Potential Anal (2018) 49:91–103
DOI 10.1007/s11118-017-9649-3

Upper Bounds for the First Stability Eigenvalue of Surfaces in 3-Riemannian Manifolds

M. Batista[1,2] · J. I. Santos[1]

Received: 12 September 2016 / Accepted: 1 August 2017 / Published online: 31 August 2017

Abstract Our target in this paper is given upper bounds for the first stability eigenvalue of closed (compact without boundary) surfaces in a 3-Riemannian manifold endowed with a smooth density function. As consequence, we deduce a topological constraint for the existence of closed stable surfaces in non-negatively curved spaces and a result of no existence of closed stable self-shrinkers of the mean curvature flow in $\mathbb{R}^3$.

Keywords Eigenvalues · Weighted mean curvature · Closed surfaces

Mathematics Subject Classifications (2010) 53C23 · 58J50

1 Introduction

The study of Riemannian manifolds endowed with a smooth density function has flourished in last few years, and a much better understanding of their analytic and geometric structure has evolved. We emphasize in that setting, the solution of Poincaré conjecture, the relaxation of the conditions for solve the Monge's problem for mass transportation, the behavior of singularities of the Ricci flow, the mean curvature flow among others, see [7, 8, 14–17, 21, 22] and references therein. Moreover, the theory of these spaces and the generalized

The first author was partially supported by CNPq, FAPEAL and CAPES/Brazil and the second author was supported by CAPES/Brazil.

✉ J. I. Santos
jissivan@gmail.com

M. Batista
mhbs@mat.ufal.br

1 IM, Universidade Federal de Alagoas, Maceió, AL, 57072-970, Brazil

2 Present address: Departament of Mathematics, Princeton University, Princeton, NJ 08540, USA

curvatures goes back to Lichnerowicz [12, 13] and more recently by Bakry and Émery [4], in context of diffusion process, and it has been very actived field in recent years.

Now we will introduce some concepts that will be used in this paper. Firstly, we recall that a *weighted manifold* is a Riemannian manifold (M^3, g) with a real-valued smooth function $f : M \to \mathbb{R}$ which is used as a density to measure geometric objects on M. Associated to this structure we have an important second order differential operator defined by

$$\Delta_f u = \Delta u - \langle \nabla u, \nabla f \rangle,$$

where $u \in C^\infty$. This operator is known as *Drift Laplacian*.

Following [4, 17, 23], the natural generalizations of the sectional, Ricci and scalar curvatures are defined by

$$\overline{Sect}_f^{2m}(X, Y) = \overline{Sect}(X, Y) + \frac{1}{2}\left(\text{Hess}\, f(X, X) - \frac{(df(X))^2}{2m}\right), \tag{1.1}$$

$$\text{Ric}_f^{2m} = \text{Ric}_f - \frac{df \otimes df}{2m}, \tag{1.2}$$

where X and Y are unit and orthogonal vectors fields tangents to M, $m > 0$, $\text{Ric}_f = \text{Ric} + \text{Hess}\, f$, and

$$S_\infty = S + 2\Delta_M f - |\overline{\nabla} f|^2, \tag{1.3}$$

that last is known as *Perelman's scalar curvature*, see [6] for a good overview.

Now, we introduce some objects related with the theory of surfaces in a weighted manifold. Let $\Sigma \subset M^3$ be a two-sided surface of M^3 and consider N an unit normal vector field globally defined on Σ. We will denote by A its second fundamental form and by H the mean curvature of Σ, that is, the trace of A.

We recall that the *weighted mean curvature*, introduced by Gromov in [10], is given by

$$H_f = H + \langle N, \overline{\nabla} f \rangle.$$

Throughout this paper, $dv_f = e^{-f} dv$ denote the weighted measure of the surface Σ, where dv is the Riemannian measure of Σ, $|\Sigma|$ and $|\Sigma|_f$ denote the area of Σ with respect to the Riemannian measure and the weighted measure of Σ, respectively. Furthermore, we will denote by K the Gaussian curvature of Σ and by $\overline{Sect}_\Sigma$ the sectional curvature of M restricted to Σ.

It is a remarkable fact that, in the variational setting, surfaces with constant weighted mean curvature are stationary points of the weighted area functional under variations that preserves the weighted volume (see [5]). Moreover, the second variation of the weighted area gives to us the weighted Jacobi operator on Σ, see [8], which is defined by

$$J_f u = \Delta_f u + (|A|^2 + \text{Ric}_f(N, N))u, \tag{1.4}$$

for any $u \in C^\infty(\Sigma)$ and $|A|^2$ is the Hilbert-Schmidt norm of A.

In this paper, encouraged by the ideas in [1–3, 18], we study some geometric aspects of surfaces with constant weighted mean curvature. More specifically, we study problems related with the first eigenvalue of the weighted Jacobi operator on closed surfaces.

We point out that the approach used in this study allows us to generalize a result obtained by Schoen and Yau on stable minimal surfaces in 3-Riemannian manifolds with nonnegative scalar curvature for the setting of manifolds with density. As far as we know, our result is new even in the Riemannian case. Now, we are able to present our main result.

It is read as follows:

Theorem 1.1 *Let (M^3, g, f) be a weighted manifold with $S_\infty \geq 6c$, for some $c \in \mathbb{R}$. Let $\Sigma^2 \subset M^3$ be a closed surface with constant weighted mean curvature H_f. Then,*

$$\lambda_1 \leq -\frac{1}{2}(H_f^2 + 6c) - \frac{4\pi(g-1)}{|\Sigma|}.$$

Moreover, equality holds if and only if Σ is totally geodesic, f is constant on Σ, $S_\infty|_\Sigma = 6c$ and K is constant.

Remark 1 In Riemannian case, $f = 0$, the estimate can be improved. See the corollary in Section 3.1.

Now we will provide the notion of stability to our context and then we will present some consequences of our result.

Definition 1 Under the above notation. We say that a surface Σ is stable if the first eigenvalue λ_1 of the Jacobi operator is nonnegative. Otherwise, we say that Σ is unstable.

The next result is a generalization of a result due to Schoen and Yau on stable minimal surfaces (see [19]) and this technique allows us to give an improvement of Theorem 2.1 in [9].

The result is the following:

Corollary 1.1 *Let (M^3, g, f) be a weighted manifold with nonnegative Perelman's scalar curvature. Let Σ be a closed stable surface with constant weighted mean curvature H_f. Then Σ is conformally equivalent to the sphere $\mathbb{S}^2$ or Σ is a totally geodesic flat torus $\mathbb{T}^2$. Moreover, if $S_\infty > 0$, then Σ is conformally equivalent to the sphere $\mathbb{S}^2$.*

Our second result is the following:

Theorem 1.2 *Let (M^3, g, f) be a weighted manifold with $\overline{Sect} \geq c$, for some $c \in \mathbb{R}$, and* Hess $f \geq \dfrac{df \otimes df}{2m}$ *(in the sense of quadratic forms). Let $\Sigma^2 \subset M^3$ be a closed surface with constant weighted mean curvature H_f. Then,*

(i) $\lambda_1 \leq -\frac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right)$, *with equality if and only if Σ is totally umbilical in M^3,* $\mathrm{Ric}(N, N) = 2c$, $df(N) = \dfrac{m}{1+m}H_f$ *on Σ and* $\mathrm{Hess}\, f(N, N) = \dfrac{df(N)^2}{2m}$;

(ii) $\lambda_1 \leq -\dfrac{H_f^2}{(1+2m)} - 4c + \dfrac{2}{|\Sigma|_f}\int_\Sigma K\, dv_f$. *Furthermore, equality holds if and only if K is constant, $\overline{Sect}_\Sigma = c$, $df(N) = \dfrac{m}{1+m}H_f$ on Σ and* $\mathrm{Hess}\, f(N, N) = \dfrac{df(N)^2}{2m}$.

Our third result reads as follows:

Theorem 1.3 *Let (M^3, g, f) be a weighted manifold with $\overline{Sect}_f^{2m} \geq c$, for some $c \in \mathbb{R}$, and $Hessf \leq \sigma \cdot g$ for some real function σ on M. Let $\Sigma^2 \subset M^3$ be a closed surface with constant weighted mean curvature H_f. Then,*

(i) $\lambda_1 \leq -\frac{1}{2}\left(\frac{H_f^2}{1+m}+4c\right)$, *with equality if and only if* Σ *is totally umbilical in* M^3, $\mathrm{Ric}_f^{2m} = 2c$ *and* $df(N) = \dfrac{m}{1+m}H_f$ *on* Σ;

(ii) $\lambda_1 \leq -\dfrac{H_f^2}{(1+2m)} - \left(4c - \dfrac{\int_\Sigma \sigma\, dv_f}{|\Sigma|_f}\right) + \dfrac{2}{|\Sigma|_f}\int_\Sigma K\, dv_f$.

Moreover, if equality holds, then $\overline{Sect}_f^{2m} = c$, $\mathrm{Ric}_f^{2m} = 2c$, $df(N) = \frac{2m}{1+2m}H_f$, *and* $|A|$ *is a constant on* Σ. *Moreover,* M^3 *has constant sectional curvature k and* e^{-f} *is the restriction of a coordinate function from the appropriate canonical embedding of* $\mathbb{Q}_k^3$ *in* $\mathbb{E}^4$, *where* $\mathbb{E}^4$ *is* $\mathbb{R}^4$ *or* $\mathbb{L}^4$.

Remark 2 We believe the hypotheses on the function f in Theorems 1.2 and 1.3 are natural, because we recovered the Riemannian case if the function is constant and also, for m large enough, we captured huge regions in the Gaussian space, which is very important in literature.

Now, we will give an application on the context of mean curvature flow. For that, we recall that a *self-shrinker* of the mean curvature flow is an oriented surface $\Sigma \subset \mathbb{R}^3$ such that

$$H = -\frac{1}{2}\langle x, N\rangle,$$

where N is an unit normal vector field on Σ. So, if we consider $\mathbb{R}^3$ endowed with the function $f(x) = \frac{|x|^2}{4}$, then a self-shrinker is a f-minimal surface in the Euclidean space. More generally, the triple $(\mathbb{R}^3, \delta_{ij}, |x|^2/4)$ is known as *Gaussian space* and the surfaces with weighted mean curvature λ are know as λ-*surfaces*.

The next result is a consequence of the proof of the Theorem 1.2 and it reads as follows:

Corollary 1.2 *All closed* λ*-surfaces in the Gaussian space are unstable. In particular, there exists no closed stable self-shrinker surfaces in* $\mathbb{R}^3$.

The paper is organized in this way: In Section 2 we give a classification of weighted manifolds with constant weighted sectional curvature, we also provide a way to describe the first eigenvalue of the weighted Jacobi operator and, to conclude the section, we rewrite the terms of the weighted Jacobi operator in an appropriate way. In Section 3 we present the proof of the results and others consequences of them.

2 Preliminaries

An important result for us is the classification of weighted manifolds with constant weighted sectional curvature. The result below follows closely the one in [23], and we include the proof here for the sake of completeness.

Lemma 1 *Let* (M^3, g, f) *be a weighted manifold. Assume that* $\overline{Sect}_f^{2m} = c$, *then M has constant sectional curvature k, for some* $k \in \mathbb{R}$. *Moreover, if f is a non constant function, then* $c = -(m-1)k$, *and* $u = e^{-f/m}$ *is the restriction of a coordinate function from the appropriate canonical embedding of a space form of curvature k,* $\mathbb{Q}_k^3$, *in* $\mathbb{E}^4$, *where* $\mathbb{E}^4$ *is* $\mathbb{R}^4$ *or* $\mathbb{L}^4$.

Proof Let X and Y be an unit and orthogonal vectors on M. Then, by Eq. 1.1, we get

$$c = \overline{Sect}\,(X, Y) + \frac{1}{2}\left(\text{Hess}\, f(X, X) - \frac{(df(X))^2}{2m}\right)$$

and

$$c = \overline{Sect}\,(Y, X) + \frac{1}{2}\left(\text{Hess}\, f(Y, Y) - \frac{(df(Y))^2}{2m}\right).$$

So, there exists a smooth function $w : M \to \mathbb{R}$ such that

$$\text{Hess}\, f - \frac{df \otimes df}{2m} = w \cdot g.$$

Then, letting $\{E_1, E_2, X\}$ be an orthonormal frame and adding up the weighted sectional curvature on the plane spanned by $\{E_i, X\}$, $i = 1, 2$, we have

$$2c = \text{Ric}(X, X) + 2w.$$

Thus, by Schur's Lemma, w is a constant function and so M has constant sectional curvature, let's say k. Defining the function $u = e^{-f/m}$, we have that

$$\text{Hess}\, u = -\frac{c-k}{m} u \cdot g. \tag{2.1}$$

So, by Lemma 1.2 in [20],

$$g = dt^2 + (u')^2 g_0, \tag{2.2}$$

where g_0 is a local metric on a surface orthogonal to ∇u (a level set of u) and u' denotes the derived of u in the direction of the gradient of u.

Computing the radial sectional curvature of the metric (2.2), we have $(c+(m-1)k)u' = 0$. Since f is non constant, we have that $c = -(m-1)k$. Moreover, as u satisfies equations (2.1) and (2.2), u is the restriction of a coordinate function from the appropriate canonical embedding of $\mathbb{Q}^3_k$ in $\mathbb{E}^4$, where $\mathbb{E}^4$ is $\mathbb{R}^4$ or $\mathbb{L}^4$. □

Now we will describe the first stability eigenvalue in an appropriate manner. For this, consider a first eigenfunction $\rho \in C^\infty(\Sigma)$ of the Jacobi operator J_f, that is, $J_f \rho = -\lambda_1 \rho$; or equivalently,

$$-\Delta_f \rho = (\lambda_1 + |A|^2 + \text{Ric}_f(N, N))\rho. \tag{2.3}$$

Furthermore, λ_1 is simple and it is characterized by

$$\lambda_1 = \inf\left\{\frac{-\int_\Sigma u J_f u\, dv_f}{\int_\Sigma u^2\, dv_f} : u \in C^\infty(\Sigma),\ u \neq 0\right\}. \tag{2.4}$$

We observe that the first eigenfunction of an elliptic second-order differential operator has a sign. Therefore, without loss of generality, we can assume that $\rho > 0$.

Thus,

$$\begin{aligned}
\Delta_f \ln \rho &= \Delta \ln \rho - \langle \nabla f, \nabla \ln \rho \rangle \\
&= \text{div}_\Sigma (\nabla \ln \rho) - \langle \nabla f, \rho^{-1} \nabla \rho \rangle \\
&= \text{div}_\Sigma (\rho^{-1} \nabla \rho) - \rho^{-1} \langle \nabla f, \nabla \rho \rangle \\
&= \rho^{-1} \text{div}_\Sigma (\nabla \rho) + \langle \nabla \rho^{-1}, \nabla \rho \rangle - \rho^{-1} \langle \nabla f, \nabla \rho \rangle \\
&= \rho^{-1} (\Delta \rho - \langle \nabla f, \nabla \rho \rangle) - \rho^{-2} |\nabla \rho|^2 \\
&= \rho^{-1} \Delta_f \rho - \rho^{-2} |\nabla \rho|^2 \\
&= -(\lambda_1 + |A|^2 + \text{Ric}_f(N, N)) - \rho^{-2} |\nabla \rho|^2.
\end{aligned} \tag{2.5}$$

Integrating the equality above on Σ with respect to the weighted measure dv_f and using the divergence theorem we have that

$$0 = -\lambda_1 |\Sigma|_f - \int_\Sigma (|A|^2 + \mathrm{Ric}_f(N, N))\, dv_f - \alpha,$$

where $\alpha_f := \int_\Sigma \rho^{-2} |\nabla \rho|^2 \, dv_f \geq 0$ defines a simple invariant that is independent of the choice of ρ, because λ_1 is simple. So,

$$\lambda_1 = -\frac{1}{|\Sigma|_f} (\alpha_f + \int_\Sigma (|A|^2 + \mathrm{Ric}_f(N, N))\, dv_f). \tag{2.6}$$

Let $\{E_i\}$ be an orthonormal frame in $T\Sigma$ and $\{a_{ij}\}$ the coefficients of A in the frame, using the Gauss equation

$$K = \overline{Sect}_\Sigma - \langle A(X), Y\rangle^2 + \langle A(X), X\rangle\langle A(Y), Y\rangle,$$

we have that

$$K - \overline{Sect}_\Sigma = a_{11}a_{22} - a_{12}^2 = \frac{1}{2}\left((a_{11} + a_{22})^2 - \sum_{i,j=1}^{2} a_{ij}^2\right) = \frac{1}{2}\left(H^2 - |A|^2\right),$$

hence

$$|A|^2 = H^2 + 2(\overline{Sect}_\Sigma - K). \tag{2.7}$$

To complete this section, we recall the traceless of the second fundamental form of Σ, that is, the tensor ϕ defined by $\phi = A - \frac{H}{2} I$, where I denotes the identity endomorphism on $T\Sigma$. We note that $tr(\phi) = 0$ and $|\phi|^2 = |A|^2 - \frac{H^2}{2} \geq 0$, with equality if and only if Σ is totally umbilical, where $|\phi|^2$ is the Hilbert-Schmidt norm.

In the literature, ϕ is know as the total umbilicity tensor of Σ. In terms of ϕ, the Jacobi operator is rewritten as

$$J_f u = \Delta_f u + \left(|\phi|^2 + \frac{H^2}{2} + \mathrm{Ric}_f(N, N)\right) u. \tag{2.8}$$

We use exactly this expression in next section to obtain an estimate of the first eigenvalue of the weighted Jacobi operator.

3 Proofs

3.1 Proof of the Theorem 1.1

We start with a straightforward computation. Let $\{e_1, e_2, e_3\}$ be an adapted referential of Σ to M. Lets rewrite the expression $|A|^2 + \mathrm{Ric}_f(N, N)$. We know that

$$\frac{S}{2} = \overline{Sect}_\Sigma + \mathrm{Ric}(e_3),$$

where S is the scalar curvature of M. By Gauss' equation (2.7), we have

$$\overline{Sect}_\Sigma = K - \frac{H^2}{2} + \frac{|A|^2}{2}.$$

Setting $e_3 = N$ and $f_3 = \langle \overline{\nabla} f, e_3\rangle$ we handle the potential of the stability operator as follows:

$$\begin{aligned} |A|^2 + \mathrm{Ric}_f(N,N) &= \frac{S}{2} - K + \frac{H^2}{2} + \frac{|A|^2}{2} + \mathrm{Hess}\, f(e_3,e_3) \\ &= \frac{1}{2}S_\infty - \Delta_M f + \frac{1}{2}|\overline{\nabla} f|^2 - K + \frac{H^2}{2} + \frac{|A|^2}{2} + \mathrm{Hess}\, f(e_3,e_3) \\ &= \frac{1}{2}S_\infty - (\Delta_\Sigma f - Hf_3 + \mathrm{Hess}\, f(e_3,e_3)) + \frac{1}{2}(|\nabla f|^2 + f_3^2) \\ &\quad - K + \frac{H^2}{2} + \frac{|A|^2}{2} + \mathrm{Hess}\, f(e_3,e_3) \\ &= \frac{1}{2}S_\infty - K - \Delta_\Sigma f + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}H_f^2 + \frac{1}{2}|A|^2. \end{aligned} \tag{3.1}$$

Integrating it with respect to Riemannian measure dv, using the divergence theorem and Gauss-Bonnet theorem we obtain

$$\int_\Sigma |A|^2 + \mathrm{Ric}_f(N,N)\, dv = 4\pi(g-1) + \frac{1}{2}\int_\Sigma (S_\infty + H_f^2 + |A|^2 + |\nabla f|^2)\, dv.$$

By the other hand, integrating (2.5) with respect to dv we obtain that

$$-\int_\Sigma \langle \frac{1}{\rho}\nabla\rho, \nabla f\rangle = -\lambda_1|\Sigma| - (\alpha + \int_\Sigma |A|^2 + \mathrm{Ric}_f(N,N)),$$

where $\alpha = \int_\Sigma \rho^{-2}|\nabla\rho|^2 dv$, and so,

$$-\int_\Sigma \left(\frac{|\nabla\rho|^2}{2\rho^2} + \frac{|\nabla f|^2}{2}\right) \le -\lambda_1|\Sigma| - (\alpha + \int_\Sigma |A|^2 + \mathrm{Ric}_f(N,N)).$$

After a simple computation we have that

$$\lambda_1 \le -\frac{1}{|\Sigma|}(\frac{\alpha}{2} + 4\pi(g-1) + \frac{1}{2}\int_\Sigma (S_\infty + H_f^2 + |A|^2).$$

By our hypothesis,

$$\lambda_1 \le -\frac{1}{2}(H_f^2 + 6c) - \frac{4\pi(g-1)}{|\Sigma|}.$$

Moreover, if equality holds then $\alpha = 0$ and thus ρ and f are constants on Σ, Σ is totally geodesic, $S_\infty|_\Sigma = 6c$ and K is constant. The reciprocal is immediate.

Now we provide a non-trivial example of a surface into a weighted manifold satisfying all hypothesis and the equality conclusion of Theorem 1.1.

Example 1 Consider the upper hemisphere

$$M = \mathbb{S}^3_+ = \{x \in \mathbb{R}^4 : |x| = 1 \text{ and } x_4 \ge 0\}$$

furnished with the standard euclidean metric and the density $f(p) = \frac{1}{2}r(p)^2$, where r is the distance to north pole. We let compute the first weighted stability eigenvalue of the great sphere $\Sigma = \partial\mathbb{S}^3_+$.

As it easy to see, Σ is totally geodesic with unit normal field $N = -\partial_r$ and the Gaussian curvature is equal to one. Moreover, a straightforward computation shows that the hessian of f is

$$Hess f = r\cot(r)(\langle\cdot,\cdot\rangle - dr\otimes dr) + dr\otimes dr.$$

Using the informations about Σ we infer that,

$$J_f = \Delta + 3,$$

where we used that $\Delta_f = \Delta$, $A = 0$ and $Ric_f(N) = 3$. So, $\lambda_1 = -3$, which is equal to term in the right hand side of the inequality of Theorem 1.1.

In Riemannian case, $f = 0$, we can improve the estimate in Theorem 1.1. The result is the following:

Corollary 3.1 *Let (M^3, g) be a Riemannian manifold with $S \geq 6c$, for some $c \in \mathbb{R}$. Let $\Sigma^2 \subset M^3$ be a closed surface with constant mean curvature H. Then,*

$$\lambda_1 \leq -\frac{3}{4}(H^2 + 4c) - \frac{4\pi(g-1)}{|\Sigma|}.$$

Moreover, equality holds if and only if Σ is totally umbilical, $S|_\Sigma = 6c$ and K is constant.

Proof The equation (3.1) can be rewrite, with $f = 0$, in the following way

$$|A|^2 + \mathrm{Ric}(N, N) = \frac{1}{2}S - K + \frac{3}{4}H^2 + \frac{1}{2}|\phi|^2.$$

After a straightforward computation we have that

$$\lambda_1 = -\frac{1}{|\Sigma|}(\alpha + 4\pi(g-1) + \frac{1}{2}\int_\Sigma (S + \frac{3}{2}H^2 + |\phi|^2),$$

and so

$$\lambda_1 \leq -\frac{3}{4}(H^2 + 4c) - \frac{4\pi(g-1)}{|\Sigma|}.$$

Moreover, if equality holds then $\alpha = 0$ and thus ρ is constant, Σ is totally umbilical, $S|_\Sigma = 6c$ and K is constant. The reciprocal is immediate. □

In the next subsection we will provide the prove of Theorem 1.2 and some consequences.

3.2 Proof of the Theorem 2.1

Using (2.7) in Eq. 2.6 we obtain that

$$\lambda_1 = -\frac{1}{|\Sigma|_f}\left\{\alpha_f - 2\int_\Sigma K\, dv_f + \int_\Sigma [H^2 + 2\overline{Sect}_\Sigma + \mathrm{Ric}_f(N, N)]\right\}.$$

So, using the definition of weighted mean curvature we have

$$\lambda_1 = -\frac{1}{|\Sigma|_f}\left\{\alpha_f - 2\int_\Sigma K\, dv_f + \int_\Sigma (H_f - \langle N, \overline{\nabla} f\rangle)^2 + \int_\Sigma [2\overline{Sect}_\Sigma + \mathrm{Ric}_f(N, N)]\right\}.$$

Moreover, we know that for all $a, b \in \mathbb{R}$ and $k > -1$, it holds that

$$(a + b)^2 \geq \frac{a^2}{1+k} - \frac{b^2}{k}, \tag{3.2}$$

with equality if and only if $b = -\frac{k}{1+k}a$. Applying that inequality with $k = 2m$, $a = H_f$, $b = -\langle \overline{\nabla} f, N\rangle$, using the definition in Eq. 1.2 we get, after a straightforward computation, that

$$\lambda_1 \leq -\frac{H_f^2}{1+2m} - \frac{1}{|\Sigma|_f}\left\{\alpha_f - 2\int_\Sigma K\, dv_f + \int_\Sigma \left(\mathrm{Ric}_f^{2m}(N,N) + 2\overline{Sect}_\Sigma\right)\right\}. \quad (3.3)$$

Using the hypotheses we obtain

$$\lambda_1 \leq -\frac{H_f^2}{1+2m} - 4c - \frac{2}{|\Sigma|_f}\int_\Sigma K\, dv_f. \quad (3.4)$$

Proof **(i)** Choosing the constant function $u = 1$ to be the test function in Eq. 2.4 to estimate λ_1, and using the expression in Eq. 2.8, we obtain that

$$\begin{aligned}
\lambda_1 &\leq \frac{-\int_\Sigma 1 J_f 1\, dv_f}{\int_\Sigma 1\, dv_f} = -\frac{1}{|\Sigma|_f}\left[\int_\Sigma |\phi|^2 + \frac{1}{2}\int_\Sigma H^2 + \int_\Sigma \mathrm{Ric}_f(N,N)\right]\\
&= -\frac{1}{|\Sigma|_f}\left[\int_\Sigma |\phi|^2 + \frac{1}{2}\int_\Sigma (H_f - \langle N, \overline{\nabla} f\rangle)^2 + \int_\Sigma \mathrm{Ric}_f(N,N)\right]\\
&\leq -\frac{1}{|\Sigma|_f}\left[\int_\Sigma |\phi|^2 + \frac{1}{2}\int_\Sigma \left(\frac{H_f^2}{1+m} - \frac{\langle N, \overline{\nabla} f\rangle^2}{m}\right) + \int_\Sigma \mathrm{Ric}_f(N,N)\right]\\
&\leq -\frac{H_f^2}{2(1+m)} - 2c - \frac{1}{|\Sigma|_f}\int_\Sigma |\phi|^2\\
&\leq -\frac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right).
\end{aligned}$$

If $\lambda_1 = -\frac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right)$, then all the inequalities above becomes equalities and consequently Σ is totally umbilical, $\mathrm{Ric}(N,N) = 2c$, $df(N) = \dfrac{m}{1+m}H_f$, and $\mathrm{Hess}\, f(N,N) = \dfrac{df(N)^2}{2m}$.

On the other hand, if Σ is totally umbilical, $\mathrm{Ric}(N,N) = 2c$, $df(N) = \dfrac{m}{1+m}H_f$ and $\mathrm{Hess}\, f(N,N) = \dfrac{df(N)^2}{2m}$, we have

$$\begin{aligned}
H &= H_f - df(N)\\
&= H_f - \frac{m}{1+m}H_f\\
&= \frac{1}{1+m}H_f,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ric}_f(N,N) &= 2c + \frac{1}{2m}(df(N))^2\\
&= 2c + \frac{m}{2(1+m)^2}H_f^2.
\end{aligned}$$

Hence,

$$\begin{aligned} J_f &= \Delta_f + \frac{H^2}{2} + 2c + \frac{m}{2(1+m)^2}H_f^2 \\ &= \Delta_f + \frac{1}{2(1+m)^2}H_f^2 + 2c + \frac{m}{2(1+m)^2}H_f^2 \\ &= \Delta_f + \frac{1}{2(1+m)}H_f^2 + 2c, \end{aligned}$$

and thus,

$$\lambda_1 = -\frac{1}{2}\left(\frac{H_f^2}{1+m} + 4c\right),$$

as desired.

(ii) Using our hypotheses, we have by Eq. 3.4 that

$$\lambda_1 \leq -\frac{H_f^2}{1+2m} - 4c - \frac{2}{|\Sigma|_f}\int_\Sigma K\, dv_f.$$

If equality holds, then $\alpha_f = 0$, $\overline{Sect}_\Sigma = c$, Hess $f(N,N) = \frac{df(N)^2}{2m}$.
Firstly, we obtain of the equation (3.2) that

$$df(N) = \frac{2m}{1+2m}H_f.$$

and so $H = \dfrac{1}{1+2m}H_f$. Moreover, $\alpha = 0$ implies $\nabla\rho = 0$ and thus using the equation (2.3) we have that $|A|^2$ is constant. Futhermore, by Eq. 2.7, we have that K is constant.

On the other hand, if K is constant, $\overline{Sect}_\Sigma = c$, Hess $f(N,N) = \dfrac{df(N)^2}{2m}$ and $df(N) = \dfrac{2m}{1+2m}H_f$, we have that

$$\mathrm{Ric}_f(N,N) = 2c + \frac{2m}{(1+2m)^2}H_f^2,$$

and so

$$\begin{aligned} J_f &= \Delta_f + |A|^2 + \mathrm{Ric}_f(N,N) \\ &= \Delta + H^2 + 2(c-K) + 2c + \frac{2m}{(1+2m)^2}H_f^2 \\ &= \Delta_f + 4c + \frac{1}{1+2m}H_f^2 - 2K, \end{aligned}$$

and this implies that

$$\lambda_1 = -4c - \frac{1}{1+2m}H_f^2 + 2K.$$

Now, using that K is constant,

$$\lambda_1 = -4c - \frac{H_f^2}{1+2m} + \frac{2}{|\Sigma|_f}\int_\Sigma K\, dv_f,$$

as desired. □

3.3 Proof of the Theorem 1.3

Before initiate the proof, we will recall the generalized sectional curvature

$$\overline{Sect}_f^{2m}(X,Y) = \overline{Sect}(X,Y) + \frac{1}{2}\left(\text{Hess}\, f(X,X) - \frac{(df(X))^2}{2m}\right),$$

where X, Y are unit and orthogonal vectors fields on M.

Moreover,

$$\text{Ric}_f^{2m}(X,X) = \sum_{i=1}^{2} \overline{Sect}_f^{2m}(X,Y_i).$$

So,

$$\text{Ric}_f^{2m}(N,N) + 2\overline{Sect}_\Sigma \geq \text{Ric}_f^{2m}(N,N) + 2\overline{Sect}_f^{2m}|_\Sigma - \text{Hess}\, f(X,X),$$

where X is a vector field on Σ.

Plugging the hypothesis Hess $f \leq \sigma g$ into Eq. 3.3 we get

$$\begin{aligned}\lambda_1 \;\leq\; & -\frac{H_f^2}{1+2m} \\ & -\frac{1}{|\Sigma|_f}\left\{\alpha_f - 2\int_\Sigma K\, dv_f + \int_\Sigma \left(\text{Ric}_f^{2m}(N,N) + 2\overline{Sect}_f^{2m} - \sigma\right) dv_f\right\}. \end{aligned} \quad (3.5)$$

Now, we are able to prove our result.

Proof The item (i) is a consequence of Theorem 1.2 (i). To second item, we use the expression in Eq. 3.5 and our hypotheses.

Now, if equality holds, then $\alpha = 0$, $\text{Ric}_f^{2m} = 2c$ and $\overline{Sect}_f^{2m} = c$. By equality in the inequality (3.2), we obtain

$$df(N) = \frac{2m}{1+2m} H_f,$$

and so

$$H = H_f - \frac{2m}{1+2m} H_f = \frac{1}{1+2m} H_f.$$

Moreover, $\alpha = 0$ imply that ρ is constant and of the equation (2.3) we have that $|A|^2$ is also a constant.

To conclude, we use the Lemma 1 to get that M^3 has constant sectional curvature and e^{-f} has the property enunciated in equality case. □

Corollary 3.2 *Let* (M^3, g, f) *be a weighted manifold with* $\overline{Sect} \geq c$. *Assume that* Hess $f \geq \dfrac{df \otimes df}{2m}$ (*in the sense of quadratic forms*). *Then,*

(i) *There is no closed stable surface with*

$$\frac{H_f^2}{1+m} + 4c > 0.$$

(ii) *If* Σ^2 *is a closed and stable surface such that* $\dfrac{H_f^2}{1+2m} + 4c < 0$, *then*

$$|\Sigma|_f \geq -2\left(\int_\Sigma K\, dv_f\right)\left(|\frac{H_f^2}{1+2m} + 4c|\right)^{-1}.$$

Proof By definition, a surface is stable if and only if $\lambda_1 \geq 0$. Thus the item (i) follows from the Theorem 1.2 (i). For the item (ii), we use the definition of stability and the Theorem 1.2 (ii). So,

$$0 \leq \lambda_1 \leq -\frac{H_f^2}{1+2m} - 4c + \frac{2}{|\Sigma|_f}\int_\Sigma K\, dv_f,$$

and thus

$$|\Sigma|_f \left|\frac{H_f^2}{1+2m} + 4c\right| \geq -2\int_\Sigma K\, dv_f.$$

□

Another consequence of the Theorem 1.2 is an improvement of the Proposition 3.2 in [11] to Σ no necessarily f-minimal.

Corollary 3.3 *Under the same assumptions of the Theorem 1.2.*

(i) *If $c > 0$, then Σ cannot be stable;*
(ii) *If $c = 0$, but $H_f \neq 0$, then Σ cannot be stable;*
(iii) *If $c = 0$ and Σ is stable, then $H_f = 0$.*

Acknowledgments The authors would like to thank the referee for very valuable comments and suggestions.

References

1. Alías, L.J., Brasil Jr., A., Sousa, L.A.M.: A characterization of Clifford Tori with constant scalar curvature one by the first stability eigenvalue. Bull. Braz. Math. Soc. (N.S.) **35**(2), 165–175 (2004)
2. Alías, L.J., Barros, A., Brasil Jr., A.: A spectral characterization of the H(r)-torus by the first stability eigenvalue. Proc. Amer. Math. Soc. **133**(3), 875–884 (2005)
3. Alías, L.J., Meroño, M.A., Ortiz, I.: On the first stability eigenvalue of constant mean curvature surfaces into homogeneous 3-manifolds. Mediterr. J. Math. **12**(1), 147–158 (2015)
4. Bakry, D., Émery, M.: Diffusions Hypercontractives. Séminaire de Probabilities, XIX, 1983/84, Lecture Notes in Math, vol. 1123, pp. 177–206. Springer, Berlin (1985)
5. Bayle, V.: Propriétées de concavité du profil isopérimétrique et applications. Thése de Doctorat (2003)
6. Chang, S.-Y.A., Gursky, M.J., Yang, P.: Conformal invariants associated to a measure. Proc. Natl. Acad. Sci. USA **103**(8), 2535–2540 (2006)
7. Cheng, X., Mejia, T., Zhou, D.: Stability and compactness for complete f-minimal surfaces. Trans. Amer. Math. Soc. **367**(6), 4041–4059 (2015)
8. Espinar, J.M.: Manifolds with density, applications and gradient Schrödinger operators. arXiv:1209.6162v6 [math.DG] (2012)
9. Fan, E.M.: Topology of three-manifolds with positive P-scalar curvature. Proc. Amer. Math. Soc. **136**(9), 3255–3261 (2008)
10. Gromov, M.: Isoperimetric of waists and concentration of maps. Geom. Funct. Anal. **13**(1), 178–215 (2003)
11. Impera, D., Rimoldi, M.: Stability properties and topology at infinity of f-minimal hypersurfaces. Geom. Dedicata **178**, 21–47 (2015)
12. Lichnerowich, A.: Variétés riemanniennes à tenseur C non négatif. C. R. Acad. Sci. Paris Sér. A-B **271**, A650–A653 (1970)
13. Lichnerowich, A.: Variétés kählériennes à première classe de Chern non negative et variétés riemanniennes à courbure de Ricci généralisée non negative. J. Diff. Geom. **6**, 47–94 (1971/72)
14. Morgan, F.: Manifolds with density. Notices Amer. Math. Soc. **52**(8), 853–858 (2005)
15. Monteanu, O., Wang, J.: Analysis of weighted Laplacian and applications to Ricci solitons. Comm. Anal. Geom. **20**(1), 55–94 (2012)
16. Monteanu, O., Wang, J.: Geometry of manifolds with densities. Adv. Math. **259**, 269–305 (2014)

17. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159v1 [math.DG] (2002)
18. Perdomo, O.: First stability eigenvalue characterization of Clifford hypersurfaces. Proc. Amer. Math. Soc. **130**, 3379–3384 (2002)
19. Schoen, R., Yau, S.T.: Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. Ann. Math. (2) **110**(1), 127–142 (1979)
20. Tashiro, Y.: Complete Riemannian manifolds and some vector fields. Trans. Amer. Math. Soc. **117**, 251–275 (1965)
21. Villani, C.: Optimal Transport Old and New, Volume 338 of Grundlehren der Mathematischen Wissenschaften. Springer, Berlin (2009)
22. Wei, G., Wylie, W.: Comparison geometry for the Bakry-Emery Ricci tensor. J. Differential Geom. **83**(2), 377–405 (2009)
23. Wylie, W.: Sectional curvature for Riemannian manifolds with density. Geom. Dedicata **178**, 151–169 (2015)