \leq \|W_{\psi,\varphi}\| \leq \left(\frac{q}{p}\right)^{\frac{1}{q}} |\psi(0)|e^{\frac{|b|^2}{2}}.$$

$\qquad\square$

Remark 3.10 Ueki [26] showed that the essential norm of $W_{\psi,\varphi}$ on Hilbert space $\mathcal{F}^2(\mathbb{C})$ is equivalent to $\limsup_{|z|\to\infty} B_\varphi(|\psi|^2)(z)$, where $B_\varphi(|\psi|^2)(z)$ is the integral transform

$$B_\varphi(|\psi|^2)(z) = \frac{1}{\pi} \int_{\mathbb{C}} |\psi(\zeta)|^2 \left| e^{\langle \varphi(\zeta), z \rangle} \right|^2 e^{-|\zeta|^2} e^{-|z|^2} dA(\zeta).$$

However, this result is difficult to use, even for composition operators, that is, when ψ is a constant function.

Our Theorem 3.8 is simpler and more effective for essential norm of $W_{\psi,\varphi}$ acting from smaller general Fock spaces $\mathcal{F}^p(\mathbb{C})$ into larger ones $\mathcal{F}^q(\mathbb{C})$. Moreover, Theorem 3.8 also give an answer to T. Le's question in [20, Remark 2.5].

4 Topological Structure

One of the recent main subjects in the study of (weighted) composition operators is related to the topological structure of the space of such operators endowed with the operator norm topology.

In a general setting, let X and Y be two spaces of holomorphic functions on a domain G. For every bounded weighted composition operator $W_{\psi,\varphi} : X \to Y$, we can easily show that $W_{\psi,\varphi}$ and the zero operator 0 belong to the same path connected component in the space of all weighted composition operators acting from X into Y via the path $T_t := W_{t\psi,\varphi}$ for $t \in [0, 1]$. Then researchers study the topological structure for the space of only nonzero weighted composition operators from X into Y. We write $\mathcal{C}(X, Y)$ for the space of composition operators and $\mathcal{C}_w(X, Y)$ for the space of nonzero weighted composition operators acting from X into Y under the operator norm topology. According to [25], the important problems in this topic were raised as follows:

(i)   Characterize the components of $\mathcal{C}(X, Y)$ and $\mathcal{C}_w(X, Y)$.
(ii) Characterize isolated points $\mathcal{C}(X, Y)$ and $\mathcal{C}_w(X, Y)$.
(iii) Characterize compact differences of (weighted) composition operators.

These questions have been intensively investigated on Bergman spaces [23], on Hardy spaces [13, 19], on the space H^∞ of bounded holomorphic functions [18, 21], on weighted Banach spaces of holomorphic functions with sup-norm [3, 22], on Hilbert Fock space $\mathcal{F}^2(\mathbb{C}^n)$ [11].

In this section we investigate the topological structure for both spaces $\mathcal{C}\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$ and $\mathcal{C}_w\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$ with $p, q \in (0, \infty)$ and give complete answers to all the mentioned-above questions.

4.1 Compact Differences

In view of Theorem 3.3 we will study the compactness of the difference of two bounded weighted composition operators acting from a smaller Fock space $\mathcal{F}^p(\mathbb{C})$ into another larger one $\mathcal{F}^q(\mathbb{C})$.

Theorem 4.1 *Let $0 < p \le q < \infty$ and W_{ψ_1,φ_1} and W_{ψ_2,φ_2} be two weighted composition operators in $\mathcal{C}_w\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$ induced respectively by nonzero entire functions $\psi_1, \psi_2 \in \mathcal{F}^q(\mathbb{C})$ and affine functions $\varphi_1(z) = a_1 z + b_1, \varphi_2(z) = a_2 z + b_2$ with $|a_1| \le 1, |a_2| \le 1$.*

Then the difference $W_{\psi_1, \varphi_1} - W_{\psi_2, \varphi_2} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ *is compact if and only if either of the following conditions is satisfied:*

(i)   *Both* W_{ψ_1, φ_1} *and* W_{ψ_2, φ_2} *are compact operators from* $\mathcal{F}^p(\mathbb{C})$ *into* $\mathcal{F}^q(\mathbb{C})$.
(ii) $\varphi_1 = \varphi_2 := \varphi$ *and* $\lim\limits_{|z| \to \infty} m_z(\psi_1 - \psi_2, \varphi) = 0$.

Proof Since $W_{\psi_1, \varphi} - W_{\psi_2, \varphi} = W_{\psi_1 - \psi_2, \varphi}$, the sufficiency follows from Theorem 3.4.

For the necessity, suppose that the difference $W_{\psi_1, \varphi_1} - W_{\psi_2, \varphi_2} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is compact. Then both W_{ψ_1, φ_1} and W_{ψ_2, φ_2} must be either compact or non-compact from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$ simultaneously.

Consider the case when both W_{ψ_1, φ_1} and W_{ψ_2, φ_2} are non-compact. From Theorem 3.4 it follows that

$$c_k = \limsup_{|z| \to \infty} m_z(\psi_k, \varphi_k) > 0 \ (k = 1, 2).$$

Then for say c_1, there exists a sequence (z_n) with $|z_n| \uparrow \infty$ as $n \to \infty$, such that

$$\lim_{n \to \infty} m_{z_n}(\psi_1, \varphi_1) = \limsup_{|z| \to \infty} m_z(\psi_1, \varphi_1) = c_1.$$

By Lemma 2.1[(i)], for all $w, z \in \mathbb{C}$,

$$
\begin{aligned}
\|W_{\psi_1, \varphi_1} k_w - W_{\psi_2, \varphi_2} k_w\|_q &\geq \left| W_{\psi_1, \varphi_1} k_w(z) - W_{\psi_2, \varphi_2} k_w(z) \right| e^{-\frac{|z|^2}{2}} \\
&= \left| \psi_1(z) e^{\overline{w}\varphi_1(z) - \frac{|w|^2 + |z|^2}{2}} - \psi_2(z) e^{\overline{w}\varphi_2(z) - \frac{|w|^2 + |z|^2}{2}} \right|.
\end{aligned}
$$

In particular, with $w = \varphi_1(z)$, the last inequality gives

$$
\begin{aligned}
\|W_{\psi_1, \varphi_1} k_{\varphi_1(z)} - W_{\psi_2, \varphi_2} k_{\varphi_1(z)}\|_q &\geq \left| \left| \psi_1(z) e^{\frac{|\varphi_1(z)|^2 - |z|^2}{2}} \right| - \left| \psi_2(z) e^{\overline{\varphi_1(z)}\varphi_2(z) - \frac{|\varphi_1(z)|^2 + |z|^2}{2}} \right| \right| \\
&= \left| m_z(\psi_1, \varphi_1) - m_z(\psi_2, \varphi_2) e^{-\frac{|\varphi_1(z)|^2 + |\varphi_2(z)|^2 - 2\operatorname{Re}\left(\overline{\varphi_1(z)}\varphi_2(z)\right)}{2}} \right| \\
&= \left| m_z(\psi_1, \varphi_1) - m_z(\psi_2, \varphi_2) e^{-\frac{|\varphi_1(z) - \varphi_2(z)|^2}{2}} \right|, \ \forall z \in \mathbb{C}.
\end{aligned}
$$

$$(4.1)$$

There are two cases for complex numbers a_1 and a_2.
Case 1: $a_1 \neq a_2$. In this case,

$$\lim_{n \to \infty} e^{-\frac{|\varphi_1(z_n) - \varphi_2(z_n)|^2}{2}} = 0.$$

From this, taking into account the inequality $m_{z_n}(\psi_2, \varphi_2) < m(\psi_2, \varphi_2)$, for every $n \in \mathbb{N}$, it follows that

$$\lim_{n \to \infty} m_{z_n}(\psi_2, \varphi_2) e^{-\frac{|\varphi_1(z_n) - \varphi_2(z_n)|^2}{2}} = 0.$$

Obviously, $W_{\psi_1, \varphi_1} - W_{\psi_2, \varphi_2}$ is a linear continuous operator on $\mathcal{O}(\mathbb{C})$. Then, by Lemma 2.3,

$$\lim_{n \to \infty} \|W_{\psi_1, \varphi_1} k_{\varphi_1(z_n)} - W_{\psi_2, \varphi_2} k_{\varphi_1(z_n)}\|_q = 0.$$

Consequently, by Eq. (4.1), we get

$$
\begin{aligned}
c_1 &= \lim_{n \to \infty} \left(m_{z_n}(\psi_1, \varphi_1) - m_{z_n}(\psi_2, \varphi_2) e^{-\frac{|\varphi_1(z_n) - \varphi_2(z_n)|^2}{2}} \right) \\
&\le \lim_{n \to \infty} \left| m_{z_n}(\psi_1, \varphi_1) - m_{z_n}(\psi_2, \varphi_2) e^{-\frac{|\varphi_1(z_n) - \varphi_2(z_n)|^2}{2}} \right| \\
&\le \lim_{n \to \infty} \| W_{\psi_1, \varphi_1} k_{\varphi_1(z_n)} - W_{\psi_2, \varphi_2} k_{\varphi_1(z_n)} \|_q = 0,
\end{aligned}
$$

which is impossible.

Case 2: $a_1 = a_2 = a$. In this case, Eq. (4.1) gives

$$
\| W_{\psi_1, \varphi_1} k_{\varphi_1(z)} - W_{\psi_2, \varphi_2} k_{\varphi_1(z)} \|_q \ge \left| m_z(\psi_1, \varphi_1) - m_z(\psi_2, \varphi_2) e^{-\frac{|b_1 - b_2|^2}{2}} \right|, \quad \forall z \in \mathbb{C}.
$$

Moreover,

$$
\liminf_{n \to \infty} m_{z_n}(\psi_2, \varphi_2) \le \limsup_{n \to \infty} m_{z_n}(\psi_2, \varphi_2) \le \limsup_{|z| \to \infty} m_z(\psi_2, \varphi_2) = c_2.
$$

Therefore,

$$
\begin{aligned}
c_1 - c_2 e^{-\frac{|b_1 - b_2|^2}{2}} &\le \limsup_{n \to \infty} \left(m_{z_n}(\psi_1, \varphi_1) - m_{z_n}(\psi_2, \varphi_2) e^{-\frac{|b_1 - b_2|^2}{2}} \right) \\
&\le \lim_{n \to \infty} \| W_{\psi_1, \varphi_1} k_{\varphi_1(z_n)} - W_{\psi_2, \varphi_2} k_{\varphi_1(z_n)} \|_q = 0.
\end{aligned}
$$

Hence,

$$
c_1 \le c_2 e^{-\frac{|b_1 - b_2|^2}{2}}.
$$

Interchanging the role of φ_1 and φ_2 in the proofs above, we also obtain

$$
c_2 \le c_1 e^{-\frac{|b_1 - b_2|^2}{2}}.
$$

Combining the last two inequalities yields

$$
c_2 \le c_2 e^{-|b_1 - b_2|^2} \le c_2,
$$

which gives $b_1 = b_2$.

Thus $\varphi_1(z) = \varphi_2(z) = \varphi(z) = az + b$, which gives $W_{\psi_1, \varphi_1} - W_{\psi_2, \varphi_2} = W_{\psi_1 - \psi_2, \varphi}$. By Theorem 3.4, $\lim_{|z| \to \infty} m_z(\psi_1 - \psi_2, \varphi) = 0$. $\qquad\square$

From this theorem we immediately get the following result for compact differences of two composition operators.

Corollary 4.2 *Let $0 < p \le q < \infty$. Then the difference of two distinct composition operators acting from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$ is compact if and only if both composition operators are compact.*

4.2 The Space $\mathcal{C}\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$

In this subsection we give a complete description of path connected and connected components and isolated points of the space $\mathcal{C}\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$.

Proposition 4.3 *Let $p, q \in (0, \infty)$ and C_φ be a compact composition operator from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$ induced by entire function $\varphi(z) = az + b$ with $|a| < 1$. Then C_φ and $C_{\varphi(0)}$ belong to the same path connected component of $\mathcal{C}\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$.*

Proof If $a = 0$ then the assertion is trivial. So we assume that $0 < |a| < 1$.

For each $s \in [0, 1]$, put $\varphi_s(z) := \varphi(sz)$, $z \in \mathbb{C}$. Then, by Corollaries 3.5 and 3.6, composition operators C_{φ_s} are all compact from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$, and $C_\varphi = C_{\varphi_1}$ and $C_{\varphi(0)} = C_{\varphi_0}$. We will show that the map

$$[0, 1] \to \mathcal{C}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})), \; s \mapsto C_{\varphi_s},$$

is continuous, that is,

$$\lim_{s \to s_0} \|C_{\varphi_s} - C_{\varphi_{s_0}}\| = 0, \; \forall s_0 \in [0, 1].$$

Case 1: $s_0 < 1$. In this case fix some $s_1 \in (s_0, 1)$. For each $s < s_1$ and each $f \in \mathcal{F}^p(\mathbb{C})$ with $\|f\|_p \leq 1$, we have

$$
\begin{aligned}
\|C_{\varphi_s} f - C_{\varphi_{s_0}} f\|_q^q &= \frac{q}{2\pi} \int_{\mathbb{C}} |f(\varphi(sz)) - f(\varphi(s_0 z))|^q e^{-\frac{q|z|^2}{2}} \, dA(z) \\
&= \frac{q}{2\pi} \int_{\mathbb{C}} \left| \int_{s_0}^{s} z(C_\varphi f)'(tz) dt \right|^q e^{-\frac{q|z|^2}{2}} \, dA(z) \\
&\leq \frac{q}{2\pi} \int_{\mathbb{C}} |s - s_0|^q |z|^q \max_{t \in <s_0, s>} |(C_\varphi f)'(tz)|^q e^{-\frac{q|z|^2}{2}} \, dA(z) \\
&\leq e^{2q} \frac{q}{2\pi} \int_{\mathbb{C}} |s - s_0|^q |z|^q \max_{t \in <s_0, s>} \left((1 + |tz|) e^{\frac{|tz|^2}{2}} \|C_\varphi f\|_q \right)^q \\
&\quad \times e^{-\frac{q|z|^2}{2}} \, dA(z) \\
&\qquad \text{(due to Lemma 2.1[(ii)])} \\
&\leq e^{2q} |s - s_0|^q \|C_\varphi f\|_q^q \frac{q}{2\pi} \int_{\mathbb{C}} |z|^q (1 + |z|)^q e^{-\frac{q(1-s_1^2)|z|^2}{2}} \, dA(z) \\
&\leq M^q \|C_\varphi\|^q \|f\|_p^q |s - s_0|^q \\
&\leq M^q \|C_\varphi\|^q |s - s_0|^q,
\end{aligned}
$$

where $<s_0, s>$ is the closed interval connecting s_0 and s and

$$M^q := e^{2q} \frac{q}{2\pi} \int_{\mathbb{C}} |z|^q (1 + |z|)^q e^{-\frac{q(1-s_1^2)|z|^2}{2}} \, dA(z) < \infty.$$

From this it follows that, for every $s < s_1$,

$$\|C_{\varphi_s} - C_{\varphi_{s_0}}\| = \sup_{\|f\|_p \leq 1} \|C_{\varphi_s} f - C_{\varphi_{s_0}} f\|_q \leq M \|C_\varphi\| |s - s_0|,$$

and the desired limit follows.

Case 2: $s_0 = 1$. Fix an arbitrary number $R > 0$. We have that, for every $f \in \mathcal{F}^p(\mathbb{C})$ with $\|f\|_p \leq 1$ and every $s \in [1/2, 1]$,

$$
\begin{aligned}
\|C_{\varphi_s} f - C_{\varphi_1} f\|_q^q &= \frac{q}{2\pi} \left(\int_{|z| \leq R} + \int_{|z| > R} \right) |f(\varphi(sz)) - f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} \, dA(z) \\
&= \mathcal{I}(R, s) + \mathcal{J}(R, s).
\end{aligned}
$$

Estimate $\mathcal{I}(R, s)$: Arguing as above in Case 1, we get

$$
\begin{aligned}
\mathcal{I}(R, s) &= \frac{q}{2\pi} \int_{|z| \leq R} \left| \int_s^1 z (C_\varphi f)'(tz) dt \right|^q e^{-\frac{q|z|^2}{2}} dA(z) \\
&\leq \frac{q}{2\pi} \int_{|z| \leq R} |1 - s|^q |z|^q \max_{t \in [s,1]} |(C_\varphi f)'(tz)|^q e^{-\frac{q|z|^2}{2}} dA(z) \\
&\leq e^{2q} |1 - s|^q \frac{q}{2\pi} \int_{|z| \leq R} |z|^q \max_{t \in [s,1]} \left((1 + |tz|) e^{\frac{|tz|^2}{2}} \|C_\varphi f\|_q \right)^q \\
&\quad \times e^{-\frac{q|z|^2}{2}} dA(z) \\
&\leq e^{2q} |1 - s|^q \|C_\varphi f\|_q^q \frac{q}{2\pi} \int_{|z| \leq R} |z|^q (1 + |z|)^q dA(z) \\
&\leq M_R^q \|C_\varphi\|^q |1 - s|^q,
\end{aligned}
$$

where

$$
M_R^q := e^{2q} \frac{q}{2\pi} \int_{|z| \leq R} |z|^q (1 + |z|)^q dA(z) < \infty.
$$

Estimate $\mathcal{J}(R, s)$: For every $f \in \mathcal{F}^p(\mathbb{C})$ with $\|f\|_p \leq 1$ and every $s \in [1/2, 1]$, using the standard inequality $(x + y)^q \leq 2^q (x^q + y^q)$ for arbitrary positive numbers x, y, q, we have

$$
\begin{aligned}
\mathcal{J}(R, s) &= \frac{q}{2\pi} \int_{|z| > R} |f(\varphi(sz)) - f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z) \\
&\leq 2^q \frac{q}{2\pi} \int_{|z| > R} \left(|f(\varphi(sz))|^q + |f(\varphi(z))|^q \right) e^{-\frac{q|z|^2}{2}} dA(z) \\
&\leq 2^q \frac{q}{2\pi} \int_{|z| > R} |f(\varphi(sz))|^q e^{-\frac{q|sz|^2}{2}} dA(z) \\
&\quad + 2^q \frac{q}{2\pi} \int_{|z| > R} |f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z) \\
&\leq 2^q \frac{q}{2\pi s^2} \int_{|z| > sR} |f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z) \\
&\quad + 2^q \frac{q}{2\pi} \int_{|z| > R} |f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z) \\
&\leq D_q \frac{q}{2\pi} \int_{|z| > R/2} |f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z),
\end{aligned}
$$

where $D_q := 5.2^q$.

We consider the following possibilities.

If $0 < p \le q < \infty$ then for every function $f \in \mathcal{F}^p(\mathbb{C})$ with $\|f\|_p \le 1$, by Lemma 2.2,

$$
\begin{aligned}
\mathcal{J}(R,s) &\le D_q \frac{q}{2\pi} \sup_{|z|>R/2} m_z(1,\varphi)^q \int_{|z|>R/2} |f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} \, dA(z) \\
&\le D_q \frac{\|f\|_q^q}{|a|^2} \sup_{|z|>R/2} m_z(1,\varphi)^q \\
&\le D_q \frac{q}{p|a|^2} \|f\|_p^q \sup_{|z|>R/2} m_z(1,\varphi)^q \\
&\le D_q \frac{q}{p|a|^2} \sup_{|z|>R/2} m_z(1,\varphi)^q.
\end{aligned}
$$

Consequently, for every $s \in [1/2, 1]$,

$$
\begin{aligned}
\|C_{\varphi_s} - C_{\varphi_1}\|^q &\le \sup_{\|f\|_p \le 1} (\mathcal{I}(R,s) + \mathcal{J}(R,s)) \\
&\le M_R^q \|C_\varphi\|^q |1-s|^q + D_q \frac{q}{p|a|^2} \sup_{|z|>R/2} m_z(1,\varphi)^q.
\end{aligned}
$$

Letting $s \to 1^-$, we obtain

$$
\limsup_{s \to 1^-} \|C_{\varphi_s} - C_{\varphi_1}\|^q \le D_q \frac{q}{p|a|^2} \sup_{|z|>R/2} m_z(1,\varphi)^q.
$$

Since $|a| < 1$, $\lim_{|z| \to \infty} m_z(1,\varphi) = 0$. Then, letting $R \to \infty$, we get

$$
\limsup_{s \to 1^-} \|C_{\varphi_s} - C_{\varphi_1}\|^q \le D_q \frac{q}{p|a|^2} \lim_{R \to \infty} \sup_{|z|>R/2} m_z(1,\varphi)^q = 0.
$$

If $0 < q < p < \infty$ then arguing as in Theorem 3.3 and using Hölder inequality, we have that, for every $f \in \mathcal{F}^p(\mathbb{C})$ with $\|f\|_p \le 1$,

$$
\begin{aligned}
\mathcal{J}(R,s) &\le D_q \frac{q}{2\pi} \int_{|z|>R/2} |f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} \, dA(z) \\
&= D_q \frac{q}{2\pi} \int_{|z|>R/2} m_z(1,\varphi)^q |f(\varphi(z))|^q e^{-\frac{q|\varphi(z)|^2}{2}} \, dA(z) \\
&\le D_q \frac{q}{2\pi} \left(\int_{|z|>R/2} |f(\varphi(z))|^p e^{-\frac{p|\varphi(z)|^2}{2}} \, dA(z) \right)^{\frac{q}{p}} \\
&\quad \times \left(\int_{|z|>R/2} m_z^{\frac{pq}{p-q}}(1,\varphi) \, dA(z) \right)^{\frac{p-q}{p}} \\
&\le D_q \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}} \|f\|_p^q \left(\int_{|z|>R/2} m_z^{\frac{pq}{p-q}}(1,\varphi) \, dA(z) \right)^{\frac{p-q}{p}} \\
&\le D_q \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}} \left(\int_{|z|>R/2} m_z^{\frac{pq}{p-q}}(1,\varphi) \, dA(z) \right)^{\frac{p-q}{p}}.
\end{aligned}
$$

Consequently, for every $s \in [1/2, 1]$,

$$\|C_{\varphi_s} - C_{\varphi_1}\|^q \le \sup_{\|f\|_p \le 1} (\mathcal{I}(R, s) + \mathcal{J}(R, s))$$

$$\le M_R^q \|C_\varphi\|^q |1 - s|^q + D_q \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}}$$

$$\times \left(\int_{|z|>R/2} m_z^{\frac{pq}{p-q}} (1, \varphi) dA(z) \right)^{\frac{p-q}{p}}.$$

Letting $s \to 1^-$ in the last inequality, we obtain

$$\limsup_{s \to 1^-} \|C_{\varphi_s} - C_{\varphi_1}\|^q \le D_q \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}}$$

$$\times \left(\int_{|z|>R/2} m_z^{\frac{pq}{p-q}} (1, \varphi) dA(z) \right)^{\frac{p-q}{p}}.$$

Since C_φ is compact from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$, by Theorem 3.3, $m_z(1, \varphi) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$. Then, letting $R \to \infty$, we get

$$\limsup_{s \to 1^-} \|C_{\varphi_s} - C_{\varphi_1}\|^q \le D_q \frac{q}{2\pi} \left(\frac{2\pi}{p|a|^2} \right)^{\frac{q}{p}}$$

$$\times \lim_{R \to \infty} \left(\int_{|z|>R/2} m_z^{\frac{pq}{p-q}} (1, \varphi) dA(z) \right)^{\frac{p-q}{p}} = 0.$$

The proof is completed.   □

Theorem 4.4 *For every $p, q \in (0, \infty)$, the set of all compact composition operators from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$ is path connected in the space $\mathcal{C}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$.*

Proof Let C_{φ_1} and C_{φ_2} be two compact composition operators from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$. By Proposition 4.3, C_{φ_1} and $C_{\varphi_1(0)}$ belong to the same path connected component and so do C_{φ_2} and $C_{\varphi_2(0)}$. We show that $C_{\varphi_1(0)}$ and $C_{\varphi_2(0)}$ are in the same path connected component.

For each $s \in [0, 1]$, put

$$\beta_s := (1 - s)\varphi_1(0) + s\varphi_2(0),$$

and

$$B_1 := \varphi_2(0) - \varphi_1(0), \quad B_2 := |\varphi_1(0)| + |\varphi_2(0)|.$$

Obviously, $|\beta_s| \le |B_2|$ for every $s \in [0, 1]$. Moreover, $C_{\varphi_1(0)} = C_{\beta_0}$ and $C_{\varphi_2(0)} = C_{\beta_1}$ and the composition operators C_{β_s} are compact from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$ for all $s \in [0, 1]$.

We now prove that the map

$$[0, 1] \to \mathcal{C}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})), \ s \mapsto C_{\beta_s}$$

is continuous. Fix $s_0 \in [0, 1]$. For every $s \in [0, 1]$ and every $f \in \mathcal{F}^p(\mathbb{C})$ with $\|f\|_p \leq 1$, using Lemma 2.1[(ii)], we have

$$
\begin{aligned}
\|C_{\beta_s} f - C_{\beta_{s_0}} f\|_q^q &= \frac{q}{2\pi} |f(\beta_s) - f(\beta_{s_0})|^q \int_{\mathbb{C}} e^{-\frac{q|z|^2}{2}} dA(z) \\
&= \left| \int_{s_0}^s \left(B_1 f'(\beta_t) \right) dt \right|^q \\
&\leq |s - s_0|^q \max_{t \in <s_0, s>} \left| B_1 f'(\beta_t) \right|^q \\
&\leq |s - s_0|^q |B_1|^q \max_{t \in <s_0, s>} |f'(\beta_t)|^q \\
&\leq |s - s_0|^q |B_1|^q \left(\|f\|_p e^2 (1 + B_2) e^{\frac{B_2^2}{2}} \right)^q \\
&\leq M^q |s - s_0|^q,
\end{aligned}
$$

where, as above, $<s_0, s>$ is the closed interval connecting s_0 and s, and

$$
M := e^2 |B_1| (1 + B_2) e^{\frac{B_2^2}{2}}.
$$

From this it follows that, for every $s \in [0, 1]$,

$$
\|C_{\beta_s} - C_{\beta_{s_0}}\| = \sup_{\|f\|_p \leq 1} \|C_{\beta_s} f - C_{\beta_{s_0}} f\|_q \leq M|s - s_0|.
$$

It implies that

$$
\lim_{s \to s_0} \|C_{\beta_s} - C_{\beta_{s_0}}\| = 0.
$$

Consequently, C_{φ_1} and C_{φ_2} are in the same path connected component of $C(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ via a path belonging to the set of all compact composition operators from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$, which completes the proof. □

From Theorem 4.4 and Corollary 3.6 we get

Corollary 4.5 *If $0 < q < p < \infty$, then the space $C(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ is path connected.*

Next for $0 < p \leq q < \infty$ we give the following result about the characterization of isolated composition operators C_φ in the space $\mathcal{C}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$.

Theorem 4.6 *Let $0 < p \leq q < \infty$ and C_φ be a bounded composition operator from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$ induced by $\varphi(z) = az + b$ with $|a| \leq 1$. The following assertions are equivalent:*

(i)   *C_φ is isolated in $\mathcal{C}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$;*
(ii) *C_φ is non-compact, that is, $|a| = 1$ and $b = 0$;*
(iii) *$\|C_\varphi - C_\phi\| \geq 1$ for all affine functions $\phi(z) = cz + d \neq \varphi(z)$ such that $C_\phi \in \mathcal{C}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$.*

Proof (i) $\implies$ (ii). By Theorem 4.4, if C_φ is an isolated composition operator in $\mathcal{C}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$, then C_φ must be non-compact. Hence, by Corollary 3.5, $|a| = 1$ and $b = 0$.

(ii) $\Longrightarrow$ (iii). Assume that $|a| = 1$ and $b = 0$. In this case, for every affine function $\phi(z) = cz + d \neq \varphi(z)$ such that $C_\phi \in \mathcal{C}\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$, by Lemma 2.1[(i)], we have that, for all $w, z \in \mathbb{C}$,

$$\|C_\varphi k_w - C_\phi k_w\|_q \geq \left|C_\varphi k_w(z) - C_\phi k_w(z)\right| e^{-\frac{|z|^2}{2}}$$
$$= \left|e^{\overline{w}\varphi(z) - \frac{|w|^2 + |z|^2}{2}} - e^{\overline{w}\phi(z) - \frac{|w|^2 + |z|^2}{2}}\right|.$$

In particular, with $w = \varphi(z)$, the last inequality gives

$$\|C_\varphi k_{\varphi(z)} - C_\phi k_{\varphi(z)}\|_q \geq \left|1 - \left|e^{\overline{az}(cz+d) - |z|^2}\right|\right|$$
$$= \left|1 - e^{\mathrm{Re}\,(\overline{a}c - 1)|z|^2 + \mathrm{Re}\,(d\overline{az})}\right|, \forall z \in \mathbb{C}.$$

Since $C_\phi \in \mathcal{C}\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$ and $\varphi(z) \neq \phi(z)$, then $|c| \leq 1$ and $c \neq a$. Hence, $\mathrm{Re}\,(\overline{a}c) < 1$. From this it follows that

$$\|C_\varphi - C_\phi\| \geq \limsup_{|z| \to \infty} \|C_\varphi k_{\varphi(z)} - C_\phi k_{\varphi(z)}\|_q$$
$$\geq \limsup_{|z| \to \infty} \left|1 - e^{\mathrm{Re}\,(\overline{a}c - 1)|z|^2 + \mathrm{Re}\,(d\overline{az})}\right| = 1.$$

(iii) $\Longrightarrow$ (i) is obvious. $\qquad\square$

From Theorems 4.4 and 4.6, we immediately get the following result.

Corollary 4.7 *Let $0 < p \leq q < \infty$. The connected component and path connected component in $\mathcal{C}\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$ are the same and they are only the set of all compact composition operators from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$.*

4.3 The Space $\mathcal{C}_w\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$

In this subsection using the results in Subsection 4.2 we obtain a complete characterization of the component structure of $\mathcal{C}_w\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$.

For $p, q \in (0, \infty)$ and $\varphi(z) = az + b$ with $|a| \leq 1$ we denote by $\mathcal{F}(\varphi, p, q)$ the set of all nonzero functions $\psi \in \mathcal{F}^q(\mathbb{C})$ such that $W_{\psi,\varphi} : \mathcal{F}^p(\mathbb{C}) \to \mathcal{F}^q(\mathbb{C})$ is bounded. Then

$$\mathcal{C}_w\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big) = \{W_{\psi,\varphi} : \psi \in \mathcal{F}(\varphi, p, q), \varphi(z) = az + b, |a| \leq 1\}.$$

Lemma 4.8 *Let $p, q \in (0, \infty)$, $\varphi(z) = az + b$ with $|a| \leq 1$ and $\psi_1, \psi_2 \in \mathcal{F}(\varphi, p, q)$. Then $W_{\psi_1,\varphi}$ and $W_{\psi_2,\varphi}$ are in the same path connected component of $\mathcal{C}_w\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$.*

Proof We can easily show that there is a complex valued continuous function $\alpha(t)$ on $[0, 1]$ such that $\alpha(0) = 0, \alpha(1) = 1$ and $u_t := (1 - \alpha(t))\psi_1 + \alpha(t)\psi_2$ are all nonzero functions in $\mathcal{F}(\varphi, p, q)$ for all $t \in [0, 1]$.

Indeed, if $\psi_2(z) = \lambda\psi_1(z)$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and all $z \in \mathbb{C}$, we can take any continuous function $\alpha(t)$ so that $\alpha(t) \neq 1/(1 - \lambda)$ for all $t \in [0, 1]$. Otherwise, we put $\alpha(t) = t$. Moreover, for each $t \in [0, 1]$, $W_{u_t,\varphi} = (1 - \alpha(t))W_{\psi_1,\varphi} + \alpha(t)W_{\psi_2,\varphi}$, and hence, $u_t \in \mathcal{F}(\varphi, p, q)$.

Obviously,

$$W_{\psi_1,\varphi} = W_{u_0,\varphi}, W_{\psi_2,\varphi} = W_{u_1,\varphi},$$

and, for every $s, t \in [0, 1]$ and $f \in \mathcal{F}^p(\mathbb{C})$,

$$W_{u_s, \varphi} f - W_{u_t, \varphi} f = (u_s - u_t) f \circ \varphi = (\alpha(s) - \alpha(t)) W_{\psi_2 - \psi_1, \varphi} f.$$

From this it follows that, for every $t \in [0, 1]$,

$$\lim_{s \to t} \| W_{u_s, \varphi} - W_{u_t, \varphi} \| = 0.$$

This means that the map

$$[0, 1] \to \mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})), \ t \mapsto W_{u_t, \varphi},$$

is continuous. The proof is completed. □

Let

$$\mathcal{S}_0 := \{\varphi(z) = az + b : |a| < 1\} \text{ and } \mathcal{S}_1 := \{\varphi(z) = az + b : |a| = 1\},$$

and

$$\mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})) := \{W_{\psi, \varphi} : \varphi \in \mathcal{S}_0, \psi \in \mathcal{F}(\varphi, p, q)\}.$$

Theorem 4.9 *Let $p, q \in (0, \infty)$ be given.*

(a) *If $0 < q < p < \infty$, then the space $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ is path connected.*
(b) *If $0 < p \leq q < \infty$, then the space $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ has the following path connected components*

$$\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})) = \mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})) \bigcup \bigcup_{\varphi \in \mathcal{S}_1} \{W_{\psi, \varphi} : \psi \in \mathcal{F}(\varphi, p, q)\}.$$

Proof (a) First we note that for every $W_{\psi, \varphi} \in \mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$, by Proposition 3.1 and Theorem 3.3,

$$m(\psi, \varphi) < \infty \text{ and } m_z(\psi, \varphi) \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA).$$

It implies that $\varphi(z) = az + b$ with $|a| < 1$. Indeed, if $|a| = 1$, then $\psi(z) = \psi(0)e^{-\bar{b}az}$ and $m_z(\psi, \varphi) = |\psi(0)|e^{\frac{|b|^2}{2}}$ for all $z \in \mathbb{C}$ which is impossible.

Then, by Corollary 3.6, $W_{1, \varphi} = C_\varphi$ also belongs to $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$. Hence, by Lemma 4.8, $W_{\psi, \varphi}$ and C_φ are in the same path connected component of $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$.

From this and Corollary 4.5 it follows that $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ is path connected.

(b) We will prove that all sets $\mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ and $\{W_{\psi, \varphi} : \psi \in \mathcal{F}(\varphi, p, q)\}$ with $\varphi \in \mathcal{S}_1$ are path connected and closed in the space $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$. Also, clearly, they are disjoint. Hence, all these sets are path connected components in $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$.

First we fix an arbitrary pair of weighted composition operators W_{ψ_1, φ_1} and W_{ψ_2, φ_2} in $\mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$. Then, by Corollary 3.5, $W_{1, \varphi_1} = C_{\varphi_1}$ and $W_{1, \varphi_2} = C_{\varphi_2}$ are compact from $\mathcal{F}^p(\mathbb{C})$ into $\mathcal{F}^q(\mathbb{C})$. Hence, by Lemma 4.8, W_{ψ_1, φ_1} and C_{φ_1} belong to the same path connected component of $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ via a path in $\mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$, and so do W_{ψ_2, φ_2} and C_{φ_2}. From this and Theorem 4.4 it follows that W_{ψ_1, φ_1} and W_{ψ_2, φ_2} belong to the same path connected component of $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ via a path in $\mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$. This means that the set $\mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ is path connected in the space $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$.

Now we show that the set $\mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ is also closed in the space $\mathcal{C}_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$. Let $(W_{\psi_n, \varphi_n})_n$ be an arbitrary sequence in $\mathcal{C}_{w,0}(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ con-

verging to $W_{\psi,\varphi}$ in $\mathcal{C}_w\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$. Suppose that $\varphi(z) = az + b$ with $|a| \leq 1$ and $\varphi_n(z) = a_n z + b_n$ with $|a_n| < 1$ for all $n \in \mathbb{N}$. By contradiction, assume that $|a| = 1$. Then

$$\psi(z) = \psi(0)e^{-\bar{b}az} \text{ and } m_z(\psi, \varphi) = |\psi(0)|e^{\frac{|b|^2}{2}}, \ \forall z \in \mathbb{C}.$$

Similarly to Eq. (4.1), for every $z \in \mathbb{C}$ and $n \in \mathbb{N}$, we have

$$\|W_{\psi,\varphi} - W_{\psi_n,\varphi_n}\| \geq \|W_{\psi,\varphi}k_{\varphi(z)} - W_{\psi_n,\varphi_n}k_{\varphi(z)}\|_q$$

$$\geq \left| m_z(\psi, \varphi) - m_z(\psi_n, \varphi_n)e^{-\frac{|\varphi(z)-\varphi_n(z)|^2}{2}} \right|$$

$$\geq |\psi(0)|e^{\frac{|b|^2}{2}} - m(\psi_n, \varphi_n)e^{-\frac{|\varphi(z)-\varphi_n(z)|^2}{2}}.$$

Since $a \neq a_n$, $\varphi(z) - \varphi_n(z) \to \infty$ as $z \to \infty$. Hence, letting $z \to \infty$ in the last inequality we get that for every $n \in \mathbb{N}$,

$$\|W_{\psi,\varphi} - W_{\psi_n,\varphi_n}\| \geq |\psi(0)|e^{\frac{|b|^2}{2}} > 0,$$

which is a contradiction. Thus, $|a| < 1$, that is, $W_{\psi,\varphi}$ belongs to $\mathcal{C}_{w,0}\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$.

Next for each $\varphi(z) = az + b \in \mathcal{S}_1$, $\psi \in \mathcal{F}(\varphi, p, q)$ if and only if $\psi(z) = \psi(0)e^{-\bar{b}az}$. Then

$$\{W_{\psi,\varphi} : \psi \in \mathcal{F}(\varphi, p, q)\} = \{W_{c\psi_0,\varphi} : c \in \mathbb{C} \setminus \{0\}\} \text{ with } \psi_0(z) := e^{-\bar{b}az}.$$

Thus, for each $\varphi \in \mathcal{S}_1$, the set $\{W_{\psi,\varphi} : \psi \in \mathcal{F}(\varphi, p, q)\}$ is path connected and closed in $\mathcal{C}_w\big(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C})\big)$.

The proof is completed. □

Finally, it should be noted that in the space $C_w(\mathcal{F}^p(\mathbb{C}), \mathcal{F}^q(\mathbb{C}))$ there does not exist an isolated weighted composition operator. Indeed, for every weighted composition operator $W_{\psi,\varphi}$,

$$\lim_{s \to 1} \|W_{s\psi,\varphi} - W_{\psi,\varphi}\| = \lim_{s \to 1} |s - 1| \|W_{\psi,\varphi}\| = 0.$$

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