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The Discrete Laplacian of a 2-Simplicial Complex

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Abstract In this paper, we introduce the notion of oriented faces especially triangles in a connected oriented locally finite graph. This framework then permits to define the Laplace operator on this structure of the 2-simplicial complex. We develop the notion of χ-completeness for the graphs, based on the cut-off functions. Moreover, we study essential self-adjointness of the discrete Laplacian from the χ-completeness geometric hypothesis.

Keywords Infinite graph · Difference operator · Laplacian on forms · Essential self-adjointness

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1 Introduction

The impact of the geometry on the essential self-adjointness of the Laplacians is studied in many areas of mathematics on Riemannian manifolds; see [4, 7, 9, 13] and also on one-dimensional simplicial complexes; see [1, 5, 8, 12, 14, 19]. Indeed, Laplacians on Riemannian manifolds and simplicial complexes share a lot of common elements. Despite of this, various geometric notions such as distance and completeness in the Riemannian framework have no immediate analog in the discrete setting. Combinatorial Laplacians

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were originally studied on graphs, beginning with *Kirchhoff* and his study of electrical networks [10]. Simplicial complexes can be viewed as generalizations of graphs, as from any graph, we can form its clique complex, a 2-simplicial complex whose faces correspond to the cliques of the graph. In this article, we take a connected oriented locally finite graph and we introduce the oriented faces especially triangles in such a way that every face is a triangle, so we can regard it as a two-dimensional simplicial complex. This work presents a more general framework for the Laplacians defined in terms of the combinatorial structure of a simplicial complex. The main result of this work gives a geometric hypothesis to ensure essential self-adjointness for the discrete Laplacian. We develop the χ-completeness hypothesis for triangulations. This hypothesis on locally finite graphs covers many situations that have been already studied in [1]. The authors prove that the χ-completeness is satisfied by graphs which are complete for some intrinsic metric, as defined in [8] and [12].

The paper is structured as follows: In the second section, we will first present the basic concepts about graphs or rather one-dimensional simplicial complexes. Next, we introduce the notion of oriented faces more particularly triangles where all the faces are triangles. This special structure of 2-simplicial complex is called triangulation. Without loss of generality, we can assume that every triangle is a face for simplicity sake. So this permits to define the Gauß-Bonnet operator $T = d + \delta$ acting on triplets of functions, 1-forms and 2-forms. After that, we define the discrete Laplacian by $L := T^2$ which admits a decomposition according to the degree

$$L := L_0 \oplus L_1 \oplus L_2.$$

In the third and fourth sections, we study the closability of the operators which are used in the following sections. Next, we get started with refer to [1] for the notion χ-completness of the graphs and we develop this geometric hypothesis for the triangulations in Definition 4.2. Moreover, we have developed it through optimal example of the "*tiangular tree*" to produce a concrete way to prove a triangulation which is not χ-complete, based on the offspring function, we refer here to [3] for this notion.

In the fifth section, we address the main results concerning essential self-adjointness for T and L. In the case of complete manifolds, there is a result of *Chernoff*; see [4], and we also have for the discrete setting; see [1], which conclude that the Dirac operator is essentially self-adjoint. As a result, they prove essential self-adjointness of the Laplace-Beltrami operator. So, we take this idea to make the relationship between T and L about the essential self-adjointness, when the triangulation is χ-complete.

In the final section, we present a particular example of a triangulation where we study the χ-completeness hypothesis. Moreover, we show that L_1 and L_2 is not necessarily essentially self-adjoint on the simple case.

We can extend the results in this paper to more general 2-simplicial complex, where the oriented faces are not necessarily triangles. Particularly we can give a more general expression of the operator d^1. More precisely, one can take 2-simplicial complexes with the number of edges of an oriented face bounded. Indeed this hypothesis is important to give a meaning of the inequality in Definition 4.2.

2 Preliminaries

2.1 The Basic Concepts

A graph $\mathcal{K}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the countable set of vertices and $\mathcal{E}$ the set of oriented edges, considered as a subset of $\mathcal{V} \times \mathcal{V}$. When two vertices x and y are connected by an edge

e, we say they are neighbors. We denote $x \sim y$ and $e = [x, y] \in \mathcal{E}$. We assume that $\mathcal{E}$ is symmetric, i.e. $[x, y] \in \mathcal{E} \Rightarrow [y, x] \in \mathcal{E}$. An oriented graph $\mathcal{K}$ is given by a partition of $\mathcal{E}$:

$$\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$$

$$(x, y) \in \mathcal{E}^- \Leftrightarrow (y, x) \in \mathcal{E}^+$$

In this case for $e = (x, y) \in \mathcal{E}^-$, we define the origin $e^- = x$, the termination $e^+ = y$ and the opposite edge $-e = (y, x) \in \mathcal{E}^+$. Let $c : \mathcal{V} \to (0, \infty)$ the weight on the vertices. We also have $r : \mathcal{E} \longrightarrow (0, \infty)$ the weight on the oriented edges with

$$\forall e \in \mathcal{E},\ r(-e) = r(e).$$

A *path* between two vertices $x, y \in \mathcal{V}$ is a finite set of oriented edges $e_1, ..., e_n, n \geq 1$ such that

$$e_1^- = x,\ e_n^+ = y \text{ and, if } n \geq 2,\ \forall j,\ 1 \leq j \leq n-1 \Longrightarrow e_j^+ = e_{j+1}^-.$$

The path is called a *cycle* or *closed* when the origin and the end are identical, *ie.* $e_1^- = e_n^+$, with $n \geq 3$. If no cycles appear more than once in a path, the path is called a *simple path*. The graph $\mathcal{K}$ is *connected* if any two vertices x and y can be connected by a path with $e_1^- = x$ and $e_n^+ = y$. We say that the graph $\mathcal{K}$ is *locally finite* if each vertex belongs to a finite number of edges. The graph $\mathcal{K}$ is *without loops* if there is not the type of edges (x, x), i.e.

$$\forall x \in \mathcal{V} \Longrightarrow (x, x) \notin \mathcal{E}.$$

2.1.1 *The Set of neighbors of $x \in \mathcal{V}$ is Denoted by*

$$\mathcal{V}(x) := \{y \in \mathcal{V} : y \sim x\}.$$

2.1.2 *The Degree of $x \in \mathcal{V}$ is by Definition $deg(x)$, the Number of Neighbors of x*

2.1.3 *The Combinatorial Distance d_{comb} on $\mathcal{K}$ is*

$$d_{comb}(x, y) = \min\{n,\ \{e_i\}_{1 \leq i \leq n} \subseteq \mathcal{E} \text{ a path between the two vertices } x \text{ and } y\}.$$

2.1.4 *Let B be a Finite Subset of $\mathcal{V}$. We Define the Edge Boundary*

$\partial_{\mathcal{E}} B$ of B by

$$\partial_{\mathcal{E}} B := \{e \in \mathcal{E} \text{ such that } \{e^-, e^+\} \cap B \neq \emptyset \text{ and } \{e^-, e^+\} \cap B^c \neq \emptyset\}.$$

In the sequel, we assume that
$\mathcal{K}$ **is without loops, connected, locally finite and oriented**

Definition 2.1 An oriented face of $\mathcal{K}$ is a surface limited by a simple closed path, considered as an element of $\mathcal{E}^n$ with $n \geqslant 3$, i.e

$$\varpi \text{ an oriented face } \Rightarrow \exists n \geq 3,\ \varpi = (e_1, e_2, ..., e_n) \in \mathcal{E}^n \text{ such that } \{e_i\}_{1 \leq i \leq n} \subseteq \mathcal{E} \text{ is a simple closed path.}$$

Let $\mathcal{F}$ be the set of all oriented faces of $\mathcal{K}$, we consider the pair $(\mathcal{K}, \mathcal{F})$ as a 2-simplicial complex, we denote it by $\mathcal{T}$. We can denote also $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$.

Remark 2.2 Care should be taken not to confuse the simple cycles and the oriented faces. Indeed, one can have simple cycles that are not oriented faces.

For a face $\varpi = (e_1, e_2, ..., e_n) \in \mathcal{F}$, we have

$$\varpi = (e_i, ..., e_n, e_1, ..., e_{i-1}) \in \mathcal{F},\ \forall 3 \leq i \leq n-1.$$

We can denote also

$$\varpi = (e_2, e_3, ..., e_n, e_1) = ... = (e_n, e_1, e_2, ..., e_{n-1}) \in \mathcal{F}.$$

Because $\mathcal{K}$ is an oriented graph, we demand

$$(e_1, e_2, ..., e_n) \in \mathcal{F} \Rightarrow (-e_n, -e_{n-1}, ..., -e_2, -e_1) \in \mathcal{F}.$$

Given $\varpi = (e_1, e_2, ..., e_n) \in \mathcal{F}$, the *opposite face* of ϖ is denoted by

$$-\varpi = (-e_n, -e_{n-1}, ..., -e_2, -e_1) \in \mathcal{F}.$$

Let B be a finite subset of $\mathcal{V}$. We define the *face boundary* $\partial_{\mathcal{F}} B$ of B by

$$\partial_{\mathcal{F}} B := \{\sigma = (e_1, e_2, ..., e_n) \in \mathcal{F},\ \exists i \text{ such that } e_i \in \partial_{\mathcal{E}} B,\ n \geqslant 3\}.$$

Definition 2.3 *(Triangulation) A triangulation* is a 2-simplicial complex such that all the faces are triangles.

Remark 2.4 In the definition of a triagulation we demand that faces are triangles. In the sequel, we assume also that each triangular cycle is an oriented face for simplicity reasons. Indeed all the results of this work can be extended easily to any triangulation.

In the sequel we will represent the oriented faces by their vertices

$$\varpi = (e_1, e_2, e_3) = [e_1^- = e_3^+, e_1^+ = e_2^-, e_2^+ = e_3^-] \in \mathcal{F}.$$

For a face $\varpi = [x, y, z] \in \mathcal{F}$. Let us set

$$\varpi = [x, y, z] = [y, z, x] = [z, x, y] \in \mathcal{F} \Rightarrow -\varpi = [y, x, z] = [x, z, y] = [z, y, x] \in \mathcal{F}.$$

To define weighted triangulations we need weights, let us give $s : \mathcal{F} \to (0, \infty)$ the weight on oriented faces such that for all $\varpi \in \mathcal{F},\ s(-\varpi) = s(\varpi)$. The weighted triangulation $(\mathcal{T}, c, r, s)$ is given by the triangulation $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$. We say that $\mathcal{T}$ is *simple* if the weights of the vertices, the edges and faces equals 1. For an edge $e \in \mathcal{E}$, we also denote the oriented face $[e^-, e^+, x]$ by (e, x), with $x \in \mathcal{V}(e^-) \cap \mathcal{V}(e^+)$. The set of vertices belonging to the edge $e \in \mathcal{E}$ is given by

$$\mathcal{F}_e := \{x \in \mathcal{V},\ (e, x) \in \mathcal{F}\} = \mathcal{V}(e^-) \cap \mathcal{V}(e^+).$$

2.2 Functions Spaces

We denote the set of 0-cochains or functions on $\mathcal{V}$ by:

$$\mathcal{C}(\mathcal{V}) = \{f : \mathcal{V} \to \mathbb{C}\}$$

and the set of functions of finite support by $\mathcal{C}_c(\mathcal{V})$.

Similarly, we denote the set of 1-cochains or 1-forms on $\mathcal{E}$ by:

$$\mathcal{C}(\mathcal{E}) = \{\varphi : \mathcal{E} \to \mathbb{C},\ \varphi(-e) = -\varphi(e)\}$$

and the set of 1-forms of finite support by $\mathcal{C}_c(\mathcal{E})$.

Moreover, we denote the set of 2-cochains or 2-forms on $\mathcal{F}$ by:

$$\mathcal{C}(\mathcal{F}) = \{\phi : \mathcal{F} \to \mathbb{C},\ \phi(-\varpi) = -\phi(\varpi)\}$$

and the set of 2-forms of finite support by $\mathcal{C}_c(\mathcal{F})$.

Let us define the Hilbert spaces $l^2(\mathcal{V}),\ l^2(\mathcal{E})$ and $l^2(\mathcal{F})$ as the sets of cochains with finite norm, we have

(a)

$$l^2(\mathcal{V}) := \{f \in \mathcal{C}(\mathcal{V});\ \sum_{x \in \mathcal{V}} c(x)|f(x)|^2 < \infty\},$$

with the inner product

$$\langle f, g\rangle_{l^2(\mathcal{V})} := \sum_{x \in \mathcal{V}} c(x) f(x) \overline{g}(x).$$

(b)

$$l^2(\mathcal{E}) := \{\varphi \in \mathcal{C}(\mathcal{E});\ \sum_{e \in \mathcal{E}} r(e)|\varphi(e)|^2 < \infty\},$$

with the inner product

$$\langle \varphi, \psi\rangle_{l^2(\mathcal{E})} := \frac{1}{2} \sum_{e \in \mathcal{E}} r(e) \varphi(e) \overline{\psi}(e).$$

(c)

$$l^2(\mathcal{F}) := \{\phi \in \mathcal{C}(\mathcal{F});\ \sum_{\varpi \in \mathcal{F}} s(\varpi)|\phi(\varpi)|^2 < \infty\},$$

with the inner product

$$\langle \phi_1, \phi_2\rangle_{l^2(\mathcal{F})} = \frac{1}{6} \sum_{[x,y,z] \in \mathcal{F}} s(x, y, z)\phi_1(x, y, z)\overline{\phi_2}(x, y, z).$$

The direct sum of the spaces $l^2(\mathcal{V})$, $l^2(\mathcal{E})$ and $l^2(\mathcal{F})$ can be considered as a new Hilbert space denoted by $\mathcal{H}$, that is

$$\mathcal{H} = l^2(\mathcal{V}) \oplus l^2(\mathcal{E}) \oplus l^2(\mathcal{F}),$$

with the norm

$$\forall F = (f, \varphi, \phi) \in \mathcal{H},\ \|F\|^2_{\mathcal{H}} = \|f\|^2_{l^2(\mathcal{V})} + \|\varphi\|^2_{l^2(\mathcal{E})} + \|\phi\|^2_{l^2(\mathcal{F})}.$$

2.3 Operators

We give in this part the expressions of the operators introduced on graphs which are already well known and we also give other operators acting on triangulations.

2.3.1 The Difference Operator

By analogy to electric networks of voltage differences across edges leading to currents [17], we define *the difference operator* $d^0 : \mathcal{C}_c(\mathcal{V}) \longrightarrow \mathcal{C}_c(\mathcal{E})$ by

$$\forall f \in \mathcal{C}_c(\mathcal{V}),\ d^0(f)(e) = f(e^+) - f(e^-).$$

2.3.2 The Co-Boundary Operator

It is the formal adjoint of d^0, denoted $\delta^0 : \mathcal{C}_c(\mathcal{E}) \longrightarrow \mathcal{C}_c(\mathcal{V})$, (see [1]) acts as

$$\forall \varphi \in \mathcal{C}_c(\mathcal{E}),\ \delta^0(\varphi)(x) = \frac{1}{c(x)} \sum_{e, e^+ = x} r(e)\varphi(e).$$

2.3.3 The Exterior Derivative

It is the operator $d^1 : \mathcal{C}_c(\mathcal{E}) \longrightarrow \mathcal{C}_c(\mathcal{F})$, given by

$$\forall \psi \in \mathcal{C}_c(\mathcal{E}),\ d^1(\psi)(x, y, z) = \psi(x, y) + \psi(y, z) + \psi(z, x).$$

2.3.4 The Co-Exterior Derivative

It is the formal adjoint of d^1, denoted $\delta^1 : \mathcal{C}_c(\mathcal{F}) \longrightarrow \mathcal{C}_c(\mathcal{E})$, which satisfies

$$\langle d^1\psi, \phi\rangle_{l^2(\mathcal{F})} = \langle \psi, \delta^1\phi\rangle_{l^2(\mathcal{E})},\ \forall(\psi, \phi) \in \mathcal{C}_c(\mathcal{E}) \times \mathcal{C}_c(\mathcal{F}). \tag{2.1}$$

Lemma 2.5 *The formal adjoint $\delta^1 : \mathcal{C}_c(\mathcal{F}) \longrightarrow \mathcal{C}_c(\mathcal{E})$, is given by*

$$\delta^1(\phi)(e) = \frac{1}{r(e)} \sum_{x \in \mathcal{F}_e} s(e, x)\phi(e, x).$$

Proof Let $(\psi, \phi) \in \mathcal{C}_c(\mathcal{E}) \times \mathcal{C}_c(\mathcal{F})$. The Eq. 2.1 gives

$$\begin{aligned}\langle d^1\psi, \phi\rangle_{l^2(\mathcal{F})} &= \frac{1}{6} \sum_{[x,y,z] \in \mathcal{F}} s(x, y, z) d^1(\psi)(x, y, z)\overline{\phi}(x, y, z)\\ &= \frac{1}{2} \sum_{[x,y,z] \in \mathcal{F}} s(x, y, z)\psi(x, y)\overline{\phi}(x, y, z)\\ &= \langle \psi, \delta^1\phi\rangle_{l^2(\mathcal{E})}.\end{aligned}$$

To justify it note that the expression of d^1 contributing to the first sum is divided into three similar parts. So it remains to show only

$$\begin{aligned}\sum_{[x,y,z] \in \mathcal{F}} s(x, y, z)\psi(x, y)\overline{\phi}(x, y, z) &= \sum_{e \in \mathcal{E}} \psi(e) \sum_{x \in \mathcal{F}_e} s(e, x)\overline{\phi}(e, x)\\ &= \sum_{e \in \mathcal{E}} r(e)\psi(e) \overline{\left(\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e} s(e, x)\phi(e, x)\right)}\end{aligned}$$

□

2.3.5 Gauß-Bonnet Operator on $\mathcal{T}$

By analogy to Riemannian geometry, we use the decomposition of the operators in [7] to define the Gauß-Bonnet operator. Let us begin by defining the operator

$$d : \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}) \circlearrowleft$$

by

$$\forall (f, \varphi, \phi) \in \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}),\ d(f, \varphi, \phi) = (0, d^0 f, d^1\varphi),$$

and δ the formal adjoint of d. Thus it satisfies

$$\langle d(f_1,\varphi_1,\phi_1),(f_2,\varphi_2,\phi_2)\rangle_{\mathcal{H}} = \langle (f_1,\varphi_1,\phi_1),\delta(f_2,\varphi_2,\phi_2)\rangle_{\mathcal{H}}, \tag{2.2}$$

for all $(f_1,\varphi_1,\phi_1),(f_2,\varphi_2,\phi_2) \in \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.

Lemma 2.6 *Let $\mathcal{T} = (\mathcal{K},\mathcal{F})$ be a triangulation. Then*

$$\delta : \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}) \circlearrowleft$$

is given by

$$\delta(f,\varphi,\phi) = (\delta^0\varphi,\delta^1\phi,0),\ \forall (f,\varphi,\phi) \in \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}).$$

Proof Let $(f_1,\varphi_1,\phi_1),(f_2,\varphi_2,\phi_2) \in \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$. Using the Eq. 2.2

$$\begin{aligned}\langle d(f_1,\varphi_1,\phi_1),(f_2,\varphi_2,\phi_2)\rangle_{\mathcal{H}} &= \langle (0,d^0f_1,d^1\varphi_1),(f_2,\varphi_2,\phi_2)\rangle_{\mathcal{H}}\\ &= \langle d^0f_1,\varphi_2\rangle_{l^2(\mathcal{E})} + \langle d^1\varphi_1,\phi_2\rangle_{l^2(\mathcal{F})}\\ &= \langle f_1,\delta^0\varphi_2\rangle_{l^2(\mathcal{V})} + \langle \varphi_1,\delta^1\phi_2\rangle_{l^2(\mathcal{E})}\\ &= \langle (f_1,\varphi_1,\phi_1),(\delta^0\varphi_2,\delta^1\phi_2,0)\rangle_{\mathcal{H}}.\end{aligned}$$

□

Definition 2.7 Let $\mathcal{T} = (\mathcal{K},\mathcal{F})$ be a triangulation, the Gauß-Bonnet operator defined as

$$T := d + \delta : \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}) \circlearrowleft$$

is given by

$$T(f,\varphi,\phi) = (\delta^0\varphi, d^0f + \delta^1\phi, d^1\varphi)$$

for all $(f,\varphi,\phi) \in \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$. Moreover, the matrix representation of T is given by $T \equiv \begin{pmatrix} 0 & \delta^0 & 0\\ d^0 & 0 & \delta^1\\ 0 & d^1 & 0\end{pmatrix}$

Lemma 2.8 *If $\mathcal{T} = (\mathcal{K},\mathcal{F})$ is a triangulation then $d^1d^0 = \delta^0\delta^1 = 0$.*

Proof Let $f \in \mathcal{C}_c(\mathcal{V})$, we have that

$$\begin{aligned} d^1(d^0f)(x,y,z) &= d^0f(x,y) + d^0f(y,z) + d^0f(z,x)\\ &= (f(y)-f(x)) + (f(z)-f(y)) + (f(x)-f(z)) = 0.\end{aligned}$$

Since $d^1d^0 = 0$ and the operator $\delta^0\delta^1$ is the formal adjoint of d^1d^0. Then $\delta^0\delta^1 = 0$. □

Before giving an important result for $f \in \mathcal{C}(\mathcal{V})$, we define the two operators $\widetilde{\ }: \mathcal{C}(\mathcal{V}) \longrightarrow \mathcal{C}(\mathcal{E})$ by $f \mapsto \widetilde{f}$ and $\approx: \mathcal{C}(\mathcal{V}) \longrightarrow \mathcal{C}(\mathcal{F})$ by $f \mapsto \widetilde{\widetilde{f}}$, where

$$\widetilde{f}(e) := \frac{1}{2}\left(f(e^+) + f(e^-)\right).$$

$$\widetilde{\widetilde{f}}(x,y,z) := \frac{1}{3}\left(\widetilde{f}(x,y) + \widetilde{f}(y,z) + \widetilde{f}(z,x)\right) = \frac{1}{3}(f(x) + f(y) + f(z)).$$

The exterior product of two 1-forms defined as $. \wedge_{disk} . : \mathcal{C}(\mathcal{E}) \times \mathcal{C}(\mathcal{E}) \longrightarrow \mathcal{C}(\mathcal{F})$, is given by:

$$\begin{aligned}(\psi \wedge_{disk} \varphi)(x,y,z) &= [\psi(z,x)+\psi(z,y)]\,\varphi(x,y)\\ &+[\psi(x,y)+\psi(x,z)]\,\varphi(y,z)\\ &+[\psi(y,z)+\psi(y,x)]\,\varphi(z,x).\end{aligned}$$

It satisfies $\psi \wedge_{disk} \varphi = -(\varphi \wedge_{disk} \psi) = -\varphi \wedge_{disk} \psi = \varphi \wedge_{disk} -\psi$, for all $\varphi, \psi \in \mathcal{C}(\mathcal{E})$.

Lemma 2.9 *(Derivation properties) Let $(f,\varphi,\phi) \in \mathcal{C}_c(\mathcal{V}) \times \mathcal{C}_c(\mathcal{E}) \times \mathcal{C}_c(\mathcal{F})$. Then*

$$d^1(\widetilde{f}\varphi)(x,y,z) = \widetilde{\widetilde{f}}(x,y,z)d^1(\varphi)(x,y,z) + \frac{1}{6}\left(d^0(f) \wedge_{disk} \varphi\right)(x,y,z). \tag{2.3}$$

$$\delta^1(\widetilde{\widetilde{f}}\phi)(e) = \widetilde{f}(e)\delta^1(\phi)(e) + \frac{1}{6r(e)}\sum_{x\in\mathcal{F}_e} s(e,x)\left[d^0(f)(e^-,x)+d^0(f)(e^+,x)\right]\phi(e,x). \tag{2.4}$$

Proof

(1) Let $(f,\varphi) \in \mathcal{C}_c(\mathcal{V}) \times \mathcal{C}_c(\mathcal{E})$, we have

$$\begin{aligned}d^1(\widetilde{f}\varphi)(x,y,z) &= \widetilde{f}(x,y)\varphi(x,y) + \widetilde{f}(y,z)\varphi(y,z) + \widetilde{f}(z,x)\varphi(z,x)\\ &= [\widetilde{f}(x,y)+\widetilde{f}(y,z)+\widetilde{f}(z,x)][\varphi(x,y)+\varphi(y,z)+\varphi(z,x)]\\ &-\left(\widetilde{f}(y,z)+\widetilde{f}(z,x)\right)\varphi(x,y) - \left(\widetilde{f}(z,x)+\widetilde{f}(x,y)\right)\varphi(y,z)\\ &-\left(\widetilde{f}(x,y)+\widetilde{f}(y,z)\right)\varphi(z,x)\\ &= \widetilde{\widetilde{f}}(x,y,z)d^1(\varphi)(x,y,z)\\ &+\left(\tfrac{2}{3}\widetilde{f}(x,y) - \tfrac{1}{3}\left[\widetilde{f}(y,z)+\widetilde{f}(z,x)\right]\right)\varphi(x,y)\\ &+\left(\tfrac{2}{3}\widetilde{f}(y,z) - \tfrac{1}{3}\left[\widetilde{f}(z,x)+\widetilde{f}(x,y)\right]\right)\varphi(y,z)\\ &+\left(\tfrac{2}{3}\widetilde{f}(z,x) - \tfrac{1}{3}\left[\widetilde{f}(x,y)+\widetilde{f}(y,z)\right]\right)\varphi(z,x).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left(\tfrac{2}{3}\widetilde{f}(x,y) - \tfrac{1}{3}\left[\widetilde{f}(y,z)+\widetilde{f}(z,x)\right]\right) &= \tfrac{1}{3}\left(\left[\widetilde{f}(x,y)-\widetilde{f}(y,z)\right]+\left[\widetilde{f}(x,y)-\widetilde{f}(z,x)\right]\right)\\ &= \tfrac{1}{6}\left(d^0(f)(z,x)+d^0(f)(z,y)\right).\end{aligned}$$

Similarly, we get

$$\left(\frac{2}{3}\widetilde{f}(y,z) - \frac{1}{3}\left[\widetilde{f}(z,x)+\widetilde{f}(x,y)\right]\right)\varphi(y,z) = \frac{1}{6}\left(d^0(f)(x,y)+d^0(f)(x,z)\right)\varphi(y,z).$$

and

$$\left(\frac{2}{3}\widetilde{f}(z,x) - \frac{1}{3}\left[\widetilde{f}(x,y)+\widetilde{f}(y,z)\right]\right)\varphi(z,x) = \frac{1}{6}\left(d^0(f)(y,z)+d^0(f)(y,x)\right)\varphi(z,x).$$

Hence, we have

$$\begin{aligned}d^1(\widetilde{f}\varphi)(x,y,z) &= \widetilde{\widetilde{f}}(x,y,z)d^1(\varphi)(x,y,z)\\ &+\tfrac{1}{6}\left[d^0(f)(z,x)+d^0(f)(z,y)\right]\varphi(x,y)\\ &+\tfrac{1}{6}\left[d^0(f)(x,y)+d^0(f)(x,z)\right]\varphi(y,z)\\ &+\tfrac{1}{6}\left[d^0(f)(y,z)+d^0(f)(y,x)\right]\varphi(z,x).\end{aligned}$$

(2) Let $(f, \phi) \in \mathcal{C}_c(\mathcal{V}) \times \mathcal{C}_c(\mathcal{F})$. Then by Lemma 2.5,

$$\begin{aligned}
\delta^1(\widetilde{\widetilde{f}}\phi)(e) &= \frac{1}{r(e)} \sum_{x\in\mathcal{F}_e} s(e,x)\widetilde{\widetilde{f}}(e,x)\phi(e,x) \\
&= \frac{1}{3r(e)} \sum_{x\in\mathcal{F}_e} s(e,x)\left(\widetilde{f}(e) + \widetilde{f}(e^+,x) + \widetilde{f}(x,e^-)\right)\phi(e,x) \\
&= \tfrac{1}{3}\widetilde{f}(e)\delta^1(\phi)(e) + \frac{1}{3r(e)} \sum_{x\in\mathcal{F}_e} s(e,x)\widetilde{f}(e^+,x)\phi(e,x) \\
&+ \frac{1}{3r(e)} \sum_{x\in\mathcal{F}_e} s(e,x)\widetilde{f}(x,e^-)\phi(e,x). \\
&= \widetilde{f}(e)\delta^1(\phi)(e) \\
&+ \frac{1}{3r(e)} \sum_{x\in\mathcal{F}_e} s(e,x)\left(\widetilde{f}(e^+,x) - \widetilde{f}(e)\right)\phi(e,x) \\
&+ \frac{1}{3r(e)} \sum_{x\in\mathcal{F}_e} s(e,x)\left(\widetilde{f}(x,e^-) - \widetilde{f}(e)\right)\phi(e,x) \\
&= \widetilde{f}(e)\delta^1(\phi)(e) \\
&+ \frac{1}{6r(e)} \sum_{x\in\mathcal{F}_e} s(e,x)\left[d^0(f)(e^-,x) + d^0(f)(e^+,x)\right]\phi(e,x).
\end{aligned}$$

□

2.3.6 Laplacian

Through the Gauß-Bonnet operator T, we can define the discrete Laplacian on $\mathcal{T}$. So, Lemma 2.8 induces the following definition

Definition 2.10 Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a triangulation, the Laplacian on $\mathcal{T}$ defined as

$$L := T^2 : \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}) \circlearrowleft$$

is given by

$$L(f, \varphi, \phi) = (\delta^0 d^0 f, (d^0\delta^0 + \delta^1 d^1)\varphi, d^1\delta^1\phi).$$

for all $(f, \varphi, \phi) \in \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.

Remark 2.11 We can write

$$L := L_0 \oplus L_1 \oplus L_2,$$

where L_0 is the discrete Laplacian acting on functions given by

$$L_0(f)(x) := \delta^0 d^0(f)(x) = \frac{1}{c(x)} \sum_{e,e^+=x} r(e)d^0(f)(e),$$

with $f \in \mathcal{C}_c(\mathcal{V})$, and where L_1 is the discrete Laplacian acting on 1-forms given by

$$\begin{aligned}
L_1(\varphi)(x,y) &:= (d^0\delta^0 + \delta^1 d^1)(\varphi)(x,y) \\
&= \frac{1}{c(y)} \sum_{e,e^+=y} r(e)\varphi(e) - \frac{1}{c(x)} \sum_{e,e^+=x} r(e)\varphi(e) \\
&+ \frac{1}{r(x,y)} \sum_{z\in\mathcal{F}_{[x,y]}} s(x,y,z)d^1(\varphi)(x,y,z),
\end{aligned}$$

with $\varphi \in \mathcal{C}_c(\mathcal{E})$, and where also L_2 is the discrete Laplacian acting on 2-forms given by

$$\begin{aligned} L_2(\phi)(x,y,z) &:= d^1\delta^1(\phi)(x,y,z) \\ &= \frac{1}{r(x,y)} \sum_{u\in\mathcal{F}_{[x,y]}} s(x,y,u)\phi(x,y,u) \\ &+ \frac{1}{r(y,z)} \sum_{u\in\mathcal{F}_{[y,z]}} s(y,z,u)\phi(y,z,u) \\ &+ \frac{1}{r(z,x)} \sum_{u\in\mathcal{F}_{[z,x]}} s(z,x,u)\phi(z,x,u), \end{aligned}$$

with $\phi \in \mathcal{C}_c(\mathcal{F})$.

Remark 2.12 The operator L_1 is called the *full Laplacian* and defined as $L_1 = L_1^- + L_1^+$, where $L_1^- = d^0\delta^0$ *(*resp. $L_1^+ = \delta^1 d^1$*)* is called the *lower Laplacian (*resp. the *upper Laplacian)*. In both articles [1] and [3], the authors denote $\Delta_0 = \delta^0 d^0$ and $\Delta_1 = d^0\delta^0$. In this work, we have $L_0 = \Delta_0$ and $L_1^- = \Delta_1$.

3 Closability

On a connected locally finite graph, the operators d^0 and δ^0 are closable (see [1]). The next Lemma proves the closability of the operators d^1 and δ^1 on a triangulation.

Lemma 3.1 *Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a weighted triangulation. Then the operators d^1 and δ^1 are closable.*

Proof

- Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence from $\mathcal{C}_c(\mathcal{E})$ and $\phi \in l^2(\mathcal{F})$ such that
$$\lim_{n\to\infty}\Big(\|\varphi_n\|_{l^2(\mathcal{E})} + \|d^1\varphi_n - \phi\|_{l^2(\mathcal{F})}\Big) = 0,$$
then for each edge e, $\varphi_n(e)$ converges to 0 and for each face ϖ, $d^1(\varphi_n)(\varpi)$ converges to $\phi(\varpi)$. But by the expression of d^1 and local finiteness of $\mathcal{T}$, for each face ϖ, $d^1(\varphi_n)(\varpi)$ converges to 0. Thus we have that $\phi = 0$.
- The same can be done for δ^1 : Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence from $\mathcal{C}_c(\mathcal{F})$ and $\varphi \in l^2(\mathcal{E})$ such that
$$\lim_{n\to\infty}\Big(\|\phi_n\|_{l^2(\mathcal{F})} + \|\delta^1\phi_n - \varphi\|_{l^2(\mathcal{E})}\Big) = 0,$$
then for each face σ, $\phi_n(\sigma)$ converges to 0 and for each edge e, $\delta^1(\phi_n)(e)$ converges to $\varphi(e)$. But by the expression of δ^1 and local finiteness of $\mathcal{T}$, for each edge e, $\delta^1(\phi_n)(e)$ converges to 0. Thus we have that $\varphi = 0$. □

The smallest extension is the closure (see [15, 18]), denoted $\overline{d^0} := d^0_{min}$ (resp. $\overline{\delta^0} := \delta^0_{min}$, $\overline{d^1} := d^1_{min}$, $\overline{\delta^1} := \delta^1_{min}$, $\overline{T} := T_{min}$, $\overline{L} := L_{min}$) has the domain

$$Dom(d^0_{min}) = \Big\{f \in l^2(\mathcal{V});\ \exists (f_n)_{n\in\mathbb{N}},\ f_n \in \mathcal{C}_c(\mathcal{V}),\ \lim_{n\to\infty}\|f_n - f\|_{l^2(\mathcal{V})} = 0,\ \lim_{n\to\infty} d^0(f_n) \text{exists in } l^2(\mathcal{E})\Big\},$$

for such an f, one puts
$$d^0_{min}(f) = \lim_{n\to\infty} d^0(f_n).$$

We notice that $d^0_{min}(f)$ is independent of the sequence $(f_n)_{n\in\mathbb{N}}$, because d^0 is closable.

The largest is $d^0_{max} = (\delta^0)^*$, the adjoint operator of δ^0_{min}, (resp. $\delta^0_{max} = (d^0)^*$, the adjoint operator of d^0_{min}).

We also note $d^1_{max} = (\delta^1)^*$, the adjoint operator of δ^1_{min}, (resp $\delta^1_{max} = (d^1)^*$, the adjoint operator of d^1_{min}).

Proposition 3.2 *Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a weighted triangulation. Then*

$$Dom(T_{min}) \subseteq Dom(d^0_{min}) \oplus \Big(Dom(\delta^0_{min}) \cap Dom(d^1_{min})\Big) \oplus Dom(\delta^1_{min}).$$

Proof Let $F = (f, \varphi, \phi) \in Dom(T_{min})$, so there exists a sequence $(F_n)_n = ((f_n, \varphi_n, \phi_n))_n \subseteq \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$ such that $\lim_{n\to\infty} F_n = F$ in $\mathcal{H}$ and $(TF_n)_{n\in\mathbb{N}}$ converges in $\mathcal{H}$. Let us denote by $l_0 = (f_0, \varphi_0, \phi_0)$ this limit. Therefore

$$\|TF_n - l_0\|^2_{\mathcal{H}} = \|\delta^0\varphi_n - f_0\|^2_{l^2(\mathcal{V})} + \|(d^0+\delta^1)(f_n, \phi_n) - \varphi_0\|^2_{l^2(\mathcal{E})} + \|d^1\varphi_n - \phi_0\|^2_{l^2(\mathcal{F})}.$$

Hence $\delta^0\varphi_n \to f_0$ and $d^1\varphi_n \to \phi_0$ respectively in $l^2(\mathcal{V})$ and in $l^2(\mathcal{E})$. So, by definition, $\varphi \in Dom(\delta^0_{min}) \cap Dom(d^1_{min})$, $f_0 = \delta^0_{min}\varphi$ and $\phi_0 = d^1_{min}\varphi$. Moreover, we combine the parallelogram identity with Lemma 2.8 to obtain the following result

$$\|(d^0+\delta^1)(f,\phi)\|^2_{l^2(\mathcal{E})} = \|d^0(f)\|^2_{l^2(\mathcal{E})} + \|\delta^1(\phi)\|^2_{l^2(\mathcal{E})}, \ \forall n \in \mathbb{N}, \ \forall (f,\phi) \in \mathcal{C}_c(\mathcal{V}) \times \mathcal{C}_c(\mathcal{F}).$$

Since $\big((d^0+\delta^1)(f_n,\phi_n)\big)_n$ converges in $l^2(\mathcal{E})$, then by completeness of $l^2(\mathcal{E})$ $\big(d^0(f_n)\big)_n$ and $\big(\delta^1(\phi_n)\big)_n$ are convergent in $l^2(\mathcal{E})$. Thus, we conclude that $f \in Dom(d^0_{min})$ and $\phi \in Dom(\delta^1_{min})$. □

Proposition 3.3 *Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a weighted triangulation. Then*

$$Dom(L_{min}) \subseteq Dom(\delta^0_{min}d^0_{min}) \oplus \Big(Dom(d^0_{min}\delta^0_{min}) \cap Dom(\delta^1_{min}d^1_{min})\Big) \oplus Dom(d^1_{min}\delta^1_{min}).$$

Proof i) We will show that $(L_0)_{min} \subseteq \delta^0_{min}d^0_{min}$. First, we note that

$$Dom(\delta^0_{min}d^0_{min}) = \{f \in Dom(d^0_{min}),\ d^0_{min}f \in Dom(\delta^0_{min})\}.$$

Let $f \in Dom((L_0)_{min})$, so there exists a sequence $(f_n)_n \subseteq \mathcal{C}_c(\mathcal{V})$ such that

$$f_n \to f \text{ in } l^2(\mathcal{V}),\ \delta^0 d^0 f_n \to (\delta^0 d^0)_{min} f \text{ in } l^2(\mathcal{V}).$$

So, $(L_0 f_n)_n$ is a Cauchy sequence. Moreover, we have

$$\begin{aligned}\forall n, m \in \mathbb{N},\ \|d^0 f_n - d^0 f_m\|^2_{l^2(\mathcal{E})} &= \langle d^0(f_n - f_m), d^0(f_n - f_m)\rangle_{l^2(\mathcal{E})} \\ &= \langle \delta^0 d^0(f_n - f_m), f_n - f_m\rangle_{l^2(\mathcal{V})} \\ &= \langle L_0(f_n - f_m), f_n - f_m\rangle_{l^2(\mathcal{V})} \\ &\le \|L_0(f_n - f_m)\|_{l^2(\mathcal{V})}\|f_n - f_m\|_{l^2(\mathcal{V})}.\end{aligned}$$

Thus $(d^0 f_n)_n$ is a Cauchy sequence because $(L_0 f_n)_n$ is a Cauchy sequence and $(f_n)_n$ is convergent. So, it is convergent in $l^2(\mathcal{E})$. By closability, we conclude that $f \in Dom(\delta^0_{min}d^0_{min})$.

ii) First, for all $\varphi, \psi \in \mathcal{C}_c(\mathcal{E})$ we have

$$\langle L_1\varphi, \psi\rangle_{l^2(\mathcal{E})} = \langle (L_1^- + L_1^+)\varphi, \psi\rangle_{l^2(\mathcal{E})} = \langle \delta^0\varphi, \delta^0\psi\rangle_{l^2(\mathcal{V})} + \langle d^1\varphi, d^1\psi\rangle_{l^2(\mathcal{F})}. \quad (3.1)$$

Using the same method as in i) with Eq. 3.1 we obtain that $(L_1^-)_{min} \subseteq d^0_{min}\delta^0_{min}$ and $(L_1^+)_{min} \subseteq \delta^1_{min}d^1_{min}$. It remains to show that we have

$$(L_1)_{min} \subseteq (L_1^-)_{min} + (L_1^+)_{min}$$

Let $\varphi \in Dom((L_1)_{min})$, so there exists a sequence $(\varphi_n)_n \subseteq \mathcal{C}_c(\mathcal{E})$ such that $\varphi = \lim_{n\to\infty} \varphi_n$ in $l^2(\mathcal{E})$ and $(L_1\varphi_n)_{n\in\mathbb{N}}$ converges in $l^2(\mathcal{E})$. Then, by the parallelogram identity with Lemma 2.8 we obtain

$$\|(L_1^- + L_1^+)(\varphi_n)\|^2_{l^2(\mathcal{E})} = \|L_1^-(\varphi_n)\|^2_{l^2(\mathcal{E})} + \|L_1^+(\varphi_n)\|^2_{l^2(\mathcal{E})},\ \forall n \in \mathbb{N}.$$

Then $\left(L_1^-(\varphi_n)\right)_n$ and $\left(L_1^+(\varphi_n)\right)_n$ are convergent in $l^2(\mathcal{E})$. Moreover, by the closability of L_1^- and L_1^+, we conclude that $\varphi \in Dom((L_1^-)_{min}) \cap Dom((L_1^+)_{min})$.

iii) Since, for any $\phi, \Theta \in \mathcal{C}_c(\mathcal{F})$, we have

$$\langle L_2\phi, \Theta\rangle_{l^2(\mathcal{F})} = \langle d^1\delta^1\phi, \Theta\rangle_{l^2(\mathcal{F})} = \langle \delta^1\phi, \delta^1\Theta\rangle_{l^2(\mathcal{E})}. \tag{3.2}$$

Using the same method as in i) with Eq. 3.2 we obtain that $(L_2)_{min} \subseteq d^1_{min}\delta^1_{min}$. □

4 Geometric Hypothesis

4.1 χ-Completeness

In this subsection, we give the geometric hypothesis for the triangulation $\mathcal{T}$. First we recall the definition of χ-completeness given in [1] for the case of graphs. A graph $\mathcal{K} = (\mathcal{V}, \mathcal{E})$ is χ-complete if there exists an increasing sequence of finite sets $(B_n)_{n\in\mathbb{N}}$ such that $\mathcal{V} = \cup_{n\in\mathbb{N}} B_n$ and there exist related functions χ_n satisfying the following three conditions:

i) $\chi_n \in \mathcal{C}_c(\mathcal{V}),\ 0 \le \chi_n \le 1$.
ii) $x \in B_n \Rightarrow \chi_n(x) = 1$.
iii) $\exists C > 0$ such that $\forall n \in \mathbb{N},\ x \in \mathcal{V}$

$$\frac{1}{c(x)} \sum_{e\in\mathcal{E}, e^\pm = x} r(e)|d^0\chi_n(e)|^2 \le C.$$

Remark 4.1 The χ-completeness is related to the notion of intrinsic metric for weighted graphs. This geometric hypothesis covers many situations that have been already studied. Particularly in [1], the authors prove that it is satisfied by locally finite graphs which are complete for some intrinsic pseudo metric, as defined in [8] and [12].

Definition 4.2 A triangulation $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ is χ-complete, if

(C_1) $\mathcal{K}$ is χ-complete.
(C_2) $\exists M > 0,\ \forall n \in \mathbb{N},\ e \in \mathcal{E}$, such that

$$\frac{1}{r(e)} \sum_{x\in\mathcal{F}_e} s(e, x)|d^0\chi_n(e^-, x) + d^0\chi_n(e^+, x)|^2 \le M.$$

For this type of 2-simplicial complexes one has

$$\forall p \in \mathbb{N},\ \exists n_p,\ n \geq n_p \Rightarrow \forall e \in \mathcal{E},\ \text{such that } e^+ \text{ or } e^- \in B_p,\ d^0\chi_n(e) = 0. \tag{4.1}$$

$$\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n \text{ if } \mathcal{E}_n := \{e \in \mathcal{E},\ e^+ \in B_n \text{ or } e^- \in B_n\}. \tag{4.2}$$

$$\forall q \in \mathbb{N},\ \exists n_q,\ n \geq n_q \Rightarrow \forall (e,x) \in \mathcal{F},\ \text{such that } e^-,\ e^+ \text{ or } x \in B_q,\ d^0\chi_n(e^\pm, x) = 0. \tag{4.3}$$

$$\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n \text{ if } \mathcal{F}_n := \{[x,y,z] \in \mathcal{F},\ x \in B_n \text{ or } y \in B_n \text{ or } z \in B_n\}. \tag{4.4}$$

$$\forall f \in l^2(\mathcal{V}),\ \|f\|^2_{l^2(\mathcal{V})} = \lim_{n\to\infty} \langle \chi_n f, f\rangle_{l^2(\mathcal{V})}. \tag{4.5}$$

$$\forall \varphi \in l^2(\mathcal{E}),\ \|\varphi\|^2_{l^2(\mathcal{E})} = \lim_{n\to\infty} \frac{1}{2}\sum_{e\in\mathcal{E}} r(e)\chi_n(e^+)|\varphi(e)|^2. \tag{4.6}$$

$$\forall \phi \in l^2(\mathcal{F}),\ \|\phi\|^2_{l^2(\mathcal{F})} = \lim_{n\to\infty} \frac{1}{6}\sum_{e\in\mathcal{E}} \widetilde{\chi}_n(e)\left(\sum_{x\in\mathcal{F}_e} s(e,x)|\phi(e,x)|^2\right). \tag{4.7}$$

$$\lim_{n\to\infty} \sum_{e\in\mathcal{E}^*(n)} r(e)|\varphi(e)|^2 = 0, \tag{4.8}$$

where

$$\mathcal{E}^*(n) := \{e \in \mathcal{E}, \exists x \in \mathcal{F}_e \text{ such that } (e^\pm, x) \in supp(d^0\chi_n)\}$$

$$\lim_{n\to\infty} \sum_{e\in\mathcal{E}} \sum_{x\in\mathcal{F}^*_e(n)} s(e,x)|\phi(e,x)|^2 = 0, \tag{4.9}$$

where

$$\forall e \in \mathcal{E},\ \mathcal{F}^*_e(n) := \{x \in \mathcal{F}_e,\ (e^\pm, x) \in supp(d^0\chi_n)\}.$$

Proposition 4.3 *Let $\mathcal{T}$ be a simple triangulation of bounded degree, i.e $\exists \lambda > 0,\ \forall x \in \mathcal{V},\ deg(x) \leq \lambda$. Then $\mathcal{T}$ is a χ-complete triangulation.*

Proof Let us consider $\mathcal{T}$ an infinite triangulation. Given $o \in \mathcal{V}$, let B_n be a ball of radius $n \in \mathbb{N}$ centered by the vertex o:

$$B_n = \{x \in \mathcal{V},\ d_{comb}(o,x) \leq n\}.$$

We set the cut-off function $\chi_n \in \mathcal{C}_c(\mathcal{V})$ as follow:

$$\chi_n(x) := \left(\frac{2n - d_{comb}(o,x)}{n} \vee 0\right) \wedge 1,\ \forall n \in \mathbb{N}^*.$$

- If $x \in B_n \Rightarrow \chi_n(x) = 1$ and $x \in B^c_{2n} \Rightarrow \chi_n(x) = 0$.
- For $e \in \mathcal{E}$, we have that

$$|d^0\chi_n(e)| \leq \frac{1}{n}\left|d_{comb}(o,e^+) - d_{comb}(o,e^-)\right| = \frac{1}{n}.$$

Hence

$$\forall x \in \mathcal{V},\quad \sum_{e\in\mathcal{E}, e^\pm = x} |d^0\chi_n(e)|^2 \leq \frac{\lambda}{n^2}$$

and

$$\forall e \in \mathcal{E},\ \sum_{x\in\mathcal{F}_e} |d^0\chi_n(e^-,x) + d^0\chi_n(e^+,x)|^2 \leq \frac{2\lambda}{n^2}.$$

□

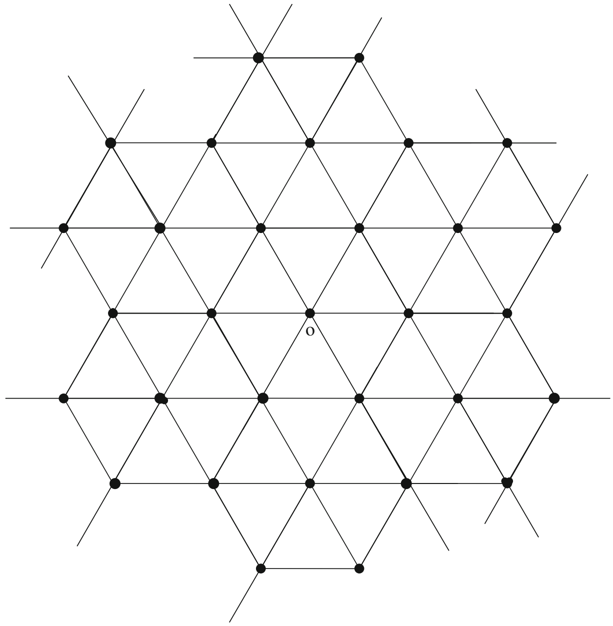

Fig. 1 An infinite 6-regular triangulation

Example 4.4 (A χ-complete triangulation)
We consider $\mathcal{T}$ a 6-regular simple triangulation, i.e. $deg(x) = 6$, $\forall x \in \mathcal{V}$. Then, by Proposition 4.3 we have that $\mathcal{T}$ is a χ-complete triangulation (Fig. 1).

Proposition 4.5 *Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a χ-complete triangulation. Then*

$$Dom\,((L_0)_{min}) = Dom(\delta^0_{min} d^0_{min}).$$

Proof In Proposition 3.3, we have already $(L_0)_{min} \subseteq \delta^0_{min} d^0_{min}$. Indeed, we will show that $\delta^0_{min} d^0_{min} \subseteq (L_0)_{min}$. Let $f \in Dom(\delta^0_{min} d^0_{min})$, by the χ-completeness of $\mathcal{T}$, we now consider a sequence $(\chi_n f)_n \subseteq \mathcal{C}_c(\mathcal{V})$. It remains to show that:

$$\lim_{n\to\infty} \|f - \chi_n f\|_{l^2(\mathcal{V})} + \|L_0(f - \chi_n f)\|_{l^2(\mathcal{V})} = 0. \tag{4.10}$$

For the first term of Eq. 4.10, since $f \in l^2(\mathcal{V})$ we have

$$\|f - \chi_n f\|^2_{l^2(\mathcal{V})} \leq \sum_{x \in B_n^c} c(x)|f(x)|^2 \to 0, \text{ when } n \to \infty.$$

For the second term of Eq. 4.10, we need a derivation formula of d^0, see [14]. Let $e \in \mathcal{E}$, for each $(f, g) \in \mathcal{C}_c(\mathcal{V}) \times \mathcal{C}_c(\mathcal{V})$ we have

$$d^0(fg)(e) = f(e^+)d^0(g)(e) + d^0(f)(e)g(e^-). \tag{4.11}$$

By the definition of L_0, we have

$$\|L_0(f-\chi_n f)\|^2_{l^2(\mathcal{V})} = \sum_{x\in\mathcal{V}} \frac{1}{c(x)} \left| \sum_{e,e^+=x} r(e) d^0((1-\chi_n)f)(e) \right|^2 .$$

Using the derivation formula (4.11), we get

$$\begin{aligned}
\|L_0(f-\chi_n f)\|^2_{l^2(\mathcal{V})} &\le 2\sum_{x\in\mathcal{V}} \frac{1}{c(x)} \left| \sum_{e,e^+=x} r(e)(1-\chi_n)(e^+) d^0(f)(e) \right|^2 \\
&\quad +2\sum_{x\in\mathcal{V}} \frac{1}{c(x)} \left| \sum_{e,e^+=x} r(e) f(e^-) d^0(\chi_n)(e) \right|^2 \\
&= 2\left(\|(1-\chi_n)L_0(f)\|^2_{l^2(\mathcal{V})} + \sum_{x\in\mathcal{V}} \frac{1}{c(x)} \left| \sum_{e,e^+=x} r(e) f(e^-) d^0(\chi_n)(e) \right|^2 \right).
\end{aligned}$$

Since $L_0(f) \in l^2(\mathcal{V})$, we have

$$\lim_{n\to\infty} \|(1-\chi_n)L_0(f)\|_{l^2(\mathcal{V})} = 0.$$

On the other hand, by the hypothesis iii) of χ-completeness and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sum_{x\in\mathcal{V}} \frac{1}{c(x)} \left| \sum_{e,e^+=x} r(e) f(e^-) d^0(\chi_n)(e) \right|^2 &\le \sum_{x\in\mathcal{V}} \frac{1}{c(x)} \left(\sum_{e,e^+=x} r(e) |d^0(\chi_n)(e)|^2 \right) \\
&\quad \left(\sum_{e\in supp(d^0\chi_n), e^+=x} r(e) |f(e^-)|^2 \right) \\
&\le \sum_{x\in\mathcal{V}} C \sum_{e\in supp(d^0\chi_n), e^+=x} r(e) |f(e^-)|^2 \\
&\le C \sum_{e\in supp(d^0\chi_n)} r(e) |f(e^-)|^2.
\end{aligned}$$

The properties (4.1) and (4.2) permit to conclude that this term tends to 0 when ∞. □

4.2 The Case of a Not χ-Complete Triangulation

In [3], the authors use the offspring function on the trees to give a counter example of a graph which is not χ-complete. The same thing for the triangulations is not always χ-complete. To prove it, we will study the triangular tree in Definition 4.6.

Let $\mathcal{T}$ a weighted triangulation, one can take any point $o \in \mathcal{V}$. Given $n \in \mathbb{N}$, we denote the spheres by

$$\mathcal{S}_n := \{x \in \mathcal{V},\ d_{comb}(o,x) = n\}.$$

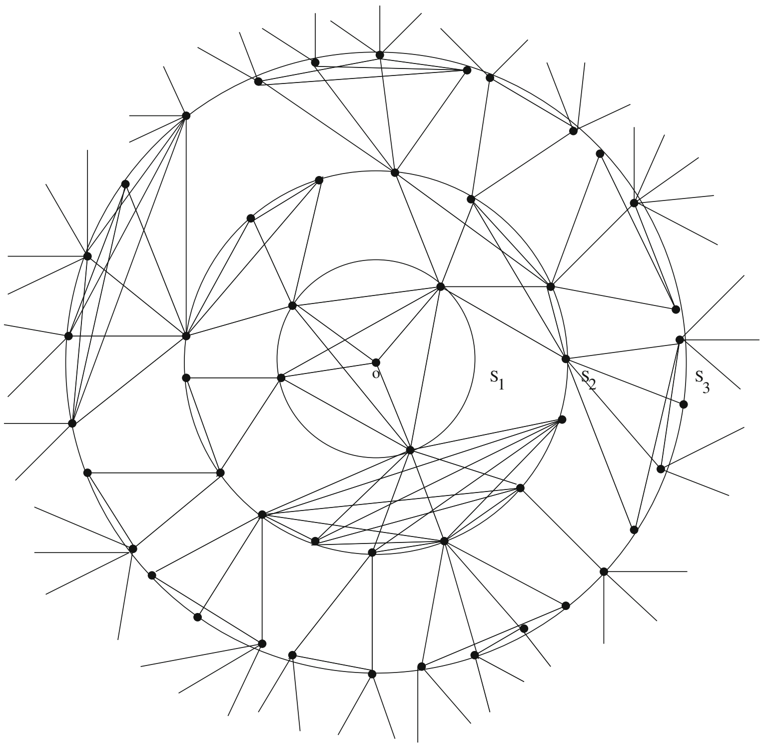

Fig. 2 A Triangular Tree

Definition 4.6 A triangular tree $\mathcal{T} = (\mathcal{V}, \mathcal{F})$ with the origin vertex o is a triangulation where $\mathcal{V} = \cup_{n\in\mathbb{N}} S_n$, such that

$$\forall x \in \mathcal{S}_n \backslash \{o\},\ \mathcal{V}(x) \cap \mathcal{S}_{n-1} = \{\overleftarrow{x}\}.$$

$$\forall x \in \mathcal{S}_n,\ y \in \mathcal{V}(x) \cap \mathcal{S}_{n+1} \Leftrightarrow \overleftarrow{y} = x.$$

$$(x, y) \in \mathcal{E} \cap (\mathcal{S}_n \backslash \{o\})^2 \Rightarrow \overleftarrow{x} = \overleftarrow{y}.$$

where $\overleftarrow{x}$ the unique vertex in $\mathcal{S}_{n-1}$, which is related with $x \in \mathcal{S}_n \backslash \{o\}$ (Fig. 2).

Let $\mathcal{T}$ be a simple triangular tree. *The offspring* of the n-th generation (see [3]) is given by

$$\mathrm{off}(n) = \frac{\#\mathcal{S}_{n+1}}{\#\mathcal{S}_n}.$$

Proposition 4.7 *Let $\mathcal{T}$ be a simple triangular tree with the origin vertex o. Assume that*

$$\sup_{n\in\mathbb{N}} \sup_{x\in\mathcal{S}_n} \frac{\#\left(\mathcal{V}(x) \cap \mathcal{S}_{n+1}\right)}{\mathit{off}(n)} < \infty.$$

Then

$$\mathcal{T} \text{ is } \chi\text{-complete} \Leftrightarrow \sum_{n\geqslant 1} \frac{1}{\sqrt{\mathit{off}(n)}} = \infty.$$

Proof

$\Rightarrow$) In a proof by contradiction, we start by assuming that $\mathcal{T}$ is χ-complete and the series converges. So, there exists a sequence $(\chi_n)_n$ included in $\mathcal{C}_c(\mathcal{V})$, satisfying

$$\exists C > 0,\ \forall n \in \mathbb{N},\ \sum_{y \sim x} |\chi_n(x) - \chi_n(y)|^2 \leq C,\ x \in \mathcal{V}.$$

Given $n, m \in \mathbb{N}$ and $x_m \in \mathcal{S}_m$. By the local finiteness of the triangulation, we find $x_{m+1} \in \mathcal{V}(x_m) \cap \mathcal{S}_{m+1}$, such that

$$|\chi_n(x_m) - \chi_n(x_{m+1})| = \min_{y \in \mathcal{V}(x_m) \cap \mathcal{S}_{m+1}} |\chi_n(x_m) - \chi_n(y)|.$$

But,

$$\sum_{y \in \mathcal{V}(x_m) \cap \mathcal{S}_{m+1}} |\chi_n(x_m) - \chi_n(y)|^2 \leq C.$$

Hence

$$|\chi_n(x_m) - \chi_n(x_{m+1})| \leq \frac{\sqrt{C}}{\sqrt{\text{off}(m)}}.$$

Moreover, by convergence of the series, there is $N \in \mathbb{N}$ such that

$$\sum_{k \geq N} \frac{1}{\sqrt{\text{off}(k)}} < \frac{1}{2\sqrt{C}}.$$

Then, by ii) of the definition of χ-completeness, there is $n_0 \in \mathbb{N}$ such that $\chi_{no}(x) = 1$ for all $d_{comb}(o, x) \leq N$. Since χ_{no} is with finite support, there is $M \in \mathbb{N}$ such that $\chi_{no}(x) = 0$ for all $d_{comb}(o, x) \geq N + M$. Therefore,

$$\begin{aligned} |\chi_{no}(x_N) - \chi_{no}(x_{N+M})| &\leq |\chi_{no}(x_N) - \chi_{no}(x_{N+1})| + \ldots.. + |\chi_{no}(x_{N+M-1}) - \chi_{no}(x_{N+M})| \\ &\leq \sqrt{C} \sum_{k=n}^{N+M-1} \frac{1}{\sqrt{\text{off}(k)}} < \frac{1}{2}. \end{aligned}$$

Since $|\chi_{no}(x_N) - \chi_{no}(x_{N+M})| = 1$, we have the contradiction.

$\Leftarrow$) We consider the cut-off function:

$$\chi_n(x) = \begin{cases} 1 & \text{if } d_{comb}(o, x) \leqslant n, \\ \max\left(0, 1 - \displaystyle\sum_{k=n}^{d_{comb}(o,x)-1} \frac{1}{\sqrt{\text{off}(k)}}\right) & \text{if } d_{comb}(o, x) > n. \end{cases}$$

Since the series diverges, χ_n is with finite support and satisfies i) and ii) of the definition of χ-completeness. Given $x \in \mathcal{S}_m$ with $m > n$, we have

$$\sum_{y \in \mathcal{V}(x) \cap \mathcal{S}_{m+1}} |\chi_n(x) - \chi_n(y)|^2 \leq \frac{\#\left(\mathcal{V}(x) \cap \mathcal{S}_{m+1}\right)}{\text{off}(m)}.$$

$$\sum_{y \in \mathcal{V}(x) \cap \mathcal{S}_m} |\chi_n(x) - \chi_n(y)|^2 = 0.$$

$$\sum_{y \in \mathcal{V}(x) \cap \mathcal{S}_{m-1}} |\chi_n(x) - \chi_n(y)|^2 = |\chi_n(x) - \chi_n(\overleftarrow{x})|^2 \leq \frac{1}{\text{off}(m-1)}.$$

On the other hand,

i) If $e \in \mathcal{S}_m \times \mathcal{S}_{m+1}$ with $m > n$, we have

$$\sum_{x \in \mathcal{F}_e} |d^0 \chi_n(e^-, x) + d^0 \chi_n(e^+, x)|^2 = \sum_{x \in \mathcal{F}_e} |2\chi_n(x) - \chi_n(e^-) - \chi_n(e^+)|^2 \leq \frac{|\mathcal{F}_e|}{\mathrm{off}(m)} \leq \frac{\#\left(\mathcal{V}(e^-) \cap \mathcal{S}_{m+1}\right)}{\mathrm{off}(m)}.$$

ii) If $e \in \mathcal{S}_m \times \mathcal{S}_m$ with $m > n$, we have

$$\begin{aligned}\sum_{x \in \mathcal{F}_e} |d^0 \chi_n(e^-, x) + d^0 \chi_n(e^+, x)|^2 &= \sum_{x \in \mathcal{F}_e} |2\chi_n(x) - \chi_n(e^-) - \chi_n(e^+)|^2 \\ &= |2\chi_n(\overleftarrow{e}) - \chi_n(e^-) - \chi_n(e^+)|^2 \\ &\leq \frac{4}{\mathrm{off}(m-1)},\end{aligned}$$

with $\overleftarrow{e}$ is a unique vertex in $\mathcal{S}_{m-1} \cap \mathcal{F}_e$.

It satisfies Definition 4.2 of χ-completeness.

□

Corollary 4.8 *Let $\mathcal{T}$ be a simple triangular tree, endowed with an origin such that*

$$\mathit{off}(n) = \#\left(\mathcal{V}(x) \cap \mathcal{S}_{n+1}\right), \text{ for all } x \in \mathcal{S}_n,$$

then $\mathcal{T}$ is χ-complete if and only if

$$\sum_{n \geqslant 1} \frac{1}{\sqrt{\mathit{off}(n)}} = \infty.$$

Example 4.9 Set $\alpha > 0$. Let $\mathcal{T}$ be a simple triangular tree, endowed with an origin such that

$$\mathit{off}(n) = \#\left(\mathcal{V}(x) \cap \mathcal{S}_{n+1}\right) = \lfloor n^\alpha \rfloor + 1, \text{ for all } x \in \mathcal{S}_n,$$

then $\mathcal{T}$ is χ-complete if only if $\alpha \leq 2$.

5 Essential Self-Adjointness

In [1], the authors use the χ-completeness hypothesis on a graph to ensure essential self-adjointness for the Gauβ-Bonnet operator and the Laplacian. In this section, with the same idea we will prove the main result, when the triangulation is χ-complete. Let us begin from

Proposition 5.1 *Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a χ-complete triangulation then the operator $d^1 + \delta^1$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.*

Proof It suffices to show that $d^1_{min} = d^1_{max}$ and $\delta^1_{min} = \delta^1_{max}$. Indeed, $d^1 + \delta^1$ is a direct sum and if $F = (\varphi, \phi) \in Dom((d^1 + \delta^1)_{max})$ then $\varphi \in Dom(d^1_{max})$ and $\phi \in Dom(\delta^1_{max})$. By hypothesis, we have $\varphi \in Dom(d^1_{min})$ and $\phi \in Dom(\delta^1_{min})$, thus $F \in Dom((d^1 + \delta^1)_{min})$.

1) Let $\varphi \in Dom(d^1_{max})$, we will show that

$$\|\varphi - \widetilde{\chi}_n \varphi\|_{l^2(\mathcal{E})} + \|d^1\left(\varphi - \widetilde{\chi}_n \varphi\right)\|_{l^2(\mathcal{F})} \to 0 \text{ when } n \to \infty.$$

By the properties (4.1) and (4.2), we know that

$$\forall p \in \mathbb{N},\ \exists n_p,\ \forall n \geq n_p,\ \|\varphi - \widetilde{\chi}_n\varphi\|^2_{l^2(\mathcal{E})} \leq \sum_{e\in\mathcal{E}_p^c} r(e)|\varphi(e)|^2$$

so $\lim_{n\longrightarrow\infty} \|\varphi - \widetilde{\chi}_n\varphi\| = 0$.

From the derivation formula (2.3) in Lemma 2.9, we have

$$\begin{aligned}
d^1(\varphi - \widetilde{\chi}_n\varphi)(e,x) &= d^1\left(\left(\widetilde{1-\chi_n}\right)\varphi\right)(e,x)\\
&= \left(1-\widetilde{\widetilde{\chi}}_n\right)(e,x)d^1(\varphi)(e,x)\\
&+\tfrac{1}{6}\left(d^0(1-\chi_n)(x,e^-) + d^0(1-\chi_n)(x,e^+)\right)\varphi(e)\\
&+\tfrac{1}{6}\left(d^0(1-\chi_n)(e) + d^0(1-\chi_n)(e^-,x)\right)\varphi(e^+,x)\\
&+\tfrac{1}{6}\left(d^0(1-\chi_n)(e^+,x) + d^0(1-\chi_n)(-e)\right)\varphi(x,e^-)\\
&= \left(1-\widetilde{\widetilde{\chi}}_n\right)(e,x)d^1(\varphi)(e,x)\\
&+\tfrac{1}{6}\left(d^0\chi_n(e^-,x) + d^0\chi_n(e^+,x)\right)\varphi(e)\\
&+\tfrac{1}{6}\left(d^0\chi_n(-e) + d^0\chi_n(x,e^-)\right)\varphi(e^+,x)\\
&+\tfrac{1}{6}\left(d^0\chi_n(x,e^+) + d^0\chi_n(e)\right)\varphi(x,e^-).
\end{aligned}$$

Since $d^1\varphi \in l^2(\mathcal{F})$, one has

$$\lim_{n\to\infty} \|\left(1-\widetilde{\widetilde{\chi}}_n\right)d^1\varphi\|_{l^2(\mathcal{F})} = 0.$$

On the other hand,

$$\begin{aligned}
\sum_{(e,x)\in\mathcal{F}} s(e,x)|\varphi(e)|^2|d^0\chi_n(e^-,x) + d^0\chi_n(e^+,x)|^2 &= \sum_{e\in\mathcal{E}}|\varphi(e)|^2\sum_{x\in\mathcal{F}_e} s(e,x)|d^0\chi_n(e^-,x)\\
&+ d^0\chi_n(e^+,x)|^2\\
&\leq M\sum_{e\in\mathcal{E}^*(n)} r(e)|\varphi(e)|^2.
\end{aligned}$$

The property (4.8) allows to conclude that this term tends to 0 as $n\to\infty$. Applying the same process to the other terms, we have

$$\begin{aligned}
\sum_{(e,x)\in\mathcal{F}} s(e,x)|\varphi(e^+,x)|^2|d^0\chi_n(-e) + d^0\chi_n(x,e^-)|^2 &= \sum_{(e^+,x)\in\mathcal{E}}|\varphi(e^+,x)|^2\sum_{y\in\mathcal{F}_{(e^+,x)}} s(e^+,x,y)\\
&|d^0\chi_n(e^+,y) + d^0\chi_n(x,y)|^2\\
&\leq M\sum_{(e^+,x)\in\mathcal{E}^*(n)} r(e^+,x)|\varphi(e^+,x)|^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{(e,x)\in\mathcal{F}} s(e,x)|\varphi(x,e^-)|^2|d^0\chi_n(x,e^+) + d^0\chi_n(e)|^2 &= \sum_{(x,e^-)\in\mathcal{E}}|\varphi(x,e^-)|^2\sum_{y\in\mathcal{F}_{(x,e^-)}} s(x,e^-,y)\\
&|d^0\chi_n(x,y) + d^0\chi_n(e^-,y)|^2\\
&\leq M\sum_{(x,e^-)\in\mathcal{E}^*(n)} r(x,e^-)|\varphi(x,e^-)|^2.
\end{aligned}$$

2) Let $\phi \in Dom(\delta^1_{max})$, we will show that

$$\|\phi - \widetilde{\widetilde{\chi}}_n\phi\|_{l^2(\mathcal{F})} + \|\delta^1(\phi - \widetilde{\widetilde{\chi}}_n\phi)\|_{l^2(\mathcal{E})} \to 0 \text{ when } n\to\infty.$$

By the properties (4.3) and (4.4), we know that

$$\begin{aligned}\|\phi - \widetilde{\widetilde{\chi}}_n\phi\|^2_{l^2(\mathcal{F})} &= \tfrac{1}{6}\sum_{(x,y,z)\in\mathcal{F}} s(x,y,z)|1-\widetilde{\widetilde{\chi}}_n(x,y,z)|^2|\phi(x,y,z)|^2\\ &\leq \sum_{(x,y,z)\in\mathcal{F}_q^c} s(x,y,z)|\phi(x,y,z)|^2 \to 0, \text{ when } n\to\infty.\end{aligned}$$

By the derivation formula (2.4) in Lemma 2.9, we have

$$\begin{aligned}\delta^1(\phi - \widetilde{\widetilde{\chi}}_n\phi)(e) &= \delta^1\left(\widetilde{\widetilde{(1-\chi_n)}}\phi\right)(e)\\ &= (1-\widetilde{\chi_n})(e)\delta^1(\phi)(e)\\ &+\frac{1}{6r(e)}\sum_{x\in\mathcal{F}_e} s(e,x)d^0(1-\chi_n)(e^-,x)\phi(e,x)\\ &+\frac{1}{6r(e)}\sum_{x\in\mathcal{F}_e} s(e,x)d^0(1-\chi_n)(e^+,x)\phi(e,x)\\ &= (1-\widetilde{\chi_n})\,(e)\delta^1(\phi)(e)\\ &+\frac{1}{6r(e)}\sum_{x\in\mathcal{F}_e} s(e,x)d^0(\chi_n)(x,e^-)\phi(e,x)\\ &+\frac{1}{6r(e)}\sum_{x\in\mathcal{F}_e} s(e,x)d^0(\chi_n)(x,e^+)\phi(e,x).\end{aligned}$$

We know that

$$\lim_{n\to\infty}\|\,(1-\widetilde{\chi_n})\,\delta^1(\phi)\| = 0$$

because $\delta^1\phi \in l^2(\mathcal{E})$. For the second and third terms, we use the inequality of Definition 4.2 and the Cauchy-Schwarz inequality. Fix $e\in\mathcal{E}$, then

$$\begin{aligned}\left|\sum_{x\in\mathcal{F}_e} s(e,x)\left(d^0(\chi_n)(x,e^-)+d^0(\chi_n)(x,e^+)\right)\phi(e,x)\right|^2 &\leq \sum_{x\in\mathcal{F}_e} s(e,x)|d^0(\chi_n)(x,e^-)\\ &\quad + d^0(\chi_n)(x,e^+)|^2\\ &\quad \times\sum_{x\in\mathcal{F}_e^*(n)} s(e,x)|\phi(e,x)|^2\\ &\leq Mr(e)\sum_{x\in\mathcal{F}_e^*(n)} s(e,x)|\phi(e,x)|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{e\in\mathcal{E}} r(e)\left|\frac{1}{r(e)}\sum_{x\in\mathcal{F}_e} s(e,x)\left(d^0(\chi_n)(x,e^-)+d^0(\chi_n)(x,e^+)\right)\phi(e,x)\right|^2\\ &\leq M\sum_{e\in\mathcal{E}}\sum_{x\in\mathcal{F}_e^*(n)} s(e,x)|\phi(e,x)|^2.\end{aligned}$$

By property (4.9), this terms tends to 0.

□

Corollary 5.2 *Let* $\mathcal{T}=(\mathcal{K},\mathcal{F})$ *be a* χ*-complete triangulation then the operator* $L_1^+\oplus L_2$ *is essentially self-adjoint on* $\mathcal{C}_c(\mathcal{E})\oplus\mathcal{C}_c(\mathcal{F})$.

Proof

First we have that $L_1^+ \oplus L_2 = \left(d^1 + \delta^1\right)^2$ and $L_1^+ \oplus L_2 \left(\mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})\right) \subseteq \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$. As Proposition 13 in [1] we prove that $d^1 + \delta^1$ is essentially self-adjoint if and only if $L_1^+ \oplus L_2$ is essentially self-adjoint. □

Theorem 5.3 *Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a χ-complete triangulation then the operator T is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.*

Proof

1^{st} *Step:* We will show that

$$Dom(T_{min}) = Dom(d^0_{min}) \oplus \left(Dom(\delta^0_{min}) \cap Dom(d^1_{min})\right) \oplus Dom(\delta^1_{min}).$$

Let $F = (f, \varphi, \phi) \in Dom(d^0_{min}) \oplus \left(Dom(\delta^0_{min}) \cap Dom(d^1_{min})\right) \oplus Dom(\delta^1_{min})$. Then there exist $(f_n)_n \subseteq \mathcal{C}_c(\mathcal{V})$ and $(\phi_n)_n \subseteq \mathcal{C}_c(\mathcal{F})$ such that:

- $f_n \to f$ in $l^2(\mathcal{V})$ and $d^0 f_n \to d^0_{min} f$ in $l^2(\mathcal{E})$.
- $\phi_n \to \phi$ in $l^2(\mathcal{F})$ and $\delta^1 \phi_n \to \delta^1_{min}\phi$ in $l^2(\mathcal{E})$

On the other hand, let $\varphi \in Dom(\delta^0_{min}) \cap Dom(d^1_{min})$. By the χ-completeness of $\mathcal{T}$, we now consider the sequence $(\widetilde{\chi}_n\varphi)_n \subseteq \mathcal{C}_c(\mathcal{E})$. It remains to show that

$$\|\varphi - \widetilde{\chi}_n\varphi\|_{l^2(\mathcal{E})} + \|\delta^0(\varphi - \widetilde{\chi}_n\varphi)\|_{l^2(\mathcal{V})} + \|d^1(\varphi - \widetilde{\chi}_n\varphi)\|_{l^2(\mathcal{F})} \to 0, \text{ when } n \to \infty.$$

The first and the third terms has already been shown in Proposition 5.1. For the following we need a derivation formula of δ^0 taken in [14]. Let $x \in \mathcal{V}$, for each $(f, \varphi) \in \mathcal{C}_c(\mathcal{V}) \times \mathcal{C}_c(\mathcal{E})$ we have

$$\delta^0(\widetilde{f}\varphi)(x) = f(x)\delta^0(\varphi)(x) - \frac{1}{2c(x)} \sum_{e, e^+ = x} r(e) d^0(f)(e)\varphi(e). \tag{5.1}$$

Therefore, by derivation formula (5.1), we get

$$\delta^0(\varphi - \widetilde{\chi}_n\varphi)(x) = (1 - \chi_n)(x)\delta^0(\varphi)(x) + \frac{1}{2c(x)} \sum_{e, e^+ = x} r(e) d^0\chi_n(e)\varphi(e).$$

As a consequence, because $\delta^0\varphi \in l^2(\mathcal{V})$, we have

$$\lim_{n\to\infty} \|(1 - \chi_n)\delta^0\varphi\|_{l^2(\mathcal{V})} = 0.$$

For the second term, we combine the property iii) of χ-completeness for a graph with the Cauchy-Schwarz inequality to obtain for all $x \in \mathcal{V}$,

$$\begin{aligned} \Big| \sum_{e, e^+ = x} r(e) d^0\chi_n(e)\varphi(e)\Big|^2 &\leq \sum_{e, e^+ = x} r(e)|d^0\chi_n(e)|^2 \sum_{e \in supp(d^0\chi_n), e^+ = x} r(e)|\varphi(e)|^2 \\ &\leq Cc(x) \sum_{e \in supp(d^0\chi_n), e^+ = x} r(e)|\varphi(e)|^2. \end{aligned}$$

So,

$$\begin{aligned} \sum_{x\in\mathcal{V}} \frac{1}{c(x)} \Big| \sum_{e, e^+ = x} r(e) d^0\chi_n(e)\varphi(e)\Big|^2 &\leq \sum_{x\in\mathcal{V}} C \sum_{e \in supp(d^0\chi_n), e^+ = x} r(e)\varphi(e)|^2 \\ &\leq C \sum_{e \in supp(d^0\chi_n)} r(e)|\varphi(e)|^2 \to 0, \text{ when } n \to \infty, \end{aligned}$$

by the properties (4.1) and (4.2).
Hence

$$F_n \to F \text{ in } \mathcal{H},\ TF_n \to T_{min}F \text{ in } \mathcal{H},$$

where $F_n = (f_n, \widetilde{\chi}_n\varphi, \phi_n)$ and $T_{min}F(f,\varphi,\phi) = (\delta^0_{min}\varphi, d^0_{min}f + \delta^1_{min}\phi, d^1_{min}\varphi)$. Then $F \in Dom(T_{min})$.

2^{th} *Step:* To show that T is essentially self-adjoint, we will prove that $T_{max} = T_{min}$. By the first step, Theorem 1 in [1] and Proposition 5.1 it remains to show that:

$$Dom(T_{max}) \subseteq Dom(d^0_{max}) \oplus \left(Dom(\delta^0_{max}) \cap Dom(d^1_{max})\right) \oplus Dom(\delta^1_{max}).$$

Let $F = (f,\varphi,\phi) \in Dom(T_{max})$ then $TF \in \mathcal{H}$. This implies that $\delta^0\varphi \in l^2(\mathcal{V})$, $d^0f + \delta^1\phi \in l^2(\mathcal{E})$ and $d^1\varphi \in l^2(\mathcal{F})$. As consequence, by the definition of δ^0_{max} and d^1_{max} we have $\varphi \in Dom(\delta^0_{max}) \cap Dom(d^1_{max})$. Moerever, by χ-completness of $\mathcal{T}$, there exists a sequence of cut-off functions $(\chi_n)_n \subseteq \mathcal{C}_c(\mathcal{V})$. Then, the parallelogram identity with Lemma 2.8 we get

$$\|d^0(\chi_n f) + \delta^1(\widetilde{\widetilde{\chi}}_n\phi)\|^2_{l^2(\mathcal{E})} = \|d^0\chi_n f\|^2_{l^2(\mathcal{E})} + \|\delta^1\widetilde{\widetilde{\chi}}_n\phi\|^2_{l^2(\mathcal{E})}.$$

Now, it remains to prove that $d^0(\chi_n f) + \delta^1(\widetilde{\widetilde{\chi}}_n\phi)$ converges in $l^2(\mathcal{E})$. Indeed, we need some formulas taken in Lemma 2.9 and [14] to give that:

$$d^0(\chi_n f) = \widetilde{\chi}_n d^0(f) + \widetilde{f}d^0(\chi_n).$$

$$\delta^1(\widetilde{\widetilde{\chi}}_n\phi)(e) = \widetilde{\chi}_n(e)\delta^1(\phi)(e) + \underbrace{\frac{1}{6r(e)}\sum_{x\in\mathcal{F}_e} s(e,x)\left[d^0(\chi_n)(e^-,x) + d^0(\chi_n)(e^+,x)\right]\phi(e,x)}_{\mathcal{I}_n(e)}.$$

Therefore, we have

$$\begin{aligned}\|d^0(f-\chi_n f) + \delta^1(\phi - \widetilde{\widetilde{\chi}}_n\phi)\|^2_{l^2(\mathcal{E})} &= \|(1-\widetilde{\chi}_n)(d^0f + \delta^1\phi) + \widetilde{f}d^0\chi_n + \mathcal{I}_n\|^2_{l^2(\mathcal{E})} \\ &\leq 3\left(\|(1-\widetilde{\chi}_n)(d^0f+\delta^1\phi)\|^2_{l^2(\mathcal{E})} + \|\widetilde{f}d^0\chi_n\|^2_{l^2(\mathcal{E})} + \|\mathcal{I}_n\|^2_{l^2(\mathcal{E})}\right)\end{aligned}$$

Because $d^0f + \delta^1\phi \in l^2(\mathcal{E})$, we have

$$\lim_{n\to\infty}\|(1-\widetilde{\chi}_n)(d^0f+\delta^1\phi)\|^2_{l^2(\mathcal{E})} = 0.$$

By Proposition 5.1 we have

$$\lim_{n\to\infty}\|\mathcal{I}_n\|^2_{l^2(\mathcal{E})} = 0.$$

Moreover, by the hypothesis iii) of χ-completeness we have

$$\begin{aligned}\|\widetilde{f}d^0\chi_n\|^2_{l^2(\mathcal{E})} &= \frac{1}{2}\sum_{e\in\mathcal{E}} r(e)|\widetilde{f}(e)d^0(\chi_n)(e)|^2\\ &\leq \sum_{e\in\mathcal{E}} r(e)|f(e^+)|^2|d^0(\chi_n)(e)|^2\\ &= \sum_{x\in\mathcal{V}}|f(x)|^2\sum_{e,e^+=x}|r(e)d^0\chi_n(e)|^2\\ &\leq C\sum_{x\in\mathcal{V}_n} c(x)|f(x)|^2\end{aligned}$$

where $\mathcal{V}_n := \{x \in \mathcal{V}, \exists e \in supp(d^0\chi_n)$ such that $e^+ = x\}$. This term tends to 0 by the property (4.2). □

Theorem 5.4 *Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a χ-complete triangulation. Then T is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$ if and only if L is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.*

Proof

Since

$$T(\mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})) \subseteq \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}),$$

using the same technique in the proof of Proposition 13 in [1], the result holds. □

Corollary 5.5 *Let $\mathcal{T} = (\mathcal{K}, \mathcal{F})$ be a χ-complete triangulation then L is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.*

6 Examples

6.1 A Triangulation with 1-Dimensional Decomposition

We now strengthen the previous example and follow ideas of [2] and [3].

Definition 6.1 *(1-dimensional decomposition)* A *1-dimensional decomposition* of the graph $\mathcal{K} = (\mathcal{V}, \mathcal{E})$ is a family of finite sets $(\mathcal{S}_n)_{n\in\mathbb{N}}$ which forms a partition of $\mathcal{V}$, that is $\mathcal{V} = \sqcup_{n\in\mathbb{N}}\mathcal{S}_n$, such that for all $x \in \mathcal{S}_n$, $y \in \mathcal{S}_m$ (Fig. 3),

$$(x, y) \in \mathcal{E} \Rightarrow |n - m| \leq 1.$$

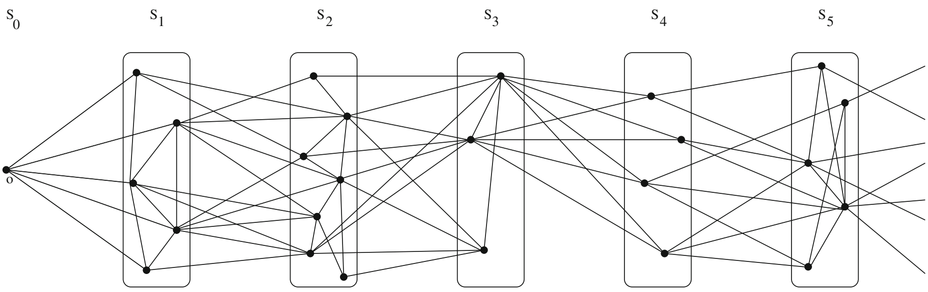

Fig. 3 A triangulation with 1-dimensional decomposition

Given such a *1-dimensional decomposition*, we write $B_n := \cup_{i=0}^{n}\mathcal{S}_i$. We set,

$$(*)\begin{cases} \deg^{\pm}_{\mathcal{S}_n}(x) := \dfrac{1}{c(x)} \displaystyle\sum_{y\in\mathcal{V}(x)\cap\mathcal{S}_{n\pm1}} r(x,y) & \text{for all } x\in\mathcal{S}_n,\\ \deg^{0}_{\mathcal{S}_n}(x) := \dfrac{1}{c(x)} \displaystyle\sum_{y\in\mathcal{V}(x)\cap\mathcal{S}_{n}} r(x,y) & \text{for all } x\in\mathcal{S}_n,\\ \deg_{\mathcal{S}_n\times\mathcal{S}_{n+1}}(e) := \dfrac{1}{r(e)} \displaystyle\sum_{x\in\mathcal{F}_e\cap(\mathcal{S}_n\cup\mathcal{S}_{n+1})} s(e,x) & \text{for all } e\in\mathcal{S}_n\times\mathcal{S}_{n+1},\\ \deg^{0}_{\mathcal{S}_n^2}(e) := \dfrac{1}{r(e)} \displaystyle\sum_{x\in\mathcal{F}_e\cap\mathcal{S}_n} s(e,x) & \text{for all } e\in\mathcal{S}_n^2,\\ \deg^{\pm}_{\mathcal{S}_n^2}(e) := \dfrac{1}{r(e)} \displaystyle\sum_{x\in\mathcal{F}_e\cap\mathcal{S}_{n\pm1}} s(e,x) & \text{for all } e\in\mathcal{S}_n^2. \end{cases}$$

We denote

$$\eta_n^{\pm} := \sup_{x\in\mathcal{S}_n} \deg^{\pm}_{\mathcal{S}_n}(x),\ \ \beta_n := \sup_{e\in\mathcal{S}_n\times\mathcal{S}_{n+1}} \deg_{\mathcal{S}_n\times\mathcal{S}_{n+1}}(e),\ \ \gamma_n^{\pm} := \sup_{e\in\mathcal{S}_n^2} \deg^{\pm}_{\mathcal{S}_n^2}(e).$$

Theorem 6.2 *Let $\mathcal{T}=(\mathcal{K},\mathcal{F})$ be a triangulation and $(\mathcal{S}_n)_{n\in\mathbb{N}}$ a 1-dimensional decomposition of the graph $\mathcal{K}$. Assume that*

$$\sum_{n\in\mathbb{N}} \frac{1}{\sqrt{\xi(n,n+1)}} = \infty,$$

with $\xi(n,n+1) = \eta_n^{+} + \eta_{n+1}^{-} + \beta_n + \gamma_n^{+} + \gamma_{n+1}^{-}$. Then $\mathcal{T}$ is χ-complete and in particular, L is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})\oplus\mathcal{C}_c(\mathcal{E})\oplus\mathcal{C}_c(\mathcal{F})$.

Proof
We set

$$\chi_n(x) = \begin{cases} 1 & \text{if } d_{comb}(o,x)\leqslant n,\\ \max\left(0, 1 - \displaystyle\sum_{k=n}^{d_{comb}(o,x)-1} \frac{1}{\sqrt{\xi(k,k+1)}}\right) & \text{if } d_{comb}(o,x) > n. \end{cases}$$

Since the series diverges, χ_n is with finite support. Note that χ_n is constant on $\mathcal{S}_n$. If $x\in\mathcal{S}_m$ with $m>n$, we have

$$\frac{1}{c(x)} \sum_{y\in\mathcal{V}(x)\cap\mathcal{S}_{m+1}} r(x,y)|\chi_n(x)-\chi_n(y)|^2 \leq \frac{\deg^{+}_{\mathcal{S}_m}(x)}{\xi(m,m+1)} \leq 1.$$

$$\frac{1}{c(x)} \sum_{y\in\mathcal{V}(x)\cap\mathcal{S}_{m}} r(x,y)|\chi_n(x)-\chi_n(y)|^2 = 0.$$

$$\frac{1}{c(x)} \sum_{y\in\mathcal{V}(x)\cap\mathcal{S}_{m-1}} r(x,y)|\chi_n(x)-\chi_n(y)|^2 \leq \frac{\deg^{-}_{\mathcal{S}_m}(x)}{\xi(m-1,m)} \leq 1.$$

On the other hand,

- If $e \in \mathcal{S}_m \times \mathcal{S}_{m+1}$, we get
$$\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e \cap (\mathcal{S}_m \cup \mathcal{S}_{m+1})} s(e,x)|(\chi_n(x) - \chi_n(e^-)) + (\chi_n(x) - \chi_n(e^+))|^2 \leq \frac{\deg^+_{\mathcal{S}_m \times \mathcal{S}_{m+1}}(x,y)}{\xi(m,m+1)} \leq 1.$$
- If $e \in \mathcal{S}_m^2$, we get
$$\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e \cap \mathcal{S}_m} s(e,x)|(\chi_n(x) - \chi_n(e^-)) + (\chi_n(x) - \chi_n(e^+))|^2 = 0$$
$$\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e \cap \mathcal{S}_{m+1}} s(e,x)|(\chi_n(x) - \chi_n(e^-)) + (\chi_n(x) - \chi_n(e^+))|^2 \leq \frac{\deg^+_{\mathcal{S}^2_m}(e)}{\xi(m,m+1)} \leq 1.$$
$$\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e \cap \mathcal{S}_{m-1}} s(e,x)|(\chi_n(x) - \chi_n(e^-)) + (\chi_n(x) - \chi_n(e^+))|^2 \leq \frac{\deg^-_{\mathcal{S}^2_m}(e)}{\xi(m-1,m)} \leq 1.$$

Then $\mathcal{T}$ is χ-complete and in particular, L is essentially self-adjoint by Corollary 5.5. □

6.2 A Triangular Tree

Let $\mathcal{T}$ be a triangular tree, endowed with an origin. Due to its structure, one can take

$$(**) \begin{cases} \deg^-_{\mathcal{S}_n}(x) := \dfrac{1}{c(x)} r(x, \overleftarrow{x}) & \text{for all } x \in \mathcal{S}_n, \\ \deg_{\mathcal{S}_n \times \mathcal{S}_{n+1}}(e) := \dfrac{1}{r(e)} \displaystyle\sum_{x \in \mathcal{F}_e \cap \mathcal{S}_{n+1}} s(e,x) & \text{for all } e \in \mathcal{S}_n \times \mathcal{S}_{n+1}, \\ \deg^-_{\mathcal{S}^2_n}(e) := \dfrac{1}{r(e)} s(e, \overleftarrow{e}) & \text{for all } e \in \mathcal{S}^2_n, \end{cases}$$

where $\overleftarrow{e}$ is a unique vertex in $\mathcal{S}_{n-1} \cap \mathcal{F}_e$.

Proposition 6.3 *Let $\mathcal{T}$ be a triangular tree with its origin o. Assume that*

$$\sum_{n \in \mathbb{N}} \frac{1}{\sqrt{\xi(n,n+1)}} = \infty,$$

with $\xi(n,n+1) = \eta^+_n + \eta^-_{n+1} + \beta_n + \gamma^-_{n+1}$. Then $\mathcal{T}$ is χ-complete and in particular, L is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.

Proof Use the same method as Theorem 6.2 with $(**)$. □

6.3 Essential Self-Adjointness on the Simple Case

In [20] and [6], they prove that L_0 is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$ when the graph is simple. But the self-adjointness property does not always hold with other operators in the simple case. We recall the operator L_1^- is not necessarily essentially self-adjoint on simple tree, see [3]. Moreover, we refer to [11] for the *adjacency matrix* $\mathcal{A}_{\mathcal{K}} = deg - L_0$ where *deg* denotes the operator of multiplication with the functions which shows that the deficiency indices of $\mathcal{A}_{\mathcal{K}}$ are infinite. In this framework, it is important to notice that L_1 and L_2 are not necessarily essentially self-adjoint on a simple triangulation.

Proposition 6.4 *Let $\mathcal{T}$ be a simple triangular tree. Assume that*

$$off(n) = \#\left(\mathcal{V}(x) \cap \mathcal{S}_{n+1}\right), \; x \in \mathcal{S}_n \tag{6.1}$$

$$n \mapsto \frac{off^2(n)}{off(n+1)} \in l^1(\mathbb{N}).$$

Then, L_1 is not essentially self-adjoint on $\mathcal{C}_c(\mathcal{E})$ and the deficiency indices are infinite.

Proof

We construct $\varphi \in l^2(\mathcal{E})\backslash\{0\}$, such that $\varphi \in Ker(L_1^* + i)$ and such that φ is equal to constant C_n on $\mathcal{S}_n \times \mathcal{S}_{n+1}$. It takes the value 0 on $\mathcal{S}_n^2$. Given the fact that $(x, y) \in \mathcal{S}_n^2$, we get

$$C_n \left(\#\left(\mathcal{V}(x) \cap \mathcal{S}_{n+1}\right) - \#\left(\mathcal{V}(y) \cap \mathcal{S}_{n+1}\right)\right) = 0.$$

It holds because of the condition (6.1). Now, we set $(x, y) \in \mathcal{S}_n \times \mathcal{S}_{n+1}$ and with the condition (6.1), we have

$$(\mathrm{off}(n) + 1 + i)C_n - \mathrm{off}(n+1)C_{n+1} - C_{n-1} = 0.$$

We can then apply Theorem 5.1 in [3] to obtain the conclusion. □

We will see now also that L_2 is not necessarily essentially self-adjoint on simple triangulation.

Proposition 6.5 *Let $\mathcal{T}$ be a simple triangulation with 1-dimensional decomposition as shown in* Fig. 4. *Assume that*

$$n \mapsto \frac{\#\mathcal{S}_{2n}}{\#\mathcal{S}_{2(n+1)}} \in l^1(\mathbb{N}). \tag{6.2}$$

Then, L_2 is not essentially self-adjoint on $\mathcal{C}_c(\mathcal{F})$.

Proof We consider the faces in Fig. 4 as follows:

$$\varpi \in \mathcal{F} \Leftrightarrow \text{ there exists } n \in \mathbb{N}\cup\{0\} \text{ such that } \varpi = (e_{2n+1}, x) \in \left(\mathcal{S}_{2n+1}^2 \times \mathcal{S}_{2n}\right) \cup \left(\mathcal{S}_{2n+1}^2 \times \mathcal{S}_{2n+2}\right).$$

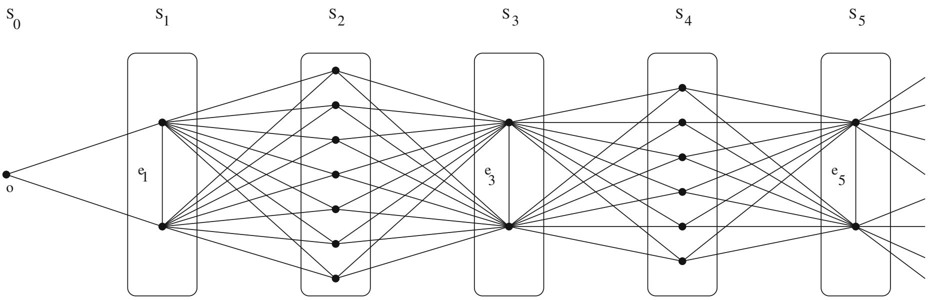

Fig. 4 A particular triangulation with 1-dimensional decomposition

Set $\phi \in l^2(\mathcal{F})\backslash\{0\}$ such that $\phi \in Ker(L_2^* + i)$. For $n \in \mathbb{N}$, it is given by

$$\phi(e_{2n+1}, x) := \begin{cases} C_{2n+2} & \text{for all } x \in \mathcal{S}_{2n+2}. \\ C_{2n} & \text{for all } x \in \mathcal{S}_{2n}. \end{cases}$$

Let $x \in \mathcal{S}_{2n+2}$, we have

$$\begin{aligned}(L_2^* + i)(\phi)(e_{2n+1}, x) &= \sum_{u \in \mathcal{F}_{e_{2n+1}}} \phi(e_{2n+1}, u) + \sum_{u \in \mathcal{F}_{(e_{2n+1}^+, x)}} \phi(e_{2n+1}^+, x, u) \\ &+ \sum_{u \in \mathcal{F}_{(x, e_{2n+1}^-)}} \phi(x, e_{2n+1}^-, u) \\ &+ i\phi(e_{2n+1}, x) = 0.\end{aligned}$$

Hence, we get

$$(\#\mathcal{S}_{2n+2} + 2 + i)\, C_{2n+2} + (\#\mathcal{S}_{2n})\, C_{2n} = 0. \tag{6.3}$$

By the Eq. 6.3, we get

$$\begin{aligned}\|\phi_{|\mathcal{S}_{2n+1}^2 \times \mathcal{S}_{2n+2}}\|^2_{l^2(\mathcal{F})} &= \frac{1}{6} \sum_{[x,y,z] \in \mathcal{S}_{2n+1}^2 \times \mathcal{S}_{2n+2}} |\phi(x, y, z)|^2 \\ &= \frac{1}{6} (\#C_{2n+2})^2 (\#\mathcal{S}_{2n+2}) \\ &= \frac{1}{6} \frac{(\#C_{2n})^2 (\#\mathcal{S}_{2n})^2}{|\#\mathcal{S}_{2n+2} + 2 + i|^2} (\#\mathcal{S}_{2n+2}) \\ &= \frac{(\#\mathcal{S}_{2n})(\#\mathcal{S}_{2n+2})}{|\#\mathcal{S}_{2n+2} + 2 + i|^2} \|\phi_{|\mathcal{S}_{2n-1}^2 \times \mathcal{S}_{2n}}\|^2_{l^2(\mathcal{F})}\end{aligned}$$

Since $\lim_{n \to \infty} \frac{\#\mathcal{S}_{2n}}{\#\mathcal{S}_{2(n+1)}} = 0$, we get by induction

$$C := \sup_{n \in \mathbb{N}^*} \|\phi_{|\mathcal{S}_{2n-1}^2 \times \mathcal{S}_{2n}}\|^2_{l^2(\mathcal{F})} < \infty.$$

Thus

$$\|\phi_{|\mathcal{S}_{2n+1}^2 \times \mathcal{S}_{2n+2}}\|^2_{l^2(\mathcal{F})} \leq C \frac{(\#\mathcal{S}_{2n})(\#\mathcal{S}_{2n+2})}{|\#\mathcal{S}_{2n+2} + 2 + i|^2}.$$

From Eq. 6.2, we conclude that $\phi \in l^2(\mathcal{F})$. By mimicking the proof of Theorem X.36 of [16] one shows that dimKer$(L_2^* + i) \geq 1$ and thus L_2 is not essentially self-adjoint on $\mathcal{C}_c(\mathcal{F})$. □

Remark 6.6 By one of Propositions 6.5 and 6.4, we conclude that L is not necessarily essentially self-adjoint on a simple triangulation.

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