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Heat Kernels of Non-symmetric Jump Processes: Beyond the Stable Case

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Abstract Let J be the Lévy density of a symmetric Lévy process in $\mathbb{R}^d$ with its Lévy exponent satisfying a weak lower scaling condition at infinity. Consider the non-symmetric and non-local operator

$$\mathcal{L}^\kappa f(x) := \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x+z) - f(x))\kappa(x,z)J(z)\,dz\,,$$

where $\kappa(x,z)$ is a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $0 < \kappa_0 \le \kappa(x,z) \le \kappa_1$, $\kappa(x,z) = \kappa(x,-z)$ and $|\kappa(x,z) - \kappa(y,z)| \le \kappa_2 |x-y|^\beta$ for some $\beta \in (0,1]$. We construct the heat kernel $p^\kappa(t,x,y)$ of $\mathcal{L}^\kappa$, establish its upper bound as well as its fractional derivative and gradient estimates. Under an additional weak upper scaling condition at infinity, we also establish a lower bound for the heat kernel p^κ.

Keywords Heat kernel estimates · Subordinate Brownian motion · Symmetric Lévy process · Non-symmetric operator · Non-symmetric Markov process

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1 Introduction

Suppose that $d \geq 1$, $\alpha \in (0, 2)$ and $\kappa(x, z)$ is a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$0 < \kappa_0 \leq \kappa(x, z) \leq \kappa_1 \, , \quad \kappa(x, z) = \kappa(x, -z) \, , \tag{1.1}$$

and for some $\beta \in (0, 1]$,

$$|\kappa(x, z) - \kappa(y, z)| \leq \kappa_2 |x - y|^\beta \, . \tag{1.2}$$

The operator

$$\mathcal{L}^\kappa_\alpha f(x) = \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x + z) - f(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} \, dz \tag{1.3}$$

is a non-symmetric and non-local stable-like operator. In the recent paper [6], Chen and Zhang studied the heat kernel of $\mathcal{L}^\kappa_\alpha$ and its sharp two-sided estimates. As the main result of the paper, they proved the existence and uniqueness of a non-negative jointly continuous function $p^\kappa_\alpha(t, x, y)$ in $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ solving the equation

$$\partial_t p^\kappa_\alpha(t, x, y) = \mathcal{L}^\kappa_\alpha p^\kappa_\alpha(t, \cdot, y)(x) \, , \quad x \neq y \, ,$$

and satisfying four properties - an upper bound, Hölder's estimate, fractional derivative estimate and continuity, cf. [6, Theorem 1.1] for details. They also proved some other properties of the heat kernel $p^\kappa_\alpha(t, x, y)$ such as conservativeness, Chapman-Kolmogorov equation, lower bound, gradient estimate and studied the corresponding semigroup. Their paper is the first one to address these questions for not necessarily symmetric non-local stable-like operators. These operators can be regarded as the non-local counterpart of elliptic operators in non-divergence form. In this context the Hölder continuity of $\kappa(\cdot, z)$ in Eq. 1.2 is a natural assumption.

The goal of this paper is to extend the results of [6] to more general operators than the ones defined in Eq. 1.3. These operators will be non-symmetric and not necessarily stable-like. We will replace the kernel $\kappa(x, z)|z|^{-d-\alpha}$ with a kernel $\kappa(x, z)J(z)$ where κ still satisfies Eqs. 1.1 and 1.2, but $J(z)$ is the Lévy density of a rather general symmetric Lévy process. Here are the precise assumptions that we make.

Let $\phi : (0, \infty) \to (0, \infty)$ be a Bernstein function without drift and killing. Then

$$\phi(\lambda) = \int_{(0,\infty)} \left(1 - e^{-\lambda t}\right) \mu(dt),$$

where μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (t \wedge 1) \mu(dt) < \infty$. Here and throughout this paper, we use the notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Without loss of generality we assume that $\phi(1) = 1$. Define $\Phi : (0, \infty) \to (0, \infty)$ by $\Phi(r) = \phi(r^2)$ and let Φ^{-1} be its inverse. The function $x \mapsto \Phi(|x|) =: \Phi(x)$, $x \in \mathbb{R}^d$, $d \geq 1$, is negative definite and hence it is the characteristic exponent of an isotropic Lévy process on $\mathbb{R}^d$. This process can be obtained by subordinating a d-dimensional Brownian motion by an independent subordinator with Laplace exponent ϕ. The Lévy measure of this process has a density $j(|y|)$ where $j : (0, \infty) \to (0, \infty)$ is the function given by

$$j(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} \mu(dt) \, .$$

Thus we have

$$\Phi(x) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(x \cdot y)) \, j(|y|) \, dy \, .$$

Note that when $\phi(\lambda) = \lambda^{\alpha/2}$, $0 < \alpha < 2$, we have $\Phi(r) = r^{\alpha}$, the corresponding subordinate Brownian motion is an isotropic α-stable process and $j(r) = c(d,\alpha)\, r^{-d-\alpha}$.

Our main assumption is the following *weak lower scaling condition at infinity*: There exist $\delta_1 \in (0, 2]$ and $a_1 \in (0, 1)$ such that

$$a_1 \lambda^{\delta_1} \Phi(r) \le \Phi(\lambda r)\,, \quad \lambda \ge 1, r \ge 1\,. \tag{1.4}$$

This condition implies that $\lim_{\lambda\to\infty} \Phi(\lambda) = \infty$ and hence $\int_{\mathbb{R}^d\setminus\{0\}} j(|y|)dy = \infty$ (i.e., the subordinate Brownian motion is not a compound Poisson process). The weak lower scaling condition at infinity governs the short-time small-space behavior of the subordinate Brownian motion. We also need a weak condition on the behavior of Φ near zero. We assume that

$$\int_0^1 \frac{\Phi(r)}{r}\, dr = C_* < \infty\,. \tag{1.5}$$

The following function will play a prominent role in the paper. For $t > 0$ and $x \in \mathbb{R}^d$ we define

$$\rho(t, x) = \rho^{(d)}(t, x) := \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right)\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d}. \tag{1.6}$$

In case when $\Phi(r) = r^{\alpha}$ we see that $\rho(t, x) = (t^{1/\alpha} + |x|)^{-d-\alpha}$. It is well known that $t(t^{1/\alpha} + |x|)^{-d-\alpha}$ is comparable to the heat kernel $p(t, x)$ of the isotropic α-stable process in $\mathbb{R}^d$. We will prove later in this paper (see Proposition 3.2) that $t\rho(t, x)$ is an upper bound of the heat kernel of the subordinate Brownian motion with characteristic exponent Φ.

We assume that $J : \mathbb{R}^d \to (0, \infty)$ is symmetric in the sense that $J(x) = J(-x)$ for all $x \in \mathbb{R}^d$ and there exists $\gamma_0 > 0$ such that

$$\gamma_0^{-1} j(|y|) \le J(y) \le \gamma_0 j(|y|), \quad \text{for all } y \in \mathbb{R}^d\,. \tag{1.7}$$

Following Eq. 1.3, we define a non-symmetric and non-local operator

$$\mathcal{L}^{\kappa} f(x) = \mathcal{L}^{\kappa,0} f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x))\kappa(x, z)J(z)\, dz := \lim_{\varepsilon\downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x)\,, \tag{1.8}$$

where

$$\mathcal{L}^{\kappa,\varepsilon} f(x) := \int_{|z|>\varepsilon} (f(x+z) - f(x))\kappa(x, z)J(z)\, dz, \quad \varepsilon > 0.$$

The following theorem is the main result of this paper.

Theorem 1.1 *Assume that* Φ *satisfies* Eqs. 1.4 *and* 1.5*, that* J *satisfies* Eq. 1.7*, and that* κ *satisfies* Eqs. 1.1 *and* 1.2*. Suppose there exists a function* $g : \mathbb{R}^d \to (0, \infty)$ *such that*

$$\lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \mathcal{L}^{\kappa} g(x)/g(x) \text{ is bounded from above.} \tag{1.9}$$

Then there exists a unique non-negative jointly continuous function $p^{\kappa}(t, x, y)$ *on* $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ *solving*

$$\partial_t p^{\kappa}(t, x, y) = \mathcal{L}^{\kappa} p^{\kappa}(t, \cdot, y)(x)\,, \quad x \ne y\,, \tag{1.10}$$

and satisfying the following properties:

(i) *(Upper bound) For every* $T \ge 1$*, there is a constant* $c_1 > 0$ *so that for all* $t \in (0, T]$ *and* $x, y \in \mathbb{R}^d$*,*

$$p^{\kappa}(t, x, y) \le c_1 t\rho(t, x - y)\,. \tag{1.11}$$

(ii) *(Fractional derivative estimate) For any* $x, y \in \mathbb{R}^d$, $x \neq y$, *the map* $t \mapsto \mathcal{L}^\kappa p^\kappa(t, \cdot, y)(x)$ *is continuous in* $(0, \infty)$, *and, for each* $T \geq 1$ *there is a constant* $c_2 > 0$ *so that for all* $t \in (0, T]$, $\varepsilon \in [0, 1]$ *and* $x, y \in \mathbb{R}^d$,

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t, \cdot, y)(x)| \leq c_2 \rho(t, x - y) . \tag{1.12}$$

(iii) *(Continuity) For any bounded and uniformly continuous function* $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y)\, dy - f(x) \right| = 0 . \tag{1.13}$$

Moreover, the constants c_1 *and* c_2 *can be chosen so that they depend only on* T, $\Phi^{-1}(T^{-1})$, d, a_1, δ_1, C_*, β, γ_0, κ_0, κ_1 *and* κ_2.

The assumption (1.9) is a quite mild one. For example, if $\int_{|z|>1} |z|^\varepsilon j(|z|) dz < \infty$ for some $\varepsilon > 0$, then the assumption (1.9) holds, see Remark 5.2 below.

Some further properties of the heat kernel $p^\kappa(t, x, y)$ are listed in the following result.

Theorem 1.2 *Suppose that the assumptions of Theorem 1.1 are satisfied.*

(1) *(Conservativeness) For all* $(t, x) \in (0, \infty) \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t, x, y)\, dy = 1 . \tag{1.14}$$

(2) *(Chapman-Kolmogorov equation) For all* $s, t > 0$ *and all* $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t, x, z) p^\kappa(s, z, y)\, dz = p^\kappa(t + s, x, y) . \tag{1.15}$$

(3) *(Joint Hölder continuity) For every* $T \geq 1$ *and* $\gamma \in (0, \delta_1) \cap (0, 1]$, *there is a constant* $c_3 = c_3(T, d, \delta_1, a_1, \beta, C_*, \Phi^{-1}(T^{-1}), \gamma_0, \kappa_0, \kappa_1, \kappa_2) > 0$ *such that for all* $0 < s \leq t \leq T$ *and* $x, x', y \in \mathbb{R}^d$,

$$\begin{aligned} |p^\kappa(s, x, y) - p^\kappa(t, x', y)| \leq\; & c_3 \left(|t - s| + |x - x'|^\gamma t\, \Phi^{-1}(t^{-1}) \right) \\ & \times (\rho(s, x - y) \vee \rho(s, x' - y)) . \end{aligned} \tag{1.16}$$

Furthermore, if the constant δ_1 *in* Eq. 1.4 *belongs to* $(2/3, 2)$ *and the constant* β *in* Eq. 1.2 *satisfies* $\beta + \delta_1 > 1$ *then* Eq. 1.16 *holds with* $\gamma = 1$.

(4) *(Gradient estimate) If* $\delta_1 \in (2/3, 2)$, *and* $\beta + \delta_1 > 1$, *then for every* $T \geq 1$, *there exists* $c_4 = c_4(T, d, \delta_1, a_1, \beta, C_*, \Phi^{-1}(T^{-1}), \gamma_0, \kappa_0, \kappa_1, \kappa_2) > 0$ *so that for all* $x, y \in \mathbb{R}^d$, $x \neq y$, *and* $t \in (0, T]$,

$$|\nabla_x p^\kappa(t, x, y)| \leq c_4 \Phi^{-1}(t^{-1}) t \rho(t, |x - y|) . \tag{1.17}$$

Note that the gradient estimate (1.17) is an improvement of the corresponding estimate [6, (4.19)] in the sense that the parameter δ_1 could be smaller than one as long as it is still larger than $2/3$ and $\beta + \delta_1 > 1$.

For $t > 0$, define the operator P_t^κ by

$$P_t^\kappa f(x) = \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y)\, dy , \quad x \in \mathbb{R}^d , \tag{1.18}$$

where f is a non-negative (or bounded) Borel function on $\mathbb{R}^d$, and let $P_0^\kappa = \mathrm{Id}$. Then by Theorems 1.1 and 1.2, $(P_t^\kappa)_{t \geq 0}$ is a Feller semigroup with the strong Feller property.

Let $C_b^{2,\varepsilon}(\mathbb{R}^d)$ be the space of bounded twice differentiable functions in $\mathbb{R}^d$ whose second derivatives are uniformly Hölder continuous. We further have

Theorem 1.3 *Suppose that the assumptions of Theorem 1.1 are satisfied.*

(1) *(Generator) Let $\varepsilon > 0$. For any $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, we have*

$$\lim_{t\downarrow 0} \frac{1}{t}\left(P_t^\kappa f(x) - f(x)\right) = \mathcal{L}^\kappa f(x)\,, \tag{1.19}$$

and the convergence is uniform.

(2) *(Analyticity) The semigroup $(P_t^\kappa)_{t\geq 0}$ of $\mathcal{L}^\kappa$ is analytic in $L^p(\mathbb{R}^d)$ for every $p \in [1,\infty)$.*

Finally, under an additional assumption, we prove by probabilistic methods a lower bound for the heat kernel $p^\kappa(t,x,y)$. The *weak upper scaling condition* means that there exist $\delta_2 \in (0,2)$ and $a_2 > 0$ such that

$$\Phi(\lambda r) \leq a_2 \lambda^{\delta_2} \Phi(r)\,, \quad \lambda \geq 1, r \geq 1\,. \tag{1.20}$$

Theorem 1.4 *Suppose that Φ satisfies* Eqs. 1.4, 1.20 and 1.5*, that J satisfies* Eq. 1.7*, and that κ satisfies* Eqs. 1.1 *and* 1.2*. Suppose also that there exists a function $g : \mathbb{R}^d \to (0,\infty)$ such that* Eq. 1.9 *holds. For every $T \geq 1$, there exists $c_5 = c_5(T,d,\delta_1,\delta_2,\gamma_0,C_*,\Phi^{-1}(T^{-1}),a_1,a_2,\beta,\kappa_0,\kappa_1,\kappa_2) > 0$ such that for all $t \in (0,T]$,*

$$p^\kappa(t,x,y) \geq c_5 \begin{cases} \Phi^{-1}(t^{-1})^d & \text{if } |x-y| \leq 3\Phi^{-1}(t^{-1})^{-1}, \\ tj(|x-y|) & \text{if } |x-y| > 3\Phi^{-1}(t^{-1})^{-1}. \end{cases} \tag{1.21}$$

In particular, for all $T, M \geq 1$, there exists $c_6 = c_6(T,d,\delta_1,\delta_2,\gamma_0,C_*,\Phi^{-1}(T^{-1}), a_1,a_2,\beta,\kappa_0,\kappa_1,\kappa_2) > 0$ for all $t \in (0,T]$ and $x,y \in \mathbb{R}^d$ with $|x-y| \leq M$,

$$p^\kappa(t,x,y) \geq c_6 t\rho(t,x-y)\,. \tag{1.22}$$

Theorems 1.1–1.4 generalize [6, Theorem 1.1]. Note that the lower bound (1.22) of $p^\kappa(t,x,y)$ is stated only for $|x-y| \leq M$. This is natural in view of the fact that Eqs. 1.4 and 1.20 only give information about short-time small-space behavior of the underlying subordinate Brownian motion. We remark in passing that, the upper bound (1.11) may not be sharp under the assumptions (1.4) and (1.5). When Φ satisfies scaling conditions both near infinity and near the origin, see [11, (H1) and (H2)], the upper bound (1.11) is sharp in the sense that the lower bound (1.22) is valid for all $x, y \in \mathbb{R}^d$.

The assumptions (1.4), (1.5), (1.9) and (1.20) are very weak conditions and they are satisfied by many subordinate Brownian motions. For the reader's convenience, we list some examples of ϕ, besides the Laplace exponent of the stable subordinator, such that $\Phi(r) = \phi(r^2)$ satisfies these assumptions.

(1) $\phi(\lambda) = \lambda^{\alpha_1} + \lambda^{\alpha_2}, 0 < \alpha_1 < \alpha_2 < 1$;
(2) $\phi(\lambda) = (\lambda + \lambda^{\alpha_1})^{\alpha_2}, \alpha_1, \alpha_2 \in (0,1)$;
(3) $\phi(\lambda) = (\lambda + m^{1/\alpha})^\alpha - m, \alpha \in (0,1), m > 0$;
(4) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1+\lambda))^{\alpha_2}, \alpha_1 \in (0,1), \alpha_2 \in (0, 1-\alpha_1]$;
(5) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1+\lambda))^{-\alpha_2}, \alpha_1 \in (0,1), \alpha_2 \in (0,\alpha_1)$;
(6) $\phi(\lambda) = \lambda/\log(1+\lambda^\alpha), \alpha \in (0,1)$.

The functions in (1)–(5) satisfy Eqs. 1.4, 1.5, 1.20 and 1.9 (see (3.1) and Remark 5.2); while the function in (6) satisfies Eqs. 1.4, 1.5 and 1.9, but does not satisfy Eq. 1.20. The function $\phi(\lambda) = \lambda/\log(1+\lambda)$ satisfies Eq. 1.4, but does not satisfy the other two conditions.

In order to prove our main results, we follow the ideas and the road-map from [6]. At many stages we encounter substantial technical difficulties due to the fact that in the stable-like case one deals with power functions while in the present situation the power functions are replaced with a quite general Φ and its variants. We also strive to simplify the proofs and streamline the presentation. In some places we provide full proofs where in [6] only an indication is given. On the other hand, we skip some proofs which would be almost identical to the corresponding ones in [6]. Below is a detailed outline of the paper with emphasis on the main differences from [6].

In Section 2 we start by introducing the basic setup, state again the assumptions, and derive some of the consequences. In Section 2.1 we discuss convolution inequalities, cf. Lemma 2.6. While in [6] these involve power functions, the most challenging task in the present setting was to find appropriate versions of these inequalities. The main new technical result here is Lemma 2.6.

In Section 3 we first study the heat kernel $p(t,x)$ of a symmetric Lévy process Z with Lévy density j_Z comparable to the Lévy density j of the subordinate Brownian motion with characteristic exponent Φ. We prove the joint Lipschitz continuity of $p(t,x)$ and then, based on a result from [10], that $t\rho(t,x)$ is the upper bound of $p(t,x)$ for all $x \in \mathbb{R}^d$ and small t, cf. Proposition 3.2. In Section 3.1, we provide some useful estimates on functions of $p(t,x)$. In Section 3.2, we specify j_Z by assuming $j_Z(z) = \mathfrak{K}(z)J(z)$, with $\mathfrak{K}$ being symmetric and bounded between two positive constants. Let $\mathcal{L}^{\mathfrak{K}}$ be the infinitesimal generator of the corresponding process and let $p^{\mathfrak{K}}$ be its heat kernel. We look at the continuous dependence of $p^{\mathfrak{K}}$ with respect to $\mathfrak{K}$. This subsection follows the ideas and proofs from [6] with additional technical difficulties.

Given a function κ satisfying Eqs. 1.1 and 1.2, we define, for a fixed $y \in \mathbb{R}^d$, $\mathfrak{K}_y = \kappa(y,\cdot)$ and denote by $p_y(t,x)$ the heat kernel of the freezing operator $\mathcal{L}^{\mathfrak{K}_y}$. Various estimates and joint continuity of $p_y(t,x)$ are shown in Section 4.1. The rest of Section 4 is devoted to constructing the heat kernel $p^\kappa(t,x,y)$ of the operator $\mathcal{L}^\kappa$. The heat kernel should have the form

$$p^\kappa(t,x,y) = p_y(t,x-y) + \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,, \tag{1.23}$$

where according to Levi's method the function $q(t,x,y)$ solves the integral equation

$$q(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x-z)q(s,z,y)\,dz\,ds\,, \tag{1.24}$$

with $q_0(t,x,y) = (\mathcal{L}^{\mathfrak{K}_x} - \mathcal{L}^{\mathfrak{K}_y})p_y(t,x-y)$. The main result is Theorem 4.5 showing existence and joint continuity of $q(t,x,y)$ satisfying Eq. 1.24. We follow [6, Theorem 3.1], and give a full proof. Joint continuity and various estimates of $p^\kappa(t,x,y)$ defined by Eq. 1.23 are given in Section 4.3.

Section 5 contains proofs of Theorems 1.1–1.4. We start with a version of a non-local maximum principle in Theorem 5.1 which is somewhat different from the one in [6, Theorem 4.1], continue with two results about the semigroup $(P_t^\kappa)_{t\ge 0}$ and then complete the proofs.

In this paper, we use the following notations. We will use ":=" to denote a definition, which is read as "is defined to be". For any two positive functions f and g, $f \asymp g$

means that there is a positive constant $c \geq 1$ so that $c^{-1} g \leq f \leq c g$ on their common domain of definition. For a set W in $\mathbb{R}^d$, $|W|$ denotes the Lebesgue measure of W in $\mathbb{R}^d$. For a function space $\mathbb{H}(U)$ on an open set U in $\mathbb{R}^d$, we let $\mathbb{H}_c(U) := \{f \in \mathbb{H}(U) : f \text{ has compact support}\}$, $\mathbb{H}_0(U) := \{f \in \mathbb{H}(U) : f \text{ vanishes at infinity}\}$ and $\mathbb{H}_b(U) := \{f \in \mathbb{H}(U) : f \text{ is bounded}\}$.

Throughout the rest of this paper, the positive constants $\delta_1, \delta_2, \gamma_0, a_1, a_2, \beta, \kappa_0, \kappa_1, \kappa_2$, C_i, $i = 0, 1, 2, \ldots$, can be regarded as fixed. In the statements of results and the proofs, the constants $c_i = c_i(a, b, c, \ldots)$, $i = 0, 1, 2, \ldots$, denote generic constants depending on $a, b, c, \ldots$, whose exact values are unimportant. They start anew in each statement and each proof. The dependence of the constants on the dimension $d \geq 1$, C_*, $\Phi^{-1}((2T)^{-1})$, $\Phi^{-1}(T^{-1})$ and γ_0 may not be mentioned explicitly.

2 Preliminaries

It is well known that the Laplace exponent ϕ of a subordinator is a Bernstein function and

$$\phi(\lambda t) \leq \lambda \phi(t) \qquad \text{for all } \lambda \geq 1, t > 0. \tag{2.1}$$

For notational convenience, in this paper, we denote $\Phi(r) = \phi(r^2)$ and without loss of generality we assume that $\Phi(1) = 1$.

Throughout this paper ϕ is the Laplace exponent of a subordinator and $\Phi(r) = \phi(r^2)$ satisfies the weak lower scaling condition (1.4) at infinity. This can be reformulated as follows: There exist $\delta_1 \in (0, 2]$ and a positive constant $a_1 \in (0, 1]$ such that for any $r_0 \in (0, 1]$,

$$a_1 \lambda^{\delta_1} r_0^{\delta_1} \Phi(r) \leq \Phi(\lambda r), \quad \lambda \geq 1, r \geq r_0. \tag{2.2}$$

In fact, suppose $r_0 \leq r < 1$ and $\lambda \geq 1$. Then, $\Phi(\lambda r) \geq a_1 \lambda^{\delta_1} r_0^{\delta_1} \Phi(1) \geq a_1 \lambda^{\delta_1} r_0^{\delta_1} \Phi(r)$ if $\lambda r > 1$, and $\Phi(\lambda r) \geq \Phi(r) \geq a_1 \lambda^{\delta_1} r_0^{\delta_1} \Phi(r)$ if $\lambda r \leq 1$.

Since ϕ is a Bernstein function and we assume (2.2), it follows that Φ is strictly increasing and $\lim_{\lambda \to \infty} \Phi(\lambda) = \infty$. We denote by $\Phi^{-1} : (0, \infty) \to (0, \infty)$ the inverse function of Φ.

From Eq. 2.1 we have

$$\Phi^{-1}(\lambda r) \geq \lambda^{1/2} \Phi^{-1}(r), \quad \lambda \geq 1, r > 0. \tag{2.3}$$

Moreover, by Eq. 2.2, Φ^{-1} satisfies the following weak upper scaling condition at infinity: For any $r_0 \in (0, 1]$,

$$\Phi^{-1}(\lambda r) \leq a_1^{-1/\delta_1} \Phi^{-1}(r_0)^{-1} \lambda^{1/\delta_1} \Phi^{-1}(r), \quad \lambda \geq 1, r \geq r_0. \tag{2.4}$$

In fact, from Eq. 2.2 we get $\Phi^{-1}(\lambda r) \leq a_1^{-1/\delta_1} \lambda^{1/\delta_1} \Phi^{-1}(r)$ for $\lambda \geq 1$ and $r \geq 1$. Suppose $r_0 \leq r < 1$. Then, $\Phi^{-1}(\lambda r) \leq 1 \leq a_1^{-1/\delta_1} \Phi^{-1}(r_0)^{-1} \lambda^{1/\delta_1} \Phi^{-1}(r)$ if $\lambda r \leq 1$, and $\Phi^{-1}(\lambda r) \leq a_1^{-1/\delta_1} \lambda^{1/\delta_1} r^{1/\delta_1} \leq a_1^{-1/\delta_1} \lambda^{1/\delta_1} \Phi^{-1}(r_0)^{-1} \Phi^{-1}(r)$ if $\lambda r > 1$.

For $t > 0$ and $x \in \mathbb{R}^d$, we define functions $r(t, x)$ and $\rho(t, x)$ by

$$r(t, x) = \Phi^{-1}(t^{-1})^d \wedge \frac{t \Phi(|x|^{-1})}{|x|^d}$$

and

$$\rho(t, x) = \rho^{(d)}(t, x) := \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right) \left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d}. \tag{2.5}$$

Note that, by [2, Lemma 17],

$$t\Phi(|x|^{-1})|x|^{-d} \geq \Phi^{-1}(t^{-1})^d \quad \text{if and only if} \quad t\Phi(|x|^{-1}) \geq 1. \tag{2.6}$$

Proposition 2.1 *For all $t > 0$ and $x \in \mathbb{R}^d$, $t\rho(t,x) \leq r(t,x) \leq 2^{d+2} t\rho(t,x)$.*

Proof

Case 1 $t\Phi(|x|^{-1}) \geq 1$. In this case, by Eq. 2.6 we have that $r(t,x) = \Phi^{-1}(t^{-1})^d$. Since $|x| \leq \frac{1}{\Phi^{-1}(t^{-1})}$, we have

$$\frac{1}{\Phi^{-1}(t^{-1})} \leq \frac{1}{\Phi^{-1}(t^{-1})} + |x| \leq \frac{2}{\Phi^{-1}(t^{-1})}. \tag{2.7}$$

This and Eq. 2.1 imply that

$$t^{-1} = \Phi(\Phi^{-1}(t^{-1})) \geq \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right) \geq \Phi(2^{-1}\Phi^{-1}(t^{-1})) \geq \frac{1}{4}\Phi(\Phi^{-1}(t^{-1}))$$
$$= \frac{1}{4}t^{-1}$$

and

$$2^{-d}\Phi^{-1}(t^{-1})^{-d} \leq \left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d} \leq \Phi^{-1}(t^{-1})^{-d}.$$

The last two displays imply that $2^{-d-2}\Phi^{-1}(t^{-1})^d \leq t\rho(t,x) \leq \Phi^{-1}(t^{-1})^d$.

Case 2 $t\Phi(|x|^{-1}) \leq 1$. In this case, by Eq. 2.6 we have that $r(t,x) = \frac{t\Phi(|x|^{-1})}{|x|^d}$. Since $|x| \geq \frac{1}{\Phi^{-1}(t^{-1})}$, we have

$$|x|^{-1} \geq \left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1} \geq 2^{-1}|x|^{-1}.$$

This with Eq. 2.1 implies that

$$\Phi(|x|^{-1}) \geq \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right) \geq \Phi(2^{-1}|x|^{-1}) \geq \frac{1}{4}\Phi(|x|^{-1}).$$

The last two displays imply the conclusion of the proposition in Case 2. □

Lemma 2.2 *Let $T \geq 1$ and $c = (2(2/a_1)^{1/\delta_1}/\Phi^{-1}((2T)^{-1}))^{d+2}$.*

(a) *For all $0 < s < t \leq T$ and $x, z \in \mathbb{R}^d$,*

$$\rho(t-s, x-z)\rho(s,z) \leq c\left(\rho(t-s,x-z) + \rho(s,z)\right)\rho(t,x). \tag{2.8}$$

(b) *For every $x \in \mathbb{R}^d$ and $0 < t/2 \leq s \leq t \leq T$, $\rho(t,x) \leq \rho(s,x) \leq 2c\rho(t,x)$.*

Proof (a) By Eq. 2.4 we have that for all $0 < t, s \leq T$,

$$\frac{1}{\Phi^{-1}((t+s)^{-1})} \leq \frac{1}{\Phi^{-1}(2^{-1}(t \vee s)^{-1})} \leq c_1\left(\frac{1}{\Phi^{-1}(t^{-1})} + \frac{1}{\Phi^{-1}(s^{-1})}\right), \tag{2.9}$$

where $c_1 = (2/a_1)^{1/\delta_1}/\Phi^{-1}((2T)^{-1}) \geq 1$.

Define $\varrho : (0,\infty) \to (0,\infty)$ by $\varrho(r) := r^d/\Phi(r^{-1})$, so that $\rho(t,x) = (\varrho(\frac{1}{\Phi^{-1}(t^{-1})} + |x|))^{-1}$. For all $a, b > 0$, $(a+b)^d \le 2^d(a \vee b)^d$ and, by Eq. 2.1, $\Phi((a+b)^{-1}) \ge \Phi(2^{-1}(a\vee b)^{-1}) \ge 4^{-1}\Phi((a\vee b)^{-1})$. Therefore, for all $a, b > 0$,

$$\varrho(a+b) \le 2^{d+2}\varrho(a\vee b) \le 2^{d+2}(\varrho(a)+\varrho(b))\,. \tag{2.10}$$

Moreover, Eq. 2.1 implies that for $r > 0$,

$$\varrho(c_1 r) = \frac{(c_1 r)^d}{\Phi(c_1^{-1}r^{-1})} \le c_1^{d+2}\frac{r^d}{\Phi(r^{-1})} = c_1^{d+2}\varrho(r)\,. \tag{2.11}$$

By using Eqs. 2.9–2.11, we have

$$\begin{aligned}
\varrho\left(\frac{1}{\Phi^{-1}(t^{-1})}+|x|\right) &\le \varrho\left(c_1\left(\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)+\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)\right)\right)\\
&\le c_1^{d+2}\varrho\left(\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)+\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)\right)\\
&\le (2c_1)^{d+2}\left(\varrho\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)+\varrho\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)\right)\,.
\end{aligned} \tag{2.12}$$

Thus we have that for $0 < s < t \le T$ and $x, z \in \mathbb{R}^d$,

$$\begin{aligned}
&(\rho(t-s,x-z)+\rho(s,z))\,\rho(t,x)\\
&= \frac{\varrho\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)+\varrho\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)}{\varrho\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)\varrho\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)}\,\frac{1}{\varrho\left(\frac{1}{\Phi^{-1}(t^{-1})}+|x|\right)}\\
&\ge (2c_1)^{-d-2}\frac{1}{\varrho\left(\frac{1}{\Phi^{-1}((t-s)^{-1})}+|x-z|\right)\varrho\left(\frac{1}{\Phi^{-1}(s^{-1})}+|z|\right)}\\
&= (2c_1)^{-d-2}\rho(t-s,x-z)\rho(s,z)\,.
\end{aligned}$$

(b) This follows from Eq. 2.12 by taking $s = t/2$, $z = 0$ and by using that ϱ is increasing. □

2.1 Convolution Inequalities

Let $B(a,b)$ be the beta function, i.e., $B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}ds$, $a, b > 0$.

Lemma 2.3 *Let $\beta, \gamma, \eta, \theta \in \mathbb{R}$ be such that $\mathbf{1}_{\beta\ge 0}(\beta/2) + \mathbf{1}_{\beta<0}(\beta/\delta_1) + 1 - \theta > 0$ and $\mathbf{1}_{\gamma\ge 0}(\gamma/2) + \mathbf{1}_{\gamma<0}(\gamma/\delta_1) + 1 - \eta > 0$. Then for every $t > 0$, we have*

$$\int_0^t u^{-\eta}\Phi^{-1}(u^{-1})^{-\gamma}(t-u)^{-\theta}\Phi^{-1}((t-u)^{-1})^{-\beta}\,du \le Ct^{1-\eta-\theta}\Phi^{-1}(t^{-1})^{-\gamma-\beta}\,. \tag{2.13}$$

Moreover, if $\beta \ge 0$ and $\gamma \ge 0$ then Eq. 2.13 *holds for all $t > 0$ with $C = B(\beta/2+1-\theta, \gamma/2+1-\eta)$.*

Proof Let I denote the integral in Eq. 2.13. By the change of variables $s = u/t$ we get that

$$I = t^{1-\eta-\theta}\int_0^1 s^{-\eta}\Phi^{-1}(t^{-1}s^{-1})^{-\gamma}(1-s)^{-\theta}\Phi^{-1}(t^{-1}(1-s)^{-1})^{-\beta}\,ds\,.$$

Since $s^{-1} \geq 1$ and $(1-s)^{-1} \geq 1$, we have by Eq. 2.3 that $\Phi^{-1}(t^{-1}s^{-1}) \geq s^{-1/2}\Phi^{-1}(t^{-1})$ and $\Phi^{-1}(t^{-1}(1-s)^{-1}) \geq (1-s)^{-1/2}\Phi^{-1}(t^{-1})$. Moreover, when $t \in (0, T]$, by Eq. 2.4 we have

$$\Phi^{-1}(t^{-1}s^{-1}) \leq a_1^{-1/\delta_1}\Phi^{-1}(T^{-1})^{-1}s^{-1/\delta_1}\Phi^{-1}(t^{-1})$$

and

$$\Phi^{-1}(t^{-1}(1-s)^{-1}) \leq a_1^{-1/\delta_1}\Phi^{-1}(T^{-1})^{-1}(1-s)^{-1/\delta_1}\Phi^{-1}(t^{-1}).$$

Hence,

$$\begin{aligned} I &\leq c_1 t^{1-\eta-\theta}\Phi^{-1}(t^{-1})^{-\gamma-\beta}\int_0^1 s^{\mathbf{1}_{\gamma\geq 0}(\gamma/2)+\mathbf{1}_{\gamma<0}(\gamma/\delta_1)-\eta}(1-s)^{\mathbf{1}_{\beta\geq 0}(\beta/2)+\mathbf{1}_{\beta<0}(\beta/\delta_1)-\theta}\,ds \\ &= C\Phi^{-1}(t^{-1})^{-\gamma-\beta}. \end{aligned}$$

When $\beta \geq 0$ and $\gamma \geq 0$ then the above inequality holds for all $t > 0$ with $c_1 = 1$ so $C = B(\beta/2+1-\theta, \gamma/2+1-\eta)$. □

Lemma 2.4 *Suppose that* $0 < t_1 \leq t_2 < \infty$. *Under the assumptions of Lemma 2.3, we have*

$$\lim_{h\to 0}\sup_{t\in[t_1,t_2]}\left(\int_0^h + \int_{t-h}^t\right)u^{-\eta}\Phi^{-1}(u^{-1})^{-\gamma}(t-u)^{-\theta}\Phi^{-1}((t-u)^{-1})^{-\beta}\,du = 0.$$

Proof Under the assumptions of this lemma, by repeating the argument in the proof of Lemma 2.3, we have that for all $t \in [t_1, t_2]$,

$$\begin{aligned} &\left(\int_0^h + \int_{t-h}^t\right)u^{-\eta}\Phi^{-1}(u^{-1})^{-\gamma}(t-u)^{-\theta}\Phi^{-1}((t-u)^{-1})^{-\beta}\,du \\ &\leq \left(t_1^{1-\eta-\theta} \vee t_2^{1-\eta-\theta}\right)\left(\Phi^{-1}(t_1^{-1})^{-\gamma-\beta} \vee \Phi^{-1}(t_2^{-1})^{-\gamma-\beta}\right) \\ &\quad \times \left(\int_0^{h/t_1} + \int_{1-h/t_1}^1\right)s^{\mathbf{1}_{\gamma\geq 0}(\gamma/2)+\mathbf{1}_{\gamma<0}(\gamma/\delta_1)-\eta}(1-s)^{\mathbf{1}_{\beta\geq 0}(\beta/2)+\mathbf{1}_{\beta<0}(\beta/\delta_1)-\theta}\,ds. \end{aligned}$$

Now the conclusion of the lemma follows immediately. □

For $\gamma, \beta \in \mathbb{R}$, we define

$$\rho_\gamma^\beta(t,x) := \Phi^{-1}(t^{-1})^{-\gamma}(|x|^\beta \wedge 1)\rho(t,x), \quad t > 0, x \in \mathbb{R}^d.$$

Note that $\rho_0^0(t,x) = \rho(t,x)$.

Remark 2.5 Recall that Φ is increasing. Thus it is straightforward to see that the following inequalities are true: for $T \geq 1$,

$$\rho_{\gamma_1}^\beta(t,x) \leq \Phi^{-1}(T^{-1})^{\gamma_2-\gamma_1}\rho_{\gamma_2}^\beta(t,x), \qquad (t,x) \in (0,T]\times\mathbb{R}^d, \quad \gamma_2 \leq \gamma_1, \tag{2.14}$$

$$\rho_\gamma^{\beta_1}(t,x) \leq \rho_\gamma^{\beta_2}(t,x), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d, \quad 0 \leq \beta_2 \leq \beta_1. \tag{2.15}$$

We record the following inequality: for every $T \geq 1$, $t \in (0, T]$ and $\beta < \delta_1$,

$$\int_{\Phi^{-1}(T^{-1})/\Phi^{-1}(t^{-1})}^{1} r^{\beta-1}\Phi(r^{-1})dr \leq \frac{1}{a_1(\delta_1-\beta)}\left(\frac{\Phi^{-1}(T^{-1})}{\Phi^{-1}(t^{-1})}\right)^{\beta}\Phi\left(\frac{\Phi^{-1}(t^{-1})}{\Phi^{-1}(T^{-1})}\right)$$
$$\leq \frac{\Phi^{-1}(T^{-1})^{\beta-2}}{a_1(\delta_1-\beta)}t^{-1}\Phi^{-1}(t^{-1})^{-\beta}. \quad (2.16)$$

The first inequality follows immediately by using the lower scaling to get that for $1 \geq r \geq \lambda^{-1}$, $\Phi(r^{-1}) \leq a_1^{-1}\lambda^{-\delta_1}r^{-\delta_1}\Phi(\lambda)$. The second inequality follows from Eq. 2.1.

For the remainder of this paper we always assume that Eq. 1.5 holds. The following result is a generalization of [6, Lemma 2.1].

Lemma 2.6 (a) *For every $T \geq 1$, there exists $c_1 = c_1(d, \delta_1, a_1, C_*, T, \Phi^{-1}(T^{-1})) > 0$ such that for $0 < t \leq T$, all $\beta \in [0, \delta_1)$ and $\gamma \in \mathbb{R}$,*

$$\int_{\mathbb{R}^d} \rho_\gamma^\beta(t,x)\,dx \leq \frac{c_1}{\delta_1-\beta}t^{-1}\Phi^{-1}(t^{-1})^{-\gamma-\beta}. \quad (2.17)$$

(b) *For every $T \geq 1$, there exists $C_0 = C_0(T) = C_0(d, \delta_1, a_1, C_*, T, \Phi^{-1}(T^{-1})) > 0$ such that for all $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 < \delta_1$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $0 < s < t \leq T$,*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_{\gamma_1}^{\beta_1}(t-s, x-z)\rho_{\gamma_2}^{\beta_2}(s,z)\,dz \\
&\leq \frac{C_0}{\delta_1-\beta_1-\beta_2}\Big((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\gamma_1-\beta_1-\beta_2}\Phi^{-1}(s^{-1})^{-\gamma_2} \\
&\quad +\Phi^{-1}((t-s)^{-1})^{-\gamma_1}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma_2-\beta_1-\beta_2}\Big)\rho(t,x) \\
&\quad +\frac{C_0}{\delta_1-\beta_1-\beta_2}(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\gamma_1-\beta_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\rho_0^{\beta_2}(t,x) \\
&\quad +\frac{C_0}{\delta_1-\beta_1-\beta_2}\Phi^{-1}((t-s)^{-1})^{-\gamma_1}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma_2-\beta_2}\rho_0^{\beta_1}(t,x). \quad (2.18)
\end{aligned}$$

(c) *Let $T \geq 1$. For all $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 < \delta_1$, and all $\theta, \eta \in [0, 1]$, $\gamma_1, \gamma_2 \in \mathbb{R}$ satisfying $\mathbf{1}_{\gamma_1 \geq 0}(\gamma_1/2) + \mathbf{1}_{\gamma_1<0}(\gamma_1/\delta_1) + \beta_1/2 + 1 - \theta > 0$ and $\mathbf{1}_{\gamma_2\geq 0}(\gamma_2/2) + \mathbf{1}_{\gamma_2<0}(\gamma_2/\delta_1) + \beta_2/2 + 1 - \eta > 0$, there exists $c_2 > 0$ such that for all $0 < t \leq T$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} (t-s)^{1-\theta}\rho_{\gamma_1}^{\beta_1}(t-s, x-z)s^{1-\eta}\rho_{\gamma_2}^{\beta_2}(s,z)\,dz\,ds \\
&\leq c_2 t^{2-\theta-\eta}\left(\rho^0_{\gamma_1+\gamma_2+\beta_1+\beta_2} + \rho^{\beta_1}_{\gamma_1+\gamma_2+\beta_2} + \rho^{\beta_2}_{\gamma_1+\gamma_2+\beta_1}\right)(t,x). \quad (2.19)
\end{aligned}$$

Moreover, when we further assume that $\gamma_1, \gamma_2 \geq 0$, we can take that

$$c_2 = 4\frac{C_0(T)}{\delta_1-\beta_1-\beta_2}B\left((\gamma_1+\beta_1)/2 + 1 - \theta, \gamma_2 + \beta_2/2 + 1 - \eta\right). \quad (2.20)$$

Proof (a) Let $c_1 = c_1(d) = d|B(0,1)|$ and $T_1 = \Phi^{-1}(T^{-1}) \le 1$. We have that for all $0 < t \le T$,

$$
\begin{aligned}
&\Phi^{-1}(t^{-1})^{\gamma} \int_{\mathbb{R}^d} \rho_\gamma^\beta(t,x)\,dx = \int_{\mathbb{R}^d} \left(|x|^\beta \wedge 1\right) \rho(t,x)\,dx \\
&\le c_1 \int_0^{T_1/\Phi^{-1}(t^{-1})} r^{\beta+d-1} \frac{\Phi\left(\left(\left(\frac{1}{\Phi^{-1}(t^{-1})}\right)^{-1}\right)\right)}{\left(\frac{T_1}{\Phi^{-1}(t^{-1})}\right)^d} dr \\
&\quad + c_1 \int_{T_1/\Phi^{-1}(t^{-1})}^1 r^{\beta-1}\Phi(r^{-1})dr + c_1 \int_1^\infty \frac{\Phi(r^{-1})}{r} dr \\
&\le \frac{c_1 T_1^\beta}{\beta+d} t^{-1} \Phi^{-1}(t^{-1})^d \Phi^{-1}(t^{-1})^{-\beta-d} \\
&\quad + c_1 \int_{T_1/\Phi^{-1}(t^{-1})}^1 r^{\beta-1}\Phi(r^{-1})dr + c_1 \int_0^1 \frac{\Phi(r)}{r} dr \\
&\le c_1 d^{-1} t^{-1} \Phi^{-1}(t^{-1})^{-\beta} + \frac{c_1 T_1^{\beta-2}}{a_1(\delta_1-\beta)} t^{-1}\Phi^{-1}(t^{-1})^{-\beta} + c_1 C_* \qquad (2.21) \\
&\le c_1 (d^{-1} + T_1^{-2} a_1^{-1} \delta_1^{-1} (\delta_1-\beta)^{-1} + C_* a_1^{-1/2} T) t^{-1} \Phi^{-1}(t^{-1})^{-\beta},
\end{aligned}
$$

where in the second to last line we used Eq. 2.16 to estimate the first term in Eq. 2.21 and used Eq. 1.5 to estimate the second term in Eq. 2.21, and in the last line we used the assumption $\beta \in [0, \delta_1)$ and the inequality $t\Phi^{-1}(t^{-1})^\beta \le t(a_1^{-1/\delta_1}(T/t)^{1/\delta_1})^\beta \le a_1^{-\beta/\delta_1} T \le a_1^{-1} T$ which follows from (2.4) with $\lambda = T/t$ and $r_0 = r = T^{-1}$.

(b) Let $c_2 = (2(2/a_1)^{1/\delta_1}/\Phi^{-1}((2T)^{-1}))^{d+2}$. As in the display after [6, (2.5)], we have that

$$
\left(|x-z|^{\beta_1} \wedge 1\right)\left(|z|^{\beta_2} \wedge 1\right) \le (|x-z|^{\beta_1+\beta_2} \wedge 1) + \left(|x-z|^{\beta_1} \wedge 1\right)\left(|x|^{\beta_2} \wedge 1\right).
$$

By using this and Eq. 2.8, we have

$$
\begin{aligned}
&\rho_{\gamma_1}^{\beta_1}(t-s, x-z)\rho_{\gamma_2}^{\beta_2}(s,z) \\
&= \Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\left(|x-z|^{\beta_1} \wedge 1\right)\left(|z|^{\beta_2} \wedge 1\right)\rho(t-s, x-z)\rho(s,z) \\
&\le c_2 \Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\left(|x-z|^{\beta_1} \wedge 1\right)\left(|z|^{\beta_2} \wedge 1\right) \\
&\qquad \times (\rho(t-s, x-z) + \rho(s,z))\,\rho(t,x) \\
&\le c_2 \Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\left\{(|x-z|^{\beta_1+\beta_2} \wedge 1) + \left(|x-z|^{\beta_1} \wedge 1\right)\left(|x|^{\beta_2} \wedge 1\right)\right\} \\
&\qquad \times \rho(t-s, x-z)\rho(t,x) \\
&\qquad + c_2 \Phi^{-1}((t-s)^{-1})^{-\gamma_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\left\{(|z|^{\beta_1+\beta_2} \wedge 1) + \left(|x|^{\beta_1} \wedge 1\right)\left(|z|^{\beta_2} \wedge 1\right)\right\} \\
&\qquad \times \rho(s,z)\rho(t,x) \\
&= c_2 \Phi^{-1}(s^{-1})^{-\gamma_2}\left\{\rho_{\gamma_1}^{\beta_1+\beta_2}(t-s, x-z)\rho(t,x) + \rho_{\gamma_1}^{\beta_1}(t-s, x-z)\rho_0^{\beta_2}(t,x)\right\} \\
&\qquad + c_2 \Phi^{-1}((t-s)^{-1})^{-\gamma_1}\left\{\rho_{\gamma_2}^{\beta_1+\beta_2}(s,z)\rho(t,x) + \rho_{\gamma_2}^{\beta_2}(s,z)\rho_0^{\beta_1}(t,x)\right\}.
\end{aligned}
$$

Since $\beta_1 + \beta_2 < \delta_1$, now Eq. 2.18 follows by integrating the above and using Eq. 2.17.

(c) By integrating Eq. 2.18 and using Lemma 2.3, we get Eq. 2.19. When we further assume that $\gamma_1, \gamma_2 \geq 0$, by integrating Eq. 2.18 and using the last part of Lemma 2.3, we get Eq. 2.19 with the constant

$$C_0\left(B\left(\frac{\gamma_1+\beta_1+\beta_2}{2}+1-\theta, \frac{\gamma_2+2}{2}+1-\eta\right)\right.$$
$$+B\left(\frac{\gamma_2+\beta_1+\beta_2}{2}+1-\eta, \frac{\gamma_1+2}{2}+1-\theta\right)$$
$$+B\left(\frac{\gamma_1+\beta_1}{2}+1-\theta, \frac{\gamma_2+2}{2}+1-\eta\right)$$
$$\left.+B\left(\frac{\gamma_2+\beta_2}{2}+1-\eta, \frac{\gamma_1+2}{2}+1-\theta\right)\right),$$

which is, using that the beta function B is symmetric and non-increasing in each variable, less than or equal to $4C_0B\left((\gamma_1+\beta_1)/2+1-\theta, \gamma_2+\beta_2/2+1-\eta\right)$. □

Lemma 2.7 *Suppose* $0 < t_1 \leq t_2 < \infty$. *For* $\beta \in (0, \delta_1/2)$,

$$\lim_{h\downarrow 0} \sup_{x,y\in\mathbb{R}^d, t\in[t_1,t_2]} \left(\int_0^h + \int_{t-h}^t\right)\int_{\mathbb{R}^d} \rho_0^\beta(t-s, x-z)(\rho_0^\beta(s, z-y)+\rho_\beta^0(s, z-y))dzds = 0.$$

Proof We first apply Lemma 2.6(b) and then use Remark 2.5, to get that for $t \in [t_1, t_2]$,

$$\int_{\mathbb{R}^d} \rho_0^\beta(t-s, x-z)(\rho_0^\beta(s, z-y)+\rho_\beta^0(s, z-y))dz$$
$$\leq c_1((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta} + s^{-1}\Phi^{-1}(s^{-1})^{-\beta})\rho(t_1, 0).$$

Now the conclusion of the lemma follows immediately from Lemma 2.4. □

3 Analysis of the Heat Kernel of $\mathcal{L}^{\mathfrak{K}}$

Throughout this paper, $Y = (Y_t, \mathbb{P}_x)$ is a subordinate Brownian motion via an independent subordinator with Laplace exponent ϕ and Lévy measure μ. The Lévy density of Y, denoted by j, is given by

$$j(x) = j(|x|) = \int_0^\infty (4\pi s)^{-d/2} e^{-|x|^2/4s}\,\mu(ds)\,.$$

It is well known that there exists $c = c(d)$ depending only on d such that

$$j(r) \leq c\frac{\phi(r^{-2})}{r^d}, \quad r > 0 \tag{3.1}$$

(see [2, (15)]). The function $r \mapsto j(r)$ is non-decreasing. Recall that we have assumed that $r \mapsto \Phi(r)(=\phi(r^2))$, the radial part of the characteristic exponent Φ of Y, satisfies the weak lower scaling condition at infinity in Eq. 2.2.

Suppose that $Z = (Z_t, \mathbb{P}_x)$ is a purely discontinuous symmetric Lévy process with characteristic exponent ψ_Z such that its Lévy measure admits a density j_Z satisfying

$$\widehat{\gamma}_0^{-1} j(|x|) \leq j_Z(x) \leq \widehat{\gamma}_0 j(|x|)\,, \qquad x \in \mathbb{R}^d\,, \tag{3.2}$$

for some $\widehat{\gamma}_0 \geq 1$. Hence, $\int_{\mathbb{R}^d} j_Z(x)dx = \infty$. The characteristic exponents of Z, respectively Y, are given by

$$\psi_Z(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) j_Z(y)\, dy\,, \qquad \Phi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) j(|y|)\, dy\,,$$

and satisfy

$$\widehat{\gamma}_0^{-1}\Phi(|\xi|) \leq \psi_Z(\xi) \leq \widehat{\gamma}_0\Phi(|\xi|)\,, \quad \xi \in \mathbb{R}^d\,. \tag{3.3}$$

Let ψ denote the radial nondecreasing majorant of the characteristic exponent of Z, i.e., $\psi(r) := \sup_{|z|\leq r} \psi_Z(z)$. Clearly

$$\widehat{\gamma}_0^{-1}\Phi(r) \leq \psi(r) \leq \widehat{\gamma}_0\Phi(r)\,, \quad r > 0, \quad \text{and} \quad \widehat{\gamma}_0^{-2}\psi(|\xi|) \leq \psi_Z(\xi) \leq \psi(|\xi|)\,, \quad \xi \in \mathbb{R}^d,$$

and thus ψ also satisfies the weak lower scaling condition at infinity in Eq. 2.2.

By Eqs. 3.1 and 3.2,

$$j_Z(x) \leq \widehat{\gamma}_0 \frac{\Phi(|x|^{-1})}{|x|^d}. \tag{3.4}$$

Moreover, for every $n \in \mathbb{Z}_+$,

$$\begin{aligned} \int_{\mathbb{R}^d} \left|\mathbb{E}\left[e^{i\xi\cdot Z_t}\right]\right| |\xi|^n\, d\xi &= \int_{\mathbb{R}^d} e^{-t\psi_Z(\xi)}|\xi|^n\, d\xi \leq \int_{\mathbb{R}^d} e^{-t\widehat{\gamma}_0^{-1}\Phi(|\xi|)}|\xi|^n\, d\xi \\ &\leq c\left(\int_0^1 r^{d-1+n} dr + \int_1^\infty r^{d-1+n} e^{-t\widehat{\gamma}_0^{-1}a_1 r^{\delta_1}}\, dr\right) < \infty\,. \end{aligned} \tag{3.5}$$

It follows from [13, Proposition 2.5(xii) and Proposition 28.1] that Z_t has a density

$$p(t,x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\psi_Z(\xi)}\, d\xi = (2\pi)^{-d/2}\int_{\mathbb{R}^d} \cos(x\cdot\xi) e^{-t\psi_Z(\xi)}\, d\xi,$$

which is infinitely differentiable in x. Let $\mathcal{L}$ be the infinitesimal generator of Z.

Lemma 3.1 (a) *For every $x \in \mathbb{R}^d$, the function $t \mapsto p(t,x)$ is differentiable and*

$$\frac{\partial p(t,x)}{\partial t} = -(2\pi)^{-d/2}\int_{\mathbb{R}^d} \cos(x\cdot\xi)\psi_Z(\xi) e^{-t\psi_Z(\xi)}\, d\xi = \mathcal{L}p(t,x)\,.$$

(b) *For every $\varepsilon > 0$ there exists a constant $c = c(d, \delta_1, a_1, \widehat{\gamma}_0, \varepsilon) > 0$ such that for all $s, t \geq \varepsilon$ and all $x, y \in \mathbb{R}^d$,*

$$|p(t,x) - p(s,y)| \leq c\,(|t-s| + |x-y|)\,.$$

Proof (a) Note that for any $t \geq 0$ and any $h \in \mathbb{R}$ such that $t + h \geq 0$,

$$\frac{p(t+h,x) - p(t,x)}{h} = (2\pi)^{-d/2}\int_{\mathbb{R}^d} \cos(x\cdot\xi) e^{-t\psi_Z(\xi)} \frac{e^{-h\psi_Z(\xi)} - 1}{h} d\xi.$$

The absolute value of the integrand is bounded by $2\widehat{\gamma}_0\Phi(|\xi|)e^{-\widehat{\gamma}_0^{-1}\Phi(|\xi|)}$ which is integrable since $\Phi(|\xi|) \leq |\xi|^2$. The claim follows from the dominated convergence theorem by letting $h \to 0$. The last equality in the statement of the lemma follows from [9, Example 4.5.5].

(b) By the triangle inequality we have that

$$\begin{aligned} |p(t,x) - p(s,y)| &\leq \int_{\mathbb{R}^d} |\cos(x\cdot\xi) - \cos(y\cdot\xi)|\, e^{-t\psi_Z(\xi)}\, d\xi \\ &+ \int_{\mathbb{R}^d} |\cos(y\cdot\xi)| \left|e^{-t\psi_Z(\xi)} - e^{-s\psi_Z(\xi)}\right| d\xi =: I_1 + I_2\,. \end{aligned}$$

Clearly, $|\cos(x\cdot\xi)-\cos(y\cdot\xi)|\le|x\cdot\xi-y\cdot\xi|\le|x-y||\xi|$, which implies that, by Eq. 3.5,

$$I_1\le|x-y|\int_{\mathbb{R}^d}|\xi|e^{-t\psi_Z(\xi)}\,d\xi\le|x-y|\int_{\mathbb{R}^d}|\xi|e^{-\varepsilon\widehat{\gamma}_0^{-1}\Phi(|\xi|)}\,d\xi=c_1(\widehat{\gamma}_0,\varepsilon)|x-y|\,.$$

In order to estimate I_2, without loss of generality we assume that $s\le t$. Then by the mean value theorem we have that

$$\left|e^{-t\psi_Z(\xi)}-e^{-s\psi_Z(\xi)}\right|\le|t-s|\psi_Z(\xi)e^{-s\psi_Z(\xi)}\le\widehat{\gamma}_0|t-s|\Phi(|\xi|)e^{-\varepsilon\widehat{\gamma}_0^{-1}\Phi(|\xi|)}\,.$$

Therefore, by Eq. 3.5,

$$I_2\le\widehat{\gamma}_0|t-s|\int_{\mathbb{R}^d}|\xi|^2e^{-\varepsilon\widehat{\gamma}_0^{-1}\Phi(|\xi|)}\,d\xi=c_2(\widehat{\gamma}_0,\varepsilon)|t-s|\,.$$

The claim follows by taking $c=c_1\vee c_2$. □

Define the Pruitt function $\mathcal{P}$ by

$$\mathcal{P}(r)=\int_{\mathbb{R}^d}\left(\frac{|x|^2}{r^2}\wedge 1\right)j(x)dx. \tag{3.6}$$

By [2, (6) and Lemma 1],

$$\frac{1}{2\widehat{\gamma}_0}\psi(r^{-1})\le\frac{1}{2}\Phi(r^{-1})\le\mathcal{P}(r)\le\frac{d\pi^2}{2}\Phi(r^{-1})\le\frac{\widehat{\gamma}_0 d\pi^2}{2}\psi(r^{-1}). \tag{3.7}$$

In this paper we will use Eq. 3.7 several times.

We next discuss the upper estimate of $p(t,x)$ and its derivatives for $0<t\le T$ and all $x\in\mathbb{R}^d$ using [10, Theorem 3].

Proposition 3.2 *For each $T\ge 1$ and $k\in\mathbb{Z}_+$, there is a constant $c=c(k,T,\widehat{\gamma}_0,d,\delta_1,a_1)\ge 1$ such that*

$$|\nabla^k p(t,x)|\le c\,t\,(\Phi^{-1}(t^{-1}))^k\rho(t,x)\,,\qquad 0<t\le T, x\in\mathbb{R}^d,$$

where ∇^k stands for the k-th order gradient with respect to the spatial variable x.

Proof First, we recall that $\int_{\mathbb{R}^d}j_Z(x)dx=\infty$. Let $f(s):=\frac{\Phi(s^{-1})}{s^d}$. Then by Eq. 3.4 we have $j_Z(x)\le C\widehat{\gamma}_0 f(|x|)$. Thus for $A\in\mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\int_A j_Z(x)dx &\le C\widehat{\gamma}_0\int_A\frac{\Phi(|x|^{-1})}{|x|^d}dx\le C\widehat{\gamma}_0\frac{\Phi(\text{dist}\,(0,A)^{-1})}{\text{dist}\,(0,A)^d}|A|\\ &\le C\widehat{\gamma}_0 f(\text{dist}\,(0,A))(\text{diam}(A))^d.\end{aligned}$$

Therefore, [10, (1)] holds with $\gamma=d$ and $M_1=C\widehat{\gamma}_0$.

Since $(s \vee |y|) - (|y|/2) \geq s/2$ for $s > 0$, using Eq. 3.7 in the last inequality we have that for $s, r > 0$,

$$\int_{|y|>r} f((s \vee |y|) - (|y|/2)) j_Z(y) dy \leq 2^d \frac{\Phi((s/2)^{-1})}{s^d} \int_{|y|>r} j_Z(y) dy$$
$$= 2^d \frac{\Phi((s/2)^{-1})}{s^d} \int_{|y|>r} \left(\frac{|y|^2}{r^2} \wedge 1 \right) j_Z(y) dy$$
$$\leq 2^{d+2} \frac{\Phi(s^{-1})}{s^d} \mathcal{P}(r) \leq 2^{d+1} \widehat{\gamma}_0 d\pi^2 f(s) \psi(r^{-1}). \tag{3.8}$$

Therefore, [10, (2)] holds with $M_1 = 2^{d+1}\widehat{\gamma}_0 d\pi^2$.

Furthermore, by Eqs. 3.3 and 2.2, for $k \in \mathbb{Z}_+$,

$$\int_{\mathbb{R}^d} e^{-t\psi_Z(\xi)} |\xi|^k \, d\xi \leq \int_{\mathbb{R}^d} e^{-t\widehat{\gamma}_0^{-1}\Phi(|\xi|)} |\xi|^k \, d\xi = d|B(0,1)| \int_0^\infty r^{d+k-1} e^{-t\widehat{\gamma}_0^{-1}\Phi(r)} \, dr$$
$$= d|B(0,1)| \int_0^\infty (\Phi^{-1}(s/t))^{d+k-1} e^{-\widehat{\gamma}_0^{-1} s} (\Phi^{-1})'(s/t) t^{-1} \, ds$$
$$\leq d|B(0,1)| \int_0^1 (\Phi^{-1}(s/t))^{d+k-1} (\Phi^{-1})'(s/t) t^{-1} \, ds$$
$$+ d|B(0,1)| \sum_{n=1}^\infty e^{-\widehat{\gamma}_0^{-1} 2^{n-1}} \int_{2^{n-1}}^{2^n} (\Phi^{-1}(s/t))^{d+k-1} (\Phi^{-1})'(s/t) t^{-1} \, ds$$
$$= \frac{d|B(0,1)|}{d+k} \int_0^1 ((\Phi^{-1}(s/t))^{d+k})' \, ds$$
$$+ \frac{d|B(0,1)|}{d+k} \sum_{n=1}^\infty e^{-\widehat{\gamma}_0^{-1} 2^{n-1}} \int_{2^{n-1}}^{2^n} ((\Phi^{-1}(s/t))^{d+k})' \, ds$$
$$\leq \frac{d|B(0,1)|}{d+k} \left((\Phi^{-1}(t^{-1}))^{d+k} + \sum_{n=1}^\infty e^{-\widehat{\gamma}_0^{-1} 2^{n-1}} (\Phi^{-1}(2^n/t))^{d+k} \right).$$

Since $t \leq T$, by Eq. 2.4 we have $\Phi^{-1}(2^n/t) \leq c_0 2^{n/\delta_1} \Phi^{-1}(t^{-1})$. Thus we see that

$$\int_{\mathbb{R}^d} e^{-t\psi_Z(\xi)} |\xi|^k \, d\xi \leq \frac{d|B(0,1)|}{d+k} (\Phi^{-1}(t^{-1}))^{d+k} \left(1 + c_0 \sum_{n=1}^\infty 2^{n(d+k)/\delta_1} e^{-\widehat{\gamma}_0^{-1} 2^{n-1}} \right)$$
$$\leq c_1 \Phi^{-1}(t^{-1})^{d+k} \leq c_2 \psi^-(t^{-1})^{d+k},$$

where $c_2 = c_2(k) > 0$ and ψ^- is the generalized inverse of ψ: $\psi^-(s) = \inf\{u \geq 0 : \psi(u) \geq s\}$. Therefore, [10, (8)] holds with the set $(0, T]$.

We have checked that the conditions in [10, Theorem 3] hold for all $k \in \mathbb{Z}_+$. Thus by [10, Theorem 3] (with $n = d + 2$ in [10, Theorem 3]), there exists $c_3(k) > 0$ such that for $t \leq T$,

$$|\nabla^k p(t,x)| \leq c_3 \psi^-(t^{-1})^k \left(\psi^-(t^{-1})^d \wedge \left(\frac{t\Phi(|x|^{-1})}{|x|^d} + \frac{\psi^-(t^{-1})^d}{(1 + |x|\psi^-(t^{-1}))^{d+2}} \right) \right)$$
$$\leq c_4 \Phi^{-1}(t^{-1})^k \left(\Phi^{-1}(t^{-1})^d \wedge \left(\frac{t\Phi(|x|^{-1})}{|x|^d} + \frac{\Phi^{-1}(t^{-1})^d}{(1 + |x|\Phi(t^{-1}))^{d+2}} \right) \right).$$

When $|x|\Phi^{-1}(t^{-1}) \geq 1$ (so that $t\Phi(|x|^{-1}) \leq 1$),

$$\frac{\Phi^{-1}(t^{-1})^d}{(1+|x|\Phi(t^{-1}))^{d+2}} \leq \frac{\Phi^{-1}(t^{-1})^d}{(|x|\Phi^{-1}(t^{-1}))^{d+2}} = |x|^{-d}\left(\frac{\Phi^{-1}(\Phi(|x|^{-1}))}{\Phi^{-1}\left(\frac{\Phi(|x|^{-1})}{t\Phi(|x|^{-1})}\right)}\right)^2 \leq |x|^{-d}(t\Phi(|x|^{-1})).$$

In the last inequality we have used Eq. 2.3. Therefore using Proposition 2.1 we conclude that for all $0 < t \leq T$ and $x \in \mathbb{R}^d$,

$$|\nabla^k p(t,x)| \leq c_4\Phi^{-1}(t^{-1})^k\left(\Phi^{-1}(t^{-1})^d \wedge \frac{t\Phi(|x|^{-1})}{|x|^d}\right) \leq c_4 2^{d+2} t\Phi^{-1}(t^{-1})^k \rho(t,x)\,.$$

□

3.1 Further Properties of $p(t,x)$

We will need the following simple inequality, cf. [6, (2.9)]: Let $a > 0$ and $x \in \mathbb{R}^d$. For every $z \in \mathbb{R}^d$ such that $|z| \leq (2a) \vee (|x|/2)$, we have

$$(a+|x+z|)^{-1} \leq 4(a+|x|)^{-1}\,. \tag{3.9}$$

Indeed, if $|z| \leq 2a$, then $a+|x| \leq a+|x+z|+|z| \leq a+|x+z|+2a \leq 4(a+|x+z|)$. If $|z| \leq |x|/2$, then $4(a+|x+z|) \geq 4a+4|x|-4|z| \geq 4a+4|x|-2|x| \geq a+|x|$.

For a function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, we define

$$\delta_f(t,x;z) := f(t,x+z)+f(t,x-z)-2f(t,x)\,. \tag{3.10}$$

Also, $f(x \pm z)$ is an abbreviation for $f(x+z)+f(x-z)$.

The following result is the counterpart of [6, Lemma 2.3].

Proposition 3.3 *For every $T \geq 1$, there exists a constant $c = c(T, d, \widehat{\gamma}_0, d, \delta_1, a_1) > 0$ such that for every $t \in (0,T]$ and $x, x', z \in \mathbb{R}^d$,*

$$\left|p(t,x)-p(t,x')\right| \leq c\left((\Phi^{-1}(t^{-1})|x-x'|) \wedge 1\right) t\left(\rho(t,x)+\rho(t,x')\right)\,, \tag{3.11}$$

$$\left|\delta_p(t,x;z)\right| \leq c\left((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1\right) t\left(\rho(t,x\pm z)+\rho(t,x)\right)\,, \tag{3.12}$$

and

$$\begin{aligned}|\delta_p(t,x;z)-\delta_p(t,x';z)| \leq{}& c\left((\Phi^{-1}(t^{-1})|x-x'|) \wedge 1\right)\left((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1\right)\\ &\times t\left(\rho(t,x\pm z)+\rho(t,x)+\rho(t,x'\pm z)+\rho(t,x')\right)\,.\end{aligned} \tag{3.13}$$

Proof (1) Note that, by Proposition 3.2 with $k = 0$, Eq. 3.11 is clearly true if $\Phi^{-1}(t^{-1})|x-y| \geq 1$. Thus we assume that $\Phi^{-1}(t^{-1})|x-y| \leq 1$. We use Proposition 3.2 for $k=1$ and

$$p(t,x)-p(t,y) = (x-y)\cdot\int_0^1 \nabla p(t,x+\theta(y-x))\,d\theta \tag{3.14}$$

to estimate $|p(t,x)-p(t,y)| \leq c_1 t\Phi^{-1}(t^{-1})|x-y|\int_0^1 \rho(t,x+\theta(y-x))d\theta$. Since $\theta|y-x| \leq 1/\Phi^{-1}(t^{-1})$, we get from Eq. 3.9 that

$$\left(\frac{1}{\Phi^{-1}(t^{-1})}+|x+\theta(y-x)|\right)^{-1} \leq 4\left(\frac{1}{\Phi^{-1}(t^{-1})}+|x|\right)^{-1}\,.$$

Therefore using Eq. 2.1 we have $|p(t,x)-p(t,y)| \le c_2|x-y|\Phi^{-1}(t^{-1})t\rho(t,x)$, $t\in(0,T]$.

(2) Note that Eq. 3.12 is clearly true if $\Phi^{-1}(t^{-1})|z|\ge 1$. In order to prove Eq. 3.12 when $\Phi^{-1}(t^{-1})|z|\le 1$ we use Eq. 3.14 twice to obtain

$$\begin{aligned}\delta_p(t,x;z) &= z\cdot\int_0^1 (\nabla p(t,x+\theta z)-\nabla p(t,x-\theta z))\,d\theta\\ &= 2(z\otimes z)\cdot\int_0^1\int_0^1 \theta\nabla^2 p(t,x+(1-2\theta')\theta z)\,d\theta'\,d\theta\,.\end{aligned}\tag{3.15}$$

Note that $|(1-2\theta')\theta z|\le|z|\le\frac{1}{\Phi^{-1}(t^{-1})}$. Hence, by Proposition 3.2 and Eq. 3.9 we get the estimate

$$\left|\theta\nabla^2 p(t,x+(1-2\theta')\theta z)\right|\le c_3\left(\Phi^{-1}(t^{-1})\right)^2 t\rho(t,x)\,.$$

Therefore, $\delta_p(t,x;z)\le c_4\left(\Phi^{-1}(t^{-1})|z|\right)^2 t\rho(t,x)$, $t\in(0,T]$.

(3) It follows from Eq. 3.12 that it suffices to prove Eq. 3.13 in the case when $\Phi^{-1}(t^{-1})|x-y|\le 1$. To do this, we start with the subcase when $\Phi^{-1}(t^{-1})|z|\le 1$ and $\Phi^{-1}(t^{-1})|x-y|\le 1$. Then by Eq. 3.15,

$$\begin{aligned}&|\delta_p(t,x;z)-\delta_p(t,y;z)|\\ &\le c_5|x-y|\cdot|z|^2\int_0^1\int_0^1\int_0^1|\nabla^3 p(t,x+(1-2\theta')\theta z+\theta''(y-x))|\,d\theta d\theta' d\theta''.\end{aligned}$$

Note that $|(1-2\theta')\theta z+\theta''(y-x))|\le\frac{2}{\Phi^{-1}(t^{-1})}$. Hence, by Proposition 3.2 and Eq. 3.9 we get

$$|\delta_p(t,x;z)-\delta_p(t,y;z)|\le c_6\Phi^{-1}(t^{-1})|x-y|(\Phi^{-1}(t^{-1})|z|)^2 t\rho(t,x)\,.$$

If $\Phi^{-1}(t^{-1})|z|\ge 1$ and $\Phi^{-1}(t^{-1})|x-y|\le 1$, then again by Proposition 3.2 and Eq. 3.9,

$$\begin{aligned}&|\delta_p(t,x;z)-\delta_p(t,y;z)|\\ &\le c_7\left(|x-y|\int_0^1|\nabla p(t,x\pm z+\theta(y-x))|\,d\theta\right.\\ &\qquad\left.+|x-y|\int_0^1|\nabla p(t,x+\theta(y-x))|\,d\theta\right)\\ &\le c_8\Phi^{-1}(t^{-1})|x-y|\,(t\rho(t,x\pm z)+t\rho(t,x))\,,\quad t\in(0,T].\end{aligned}$$

□

The following result is the counterpart of [6, Theorem 2.4].

Theorem 3.4 *For every $T\ge 1$, there exists a constant $c=c(T,d,\widehat{\gamma}_0,d,\delta_1,a_1)>0$ such that for all $t\in(0,T]$ and all $x,x'\in\mathbb{R}^d$,*

$$\int_{\mathbb{R}^d}\left|\delta_p(t,x;z)\right| j(|z|)\,dz\le c\rho(t,x)\tag{3.16}$$

and

$$\int_{\mathbb{R}^d}\left|\delta_p(t,x;z)-\delta_p(t,x';z)\right| j(|z|)\,dz\le c\left((\Phi^{-1}(t^{-1})|x-x'|)\wedge 1\right)(\rho(t,x)+\rho(t,x'))\,.\tag{3.17}$$

Proof By Eq. 3.12 we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\delta_p(t,x;z)\right| j(|z|)\,dz \\
&\le c_0 \int_{\mathbb{R}^d} \left((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1\right) t\,(\rho(t,x\pm z)+\rho(t,x))\, j(|z|)\,dz \\
&= c_0 \left(\int_{\mathbb{R}^d} \left((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1\right) t\rho(t,x\pm z) j(|z|)\,dz + t\rho(t,x)\mathcal{P}(1/\Phi^{-1}(t^{-1}))\right) \\
&=: c_0\,(I_1+I_2)\,.
\end{aligned} \tag{3.18}$$

Clearly by Eq. 3.7, $I_2 \le c_1 t\rho(t,x)\Phi(\Phi^{-1}(t^{-1})) = c_1\rho(t,x)$. Next,

$$\begin{aligned}
I_1 &= \Phi^{-1}(t^{-1})^2 \int_{\Phi^{-1}(t^{-1})|z|\le 1} |z|^2 t\rho(t,x\pm z) j(|z|)\,dz \\
&\quad + \int_{\Phi^{-1}(t^{-1})|z|>1} t\rho(t,x\pm z) j(|z|)\,dz \\
&=: I_{11}+I_{12}\,.
\end{aligned}$$

By using Eq. 3.9 in the first inequality below and Eq. 3.7 in the third, we further have

$$\begin{aligned}
I_{11} &\le 4^{d+1} t\rho(t,x) \int_{|z|\le \frac{1}{\Phi^{-1}(t^{-1})}} ((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1) j(|z|)\,dz \\
&\le 4^{d+1} t\rho(t,x)\mathcal{P}(1/\Phi^{-1}(t^{-1})) \le c_2\rho(t,x)\,.
\end{aligned}$$

Next, we have

$$\begin{aligned}
I_{12} &\le t \int_{|z|>\frac{1}{\Phi^{-1}(t^{-1})}} \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})}\right)^{-1}\right)\left(\frac{1}{\Phi(t^{-1})}\right)^{-d} j(|z|)\,dz \\
&= \Phi^{-1}(t^{-1})^d \int_{|z|>\frac{1}{\Phi^{-1}(t^{-1})}} ((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1) j(|z|)\,dz \\
&\le \Phi^{-1}(t^{-1})^d \mathcal{P}(1/\Phi^{-1}(t^{-1})) \le c_3 \Phi^{-1}(t^{-1})^d \Phi(\Phi^{-1}(t^{-1})) = c_3\Phi^{-1}(t^{-1})^d t^{-1}\,,
\end{aligned}$$

where in the last line we used Eq. 3.7. If $|x| \le 2/\Phi^{-1}(t^{-1})$, we have that

$$\rho(t,x) \ge \Phi\left(\left(\frac{3}{\Phi^{-1}(t^{-1})}\right)^{-1}\right)\left(\frac{3}{\Phi(t^{-1})}\right)^{-d} \ge c_4 t^{-1}\Phi^{-1}(t^{-1})^d,$$

implying that $I_{12} \le c_5\rho(t,x)$.

If $|x| > 2/\Phi^{-1}(t^{-1})$, then by Eq. 3.7,

$$\begin{aligned}
I_{12} &= \left(\int_{\frac{|x|}{2} \geq |z| > \frac{1}{\Phi^{-1}(t^{-1})}} + \int_{|z| > \frac{|x|}{2}} \right) t\rho(t, x \pm z) j(|z|)\, dz \\
&\leq c_6 \left(t\rho(t,x) \int_{\frac{|x|}{2} \geq |z| > \frac{1}{\Phi^{-1}(t^{-1})}} j(|z|)\, dz + j(|x|/2) \int_{|z| > \frac{|x|}{2}} t\rho(t, x \pm z)\, dz \right) \\
&\leq c_7 \left(t\rho(t,x) \int_{|z| > \frac{1}{\Phi^{-1}(t^{-1})}} j(|z|)\, dz + \frac{\Phi(2|x|^{-1})}{|x|^d} \int_{\mathbb{R}^d} t\rho(t, x \pm z)\, dz \right) \\
&\leq c_7 \left(t\rho(t,x) \mathcal{P}(1/\Phi^{-1}(t^{-1})) + \frac{\Phi(|x|^{-1})}{|x|^d} \right) \\
&\leq c_8 \left(\rho(t,x) + \frac{\Phi(|x|^{-1})}{|x|^d} \right) \leq c_9 \rho(t,x)\,,
\end{aligned}$$

where in the last line the second term is estimated by a constant times the first term in view of the assumption that $|x| > 2/\Phi^{-1}(t^{-1})$. This finishes the proof of Eq. 3.16.

Next, by Eq. 3.13 we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \left| \delta_p(t,x;z) - \delta_p(t,x';z) \right| j(|z|)\, dz &\leq c_{10} \left((\Phi^{-1}(t^{-1})|x - x'|) \wedge 1 \right) \\
&\times \left\{ \int_{\mathbb{R}^d} \left((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1 \right) \left(t\rho(t, x \pm z) + t\rho(t, x' \pm z) \right) j(|z|)\, dz \right. \\
&\left. + \left(t\rho(t,x) + t\rho(t,x') \right) \int_{\mathbb{R}^d} \left((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1 \right) j(|z|)\, dz \right\} \\
&\leq c_{11} \left((\Phi^{-1}(t^{-1})|x - x'|) \wedge 1 \right) t^{-1} \left(t\rho(t,x) + t\rho(t,x') \right)\,,
\end{aligned}$$

where the last line follows by using the estimates of the integrals I_1 and I_2 from the first part of the proof. □

3.2 Continuous Dependence of Heat Kernels with Respect to $\mathfrak{K}$

Recall that $J : \mathbb{R}^d \to (0, \infty)$ is a symmetric function satisfying Eq. 1.7. We now specify the jumping kernel j_Z. Let $\mathfrak{K} : \mathbb{R}^d \to (0, \infty)$ be a symmetric function, that is, $\mathfrak{K}(z) = \mathfrak{K}(-z)$. Assume that there are $0 < \kappa_0 \leq \kappa_1 < \infty$ such that

$$\kappa_0 \leq \mathfrak{K}(z) \leq \kappa_1\,, \qquad \text{for all } z \in \mathbb{R}^d\,. \tag{3.19}$$

Let $j^{\mathfrak{K}}(z) := \mathfrak{K}(z)J(z)$, $z \in \mathbb{R}^d$. Then $j^{\mathfrak{K}}$ satisfies Eq. 3.2 with $\widehat{\gamma_0} = \gamma_0(\kappa_1 \vee \kappa_0^{-1})$. The infinitesimal generator of the corresponding symmetric Lévy process $Z^{\mathfrak{K}}$ is given by

$$\begin{aligned}
\mathcal{L}^{\mathfrak{K}} f(x) &= \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x)) \mathfrak{K}(z) J(z)\, dz \\
&= \frac{1}{2} \text{p.v.} \int_{\mathbb{R}^d} \delta_f(x;z) \mathfrak{K}(z) J(z)\, dz\,.
\end{aligned} \tag{3.20}$$

We note in passing that, when $f \in C_b^2(\mathbb{R}^d)$, it is not necessary to take the principal value in the last line above. The transition density of $Z^{\mathfrak{K}}$ (i.e., the heat kernel of $\mathcal{L}^{\mathfrak{K}}$) will be denoted by $p^{\mathfrak{K}}(t,x)$. Then by Lemma 3.1,

$$\frac{\partial p^{\mathfrak{K}}(t,x)}{\partial t} = \mathcal{L}^{\mathfrak{K}} p^{\mathfrak{K}}(t,x)\,, \qquad \lim_{t\to 0} p^{\mathfrak{K}}(t,x) = \delta_0(x)\,. \tag{3.21}$$

We will need the following observation for the next result. The inequality Eq. 2.4 implies that there exists a constant $c(\kappa_0) \geq 1$ such that

$$\Phi^{-1}((\kappa_0 t/2)^{-1}) \leq a_1^{-1/\delta_1}\Phi^{-1}(T^{-1})^{-1}(1\vee(\kappa_0/2))^{1/\delta_1}\Phi^{-1}(t^{-1}) \qquad \text{for all } t\in(0,T].$$

Consequently, for all $z \in \mathbb{R}^d$ and $t \in (0, T]$,

$$\left(\Phi^{-1}((\kappa_0 t/2)^{-1})|z|\right)\wedge 1 \leq a_1^{-1\delta_1}\Phi^{-1}(T^{-1})^{-1}(1\vee(\kappa_0/2))^{1/\delta_1}\left(\left(\Phi^{-1}(t^{-1})|z|\right)\wedge 1\right). \tag{3.22}$$

The following result is the counterpart of [6, Theorem 2.5], and in its proof we follow the proof of [6, Theorem 2.5] with some modifications.

Theorem 3.5 *For every $T \geq 1$, there exists a constant $c > 0$ depending on $T, d, \kappa_0, \kappa_1, \gamma_0, a_1$ and δ_1 such that for any two symmetric functions $\mathfrak{K}_1$ and $\mathfrak{K}_2$ in $\mathbb{R}^d$ satisfying* Eq. 3.19, *every $t \in (0, T]$ and $x \in \mathbb{R}^d$, we have*

$$\left|p^{\mathfrak{K}_1}(t,x) - p^{\mathfrak{K}_2}(t,x)\right| \leq c\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\, t\rho(t,x)\,, \tag{3.23}$$

$$\left|\nabla p^{\mathfrak{K}_1}(t,x) - \nabla p^{\mathfrak{K}_2}(t,x)\right| \leq c\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \Phi^{-1}(t^{-1}) t\rho(t,x) \tag{3.24}$$

and

$$\int_{\mathbb{R}^d}\left|\delta_{p^{\mathfrak{K}_1}}(t,x;z) - \delta_{p^{\mathfrak{K}_2}}(t,x;z)\right| j(|z|)\,dz \leq c\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty \rho(t,x)\,. \tag{3.25}$$

Proof (i) Using Eq. 3.21 in the second and third lines, the fact $\mathcal{L}^{\mathfrak{K}_1}$ is self-adjoint in the fourth and fifth lines, we have

$$\begin{aligned}
p^{\mathfrak{K}_1}(t,x) - p^{\mathfrak{K}_2}(t,x) &= \int_0^t \frac{d}{ds}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)p^{\mathfrak{K}_2}(t-s,y-x)\,dy\right)ds \\
&= \int_0^t \left(\int_{\mathbb{R}^d}\left(\mathcal{L}^{\mathfrak{K}_1}p^{\mathfrak{K}_1}(s,\cdot)(y)p^{\mathfrak{K}_2}(t-s,y-x)\right.\right. \\
&\quad \left.\left. - p^{\mathfrak{K}_1}(s,y)\mathcal{L}^{\mathfrak{K}_2}p^{\mathfrak{K}_2}(t-s,\cdot)(y-x)\right)dy\right)ds \\
&= \int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)\left(\mathcal{L}^{\mathfrak{K}_1}-\mathcal{L}^{\mathfrak{K}_2}\right)p^{\mathfrak{K}_2}(t-s,\cdot)(y-x)dy\right)ds \\
&\quad + \int_{t/2}^t\left(\int_{\mathbb{R}^d}\left(\mathcal{L}^{\mathfrak{K}_1}-\mathcal{L}^{\mathfrak{K}_2}\right)p^{\mathfrak{K}_1}(s,\cdot)(y)p^{\mathfrak{K}_2}(t-s,y-x)dy\right)ds \\
&= \frac{1}{2}\int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)\left(\int_{\mathbb{R}^d}\delta_{p^{\mathfrak{K}_2}}(t-s,x-y;z)(\mathfrak{K}_1(z)-\mathfrak{K}_2(z))J(z)dz\right)dy\right)ds \\
&\quad + \frac{1}{2}\int_{t/2}^t\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_2}(t-s,x-y)\left(\int_{\mathbb{R}^d}\delta_{p^{\mathfrak{K}_1}}(s,y;z)(\mathfrak{K}_1(z)-\mathfrak{K}_2(z))J(z)dz\right)dy\right)ds.
\end{aligned}$$

By using Eq. 3.16, Proposition 3.2 and the convolution inequality Eq. 2.19, we have

$$\begin{aligned}
&\int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)\left(\int_{\mathbb{R}^d}\delta_{p^{\mathfrak{K}_2}}(t-s,x-y;z)(\mathfrak{K}_1(z)-\mathfrak{K}_2(z))J(z)dz\right)dy\right)ds\\
&+\int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_2}(s,x-y)\left(\int_{\mathbb{R}^d}\delta_{p^{\mathfrak{K}_1}}(t-s,y;z)(\mathfrak{K}_1(z)-\mathfrak{K}_2(z))J(z)dz\right)dy\right)ds\\
\le\ &\widehat{\gamma}_0\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\left(\int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_1}(s,y)\left(\int_{\mathbb{R}^d}\left|\delta_{p^{\mathfrak{K}_2}}(t-s,x-y;z)\right|j(|z|)dz\right)dy\right)ds\right.\\
&\left.+\int_0^{t/2}\left(\int_{\mathbb{R}^d} p^{\mathfrak{K}_2}(s,x-y)\left(\int_{\mathbb{R}^d}\left|\delta_{p^{\mathfrak{K}_1}}(t-s,y;z)\right|j(|z|)dz\right)dy\right)ds\right)\\
\le\ &c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\int_0^{t/2}\int_{\mathbb{R}^d} s\left(\rho(s,y)\rho(t-s,x-y)+\rho(s,x-y)\rho(t-s,y)\right)dy\,ds\\
\le\ &2c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty t^{-1}\int_0^{t}\int_{\mathbb{R}^d} s(t-s)(\rho(s,y)\rho(t-s,x-y)\\
&+\rho(s,x-y)\rho(t-s,y))dyds\\
\le\ &c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\, t\rho(t,x),\qquad \text{for all } t\in(0,T],\ x\in\mathbb{R}^d.
\end{aligned}$$

(ii) Set $\widehat{\mathfrak{K}}_i(z):=\mathfrak{K}_i(z)-\kappa_0/2,\quad i=1,2$. It is straightforward to see that $p^{\kappa_0/2}(t,x)=p^1(\kappa_0t/2,x)$. Thus, by the construction of the Lévy process we have that for $i=1,2$,

$$p^{\mathfrak{K}_i}(t,x)=\int_{\mathbb{R}^d}p^{\kappa_0/2}(t,x-y)p^{\widehat{\mathfrak{K}}_i}(t,y)\,dy=\int_{\mathbb{R}^d}p^1(\kappa_0t/2,x-y)p^{\widehat{\mathfrak{K}}_i}(t,y)\,dy. \tag{3.26}$$

By Eq. 3.26, Proposition 3.2, Eqs. 3.23, 2.18 in the penultimate line (with $t, 2t$ instead of s, t), and Lemma 2.2(b) in the last line, we have that for all $t\in(0,T]$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}
&\left|\nabla p^{\mathfrak{K}_1}(t,x)-\nabla p^{\mathfrak{K}_2}(t,x)\right|=\left|\int_{\mathbb{R}^d}\nabla p^1\,(\kappa_0t/2,x-y)\,(p^{\widehat{\mathfrak{K}}_1}(t,y)-p^{\widehat{\mathfrak{K}}_2}(t,y))dy\right|\\
\le\ &c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\Phi^{-1}(t^{-1})t^2\int_{\mathbb{R}^d}\rho(t,x-y)\rho(t,y)\,dy\\
\le\ &c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\Phi^{-1}(t^{-1})t\rho(t,y)\,.
\end{aligned}$$

(iii) By using Eqs. 3.26, 3.12, Lemma 2.6(b) and Eq. 3.23, we have

$$\begin{aligned}
&\left|\delta_{p^{\mathfrak{K}_1}}(t,x;z)-\delta_{p^{\mathfrak{K}_2}}(t,x;z)\right|\\
=\ &\left|\int_{\mathbb{R}^d}\delta_{p^1}\,(\kappa_0t/2,x-y;z)\left(p^{\widehat{\mathfrak{K}}_1}(t,y)-p^{\widehat{\mathfrak{K}}_2}(t,y)\right)dy\right|\\
\le\ &c_1\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\left((\Phi^{-1}(t^{-1})|z|)^2\wedge 1\right)t^2\int_{\mathbb{R}^d}(\rho(t,x-y\pm z)\\
&+\rho(t,x-y))\rho(t,y)dy\\
\le\ &c_2\|\mathfrak{K}_1-\mathfrak{K}_2\|_\infty\left((\Phi^{-1}(t^{-1})|z|)^2\wedge 1\right)t\left(\rho(t,x\pm z)+\rho(t,x)\right)\,.
\end{aligned}$$

Now we have

$$\begin{aligned}\int_{\mathbb{R}^d} \left|\delta_{p^{\mathfrak{K}_1}}(t,x;z) - \delta_{p^{\mathfrak{K}_2}}(t,x;z)\right| j(|z|)\,dz \;\le\; c_2\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \\ \times \int_{\mathbb{R}^d} \Big((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1\Big)\, t\,(\rho(t,x\pm z) + \rho(t,x))\;\; j(|z|)\,dz \\ = c_2\|\mathfrak{K}_1 - \mathfrak{K}_2\|_\infty \int_{\mathbb{R}^d} \Big((\Phi^{-1}(t^{-1})|z|)^2 \wedge 1\Big)\, t\,(\rho(t,x\pm z) + \rho(t,x))\;\; j(|z|)\,dz,\end{aligned}$$

which is the same as Eq. 3.18 and was estimated in the proof of Theorem 3.4 by $c_3\rho(t,x)$. This finishes the proof. □

4 Levi's Construction of Heat Kernels

For the remainder of this paper, we always assume that $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ is a Borel function satisfying Eqs. 1.1 and 1.2, that Φ satisfies Eqs. 1.4 and 1.5 and that J satisfies Eq. 1.7. Throughout the remaining part of this paper, β is the constant in Eq. 1.2.

For a fixed $y \in \mathbb{R}^d$, let $\mathfrak{K}_y(z) = \kappa(y,z)$ and let $\mathcal{L}^{\mathfrak{K}_y}$ be the freezing operator

$$\begin{aligned}\mathcal{L}^{\mathfrak{K}_y} f(x) \;=\; \mathcal{L}^{\mathfrak{K}_y,0} f(x) = \lim_{\varepsilon\downarrow 0} \mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x), \quad \text{where } \mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x) \\ = \int_{|z|>\varepsilon} \delta_f(x;z)\kappa(y,z)J(z)dz. \end{aligned} \tag{4.1}$$

Let $p_y(t,x) = p^{\mathfrak{K}_y}(t,x)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_y}$. Note that $x \mapsto p_y(t,x)$ is in $C_0^\infty(\mathbb{R}^d)$ and satisfies Eq. 3.21.

4.1 Estimates on $p_y(t, x-y)$

The following result is the counterpart of [6, Lemmas 3.2 and 3.3].

Lemma 4.1 *For every $T \ge 1$ and $\beta_1 \in (0,\delta_1) \cap (0,\beta]$, there exists a constant $c = c(T, d, \delta_1, \beta_1, \kappa_0, \kappa_1, \kappa_2, \gamma_0) > 0$ such that for all $x \in \mathbb{R}^d$ and $t \in (0,T]$,*

$$\left|\int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_y(t,\cdot)(x-y)\,dy\right| \;\le\; c\,t^{-1}\,\Phi^{-1}(t^{-1})^{-\beta_1}\,, \textit{ for all } \varepsilon \in [0,1], \tag{4.2}$$

$$\left|\int_{\mathbb{R}^d} \partial_t p_y(t,x-y)\,dy\right| \;\le\; c\,t^{-1}\,\Phi^{-1}(t^{-1})^{-\beta_1}, \tag{4.3}$$

$$\left|\int_{\mathbb{R}^d} \nabla p_y(t,\cdot)(x-y)\,dy\right| \le c\,\Phi^{-1}(t^{-1})^{1-\beta_1}\,. \tag{4.4}$$

Furthermore, we have

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} p_y(t,x-y)\,dy - 1\right| = 0\,. \tag{4.5}$$

Proof Choose $\gamma \in (0, \delta_1 - \beta_1) \cap (0, 1]$. Since $\int_{\mathbb{R}^d} p_z(t, \xi - y)dy = 1$ for every $\xi, z \in \mathbb{R}^d$, by the definition of δ_{p_x} we have $\int_{\mathbb{R}^d} \delta_{p_x}(t, x - y; w)dy = 0$. Therefore, using this, Eqs. 1.1, 1.7 and 3.25, for $\varepsilon \in [0, 1]$ and $t \in (0, T]$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x, \varepsilon} p_y(t, \cdot)(x - y)\, dy \right| \\
&= \left| \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \left(\delta_{p_y}(t, x - y; w) - \delta_{p_x}(t, x - y; w) \right) \kappa(x, w) J(w)\, dw \right) dy \right| \\
&\le \kappa_1 \gamma_0 \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \left| \delta_{p_y}(t, x - y; w) - \delta_{p_x}(t, x - y; w) \right| j(|w|)\, dw \right) dy \\
&\le c_1 \int_{\mathbb{R}^d} \| \kappa(y, \cdot) - \kappa(x, \cdot) \|_\infty \rho(t, x - y)\, dy \\
&\le c_1 \kappa_2 \int_{\mathbb{R}^d} \left(|x - y|^{\beta_1} \wedge 1 \right) \rho(t, x - y)\, dy \le c_2 t^{-1} \Phi^{-1}(t^{-1})^{-\beta_1}.
\end{aligned}$$

Here the last line follows from Eqs. 1.2 and 2.17 since $\beta_1 + \gamma \in (0, \delta_1)$.

For Eq. 4.3, by using Eqs. 3.16 and 4.2 in the third line, we get, for $t \in (0, T]$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \partial_t p_y(t, x - y)\, dy \right| = \left| \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_y} p_y(t, \cdot)(x - y)\, dy \right| \\
&\le \left| \int_{\mathbb{R}^d} \left(\mathcal{L}^{\mathfrak{K}_x} - \mathcal{L}^{\mathfrak{K}_y} \right) p_y(t, \cdot)(x - y)\, dy \right| + \left| \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x} p_y(t, \cdot)(x - y)\, dy \right| \\
&\le c_3 \int_{\mathbb{R}^d} \rho_0^{\beta_1}(t, x - y)\, dy + c_2 t^{-1} \Phi^{-1}(t^{-1})^{-\beta_1} \le c_4 t^{-1} \Phi^{-1}(t^{-1})^{-\beta_1}.
\end{aligned}$$

Here we have used Eq. 2.17 in the last inequality.

For Eq. 4.4, by Eq. 3.24 we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \nabla p_y(t, \cdot)(x - y)\, dy \right| = \left| \int_{\mathbb{R}^d} \left(\nabla p_y(t, \cdot) - \nabla p_x(t, \cdot) \right) (x - y)\, dy \right| \\
&\le c_5 \int_{\mathbb{R}^d} \| \kappa(x, \cdot) - \kappa(y, \cdot) \|_\infty t \Phi^{-1}(t^{-1}) \rho(t, x - y)\, dy \\
&\le c_6 \int_{\mathbb{R}^d} \left(|x - y|^{\beta_1} \wedge 1 \right) t \Phi^{-1}(t^{-1}) \rho(t, x - y)\, dy \\
&= t \Phi^{-1}(t^{-1}) \int_{\mathbb{R}^d} \rho_0^{\beta_1}(t, x - y)\, dy \\
&\le c_7 t \Phi^{-1}(t^{-1}) t^{-1} \Phi^{-1}(t^{-1})^{-\beta_1} = \Phi^{-1}(t^{-1})^{1-\beta_1}.
\end{aligned}$$

In the last inequality we used Lemma 2.6(a) which requires that $\beta_1 + \gamma \in (0, \delta_1)$.

Finally, by using Eq. 3.23 in the second line and Eq. 2.17 in the last inequality, we get

$$\begin{aligned}
&\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_y(t, x - y)\, dy - 1 \right| \le \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| p_y(t, x - y) - p_x(t, x - y) \right| dy \\
&\le c_8 \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \| \kappa(y, \cdot) - \kappa(x, \cdot) \|_\infty t \rho(t, x - y)\, dy \\
&\le c_9 t \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \rho_0^{\beta_1}(t, x - y)\, dy \le c_{10} \Phi^{-1}(t^{-1})^{-\beta_1}, \quad t \in (0, T].
\end{aligned}$$

□

Lemma 4.2 *The function $p_y(t, x)$ is jointly continuous in (t, x, y).*

Proof By the triangle inequality, we have

$$|p_{y_1}(t_1, x_1) - p_{y_2}(t_2, x_2)| \le |p_{y_1}(t_1, x_1) - p_{y_2}(t_1, x_1)| + |p_{y_2}(t_1, x_1) - p_{y_2}(t_2, x_2)|.$$

Applying Eqs. 3.23 and 1.2 to the first term on the right hand side and Lemma 3.1(b) to the second term on the right hand side, we immediately get the desired joint continuity. □

4.2 Construction of $q(t, x, y)$

For $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ define

$$\begin{aligned} q_0(t, x, y) &:= \frac{1}{2} \int_{\mathbb{R}^d} \delta_{p_y}(t, x-y; z)\, (\kappa(x, z) - \kappa(y, z))\, J(z)\, dz \\ &= \left(\mathcal{L}^{\mathfrak{K}_x} - \mathcal{L}^{\mathfrak{K}_y}\right) p_y(t, \cdot)(x-y). \end{aligned} \tag{4.6}$$

In the next lemma we collect several estimates on q_0 that will be needed later on.

Lemma 4.3 *For every $T \ge 1$ and $\beta_0 \in (0, \beta]$, there is a constant $C_1 \ge 1$ depending on $d, \delta_1, \kappa_0, \kappa_1, \kappa_2, \gamma, T$ and $\Phi^{-1}(T^{-1})$ such that for $t \in (0, T]$ and $x, x', y, y' \in \mathbb{R}^d$,*

$$|q_0(t, x, y)| \le C_1(|x-y|^{\beta_0} \wedge 1)\rho(t, x-y) = C_1 \rho_0^{\beta_0}(t, x-y), \tag{4.7}$$

and for all $\gamma \in (0, \beta_0)$,

$$\begin{aligned} &|q_0(t, x, y) - q_0(t, x', y)| \\ &\le C_1 \left(|x-x'|^{\beta_0-\gamma} \wedge 1\right) \left\{ \left(\rho_\gamma^0 + \rho_{\gamma-\beta_0}^{\beta_0}\right)(t, x-y) + \left(\rho_\gamma^0 + \rho_{\gamma-\beta_0}^{\beta_0}\right)(t, x'-y) \right\} \end{aligned} \tag{4.8}$$

and

$$|q_0(t, x, y) - q_0(t, x, y')| \le C_1 \Phi^{-1}(t^{-1})^{\beta_0} \left(|y-y'|^{\beta_0} \wedge 1\right) \left(\rho(t, x-y) + \rho(t, x-y')\right). \tag{4.9}$$

Proof (a) Equation 4.7 follows from Eqs. 3.16 and 1.2.
(b) By Eqs. 4.7 and 2.14, we have that for $t \in (0, T]$,

$$|q_0(t, x, y)| \le c_0 \rho_0^{\beta_0}(t, x-y) \le c_0 \Phi^{-1}(T^{-1})^{\gamma-\beta_0} \rho_{\gamma-\beta_0}^{\beta_0}(t, x-y),$$

which proves Eq. 4.8 for $|x-x'| \ge 1$. Now suppose that $1 \ge |x-x'| \ge \Phi^{-1}(t^{-1})^{-1}$. Then, by Eq. 4.7, for $t \in (0, T]$,

$$|q_0(t, x, y)| \le c_1 \left(\Phi^{-1}(t^{-1})\right)^{-(\beta_0-\gamma)} \rho_{\gamma-\beta_0}^{\beta_0}(t, x-y) \le c_1 |x-x'|^{\beta_0-\gamma} \rho_{\gamma-\beta_0}^{\beta_0}(t, x-y),$$

and the same estimate is valid for $|q_0(t,x',y)|$. By adding we get Eq. 4.8 for this case. Finally, assume that $|x-x'| \le 1 \wedge \Phi^{-1}(t^{-1})^{-1}$. Then, by Eqs. 1.7, 1.2 and 3.17, for $t \in (0,T]$,

$$\begin{aligned}
|q_0(t,x,y) - q_0(t,x',y)| &= \Big| \int_{\mathbb{R}^d} \delta_{p_y}(t,x-y;z)(\kappa(x,z)-\kappa(y,z))J(z)\,dz \\
&\quad - \int_{\mathbb{R}^d} \delta_{p_y}(t,x'-y;z)(\kappa(x',z)-\kappa(y,z))J(z)\,dz \Big| \\
&\le \gamma_0 \int_{\mathbb{R}^d} |\delta_{p_y}(t,x-y;z) - \delta_{p_y}(t,x'-y;z)|\,|\kappa(x,z)-\kappa(y,z)| j(|z|)\,dz \\
&\quad + \gamma_0 \int_{\mathbb{R}^d} |\delta_{p_y}(t,x'-y;z)||\kappa(x,z)-\kappa(x',z)| j(|z|)\,dz \\
&\le \gamma_0\kappa_2 \left(|x-y|^{\beta_0} \wedge 1\right) \int_{\mathbb{R}^d} |\delta_{p_y}(t,x-y;z) - \delta_{p_y}(t,x'-y;z)| j(|z|)\,dz \\
&\quad + \gamma_0\kappa_2 \left(|x-x'|^{\beta_0} \wedge 1\right) \int_{\mathbb{R}^d} |\delta_{p_y}(t,x'-y;z)| j(|z|)\,dz \\
&\le c_2 \left(|x-y|^{\beta_0} \wedge 1\right) \left(\rho(t,x-y) + \rho(t,x'-y)\right) + c_2|x-x'|^{\beta_0}\rho(t,x'-y).
\end{aligned}$$

By using the definition of $\rho(t,x'-y)$, the obvious equality $x'-y = (x-y)+(x'-x)$, the assumption that $|x-x'| \le \Phi^{-1}(t^{-1})^{-1}$ and Eq. 3.9, we conclude that $\rho_0^\beta(t,x'-y) \le 4\rho_0^\beta(t,x-y)$. Thus, it follows that for $t \in (0,T]$,

$$\begin{aligned}
|q_0(t,x,y) - q_0(t,x',y)| &\le 5\,c_2\,\rho_0^{\beta_0}(t,x-y) + c_2|x-x'|^{\beta_0}\rho(t,x'-y) \\
&\le 5\,c_2\,|x-x'|^{\beta_0-\gamma}\,\rho_{\gamma-\beta_0}^{\beta_0}(t,x-y) \\
&\quad + c_2|x-x'|^{\beta_0-\gamma}\,\rho_\gamma^0(t,x'-y)\,.
\end{aligned}$$

(c) First note that

$$\begin{aligned}
&q_0(t,x,y) - q_0(t,x,y') \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \delta_{p^y}(t,x-y;z)\left(\kappa(y',z)-\kappa(y,z)\right) J(z)\,dz \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d} \left(\delta_{p^y}(t,x-y;z) - \delta_{p^y}(t,x-y';z)\right)\left(\kappa(x,z)-\kappa(y',z)\right) J(z)\,dz \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d} \left(\delta_{p^y}(t,x-y';z) - \delta_{p^{y'}}(t,x-y';z)\right)\left(\kappa(x,z)-\kappa(y',z)\right) J(z)\,dz \\
&=: I_1 + I_2 + I_3\,.
\end{aligned}$$

It follows from Eqs. 1.2, 1.7 and 3.16 that for $t \in (0,T]$,

$$|I_1| \le c_1 \left(|y-y'|^{\beta_0} \wedge 1\right) \int_{R^d} \left|\delta_{p^y}(t,x-y;z)\right| j(|z|)\,dz \le c_2 \left(|y-y'|^{\beta_0} \wedge 1\right) \rho(t,x-y)\,,$$

which is smaller than or equal to the right-hand side in Eq. 4.9 since $\Phi^{-1}(t^{-1}) \ge \Phi^{-1}(T^{-1})$. By Eqs. 1.1, 1.7 and 3.17 we get that

$$\begin{aligned}
|I_2| &\le c_1 \int_{\mathbb{R}^d} \left|\delta_{p^y}(t,x-y;z) - \delta_{p^y}(t,x-y';z)\right| j(|z|)\,dz \\
&\le c_2 \left((\Phi^{-1}(t^{-1})|y-y'|) \wedge 1\right) \left(\rho(t,x-y) + \rho(t,x-y')\right) \\
&\le c_2\,\Phi^{-1}(T^{-1})^{-\beta_0}\Phi^{-1}(t^{-1})^{\beta_0} \left(|y-y'|^{\beta_0} \wedge 1\right) \left(\rho(t,x-y) + \rho(t,x-y')\right).
\end{aligned}$$

Finally, by Eqs. 1.1, 1.2, 1.7 and 3.25, for $t \in (0, T]$,

$$\begin{aligned} |I_3| &\le c_1 \int_{\mathbb{R}^d} \left| \delta_{p^y}(t, x-y'; z) - \delta_{p^{y'}}(t, x-y'; z) \right| j(|z|)\, dz \\ &\le c_3 \, \|\kappa(y,\cdot) - \kappa(y',\cdot)\|_\infty \rho(t, x-y') \le c_4 \left(|y-y'|^{\beta_0} \wedge 1\right) \rho(t, x-y') . \quad \square \end{aligned}$$

Lemma 4.4 *The function $q_0(t, x, y)$ is jointly continuous in (t, x, y).*

Proof It follows from Lemma 4.2 that $(t, x, y) \mapsto p_y(t, x-y)$ is jointly continuous and hence also that $\delta_{p_y}(t, x-y; z)$ is jointly continuous in (t, x, y). To prove the joint continuity of $q_0(t, x, y)$, let $(t_n, x_n, y_n) \to (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ and assume that $0 < \varepsilon \le t_n \le T$. The integrands will converge because of the joint continuity of δ_{p_y} and continuity of κ in the first variable. Moreover, by Eq. 3.12,

$$\begin{aligned} &\left|\delta_{p_{y_n}}(t_n, x_n - y_n; z)\right| |\kappa(x_n, z) - \kappa(y_n, z)| j(|z|) \\ &\le c_1 \left((\Phi^{-1}(t_n^{-1})|z|^2) \wedge 1\right) T \left(\rho(t_n, x_n - y_n \pm z) + \rho(t_n, x_n, y_n)\right) j(|z|) \\ &\le c_2 \rho(\varepsilon, 0) \left((\Phi^{-1}(\varepsilon^{-1})|z|^2) \wedge 1\right) j(|z|). \end{aligned}$$

Since the right-hand side is integrable on $\mathbb{R}^d$, the joint continuity follows by use of the dominated convergence theorem. $\square$

For $n \in \mathbb{N}$, we inductively define

$$q_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} q_0(t-s, x, z) q_{n-1}(s, z, y)\, dz\, ds, \quad (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d . \tag{4.10}$$

The following result is the counterpart of [6, Theorem 3.1].

Theorem 4.5 *The series $q(t, x, y) := \sum_{n=0}^\infty q_n(t, x, y)$ is absolutely and locally uniformly convergent on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and solves the integral equation*

$$q(t, x, y) = q_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s, x, z) q(s, z, y)\, dz\, ds . \tag{4.11}$$

Moreover, $q(t, x, y)$ is jointly continuous in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and has the following estimates: for every $T \ge 1$ and $\beta_2 \in (0, \beta] \cap (0, \delta_1/2)$ there is a constant $C_2 = C_2(T, d, \delta_1, \kappa_0, \kappa_1, \kappa_2, \beta_2, \gamma_0) > 0$ such that

$$|q(t, x, y)| \le C_2 \left(\rho_0^{\beta_2} + \rho_{\beta_2}^0\right)(t, x-y), \quad (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d , \tag{4.12}$$

and for any $\gamma \in (0, \beta_2)$ and $T \ge 1$ there is a constant $C_3 = C_3(T, d, \delta_1, \gamma, \kappa_0, \kappa_1, \kappa_2, \gamma_0, \beta_2) > 0$ such that for all $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} &|q(t, x, y) - q(t, x', y)| \\ &\le C_3 \left(|x-x'|^{\beta_2 - \gamma} \wedge 1\right) \left(\left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2}\right)(t, x-y) + \left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2}\right)(t, x'-y)\right). \end{aligned} \tag{4.13}$$

Proof This proof follows the main idea of the proof of [6, Theorem 3.1], except that we give a full proof of the joint continuity in Step 2. We give the details for the readers' convenience. In this proof, $T \geq 1$ is arbitrary.

Step 1: By Eqs. 4.7, 2.19 and 2.20, we have that

$$\begin{aligned}|q_1(t,x,y)| &\leq C_1^2 \int_0^t \int_{\mathbb{R}^d} \rho_0^{\beta_2}(t-s,x-y-u)\rho_0^{\beta_2}(s,u)\,du\,ds \\ &\leq 8C_0C_1^2 B(\beta_2/2,\beta_2/2)\left(\rho_{2\beta_2}^0+\rho_{\beta_2}^{\beta_2}\right)(t,x-y), \quad t \leq T\,.\end{aligned}$$

Let $C = 2^4 C_0 C_1^2$ and we claim that for $n \geq 1$ and $t \leq T$,

$$|q_n(t,x,y)| \leq \gamma_n \left(\rho_{(n+1)\beta_2}^0 + \rho_{n\beta_2}^{\beta_2}\right)(t,x-y) \tag{4.14}$$

with

$$\gamma_n = C^{n+1}\prod_{j=1}^n B(\beta_2/2, j\beta_2/2)\,.$$

We have seen that Eq. 4.14 is valid for $n=1$. Suppose that it is valid for n. Then by using Eqs. 2.19, 2.20, 2.14 and 2.15, we have that for $t \leq T$,

$$\begin{aligned}|q_{n+1}(t,x,y)| &\leq \int_0^t \int_{\mathbb{R}^d} |q_0(t-s,x,z)|\,|q_n(s,z,y)|\,dz\,ds \\ &\leq C_1\gamma_n \int_0^t \int_{\mathbb{R}^d} \rho_0^{\beta_2}(t-s,x-z)\left(\rho_{(n+1)\beta_2}^0+\rho_{n\beta_2}^{\beta_2}\right)(s,z-y)\,dz\,ds \\ &\leq 2^4 C_0 C_1 \gamma_n B\left(\frac{\beta_2}{2},\frac{(n+1)\beta_2}{2}\right)\left(\rho_{(n+2)\beta_2}^0+\rho_{(n+1)\beta_2}^{\beta_2}\right)(t,x-y) \\ &\leq \gamma_{n+1}\left(\rho_{(n+2)\beta_2}^0+\rho_{(n+1)\beta_2}^{\beta_2}\right)(t,x-y)\,.\end{aligned}$$

Thus Eq. 4.14 is valid. Since

$$\sum_{n=0}^\infty \gamma_n \Phi^{-1}(T^{-1})^{-(n+1)\beta_2} = \sum_{n=0}^\infty \frac{\left(\Phi^{-1}(T^{-1})^{-\beta_2} C\Gamma\left(\frac{\beta_2}{2}\right)\right)^{n+1}}{\Gamma\left(\frac{(n+1)\beta_2}{2}\right)} =: C_2 < \infty\,,$$

by using Eqs. 2.14 and 2.15 in the second line, it follows that for $t \leq T$,

$$\begin{aligned}\sum_{n=0}^\infty |q_n(t,x,y)| &\leq \sum_{n=0}^\infty \gamma_n \left(\rho_{(n+1)\beta_2}^0+\rho_{n\beta_2}^{\beta_2}\right)(t,x-y) \\ &\leq \sum_{n=0}^\infty \gamma_n \Phi^{-1}(T^{-1})^{-(n+1)\beta_2}\left(\rho_{\beta_2}^0+\rho_0^{\beta_2}\right)(t,x-y) = C_2\left(\rho_{\beta_2}^0+\rho_0^{\beta_2}\right)(t,x-y)\,.\end{aligned}$$

This proves that $\sum_{n=0}^\infty q_n(t,x,y)$ is absolutely and uniformly convergent on $[\varepsilon, T]\times\mathbb{R}^d\times\mathbb{R}^d$ for all $\varepsilon \in (0,1)$ and $T \geq 1$, hence $q(t,x,y)$ is well defined. Further, by Eq. 4.10,

$$\sum_{n=0}^{m+1} q_n(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)\sum_{n=0}^m q_n(s,z,y)\,dz\,ds\,,$$

and Eq. 4.11 follows by taking the limit of both sides as $m \to \infty$.

Step 2: The joint continuity of $q_0(t,x,y)$ was shown in Lemma 4.4. We now prove the joint continuity of $q_1(t,x,y)$. For any $x,y\in\mathbb{R}^d$ and $t,h>0$, we have

$$\begin{aligned} &q_1(t+h,x,y)-q_1(t,x,y)\\ &=\int_t^{t+h}\int_{\mathbb{R}^d} q_0(t+h-s,x,z)q_0(s,z,y)dzds\\ &\quad+\int_0^t\int_{\mathbb{R}^d}(q_0(t+h-s,x,z)-q_0(t-s,x,z))\,q_0(s,z,y)dzds. \end{aligned} \tag{4.15}$$

It follows from Eq. 4.7 that, there exists $c_1=c_1(T)>0$ such that, for $0<h\le t/4$ and $t+h\le T$,

$$\begin{aligned} &\sup_{x,y\in\mathbb{R}^d}\left|\int_t^{t+h}\int_{\mathbb{R}^d} q_0(t+h-s,x,z)q_0(s,z,y)dzds\right|\\ &\le c_1\sup_{x,y\in\mathbb{R}^d}\int_t^{t+h}\int_{\mathbb{R}^d}\rho_0^{\beta_2}(t+h-s,x-z)\rho_0^{\beta_2}(s,z-y)dzds\\ &=c_1\sup_{x,y\in\mathbb{R}^d}\int_0^{h}\int_{\mathbb{R}^d}\rho_0^{\beta_2}(r,x-z)\rho_0^{\beta_2}(t+h-r,z-y)dzdr\\ &\le c_1\int_0^{h}\sup_{x,y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_0^{\beta_2}(r,x-z)\rho_0^{\beta_2}(t-r,z-y)dzdr. \end{aligned}$$

Now applying Lemma 2.6(b), we get

$$\begin{aligned} &\sup_{x,y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_0^{\beta_2}(r,x-z)\rho_0^{\beta_2}(t-r,z-y)dz\\ &\le c_2((t-r)^{-1}\Phi^{-1}((t-r)^{-1})^{-\beta_2}+r^{-1}\Phi^{-1}(r^{-1})^{-\beta_2})\rho(t,0). \end{aligned}$$

It follows from Lemma 2.3 that the right-hand side of the display above is integrable in $r\in(0,t)$, so by the dominated convergence theorem, we get

$$\lim_{h\downarrow 0}\sup_{x,y\in\mathbb{R}^d}\left|\int_t^{t+h}\int_{\mathbb{R}^d} q_0(t+h-s,x,z)q_0(s,z,y)dzds\right|=0. \tag{4.16}$$

Using Eq. 4.7 again, we get that for $s\in(0,t]$,

$$\begin{aligned} &|\,(q_0(t+h-s,x,z)-q_0(t-s,x,z))\,q_0(s,z,y)|\\ &\le c_3\left(\rho_0^{\beta_2}(t+h-s,x-z)+\rho_0^{\beta_2}(t-s,x-z)\right)\rho_0^{\beta_2}(s,z-y)\\ &\le c_4\rho_0^{\beta_2}(t-s,x-z)\rho_0^{\beta_2}(s,z-y). \end{aligned}$$

It follows from Lemma 2.6(c) that

$$\int_0^t\int_{\mathbb{R}^d}\rho_0^{\beta_2}(t-s,x-z)\rho_0^{\beta_2}(s,z-y)dzds\le c_5(\rho_0^{2\beta_2}(t,0)+\rho_{\beta_2}^{\beta_2}(t,0))<\infty,$$

thus we can use the dominated convergence theorem to get that, by the continuity of q_0,

$$\lim_{h\downarrow 0}\int_0^t\int_{\mathbb{R}^d}(q_0(t+h-s,x,z)-q_0(t-s,x,z))\,q_0(s,z,y)dzds=0. \tag{4.17}$$

It follows from Eq. 4.9 that for $s \in (0, T]$,

$$\begin{aligned} &|q_0(s,z,y) - q_0(s,z,y')| \\ &\leq c_6 \left((\Phi^{-1}(s^{-1})|y-y'|)^{\beta_2} \wedge 1 \right) \left(\rho(s, z-y) + \rho(s, z-y') \right). \end{aligned}$$

Now we fix $0 < t_1 \leq t_2 \leq T$. Then for any $\varepsilon \in (0, t_1/4)$, $t \in [t_1, t_2]$ and $s \in [\varepsilon, t]$,

$$\begin{aligned} &|q_0(t-s,x,z) \left(q_0(s,z,y) - q_0(s,z,y') \right)| \\ &\leq c_7 \left((\Phi^{-1}(\varepsilon^{-1})|y-y'|)^{\beta_2} \wedge 1 \right) \rho_0^{\beta_2}(t-s,x,z) \left(\rho(s, z-y) + \rho(s, z-y') \right). \end{aligned}$$

By Lemma 2.6(c), we have

$$\sup_{x,y,y' \in \mathbb{R}^d, t \in [t_1,t_2]} \int_0^t \int_{\mathbb{R}^d} \rho_0^{\beta_2}(t-s,x,z) \left(\rho(s,z-y) + \rho(s,z-y') \right) dz ds < \infty.$$

Thus

$$\lim_{y' \to y} \sup_{x \in \mathbb{R}^d, t \in [t_1,t_2]} \int_\varepsilon^t \int_{\mathbb{R}^d} |q_0(t-s,x,z) \left(q_0(s,z,y) - q_0(s,z,y') \right)| dz ds = 0.$$

Consequently, for each $0 < t_1 < t_2 \leq T$ and $\varepsilon \in (0, t_1/4)$, the family of functions

$$\left\{ \int_\varepsilon^t \int_{\mathbb{R}^d} q_0(t-s,x,z) q_0(s,z,\cdot) dz ds : x \in \mathbb{R}^d, t \in [t_1,t_2] \right\}$$

is equi-continuous. By combining Eq. 4.7 and Lemma 2.7, we get that

$$\lim_{\varepsilon \to 0} \sup_{x,y \in \mathbb{R}^d, t \in [t_1,t_2]} \left(\int_0^\varepsilon + \int_{t-\varepsilon}^t \right) \int_{\mathbb{R}^d} q_0(t-s,x,z) q_0(s,z,y) dz ds = 0. \quad (4.18)$$

Therefore the family

$$\left\{ \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z) q_0(s,z,\cdot) dz ds : x \in \mathbb{R}^d, t \in [t_1,t_2] \right\} \quad (4.19)$$

is equi-continuous.

Similarly, by using Eq. 4.8, we can show that, for each $0 < t_1 < t_2 \leq T$ and $\varepsilon \in (0, t_1/4)$, the family of functions

$$\left\{ \int_0^{t-\varepsilon} \int_{\mathbb{R}^d} q_0(t-s,\cdot,z) q_0(s,z,y) dz ds : y \in \mathbb{R}^d, t \in [t_1,t_2] \right\}$$

is equi-continuous. Combining this with Eq. 4.18, we get the family of functions

$$\left\{ \int_0^t \int_{\mathbb{R}^d} q_0(t-s,\cdot,z) q_0(s,z,y) dz ds : y \in \mathbb{R}^d, t \in [t_1,t_2] \right\} \quad (4.20)$$

is equi-continuous.

Now combining the continuity of $t \to q_1(t,x,y)$ (by Eqs. 4.16 and 4.17) and the equi-continuities of the families Eqs. 4.19 and 4.20, we immediately get the joint continuity of q_1.

The joint continuity of $q_n(t,x,y)$ can be proved by induction by using the estimate Eq. 4.14 of q_n and Lemma 2.7. The joint continuity of $q(t,x,y)$ follows immediately.

Step 3: By replacing α by 2 and β by β_2, this step is exactly the same as Step 4 in [6]. □

4.3 Properties of $\phi_y(t,x)$

Let

$$\phi_y(t,x,s) := \int_{\mathbb{R}^d} p_z(t-s, x-z)q(s,z,y)\,dz, \quad x \in \mathbb{R}^d,\ 0<s<t \tag{4.21}$$

and

$$\phi_y(t,x) := \int_0^t \phi_y(t,x,s)\,ds = \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,. \tag{4.22}$$

The following result is the counterpart of [6, Lemma 3.5].

Lemma 4.6 *For all* $x, y \in \mathbb{R}^d$, $x \neq y$, *the mapping* $t \mapsto \phi_y(t,x)$ *is absolutely continuous on* $(0,\infty)$ *and*

$$\partial_t \phi_y(t,x) = q(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_z} p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds, \quad t \in (0,\infty)\,. \tag{4.23}$$

Proof *Step 1:* Here we prove that for any $T \geq 1$, $t \in (0,T]$ and $s \in (0,t)$,

$$\partial_t \phi_y(t,x,s) = \int_{\mathbb{R}^d} \partial_t p_z(t-s,x-z)q(s,z,y)\,dz\,. \tag{4.24}$$

Let $|\varepsilon| < (t-s)/2$. We have that

$$\frac{\phi_y(t+\varepsilon,x,s)-\phi_y(t,x,s)}{\varepsilon} = \int_{\mathbb{R}^d}\left(\int_0^1 \partial_t p_z(t+\theta\varepsilon-s,x,z)\,d\theta\right) q(s,z,y)\,dz\,.$$

By using Eqs. 1.7, 3.21, 3.16 and 3.20, we have,

$$\begin{aligned} |\partial_t p_z(t+\theta\varepsilon-s,x-z)| &= \left|\mathcal{L}^{\mathfrak{K}_z} p_z(t+\theta\varepsilon-s,\cdot)(x-z)\right| \\ &\leq \frac{1}{2}\gamma_0 \int_{\mathbb{R}^d} |\delta_{p_z}(t+\theta\varepsilon-s,x-z;w)|\kappa(z,w)j(|w|)\,dw \\ &\leq c_1\rho(t+\theta\varepsilon-s,x-z) \leq c_2\rho(t-s,x-z)\,. \end{aligned}$$

In the last inequality we used that $|\varepsilon| < (t-s)/2$ and applied Lemma 2.2(b). Together with Eq. 4.12 this gives that for any $\beta_2 \in (0,\beta)\cap(0,\delta_1/2)$ and $t \in (0,T]$

$$\begin{aligned} |\partial_t p_z(t+\theta\varepsilon-s,x-z)q(s,z,y)| &\leq c_3(T)\rho(t-s,x-z)\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(s,z-y) \\ &=: g(z)\,. \end{aligned}$$

By Eq. 2.18, we see that $\int_{\mathbb{R}^d} g(z)\,dz < \infty$. Thus, by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\frac{\phi_y(t+\varepsilon,x,s)-\phi_y(t,x,s)}{\varepsilon} = \int_{\mathbb{R}^d} \partial_t p_z(t-s,x-z)q(s,z,y)\,dz\,,$$

proving Eq. 4.24.

Step 2: Here we prove that for all $x \neq y$ and $t \in (0,T]$, $T \geq 1$,

$$\int_0^t \int_0^r \left|\partial_r \phi_y(r,x,s)\right|\,ds\,dr \leq c_1(T)\,t\,\frac{\Phi(|x-y|^{-1})}{|x-y|^d} < +\infty\,. \tag{4.25}$$

By Eq. 4.24 we have

$$\begin{aligned}\left|\partial_r \phi_y(r,x,s)\right| \le & \int_{\mathbb{R}^d} |\partial_r p_z(r-s,x-z)|\,|q(s,z,y)-q(s,x,y)|\,dz \\ & + |q(s,x,y)| \left| \int_{\mathbb{R}^d} \partial_r p_z(r-s,x-z)\,dz \right| =: Q_y^{(1)}(r,x,s) \\ & + Q_y^{(2)}(r,x,s)\,.\end{aligned}$$

For $Q_y^{(1)}(r,x,s)$, by Eqs. 4.13, 3.20, 3.16 and Lemma 2.6(a) and (c), for $\beta_2 \in (0,\delta_1/2)\cap(0,\beta]$ and $\gamma \in ((2-\delta_1)\beta_2/2, \beta_2)$,

$$\begin{aligned}
&\int_0^t \int_0^r Q_y^{(1)}(r,x,s)\,ds\,dr \\
&\le c_2 \int_0^t \int_0^r \int_{\mathbb{R}^d} \left| \mathcal{L}^{\mathfrak{K}_z} p_z(r-s,x-z) \right| \left(|x-z|^{\beta_2-\gamma} \wedge 1 \right) \\
&\quad \times \left\{ \left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2} \right)(s,x-y) + \left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2} \right)(s,z-y) \right\} dz\,ds\,dr \\
&\le c_3 \int_0^t \int_0^r \left(\int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(r-s,x-z)dz \right) \left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2} \right)(s,x-y)\,ds\,dr \\
&\quad + c_3 \int_0^t \int_0^r \int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(r-s,x-z) \left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2} \right)(s,z-y)\,dz\,ds\,dr \\
&\le c_4 \int_0^t \int_0^r (r-s)^{-1} \Phi^{-1}((r-s)^{-1})^{\gamma-\beta_2} \left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2} \right)(s,x-y)\,ds\,dr \\
&\quad + c_4 \int_0^t \left(\rho_{\beta_2}^0 + \rho_0^{\beta_2} + \rho_\gamma^{\beta_2-\gamma} \right)(r,x-y)\,dr \\
&\le c_4 \frac{\Phi(|x-y|^{-1})}{|x-y|^d} \int_0^t \int_0^r (r-s)^{-1} \Phi^{-1}((r-s)^{-1})^{\gamma-\beta_2} \\
&\times \left(\Phi^{-1}(s^{-1})^{-\gamma} + \Phi^{-1}(s^{-1})^{\beta_2-\gamma} \right) ds\,dr \\
&\quad + c_4 \frac{\Phi(|x-y|^{-1})}{|x-y|^d} \int_0^t \left(\Phi^{-1}(r^{-1})^{-\beta_2} + 1 + \Phi^{-1}(r^{-1})^{-\gamma} \right) dr \\
&\le c_5 \frac{\Phi(|x-y|^{-1})}{|x-y|^d} \int_0^t \left(\Phi^{-1}(r^{-1})^{-\beta_2} + 1 + \Phi^{-1}(r^{-1})^{-\gamma} \right) dr \\
&\le c_6 t \frac{\Phi(|x-y|^{-1})}{|x-y|^d} < +\infty\,.
\end{aligned}$$

The second to last inequality follows from Lemma 2.3.

For $Q_y^{(2)}$, by Eqs. 4.3, 4.12 and Lemma 2.3 we have

$$\begin{aligned}
&\int_0^t \int_0^r Q_y^{(2)}(r,x,s)\,dr\,ds \le c_7 \int_0^t \int_0^r \left(\rho_{\beta_2}^0 + \rho_0^{\beta_2} \right)(s,x-y)(r-s)^{-1}\Phi^{-1} \\
&\times ((r-s)^{-1})^{-\beta_2}\,ds\,dr \\
&\le 2c_7 \frac{\Phi(|x-y|^{-1})}{|x-y|^d} \int_0^t \left(\int_0^r (r-s)^{-1} \Phi^{-1}((r-s)^{-1})^{-\beta_2}\,ds \right) dr \\
&\le c_8 t \frac{\Phi(|x-y|^{-1})}{|x-y|^d} < +\infty\,.
\end{aligned}$$

Step 3: We claim that for fixed $s > 0$ and $x, y \in \mathbb{R}^d$,

$$\lim_{t \downarrow s} \phi_y(t, x, s) = q(s, x, y)\,. \tag{4.26}$$

Assume $t \le T$, $T \ge 1$. For any $\delta > 0$ we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} p_z(t-s, x-z)\,(q(s,z,y) - q(s,x,y))\,dz \right| \\
&\le \int_{|x-z|\le\delta} p_z(t-s, x-z)\,|q(s,z,y) - q(s,x,y)|\,dz \\
&+ \int_{|x-z|>\delta} p_z(t-s, x-z)\,(|q(s,z,y)| + |q(s,x,y)|)\,dz =: J_1(\delta, t, s) \\
&+ J_2(\delta, t, s)\,.
\end{aligned}$$

By Eq. 4.13, for any $\varepsilon > 0$ there exists $\delta = \delta(s, x, y, T) > 0$ such that if $|z - x| \le \delta$, then $|q(s,z,y) - q(s,x,y)| \le \varepsilon$. Therefore, by Proposition 3.2 and Lemma 2.6(a),

$$J_1(\delta, t, s) \le \varepsilon \int_{\mathbb{R}^d} p_z(t-s, x-z)\,dz \le \varepsilon(t-s) \int_{\mathbb{R}^d} \rho(t-s, z)\,dz \le c_1 \varepsilon\,.$$

For $J_2(\delta, t, s)$, since $p_z(t-s, x-z) \le c_2(t-s)\rho(t-s, x-z) \le c_2(t-s)\rho(0, x-z)$, by Eq. 4.12 we have

$$\begin{aligned}
J_2(\delta, t, s) \;\le\; c_3(t-s) \Bigg(& \frac{\Phi(\delta^{-1})}{\delta^d} \int_{\mathbb{R}^d} \rho(s, z-y)\,dz \\
& + \rho(s, x-y) \int_{|x-z|>\delta} \frac{\Phi(|x-z|^{-1})}{|x-z|^d}\,dz \Bigg)
\end{aligned}$$

where $c_3 = c_3(T) > 0$ is independent of t. By Eq. 2.17, the term in parenthesis is finite. Hence, the last line converges to 0 as $t \downarrow s$. This and Eq. 4.5 prove Eq. 4.26.

Step 4: By Eq. 4.26, we have that

$$\phi_y(t, x, s) - q(s, x, y) = \int_s^t \partial_r \phi_y(r, x, s)\,dr\,.$$

Integrating both sides with respect to s from 0 to t, using first Eq. 4.25 and Fubini's theorem, and then Eqs. 4.24 and 3.21, we get

$$\begin{aligned}
\phi_y(t, x) - \int_0^t q(s, x, y)\,ds &= \int_0^t \int_s^t \partial_r \phi_y(r, x, s)\,dr\,ds = \int_0^t \int_0^r \partial_r \phi_y(r, x, s)\,ds\,dr \\
&= \int_0^t \int_0^r \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_z} p_z(r-s, \cdot)(x-z) q(s, z, y)\,dz\,ds\,dr\,.
\end{aligned}$$

This proves that $t \mapsto \phi_y(t, x)$ is absolutely continuous and gives its Radon-Nykodim derivative (4.23). □

The following result is the counterpart of [6, Lemma 3.6].

Lemma 4.7 *For all $t > 0$, $x \neq y$ and $\varepsilon \in [0, 1]$, we have*

$$\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds \tag{4.27}$$

and

$$t \mapsto \mathcal{L}^{\mathfrak{K}_x} p_y(t, x-y) \text{ and } t \mapsto \mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x) \text{ are continuous on } (0,\infty)\,. \tag{4.28}$$

Furthermore, if $\beta + \delta_1 > 1$ and $\delta_1 \in (2/3, 2)$ we also have

$$\nabla_x \phi_y(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,ds. \tag{4.29}$$

Proof Fix $x \neq y$ and $T \geq 1$. In this proof we assume $0 < t < T$ and all the constants will depend on T, but independent of s and t.

(a) By Eqs. 1.7, 1.1, 3.16, 4.12 and Lemma 2.6(b), for each $s \in (0, t)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\delta_{p_z}(t-s, x-z; w)|\kappa(x,w)J(w)dw|q(s,z,y)|dz \\
&\leq c_1 \int_{\mathbb{R}^d} \rho(t-s, x-z)\rho(s, z-y)dz < \infty.
\end{aligned} \tag{4.30}$$

Thus we can use Fubini's theorem so that from Eq. 4.21 we have that for $s \in (0, t)$,

$$\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x) = \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz, \quad \varepsilon \in [0,1]\,. \tag{4.31}$$

Let $\beta_2 \in (0, \delta_1/2) \cap (0, \beta]$ and $\gamma \in (0, \beta_2)$. By the definition of ϕ_y, Eq. 4.21, and Fubini's theorem, using the notation (3.10) we have for $\varepsilon \in (0, 1]$ and $s \in (0, t)$,

$$\begin{aligned}
&\left|\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x)\right| \\
&= \frac{1}{2}\left|\int_{|w|>\varepsilon}\left(\int_{\mathbb{R}^d} \delta_{p_z}(t-s,x-z;w)q(s,z,y)\,dz\right)\kappa(x,w)J(w)\,dw\right| \\
&= \frac{1}{2}\left|\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon} \delta_{p_z}(t-s,x-z;w)\kappa(x,w)J(w)\,dw\right)q(s,z,y)\,dz\right| \\
&\leq \frac{1}{2}\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon} |\delta_{p_z}(t-s,x-z;w)|\kappa(x,w)J(w)\,dw\right)|q(s,z,y)-q(s,x,y)|\,dz \\
&\quad +\frac{1}{2}\left|\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon} \delta_{p_z}(t-s,x-z;w)\kappa(x,w)J(w)\,dw\right)dz\right|\,|q(s,x,y)|\,.
\end{aligned}$$

By using Eqs. 1.7, 3.16, 4.2, 4.12 and 4.13 first and then using Lemma 2.6(a)–(b), we have that for $\varepsilon \in (0, 1]$ and $s \in (0, t)$,

$$
\begin{aligned}
&\left|\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x)\right| \\
&\leq c_2 \int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\,dz \\
&\quad +c_2\left(\int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\,dz\right)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y) \\
&\quad +c_2(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2}\left(\rho_0^{\beta_2}(s,x-y)+\rho_{\beta_2}^0(s,x-y)\right) \\
&\leq c_2 \int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\rho_\gamma^0(s,z-y)\,dz \\
&\quad +c_2 \int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\rho_{\gamma-\beta_2}^{\beta_2}(s,z-y)\,dz \\
&\quad +c_3(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y) \\
&\quad +c_3(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2}\left(\rho_0^{\beta_2}(s,x-y)+\rho_{\beta_2}^0(s,x-y)\right) \\
&\leq c_4\Big((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-2\beta_2}\Phi^{-1}(s^{-1})^{\beta_2-\gamma} \\
&\quad +(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}\Phi^{-1}(s^{-1})^{\beta_2-\gamma} \\
&\qquad +(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}\Phi^{-1}(s^{-1})^{-\gamma}+(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2} \\
&\qquad +s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}+s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}\Big)\rho(0,x-y) \\
&\leq c_5(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}\rho(0,x-y).
\end{aligned}
\tag{4.32}
$$

In the last inequality above we have used the inequality

$$
\Phi^{-1}(s^{-1})^{\beta_2} \leq a_1^{-\beta_2/\delta_1}\Phi^{-1}(T^{-1})^{-\beta_2}s^{-\beta_2/\delta_1} \leq a_1^{-\beta_2/\delta_1}\Phi^{-1}(T^{-1})^{-\beta_2}T^{1-\beta_2/\delta_1}s^{-1}.
$$

Using the fact that $x \neq y$ and Lemma 2.3 we see that the term on the right hand side of Eq. 4.32 is integrable in $s \in (0, t)$. Moreover, by Eqs. 1.1, 1.7, 4.12 and Proposition 3.2,

$$
\begin{aligned}
&\int_{|w|>\varepsilon}\int_0^t |\delta_{\phi_y}(t,x,s;w)|\kappa(x,w)J(w)\,ds\,dw \\
&\leq 2\kappa_1\gamma_0C_2\int_{|w|>\varepsilon}\int_0^t\int_{\mathbb{R}^d} p_z(t-s,x-z)(\rho_0^{\beta_2}(s,z-y)+\rho_{\beta_2}^0(s,z-y))dz j(|w|)\,ds\,dw \\
&\quad +\kappa_1\gamma_0C_2\int_{|w|>\varepsilon}\int_0^t\int_{\mathbb{R}^d} p_z(t-s,x\pm w-z)(\rho_0^{\beta_2}(s,z-y) \\
&\quad +\rho_{\beta_2}^0(s,z-y))dz j(|w|)\,ds\,dw \\
&\leq c_6\int_{|w|>\varepsilon} j(|w|)dw\int_0^t (t-s)\left(\int_{\mathbb{R}^d}\rho(t-s,x-z)(\rho_0^{\beta_2}(s,z-y)+\rho_{\beta_2}^0(s,z-y))dz\right)ds \\
&\quad +c_6 j(\varepsilon)\int_0^t\int_{\mathbb{R}^d}(t-s)\left(\int_{\mathbb{R}^d}\rho(t-s,x\pm w-z)dw\right)(\rho_0^{\beta_2}(s,z-y) \\
&\quad +\rho_{\beta_2}^0(s,z-y))dz\,ds,
\end{aligned}
\tag{4.33}
$$

which is, by Lemma 2.6(a)–(b), less than or equal to

$$c_7(\varepsilon)\left(\int_0^t s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}\rho(t,x-y)\,ds+\int_0^t\int_{\mathbb{R}^d}(\rho_0^{\beta_2}(s,z-y)+\rho_{\beta_2}^0(s,z-y))dz\,ds\right)$$
$$\leq c_8(\varepsilon)\left(\int_0^t s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}ds\rho(t,x-y)+\int_0^t s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}ds\right)<\infty. \tag{4.34}$$

Thus we can apply Fubini's theorem to see that, by Eqs. 4.31, 4.27 holds for $\varepsilon\in(0,1]$. Moreover, by Fubini's theorem and the dominated convergence theorem in the first equality and the second equality below respectively:

$$\mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x)=\lim_{\varepsilon\downarrow 0}\int_0^t\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x)\,ds=\int_0^t\lim_{\varepsilon\downarrow 0}\mathcal{L}^{\mathfrak{K}_x,\varepsilon}\phi_y(t,\cdot,s)(x)\,ds\,,$$

which together with Eq. 4.31 yields Eq. 4.27 for $\varepsilon=0$.

(b) Now we prove Eq. 4.28. Note that, by Lemma 3.1(b), $t\mapsto\delta_{p_y}(t,x-y;z)=p_y(t,x-y+z)+p_y(t,x-y-z)-2p_y(t,x-y)$ is continuous. Let $\varepsilon\in(0,t)$. By Eq. 3.12,

$$\begin{aligned}|\delta_{p_y}(t,x-y;z)|&\leq c_{11}\left(\Phi^{-1}(t^{-1})|z|^2\wedge 1\right)t\,(\rho(t,x-y\pm z)+\rho(t,x-y))\\&\leq c_{12}\frac{t}{\varepsilon}\left(\Phi^{-1}(\varepsilon^{-1})|z|^2\wedge 1\right)\varepsilon\,(\rho(\varepsilon,x-y\pm z)+\rho(\varepsilon,x-y))\,.\end{aligned}$$

By Eq. 1.7 and the proof of Eq. 3.16 we see that the right-hand side multiplied by $\kappa(x,z)J(z)$ is integrable with respect to dz. This shows that the family $\{\delta_{p_y}(t,x-y;z)\kappa(x,z)J(z):t\in(\varepsilon,T)\}$ is dominated by an integrable function. Now by the dominated convergence theorem we see that $t\mapsto\mathcal{L}^{\mathfrak{K}_x}p_y(t,x-y)$ is continuous on $(0,T]$.

Let $\beta_2\in(0,\delta_1/2)\cap(0,\beta]$ and $\gamma\in(0,\beta_2)$. By Eq. 4.32,

$$\left|\mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x,s)\right|\leq c_5(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}\rho(0,x-y)\,. \tag{4.35}$$

Note that for $0<t\leq t+h\leq T$,

$$\begin{aligned}&\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x)-\mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x)\\&=\int_t^{t+h}\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s)ds+\int_0^t\left(\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s)-\mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x,s)\right)ds.\end{aligned} \tag{4.36}$$

When $h\leq t/2$, by Eqs. 2.3 and 2.4, we have

$$\begin{aligned}&\int_t^{t+h}(t+h-s)^{-1}\Phi^{-1}((t+h-s)^{-1})^{\gamma-\beta_2}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}ds\\&=\int_0^h r^{-1}\Phi^{-1}(r^{-1})^{\gamma-\beta_2}(t+h-r)^{-1}\Phi^{-1}((t+h-r)^{-1})^{-\gamma}dr\\&\leq c_{13}\int_0^h r^{-1}\Phi^{-1}(r^{-1})^{\gamma-\beta_2}(t-r)^{-1}\Phi^{-1}((t-r)^{-1})^{-\gamma}dr\,,\end{aligned}$$

and so by Lemma 2.4 and Eq. 4.35 we get

$$\lim_{h\to 0}\int_t^{t+h}\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s)ds=0. \tag{4.37}$$

Note that, by Eq. 4.30 we can apply the dominated convergence theorem and use the continuity of $t \mapsto \mathcal{L}^{\mathfrak{K}_x} p_y(t, x-y)$ so that for each $s \in (0,t)$,

$$\begin{aligned}
&\lim_{h\to 0}(\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s) - \mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x,s)) \\
&= \int_{\mathbb{R}^d} \lim_{h\to 0}(\mathcal{L}^{\mathfrak{K}_x} p_z(t+h-s,\cdot)(x-z) - \mathcal{L}^{\mathfrak{K}_x} p_z(t-s,\cdot)(x-z))q(s,z,y)\,dz \\
&= 0. \qquad (4.38)
\end{aligned}$$

By Lemma 2.3, $s \mapsto (t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{\gamma-\beta_2}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}$ is integrable in $(0,t)$, so using Eq. 4.35, we can apply the dominated convergence theorem and use Eq. 4.38 to get that

$$\lim_{h\to 0}\int_0^t (\mathcal{L}^{\mathfrak{K}_x}\phi_y(t+h,x,s) - \mathcal{L}^{\mathfrak{K}_x}\phi_y(t,x,s))ds = 0. \qquad (4.39)$$

Combining Eqs. 4.37–4.39 we get the desired continuity.

(c) Finally we show Eq. 4.29. Since $\beta+\delta_1 > 1$ and $\delta_1 \in (2/3, 2)$, we can and will choose $\beta_2 \in (0 \vee (1-\delta_1), \delta_1/2) \cap (0,\beta]$ and $\gamma \in (0, \beta_2 \wedge (\beta_2+\delta_1-1) \wedge (\delta_1 - 2\beta_2))$. For example, one can take $\beta_2 = \beta \wedge (1/3)$.

For each fixed $0 < s < t$ and $he_i = (0,\ldots,h,\ldots,0) \in \mathbb{R}^d$ with $|h| \le 1/(2\Phi^{-1}((t-s)^{-1}))$, by Eqs. 3.11, 3.9, 2.1 and 4.12 we have

$$\begin{aligned}
&\frac{1}{h}|p_z(t-s,x-z+he_i) - p_z(t-s,x-z)|\,|q(s,z,y)| \\
&\le c\frac{1}{h}\left((\Phi^{-1}((t-s)^{-1})|h|)\wedge 1\right)(t-s)(\rho(t-s,x-z+he_i) \\
&\quad +\rho(t-s,x-z))|q(s,z,y)| \\
&\le 2^{d+2}c(t-s)\Phi^{-1}((t-s)^{-1})\rho(t-s,x-z)(\rho_0^{\beta_2}+\rho_{\beta_2}^0)(s,z-y) \qquad (4.40)
\end{aligned}$$

which is integrable in $z \in \mathbb{R}^d$ by Lemma 2.6(b). Thus we can use the dominated convergence theorem and Eq. 4.21 to get that for $s \in (0,t)$,

$$\partial_i\phi_y(t,\cdot,s)(x) = \int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(x-z)q(s,z,y)\,dz\,. \qquad (4.41)$$

Let

$$\begin{aligned}
\partial_i\phi_y(t,\cdot,s)(w) &= \int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(w-z)q(s,z,y)\,dz \\
&= \mathbf{1}_{[t/2,t)}(s)\int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(w-z)\,(q(s,z,y)-q(s,w,y))dz \\
&\quad + \mathbf{1}_{[t/2,t)}(s)\int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(w-z)\,q(s,w,y)dz \\
&\quad + \mathbf{1}_{(0,t/2)}(s)\int_{\mathbb{R}^d} \partial_i p_z(t-s,\cdot)(w-z)\,q(s,z,y)dz \\
&=: \mathbf{1}_{[t/2,t)}(s)R_1(t,s,w,y) + \mathbf{1}_{[t/2,t)}(s)R_2(t,s,w,y) \\
&\quad + \mathbf{1}_{(0,t/2)}(s)R_3(t,s,w,y)\,. \qquad (4.42)
\end{aligned}$$

Let $x' \in B(x, |x-y|/4)$. Then it follows from Proposition 3.2 and Eq. 4.13 that for $s \in [t/2, t)$,

$$
\begin{aligned}
&\left|R_1(t,s,x',y)\right| \\
&\le \int_{\mathbb{R}^d} |\partial_i p_z(t-s,\cdot)(x'-z)||q(s,z,y)-q(s,x',y)|dz \\
&\le \int_{\mathbb{R}^d} \Big((t-s)\Phi^{-1}((t-s)^{-1})\rho(t-s,x'-z)\big(|x'-z|^{\beta_2-\gamma}\wedge 1\big)\Big(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\Big)(s,x'-y) \\
&\qquad +(t-s)\Phi^{-1}((t-s)^{-1})\rho(t-s,x'-z)\big(|x'-z|^{\beta_2-\gamma}\wedge 1\big)\Big(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\Big)(s,z-y)\Big)\, dz \\
&= (t-s)\left(\int_{\mathbb{R}^d}\rho_{-1}^{\beta_2-\gamma}(t-s,x'-z)dz\right)\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x'-y). \\
&\qquad +(t-s)\int_{\mathbb{R}^d}\rho_{-1}^{\beta_2-\gamma}(t-s,x'-z)\rho_\gamma^0(s,z-y)dz \\
&\qquad +(t-s)\int_{\mathbb{R}^d}\rho_{-1}^{\beta_2-\gamma}(t-s,x'-z)\rho_{\gamma-\beta_2}^{\beta_2}(s,z-y)dz \\
&\le c_9\Big(\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\left(\rho_\gamma^0+\rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x'-y) \\
&\qquad +\Big(\Phi^{-1}((t-s)^{-1})^{1-2\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma+\beta_2}+\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma} \\
&\qquad +(t-s)s^{-1}\Phi^{-1}(s^{-1})(\Phi^{-1}(s^{-1})^{-\gamma}+\Phi^{-1}(s^{-1})^{-\beta_2})\Big)\rho(t,x'-y)\Big)\Big) \\
&\le c_{10}\Big(\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma+\beta_2} \\
&\qquad +\Phi^{-1}((t-s)^{-1})^{1-2\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma+\beta_2}+\Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma} \\
&\qquad +(t-s)s^{-1}\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\gamma}\Big)\rho(t,(x-y)/2).
\end{aligned} \tag{4.43}
$$

Here the third inequality follows from Lemma 2.6(a)–(b). Since $\delta_1 > 2/3 > 1/2$ and $\gamma < \delta_1 + \beta_2 - 1$, using Lemma 2.3 (so that $\int_{t/2}^t \Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma} ds$ and $\int_{t/2}^t (t-s)\Phi^{-1}((t-s)^{-1}) ds$ are finite) it is straightforward to see that the function on the right-hand side above is integrable in s over $[t/2, t)$.

Next, for $s \in [t/2, t)$, using Eq. 4.12 in the second and Eq. 4.4 in the third line below,

$$
\begin{aligned}
\left|R_2(t,s,x',y)\right| &= \left|\int_{\mathbb{R}^d}\partial_i p_z(t-s,\cdot)(x'-z)\,dz\right| q(s,x',y) \\
&\le \left|\int_{\mathbb{R}^d}\partial_i p_z(t-s,\cdot)(x'-z)\,dz\right|\left(\rho_0^{\beta_2}+\rho_{\beta_2}^0\right)(s,x',y) \\
&\le c\Phi^{-1}((t-s)^{-1})^{1-\beta_2}\rho(t,x'-y) \\
&\le c\Phi^{-1}((t-s)^{-1})^{1-\beta_2}\rho(t,(x-y)/2).
\end{aligned} \tag{4.44}
$$

Since $\int_{t/2}^t \Phi^{-1}((t-s)^{-1})^{1-\beta_2}\, ds < \infty$ because $\beta_2+\delta_1 > 1$, the right-hand side above is integrable in s over $[t/2, t)$.

Finally for $s \in (0, t/2]$, since $\beta_2 < \delta_1/2$,

$$
\begin{aligned}
&\left|R_3(t,s,x',y)\right| \le \int_{\mathbb{R}^d} |\partial_i p_z(t-s,\cdot)(x'-z)|q(s,z,y)|dz \\
&\le c\int_{\mathbb{R}^d}(t-s)\Phi^{-1}((t-s)^{-1})\rho(t-s,x'-z)\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(s,z-y)dz \\
&= c(t-s)\int_{\mathbb{R}^d}\rho_{-1}(t-s,x'-z)\left(\rho^0_{\beta_2}+\rho^{\beta_2}_0\right)(s,z-y)dz \\
&\le c(t-s)\Big((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{1-\beta_2}+(t-s)^{-1}\Phi^{-1}((t-s)^{-1}) \\
&\quad +(t-s)^{-1}\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\beta_2}+\Phi^{-1}((t-s)^{-1})s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}\Big)\rho(t,x'-y) \\
&\le c\Big(\Phi^{-1}((t-s)^{-1})+\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\beta_2} \\
&\quad +(t-s)\Phi^{-1}((t-s)^{-1})s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}\Big)\rho(t,x'-y),
\end{aligned}
\tag{4.45}
$$

which is integrable using Lemma 2.3.

Hence we can use the dominated convergence theorem and Eq. 4.41 to conclude that

$$
\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\left(\phi_y(t,x+w)-\phi_y(t,x)\right)=\lim_{h\to 0}\int_0^t\int_0^1\partial_i\phi_y(t,\cdot,s)(x+\theta w)\,d\theta ds ds \\
&=\int_0^t\partial_i\phi_y(t,\cdot,s)(x)ds=\int_0^t\int_{\mathbb{R}^d}\partial_i p_z(t-s,\cdot)(x-z)q(s,z,y)\,dzds,
\end{aligned}
$$

which gives Eq. 4.29.

□

4.4 Estimates and Smoothness of $p^\kappa(t,x,y)$

Now we define and study the function

$$
\begin{aligned}
p^\kappa(t,x,y) := p_y(t,x-y)+\phi_y(t,x) &= p_y(t,x-y) \\
&+\int_0^t\int_{\mathbb{R}^d}p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,.
\end{aligned}
\tag{4.46}
$$

Lemma 4.8 *(1) For every $T \ge 1$ and $\beta_2 \in (0,\beta]\cap(0,\delta_1/2)$, there is a constant $c_1 = c_1(T,d,\delta_1,\beta_2,\gamma,\kappa_0,\kappa_1,\kappa_2) > 0$ so that for all $t \in (0,T]$ and $x,y \in \mathbb{R}^d$, $p^\kappa(t,x,y) \le c_1 t\rho(t,x-y)$. (2) For any $\gamma \in (0,\delta_1)\cap(0,1]$ and $T \ge 1$ there exists $c_2 = c_2(T,d,\delta_1,\beta_2,\gamma,\kappa_0,\kappa_1,\kappa_2) > 0$ such that for all $x,x',y \in \mathbb{R}^d$ and $t \in (0,T]$,*

$$
\left|p^\kappa(t,x,y)-p^\kappa(t,x',y)\right| \le c_2|x-x'|^\gamma\ t\left(\rho^0_{-\gamma}(t,x-y)+\rho^0_{-\gamma}(t,x'-y)\right).
$$

Proof Throughout this proof we assume that $x,x',y \in \mathbb{R}^d$ and $t \in (0,T]$.

(1) By the estimate of p_z (Proposition 3.2), Eq. 4.12, Lemma 2.6(c), Eqs. 2.14 and 2.15, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z)|q(s,z,y)|\, dz\, ds \\
&\le c_1 \int_0^t \int_{\mathbb{R}^d} (t-s)\rho(t-s, x-z)\left(\rho^0_{\beta_2} + \rho^{\beta_2}_0\right)(s, z-y)\, dz\, ds \\
&\le c_2 t \left(\rho^0_{\beta_2} + \rho^{\beta_2}_0\right)(t, x-y) \\
&\le 2\Phi^{-1}(T^{-1})^{-\beta_2} c_2 t\rho(t, x-y), \qquad \text{for all } t \in (0, T]. \qquad (4.47)
\end{aligned}$$

Therefore, $p^\kappa(t, x, y) \le p_y(t, x-y) + |\phi_y(t, x)| \le c_4 t\rho(t, x-y)$.

(2) We have by Eq. 3.11 and the fact that $\gamma \le 1$,

$$\begin{aligned}
|p_z(t, x-z) - p_z(t, x'-z)| &\le c_1|x-x'|^\gamma t\Phi^{-1}(t^{-1})^\gamma \left(\rho(t, x-z) + \rho(t, x'-z)\right) \\
&= c_1|x-x'|^\gamma t\left(\rho^0_{-\gamma}(t, x-z) + \rho^0_{-\gamma}(t, x'-z)\right).
\end{aligned}$$

Thus, by Eq. 4.12 and a change of the variables, we further have

$$\begin{aligned}
&|\phi_y(t,x) - \phi_y(t,x')| \le \int_0^t \int_{\mathbb{R}^d} |p_z(t-s, x-z) - p_z(t-s, x'-z)|\, |q(s,z,y)|\, dz\, ds \\
&\le c_2|x-x'|^\gamma \int_0^t \int_{\mathbb{R}^d} (t-s)\left(\rho^0_{-\gamma}(t-s, x-z) + \rho^0_{-\gamma}(t-s, x'-z)\right)\left(\rho^{\beta_2}_0 + \rho^0_{\beta_2}\right) \\
&\quad \times (s, z-y)\, dz\, ds \\
&\le c_3|x-x'|^\gamma t\left(\rho^0_{-\gamma+\beta_2}(t, x-y) + \rho^{\beta_2}_{-\gamma}(t, x-y) + \rho^0_{-\gamma+\beta_2}(t, x'-y)\right. \\
&\quad \left. + \rho^{\beta_2}_{-\gamma}(t, x'-y)\right) \\
&\le 2c_3\Phi^{-1}(T^{-1})^{-\beta_2}|x-x'|^\gamma t\left(\rho^0_{-\gamma}(t, x-y) + \rho^0_{-\gamma}(t, x'-y)\right), \qquad \text{for all } t \in (0, T].
\end{aligned}$$

Since $\gamma \in (0, \delta_1)$, the penultimate inequality follows from Eq. 2.19 (with $\theta = 0$), and the last inequality by Eqs. 2.14 and 2.15. The claim of the lemma follows by combining the two estimates. □

The following result is the counterpart of [6, Lemma 3.7].

Lemma 4.9 *The function* $p^\kappa(t, x, y)$ *defined in* Eq. 4.46 *is jointly continuous on* $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof The joint continuity of $p_y(t, x-y)$ was shown in Lemma 4.2. For $\phi_y(t, x)$ we use Eq. 4.22 and the joint continuity of $q(s, z, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ together with the dominated convergence theorem. This is justified by the estimates $p_z(t-s, x-z) \le c_1(t-s)\rho(t-s, x-z)$ and Eq. 4.12 which yield that $|p_z(t-s, x-z)q(s, z, y)| \le c_2(t-s)\rho(t-s)\left(\rho^{\beta_2}_0 + \rho^0_{\beta_2}\right)(s, z-y)$ for $\beta_2 \in (0, \beta] \cap (0, \delta_1/2)$. The latter function is integrable over $(0, t] \times \mathbb{R}^d$ with respect to $ds\, dz$ by Lemma 2.6. □

Now we define the operator $\mathcal{L}^\kappa$ as in Eq. 1.8 which can be rewritten as

$$\mathcal{L}^\kappa f(x) = \mathcal{L}^{\kappa,0} f(x) = \lim_{\varepsilon\downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x), \quad \text{where } \mathcal{L}^{\kappa,\varepsilon} f(x) = \frac{1}{2}\int_{|z|>\varepsilon} \delta_f(x;z)\kappa(x,z)J(z)\,dz. \tag{4.48}$$

Note that for a fixed $x \in \mathbb{R}^d$, it holds that $\mathcal{L}^\kappa f(x) = \mathcal{L}^{\mathfrak{K}_x} f(x)$. This will be used later on.

The following result is the counterpart of [6, Lemma 4.2].

Lemma 4.10 *For every $T \geq 1$, there is a constant $c_1 = c_1(T, d, \delta_1, a_1, \beta, C_*, \gamma_0, \kappa_0, \kappa_1, \kappa_2) > 0$ such that for all $\varepsilon \in [0, 1]$,*

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t,\cdot,y)(x)| \leq c_1\rho(t, x-y), \qquad \text{for all } t \in (0,T] \text{ and } x, y \in \mathbb{R}^d, x \neq y \tag{4.49}$$

and if $\beta + \delta_1 > 1$ and $\delta_1 \in (2/3, 2)$ we also have

$$\left|\nabla_x p^\kappa(t,x,y)\right| \leq c_1 t\Phi^{-1}(t^{-1})\rho(t,x-y) \quad \text{for all } t \in (0,T] \text{ and } x, y \in \mathbb{R}^d, x \neq y\,. \tag{4.50}$$

Proof By Eq. 3.16 and the fact that for fixed x, $\mathcal{L}^{\kappa,\varepsilon} f(x) = \mathcal{L}^{\mathfrak{K}_x,\varepsilon} f(x)$ for $\varepsilon \in [0, 1]$, we see that

$$|\mathcal{L}^\kappa p_y(t,\cdot)(x-y)| \leq c_1\rho(t,x-y), \qquad \text{for all } t \in (0,T] \text{ and } \varepsilon \in [0,1].$$

Let $\varepsilon \in [0, 1]$. By recalling the definition (4.22) of ϕ_y and using Eq. 4.27, we have

$$\begin{aligned}
\mathcal{L}^{\kappa,\varepsilon}\phi_y(t,x) &= \int_{t/2}^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_z(t-s,\cdot)(x-z)\,(q(s,z,y) - q(s,x,y))\,dz\,ds \\
&\quad + \int_{t/2}^t \left(\int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_z(t-s,\cdot)(x-z)\,dz\right) q(s,x,y)\,ds \\
&\quad + \int_0^{t/2} \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x,\varepsilon} p_z(t-s,\cdot)(x-z) q(s,z,y)\,dz\,ds \\
&=: Q_1(t,x,y) + Q_2(t,x,y) + Q_3(t,x,y)\,.
\end{aligned}$$

Let $\beta_2 \in (0, \delta_1/2) \cap (0, \beta]$. For $Q_1(t,x,y)$ we use Eq. 3.16, Lemmas 2.2(b), 2.3 and 2.6(a) and (c) to get that for any $\gamma \in ((2-\delta_1)\beta_2/2, \beta_2)$,

$$\begin{aligned}
|Q_1(t,x,y)| &\leq c_1 \int_{t/2}^t \left(\int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\,dz\right)\left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2}\right)(s,x-y)\,ds \\
&\quad + c_1 \int_{t/2}^t \int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\,dz\,ds \\
&\leq c_2\left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2}\right)(t,x-y)\int_0^t \int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\,dzds \\
&\quad + c_1 \int_0^t \int_{\mathbb{R}^d} \rho_0^{\beta_2-\gamma}(t-s,x-z)\left(\rho_\gamma^0 + \rho_{\gamma-\beta_2}^{\beta_2}\right)(s,z-y)\,dz\,ds \\
&\leq c_3\rho_{\gamma-\beta_2}^0(t,x-y)\Phi^{-1}(t^{-1})^{-\beta_2-\gamma} + c_3\left(\rho_{\beta_2}^0 + \rho_\gamma^{\beta_2-\gamma} + \rho_0^{\beta_2}\right)(t,x-y) \\
&\leq c_4\rho(t,x-y), \qquad \text{for all } t \in (0,T]\,,
\end{aligned}$$

where the last two lines follow from Eqs. 2.14 and 2.15.

For $Q_2(t,x,y)$, by Eqs. 4.2, 4.12, Lemmas 2.2(b), 2.3, Eqs. 2.14 and 2.15,

$$\begin{aligned}
|Q_2(t,x,y)| &\le c_5 \int_{t/2}^{t} (t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2} \left(\rho_{\beta_2}^{0} + \rho_0^{\beta_2}\right)(s, x-y)\, ds \\
&\le c_6 \left(\rho_{\beta_2}^{0} + \rho_0^{\beta_2}\right)(t, x-y) \int_0^t (t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\beta_2}\, ds \\
&\le c_7 \rho(t, x-y)\Phi^{-1}(t^{-1})^{-\beta_2} \le c_7 \Phi^{-1}(T^{-1})^{-\beta_2}\rho(t,x-y), \quad \text{for all } t \in (0, T].
\end{aligned}$$

For $Q_3(t,x,y)$, by Eqs. 3.16, 4.12, Lemma 2.6(c), Eqs. 2.14 and 2.15,

$$\begin{aligned}
|Q_3(t,x,y)| &\le c_7 \int_0^{t/2}\int_{\mathbb{R}^d} \rho(t-s, x-z)\left(\rho_{\beta_2}^{0} + \rho_0^{\beta_2}\right)(s, z-y)\, dz\, ds \\
&\le 2\frac{c_7}{t}\int_0^{t}\int_{\mathbb{R}^d} (t-s)\rho(t-s, x-z)\left(\rho_{\beta_2}^{0} + \rho_0^{\beta_2}\right)(s, z-y)\, dz\, ds \\
&\le c_8 \left(\rho_{\beta_2}^{0} + \rho_0^{\beta_2}\right)(t, x-y) \le 2c_8 \Phi^{-1}(T^{-1})^{-\beta}\rho(t, x-y).
\end{aligned}$$

Combining the above calculations and Eq. 4.46 we obtain Eq. 4.49.

(ii) Since $\beta + \delta_1 > 1$ and $\delta_1 \in (2/3, 2)$, we can and will choose $\beta_2 \in (0 \vee (1-\delta_1), \delta_1/2) \cap (0, \beta]$ and $\gamma \in (0, \beta_2 \wedge (\beta_2 + \delta_1 - 1) \wedge (\delta_1 - 2\beta_2))$. By Eqs. 4.29 and 4.42–4.45 we have

$$\begin{aligned}
|\nabla_x \phi_y(t,x)| \le c_1 \rho(t, x-y) \Bigg(&\int_0^{t/2} \Phi^{-1}((t-s)^{-1}) + \Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\beta_2} \\
&+ (t-s)\Phi^{-1}((t-s)^{-1}) s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2} ds \\
+ &\int_{t/2}^{t} \Phi^{-1}((t-s)^{-1})^{1-\beta_2} + \Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\gamma+\beta_2} \\
&+ \Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma}\Phi^{-1}(s^{-1})^{-\beta_2} \\
&+ (t-s)s^{-1}\Phi^{-1}((t-s)^{-1})\Phi^{-1}(s^{-1})^{-\gamma} ds \Bigg). \qquad (4.51)
\end{aligned}$$

Since $\beta + \delta_1 > 1$, $\delta_1 > 2/3 > 1/2$ and $\gamma < \delta_1 + \beta_2 - 1$, using Lemma 2.3 we see that $\int_{t/2}^{t} \Phi^{-1}((t-s)^{-1})^{1-\beta_2} ds \le c_2 t \Phi^{-1}(t^{-1})^{1-\beta_2}$, $\int_{t/2}^{t} \Phi^{-1}((t-s)^{-1})^{1-\beta_2+\gamma} ds \le c_3 t \Phi^{-1}(t^{-1})^{1-\beta_2+\gamma}$ and $\int_0^t (t-s)\Phi^{-1}((t-s)^{-1}) ds \le c_4 t^2 \Phi^{-1}(t^{-1})$. Thus, by Lemma 2.3, Eq. 4.51 is bounded above by $c_5 t \Phi^{-1}(t^{-1})\rho(t, x-y)$. Now, Eq. 4.50 follows immediately from this, Eqs. 4.46, 4.29 and Proposition 3.2. □

We will also need the following corollary, which follows from Eq. 4.28.

Corollary 4.11 *For $x \neq y$, the function $t \mapsto \mathcal{L}^{\kappa} p^{\kappa}(t,x,y)$ is continuous on* $(0, \infty)$.

5 Proofs of Main Results

5.1 A Nonlocal Maximum Principle

We first establish a somewhat different version of [6, Theorem 4.1].

Theorem 5.1 *Suppose there exists a function* $g : \mathbb{R}^d \to (0, \infty)$ *such that* Eq. 1.9 *holds. Let* $T > 0$ *and* $u \in C_b([0, T] \times \mathbb{R}^d)$ *be such that*

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} |u(t, x) - u(0, x)| = 0\,, \tag{5.1}$$

and for each $x \in \mathbb{R}^d$,

$$t \mapsto \mathcal{L}^\kappa u(t, x) \text{ is continuous on } (0, T]. \tag{5.2}$$

Suppose that $u(t, x)$ *satisfies the following inequality: for all* $(t, x) \in (0, T] \times \mathbb{R}^d$,

$$\partial_t u(t, x) \le \mathcal{L}^\kappa u(t, x)\,. \tag{5.3}$$

Then for all $t \in (0, T)$,

$$\sup_{x \in \mathbb{R}^d} u(t, x) \le \sup_{x \in \mathbb{R}^d} u(0, x)\,. \tag{5.4}$$

Proof Choose $a > 0$ such that

$$\mathcal{L}^\kappa g(x) \le a g(x), \quad \text{for all } x \in \mathbb{R}^d. \tag{5.5}$$

Let $\delta, \varepsilon > 0$ and $u^\delta_\varepsilon(t, x) := u(t, x) - \delta(t - \varepsilon + e^{at} g(x))$. Then by Eqs. 5.3 and 5.5, for all $(t, x) \in (0, T] \times \mathbb{R}^d$, we have

$$\begin{aligned} \partial_t u^\delta_\varepsilon(t, x) &= \partial_t u(t, x) - \delta(1 + a e^{at} g(x)) \le \mathcal{L}^\kappa u(t, x) - \delta - \delta a e^{at} g(x) \\ &= \mathcal{L}^\kappa u^\delta_\varepsilon(t, x) - \delta + \delta e^{at}(\mathcal{L}^\kappa g(x) - a g(x)) \le \mathcal{L}^\kappa u^\delta_\varepsilon(t, x) - \delta. \end{aligned} \tag{5.6}$$

Since $u \in C_b([0, T] \times \mathbb{R}^d)$, by letting $\delta \to 0$ and $\varepsilon \to 0$, it suffices to show that

$$\sup_{x \in \mathbb{R}^d} u^\delta_\varepsilon(t, x) \le \sup_{x \in \mathbb{R}^d} u^\delta_\varepsilon(\varepsilon, x), \quad t \in (\varepsilon, T]\,. \tag{5.7}$$

Fix $\delta, \varepsilon > 0$ and suppose that Eq. 5.7 does not hold. Then, by the continuity of u^δ_ε and the fact that $\lim_{x \to \infty} u^\delta_\varepsilon(t, x) = -\infty$ (which is a consequence of Eq. 1.9), there exist $t_0 \in (\varepsilon, T]$ and $x_0 \in \mathbb{R}^d$ such that

$$\sup_{t \in (\varepsilon, T], x \in \mathbb{R}^d} u^\delta_\varepsilon(t, x) = u^\delta_\varepsilon(t_0, x_0). \tag{5.8}$$

Thus by Eq. 5.6, for $h \in (0, t_0 - \varepsilon)$,

$$0 \le \frac{1}{h}(u^\delta_\varepsilon(t_0, x_0) - u^\delta_\varepsilon(t_0 - h, x_0)) = \frac{1}{h}\int_{t_0 - h}^{t_0} \partial_t u^\delta_\varepsilon(s, x_0) ds \le \frac{1}{h}\int_{t_0 - h}^{t_0} \mathcal{L}^\kappa u^\delta_\varepsilon(s, x_0) ds - \delta.$$

Letting $h \to 0$ and using Eqs. 5.2 and 5.8 we get

$$\begin{aligned} 0 \;&\le\; \mathcal{L}^\kappa u^\delta_\varepsilon(t_0, x_0) - \delta \\ &= \text{p.v.} \int_{\mathbb{R}^d} \left(u^\delta_\varepsilon(t_0, x_0 + z) - u^\delta_\varepsilon(t_0, x_0)\right) \kappa(x_0, z) J(z)\, dz - \delta \le -\delta, \end{aligned}$$

which gives a contradiction. Therefore Eq. 5.7 holds. □

Remark 5.2 Suppose that $\int_{|z|>1} |z|^\varepsilon j(|z|) dz < \infty$ for some $\varepsilon > 0$. Let $g(x) = (1+|x|^2)^{\varepsilon/2}$. Note that

$$|\partial_{i,j} g(x)| \le c_1(1 + |x|)^{\varepsilon - 2}, \quad i, j = 1, \dots, d. \tag{5.9}$$

By Eqs. 5.9 and 3.7, we have that for $|x| \le 1$,

$$
\begin{aligned}
|\mathcal{L}^\kappa g(x)| &\le \gamma_0 \int_{|z|\le 1} |\delta_g(x;z)| j(|z|) dz + \gamma_0 g(x) \int_{|z|>1} j(|z|) dz \\
&\quad + \gamma_0 \int_{|z|>1} g(x \pm z) j(|z|) dz \\
&\le c_2 \left(\int_{|z|\le 1} |z|^2 j(|z|) dz + \int_{|z|>1} j(|z|) dz + \int_{|z|>1} |z|^\varepsilon j(|z|) dz \right) \\
&\le c_3 \le c_3 g(x). \qquad (5.10)
\end{aligned}
$$

If $|x| > 1$, then by Eqs. 5.9 and 3.7,

$$
\begin{aligned}
|\mathcal{L}^\kappa g(x)| &\le \gamma_0 \int_{|z|\le |x|} |\delta_g(x;z)| j(|z|) dz + \gamma_0 g(x) \int_{|z|>|x|} j(|z|) dz \\
&\quad + \gamma_0 \int_{|z|>|x|} g(x \pm z) j(|z|) dz \\
&\le c_3 \left(\int_{|z|\le |x|} |x|^{\varepsilon-2} |z|^2 j(|z|) dz + g(x) \int_{|z|>1} j(|z|) dz + \int_{|z|>|x|} |z|^\varepsilon j(|z|) dz \right) \\
&\le c_4 \left(|x|^\varepsilon \int_{\mathbb{R}^d} ((|z|/|x|)^2 \wedge 1) j(|z|) dz + g(x) + 1 \right) \le c_5 g(x). \qquad (5.11)
\end{aligned}
$$

Therefore g satisfies Eq. 1.9.

5.2 Properties of the Semigroup $(P_t^\kappa)_{t\ge 0}$

Define

$$
P_t^\kappa f(x) = \int_{\mathbb{R}^d} p^\kappa(t,x,y) f(y) dy.
$$

Lemma 5.3 *For any bounded function f, we have*

$$
\mathcal{L}^\kappa P_t^\kappa f(x) = \int_{\mathbb{R}^d} \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x) f(y) dy\,. \qquad (5.12)
$$

Proof By the same computation as in the proof of Eq. 3.16 we have that for all $t \le T$, $T \ge 1$, and $\varepsilon > 0$,

$$
\begin{aligned}
& t \int_{|z|>\varepsilon} \rho(t, x \pm z) j(|z|)\, dz \\
&\le \int_{\Phi^{-1}(t^{-1})|z|\le 1, |z|>\varepsilon} t\rho(t, x \pm z) j(|z|)\, dz + \int_{\Phi^{-1}(t^{-1})|z|>1} t\rho(t, x \pm z) j(|z|)\, dz \\
&\le c_1 4^{d+1} t\rho(t,x) \int_{|z|>\varepsilon} j(|z|)\, dz + c_1 \rho(t,x),
\end{aligned}
$$

thus by Lemma 4.8(1),

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon}\left|p^\kappa(t,x\pm w,y)-2p^\kappa(t,x,y)\right|\kappa(x,w)J(w)\,dw\right)dy\\
&\le 2\gamma_0\kappa_1\int_{\mathbb{R}^d}\int_{|w|>\varepsilon}|p^\kappa(t,x,y)|j(|w|)\,dwdy+\gamma_0\kappa_1\int_{\mathbb{R}^d}\int_{|w|>\varepsilon}|p^\kappa(t,x\pm w,y)|j(|w|)\,dwdy\\
&\le c_2t\left(\int_{|w|>\varepsilon}j(|w|)\,dw\right)\int_{\mathbb{R}^d}\rho(t,x-y)dy+c_2t\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon}\rho(t,x\pm w-y)j(|w|)\,dw\right)dy\\
&<\infty.
\end{aligned}$$

Thus by Fubini's theorem, for all for bounded function f and $\varepsilon\in(0,1]$,

$$\mathcal{L}^{\kappa,\varepsilon}P_t^\kappa f(x)=\int_{\mathbb{R}^d}\mathcal{L}^{\kappa,\varepsilon}p^\kappa(t,\cdot,y)(x)f(y)dy.$$

Now, Eq. 5.12 follows from this, Eq. 4.49 and the dominated convergence theorem. □

The following result is the counterpart of [6, Lemma 4.4].

Lemma 5.4 (a) *For any $p\in[1,\infty]$, there exists a constant $c=c(p,d,\delta_1,\beta,\kappa_0,\kappa_1,\kappa_2)>0$ such that for all $f\in L^p(\mathbb{R}^d)$ and $t>0$,*

$$\|\mathcal{L}^\kappa P_t^\kappa f\|_p\le ct^{-1}\|f\|_p\,. \tag{5.13}$$

(b) *If $f\in L^\infty(\mathbb{R}^d)$, $t\mapsto\mathcal{L}^\kappa P_t^\kappa f$ is a continuous function on $(0,\infty)$.*
(c) *For any $p\in[1,\infty)$ and $f\in L^p(\mathbb{R}^d)$, $t\mapsto\mathcal{L}^\kappa P_t^\kappa f$ is continuous from $(0,\infty)$ into $L^p(\mathbb{R}^d)$.*

Proof (a) Let $p\in[1,\infty]$. By Eq. 5.12, Lemma 4.10, Young's inequality and Lemma 2.6(a), we have that for all $f\in L^p(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
\|\mathcal{L}^\kappa P_t^\kappa f\|_p&\le c_1\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\rho(t,x-y)|f(y)|\,dy\right|^p dx\right)^{1/p}\\
&\le c_1\|\rho(t,\cdot)\|_1\,\|f\|_p\le c_2t^{-1}\|f\|_p\,.
\end{aligned}$$

Inequality Eq. 5.13 for $f\in L^p(\mathbb{R}^d)$ now follows by a standard density argument.

(b) For any $\varepsilon\in(0,1)$, by Lemma 4.10 we have for $x\ne y$,

$$\sup_{t\in(\varepsilon,T)}\left|\mathcal{L}^\kappa p^\kappa(t,x,y)\right|\le c\sup_{t\in(\varepsilon,T)}\rho(t,x-y)\le c\rho(\varepsilon,x-y)\,.$$

Assume that f is bounded and measurable. By Corollary 4.11, $t\mapsto\mathcal{L}^\kappa p^\kappa(t,x,y)f(y)$ is continuous for $x\ne y$. By the above display, the family $\{\mathcal{L}^\kappa p^\kappa(t,x,y)f(y):t\in(\varepsilon,1)\}$ is bounded by the integrable function $\rho(\varepsilon,x-y)|f(y)|$. Now it follows from the dominated convergence theorem and Eq. 5.12 that $t\mapsto\mathcal{L}^\kappa P_t^\kappa f(x)$ is continuous.

(c) Let $p\in[1,\infty)$. When $f\in L^p(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, the claim follows similarly as (b) by using Eq. 5.12 and the domination by the L^p-function $\int_{\mathbb{R}^d}\rho(\varepsilon,x-y)f(y)\,dy$. The claim for $f\in L^p(\mathbb{R}^d)$ now follows by standard density argument and Eq. 5.13. □

Remark 5.5 Note that Lemma 5.4 uses only the following properties of $p^\kappa(t,x,y)$: Eq. 5.12, $|\mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)|\le c_1(T)\rho(t,x-y)$ for $t\in(0,T]$ and $t\mapsto\mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)$ is continuous on $(0,T]$. Moreover, Lemma 5.3 uses only the following properties of

$p^\kappa(t,x,y)$: $p^\kappa(t,\cdot,y)(x) \le c_2(T)t\rho(t,x-y)$ and $|\mathcal{L}^{\kappa,\varepsilon}p^\kappa(t,\cdot,y)(x)| \le c_3(T)\rho(t,x-y)$ for $\varepsilon \in [0,1]$ and $t \in (0,T]$.

The following result is the counterpart of [6, Lemma 4.3].

Lemma 5.6 *For any bounded Hölder continuous function $f \in C_b^\eta(\mathbb{R}^d)$, we have*

$$\mathcal{L}^\kappa\left(\int_0^t P_s^\kappa f(\cdot)ds\right)(x) = \int_0^t \mathcal{L}^\kappa P_s^\kappa f(x)ds\,, \quad x \in \mathbb{R}^d\,. \tag{5.14}$$

Proof Define

$$T_t f(x) = \int_{\mathbb{R}^d} p_y(t,x-y)f(y)dy, \quad S_t f(x) = \int_{\mathbb{R}^d} q(t,x,y)f(y)dy$$

and

$$R_t f(x) = \int_0^t T_{t-s}S_s f(x)ds.$$

Then, by Fubini's theorem and Eq. 4.12, for all for bounded function f,

$$P_t^\kappa f(x) = T_t f(x) + R_t f(x). \tag{5.15}$$

We now assume $\varepsilon \in (0,1]$ and $0 < s < t \le T$, $T \ge 1$. Suppose that $|f(x)-f(y)| \le c_1(|x-y|^\eta \wedge 1)$. Without loss of generality we may and will assume that $\eta < \beta$. By Fubini's theorem, Eqs. 1.7, 1.1 and 3.16,

$$\mathcal{L}^{\kappa,\varepsilon}T_t f(x) = \int_{\mathbb{R}^d} \mathcal{L}^{\kappa,\varepsilon}p_z(s,\cdot)(x-z)f(z)\,dz.$$

Thus,

$$\begin{aligned}|\mathcal{L}^{\kappa,\varepsilon}T_s f(x)| \le& \int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon}|\delta_{p_z}(s,x-z;w)|\kappa(x,w)J(w)\,dw\right)|f(z)-f(x)|\,dz\\ &+\left|\int_{\mathbb{R}^d}\left(\int_{|w|>\varepsilon}\delta_{p_z}(s,x-z;w)\kappa(x,w)J(w)\,dw\right)dz\right||f(x)|\,.\end{aligned}$$

By using Eqs. 1.7, 3.16, 4.2 and 2.17, for any $\beta_1 \in (0,\delta_1)\cap(0,\beta]$, $|\mathcal{L}^{\kappa,\varepsilon}T_s f(x)|$ is bounded by

$$\begin{aligned}&c_1\int_{\mathbb{R}^d}\rho(s,x-z)\left(|x-z|^\eta\wedge 1\right)dz + c_1\,s^{-1}\Phi^{-1}(s^{-1})^{-\beta_1}\\ &\le c_2\,s^{-1}\Phi^{-1}(s^{-1})^{-\eta} + c_1\,s^{-1}\Phi^{-1}(s^{-1})^{-\beta_1}\,,\end{aligned}$$

and the right hand side is integrable by Lemma 2.3. Thus by the dominated convergence theorem and Fubini's theorem,

$$\mathcal{L}^\kappa\int_0^t T_s f(x)\,ds = \lim_{\varepsilon\downarrow 0}\mathcal{L}^{\kappa,\varepsilon}\int_0^t T_s f(x)\,ds = \int_0^t \lim_{\varepsilon\downarrow 0}\mathcal{L}^{\kappa,\varepsilon}T_s f(x)ds = \int_0^t \mathcal{L}^\kappa T_s f(x)\,ds\,. \tag{5.16}$$

It follows from Eqs. 4.13, 2.17 and the boundedness of f that for any $\beta_2 \in (0,\beta]\cap(0,\delta_1/2)$ and $\gamma \in (0,\beta_2)$, we have

$$|S_s f(x) - S_s f(x')| \le c_3 s^{-1}\Phi^{-1}(s^{-1})^{-\gamma}\left(|x-x'|^{\beta_2-\gamma}\wedge 1\right). \tag{5.17}$$

It follows from Eqs. 4.12, 2.17 and the boundedness of f that

$$|S_s f(x)| \le c_4 s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}. \tag{5.18}$$

We use Lemma 4.8(1) and Fubini's theorem in the first line below, which can be justified by an argument similar to Eqs. 4.33 and 4.34:

$$
\begin{aligned}
&|\mathcal{L}^{\kappa,\varepsilon} R_s f(x)| \\
&\leq \int_0^s \left| \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \delta_{p_z}(s-r, x-z; w)\kappa(x,w)J(w)\,dw \right) S_r f(z)\,dz \right| dr \\
&\leq \int_0^s \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} |\delta_{p_z}(s-r, x-z; w)|\kappa(x,w)J(w)\,dw \right) |S_r f(z) - S_r f(x)|\,dzdr \\
&\quad + \int_0^s \left| \int_{\mathbb{R}^d} \left(\int_{|w|>\varepsilon} \delta_{p_z}(s-r, x-z; w)\kappa(x,w)J(w)\,dw \right) dz \right| |S_r f(x)| dr\,.
\end{aligned}
$$

By using Eqs. 1.7, 3.16, 4.2, 2.17, 5.17, 5.18 and Lemma 2.3, we further have that

$$
\begin{aligned}
|\mathcal{L}^{\kappa,\varepsilon} R_s f(x)| &\leq c_5 \int_0^s \int_{\mathbb{R}^d} \rho(s-r, x-z) r^{-1}\Phi^{-1}(r^{-1})^{-\gamma} \left(|x-z|^{\beta_2-\gamma} \wedge 1\right) dzdr \\
&\quad + c_5 \int_0^s r^{-1}\Phi^{-1}(r^{-1})^{-\beta_2} dr \\
&\leq c_6 \int_0^s (s-r)^{-1}\Phi^{-1}((s-r)^{-1})^{-(\beta_2-\gamma)} r^{-1}\Phi^{-1}(r^{-1})^{-\gamma} dr + c_5 \int_0^s r^{-1}\Phi^{-1}(r^{-1})^{-\beta_2} dr \\
&\leq c_7 s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2} + c_5 \Phi^{-1}(s^{-1})^{-\beta_2} = 2c_7 s^{-1}\Phi^{-1}(s^{-1})^{-\beta_2}\,,
\end{aligned}
$$

and the right hand side is integrable by Lemma 2.3. This justifies the use of the dominated convergence theorem in the second line of the following calculation:

$$
\mathcal{L}^{\kappa} \int_0^t R_s f(x)\,ds = \lim_{\varepsilon\downarrow 0} \mathcal{L}^{\kappa,\varepsilon} \int_0^t R_s f(x)\,ds = \int_0^t \lim_{\varepsilon\downarrow 0} \mathcal{L}^{\kappa,\varepsilon} R_s f(x)\,ds = \int_0^t \mathcal{L}^{\kappa} R_s f(x)\,ds\,. \tag{5.19}
$$

Combining Eq. 5.19 with 5.16 and 5.15, we arrive at the conclusion of this lemma. □

5.3 Proofs of Theorems 1.1–1.3

Proof of Theorem 1.1. By using Lemma 4.6 in the second equality, Eq. 4.6 in the third, Eq. 4.11 in the fourth, Eq. 4.6 in the fifth, and Lemma 4.7 in the sixth equality, we have

$$
\begin{aligned}
\partial_t p^\kappa(t,x,y) &= \partial_t p_y(t, x-y) + \partial_t \phi_y(t,x) \\
&= \mathcal{L}^{\mathfrak{K}_y} p_y(t,x-y) + \left(q(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_z} p_z(t-s,\cdot)(x-z) q(s,z,y)\,dz\,ds \right) \\
&= \left(\mathcal{L}^{\mathfrak{K}_x} p_y(t,x-y) - q_0(t,x,y) \right) \\
&\quad + \left(q(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_z} p_z(t-s,\cdot)(x-z) q(s,z,y)\,dz\,ds \right) \\
&= \mathcal{L}^{\mathfrak{K}_x} p_y(t,x-y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x-z) q(s,z,y)\,dz\,ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_z} p_z(t-s,\cdot)(x-z) q(s,z,y)\,dz\,ds \\
&= \mathcal{L}^{\mathfrak{K}_x} p_y(t,x-y) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^{\mathfrak{K}_x} p_z(t-s,\cdot)(x-z) q(s,z,y)\,dz\,ds \\
&= \mathcal{L}^\kappa p^\kappa(t,x,y)\,.
\end{aligned}
$$

Thus Eq. 1.10 holds. The joint continuity of $p^\kappa(t,x,y)$ is proved in Lemma 4.9. Further, if we apply the maximum principle, Theorem 5.1, to $u_f(t,x) := P_t^\kappa f(x)$ with $f \in C_c^\infty(\mathbb{R}^d)$ and $f \le 0$, we get $u_f(t,x) \le 0$ for all $t \in (0,T]$ and all $x \in \mathbb{R}^d$. This implies that $p^\kappa(t,x,y) \ge 0$.

(i) Equation 1.11 is proved in Lemma 4.8(1).

(ii) The estimate Eq. 1.12 is given in Eq. 4.49, while continuity of $t \mapsto \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)$ is proven in Corollary 4.11.

(iii) Let f be a bounded and uniformly continuous function. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $|x-y| < \delta$. By Eqs. 4.5, 1.5, 2.17 and the estimate for $p_y(t, x-y)$ in Proposition 3.2 we have

$$\begin{aligned}
&\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} p_y(t,x-y)f(y)\,dy - f(x)\right| \\
&= \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} p_y(t,x-y)f(y)\,dy - \int_{\mathbb{R}^d} p_y(t,x-y)f(x)\,dy\right| \\
&\le c_1 \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} t\rho(t,x-y)\,|f(y)-f(x)|\,dy \\
&\le \varepsilon c_1 \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<\delta} t\rho(t,x-y)dy + 2c_1\|f\|_\infty \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\ge\delta} t\rho(t,x-y)dy \\
&\le c_2\varepsilon \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} t\rho(t,x-y)dy + 2c_1\|f\|_\infty \lim_{t\downarrow 0} t \sup_{x\in\mathbb{R}^d}\int_{|x-y|\ge\delta} \frac{\Phi(|x-y|^{-1})}{|x-y|^d}dy \\
&\le c_2\varepsilon + 2c_1\|f\|_\infty \lim_{t\downarrow 0} t\int_{|z|\ge\delta}\frac{\Phi(|z|^{-1})}{|z|^d}dz = c_2\varepsilon\,.
\end{aligned}$$

This implies that

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} p_y(t,x-y)f(y)\,dy - f(x)\right| = 0\,. \tag{5.20}$$

Further, by Eqs. 4.47 and 2.17, for any $\beta_2 \in (0,\beta] \cap (0,\delta_1)$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)\,dz\,ds\,f(y)dy\right| \\
&\le c_3\,\|f\|_\infty\, t\int_{\mathbb{R}^d}\left(\rho_0^{\beta_2} + \rho_{\beta_2}^0\right)(t,x-y)\,dy \le c_4\,\Phi^{-1}(t^{-1})^{-\beta_2} \longrightarrow 0\,, \quad t\downarrow 0\,.
\end{aligned}$$

The claim now follows from this, Eqs. 4.46 and 5.20.

Uniqueness of the kernel satisfying Eqs. 1.10–1.13 Let $\widetilde{p}^\kappa(t,x,y)$ be another non-negative jointly continuous kernel satisfying Eqs. 1.10–1.13. For any function $f \in C_c^\infty(\mathbb{R}^d)$, define $\widetilde{u}_f(t,x) := \int_{\mathbb{R}^d}\widetilde{p}^\kappa(t,x,y)f(y)\,dy$. By the joint continuity of $\widetilde{p}^\kappa(t,x,y)$, (i) and (iii) we have that

$$\widetilde{u}_f \in C_b([0,T]\times\mathbb{R}^d), \qquad \lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|\widetilde{u}_f(t,x) - f(x)| = 0\,.$$

By Lemma 5.3 and Remark 5.5,

$$\mathcal{L}^\kappa\widetilde{u}_f(t,x) = \int_{\mathbb{R}^d}\mathcal{L}^\kappa\,\widetilde{p}^\kappa(t,x,y)f(y)\,dy \quad\text{and}\quad \mathcal{L}^\kappa u_f(t,x) = \int_{\mathbb{R}^d}\mathcal{L}^\kappa\,p^\kappa(t,x,y)f(y)\,dy. \tag{5.21}$$

Moreover, by Lemma 5.4 and Remark 5.5, $t \mapsto \mathcal{L}^\kappa u_f(t,x)$ and $t \mapsto \mathcal{L}^\kappa \widetilde{u}_f(t,x)$ are continuous on $(0,T]$. Here and in Eq. 5.21 we use that $\widetilde{p}^\kappa$ satisfies (i)–(ii).

Let $w(t,x) := u_f(t,x) - \widetilde{u}_f(t,x)$. Then $w(0,x) = 0$, $\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}|w(t,x) - w(0,x)| = 0$, and $t \mapsto \mathcal{L}^\kappa w(t,x)$ is continuous on $(0,T]$. Note that by Eqs. 1.12 and 1.10,

$$|\partial_t p^\kappa(t,x,y)| + |\partial_t \widetilde{p}^\kappa(t,x,y)| \le c_5\rho(t,x-y), \quad t\in(0,T]\,.$$

Thus, by the dominated convergence theorem,

$$\partial_t\widetilde{u}_f(t,x) = \int_{\mathbb{R}^d}\partial_t\widetilde{p}^\kappa(t,x,y)f(y)\,dy \quad \text{and} \quad \partial_t u_f(t,x) = \int_{\mathbb{R}^d}\partial_t p^\kappa(t,x,y)f(y)\,dy.$$

By this, Eqs. 1.10 and 5.21, we have $\partial_t w(t,x) = \mathcal{L}^\kappa w(t,x)$. Hence, all the assumptions of Theorem 5.1 are satisfied and we can conclude that for every $t\in(0,T]$, $\sup_{x\in\mathbb{R}^d} w(t,x) \le \sup_{x\in\mathbb{R}^d} w(0,x) = 0$. By applying the theorem to $-w$ we get that $w(t,x) = 0$ for all $t\in(0,T]$ and every $x\in\mathbb{R}^d$. Hence, $u_f = \widetilde{u}_f$ for every $f\in C_c^\infty(\mathbb{R}^d)$, which implies that $\widetilde{p}^\kappa(t,x,y) = p^\kappa(t,x,y)$.

The last statement of the theorem about the dependence of constants c_1 and c_2 has been already proved in the results above.

Proof of Theorem 1.2 (1) The constant function $u(t,x) = 1$ solves $\partial_t u(t,x) = \mathcal{L}^\kappa u(t,x)$, hence applying Theorem 5.1 to $\pm(P_t^\kappa 1(x) - 1)$ we get that $P_t^\kappa 1(x) \equiv 1$ proving Eq. 1.14.

(2) Same as the proof of [6, Theorem 1.1(3)].

(3) By Eqs. 1.10 and 1.12 we see that $|\partial_t p^\kappa(t,x,y)| \le c_2\rho(t,x-y)$ for $t\in(0,T]$ and $x\ne y$. Hence by the mean value theorem, for $0<s\le t\le T$ and $x\ne y$,

$$\left|p^\kappa(s,x,y) - p^\kappa(t,x,y)\right| \le c_2|t-s|\rho(s,x-y)\,. \tag{5.22}$$

Let $\gamma\in(0,\delta_1)\cap(0,1]$. By Lemma 4.8 and by the definition of ρ_{-1}^0, we have that for every $t\in(0,T]$,

$$\begin{aligned}|p^\kappa(t,x,y) - p^\kappa(t,x',y)| &\le c_1|x-x'|^\gamma\Phi^{-1}(t^{-1})t\left(\rho(t,x-y)+\rho(t,x'-y)\right)\\ &\le 2c_1|x-x'|^\gamma\Phi^{-1}(t^{-1})t\left(\rho(t,x-y)\vee\rho(t,x'-y)\right)\,.\end{aligned} \tag{5.23}$$

By use of the triangle inequality, this together with Eq. 5.22 implies the first claim.

By Eq. 1.11, if $\Phi^{-1}(t^{-1})|x-x'|\ge 1$,

$$\begin{aligned}&|p^\kappa(t,x,y) - p^\kappa(t,x',y)| \le p^\kappa(t,x,y) + p^\kappa(t,x',y)\\ &\le c_1 t\left(\rho(t,x-y)+\rho(t,x'-y)\right)\\ &\le 2c_1|x-x'|\Phi^{-1}(t^{-1})t\left(\rho(t,x-y)\vee\rho(t,x'-y)\right)\,.\end{aligned} \tag{5.24}$$

Suppose $\Phi^{-1}(t^{-1})|x-x'|\ge 1$, $\beta+\delta_1>1$ and $\delta_1\in(2/3,2)$. Then by Eq. 4.50

$$\begin{aligned}&|p^\kappa(t,x,y) - p^\kappa(t,x',y)| \le |x-x'|\cdot\int_0^1|\nabla p(t,x+\theta(x'-x),y)|\,d\theta\\ &\le ct\Phi^{-1}(t^{-1})|x-x'|\int_0^1\rho(t,(x-y)+\theta(x'-x))d\theta.\end{aligned} \tag{5.25}$$

Since $\theta|x'-x|\le 1/\Phi^{-1}(t^{-1})$, from Eq. 5.25 we have

$$\begin{aligned}&|p^\kappa(t,x,y) - p^\kappa(t,x',y)| \le ct\Phi^{-1}(t^{-1})|x-x'|\rho(t,x-y)\\ &\le ct\Phi^{-1}(t^{-1})|x-x'|\left(\rho(t,x-y)\vee\rho(t,x'-y)\right).\end{aligned} \tag{5.26}$$

Equations 5.22, 5.24 and 5.26 imply the second claim.
(4) This follows immediately from the second part of Lemma 4.10. □

Proof of Theorem 1.3 (1) We first claim that for $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, $\mathcal{L}^\kappa f$ is bounded Hölder continuous. We will use results from [1]. For $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$ and $x, z \in \mathbb{R}^d$, let

$$E_z f(x) = f(x+z) - f(x) \quad \text{and} \quad F_z f(x) = f(x+z) - f(x) - \nabla f(x)\cdot z.$$

Using the assumption that $\kappa(y,z) = \kappa(y,-z)$, we have

$$\mathcal{L}^{\mathfrak{K}_y} f(x) = \int_{|z|<1} F_z f(x)\kappa(y,z)J(z)dz + \int_{|z|\geq 1} E_z f(x)\kappa(y,z)J(z)dz.$$

Thus, $\mathcal{L}^\kappa f$ is bounded by Eqs. 1.7 and 1.1. Moreover, using Eqs. 1.2, 1.7 and [1, Theorem 5.1 (b) and (e)] with $\gamma = 2+\varepsilon$,

$$\begin{aligned}
&|\mathcal{L}^\kappa f(x) - \mathcal{L}^\kappa f(y)| \\
&\leq \left|\int_{\mathbb{R}^d} \delta_f(x;z)(\kappa(x,z)-\kappa(y,z))J(z)dz\right| + |\mathcal{L}^{\mathfrak{K}_y} f(x) - \mathcal{L}^{\mathfrak{K}_y} f(y)| \\
&\leq c_1(|x-y|^\beta \wedge 1)\int_{\mathbb{R}^d}(|z|^2\wedge 1)j(|z|)dz + c_1\int_{|z|<1}|F_z f(x) - F_z f(y)|\kappa(y,z)j(|z|)dz \\
&\quad + c_1\int_{|z|\geq 1}|E_z f(x) - E_z f(y)|\kappa(y,z)j(|z|)dz \\
&\leq c_2|x-y|^\beta + c_2\left(\int_{|z|<1}|z|^2 j(|z|)dz\right)|x-y|^\varepsilon + c_2\left(\int_{|z|\geq 1} j(|z|)dz\right)|x-y|.
\end{aligned}$$

Thus we have proved the claim.

For $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, we define $u(t,x) := f(x) + \int_0^t P_s^\kappa \mathcal{L}^\kappa f(x)\,ds$. Note that

$$|u(t,x) - u(0,x)| \leq \int_0^t |P_s^\kappa \mathcal{L}^\kappa f(x)|ds \leq t\|\mathcal{L}^\kappa f\|_\infty.$$

Thus Eq. 5.1 holds. Since $\mathcal{L}^\kappa f$ is bounded Hölder continuous, we can use Eq. 5.14 (together with Eqs. 1.12, 1.10 and 5.21) to get $\mathcal{L}^\kappa P_s^\kappa \mathcal{L}^\kappa f(x) = \partial_s\left(P_s^\kappa \mathcal{L}^\kappa f\right)(x))$ and obtain

$$\begin{aligned}
\mathcal{L}^\kappa u(t,x) &= \mathcal{L}^\kappa f(x) + \int_0^t \mathcal{L}^\kappa P_s^\kappa \mathcal{L}^\kappa f(x)\,ds \\
&= \mathcal{L}^\kappa f(x) + \int_0^t \partial_s\left(P_s^\kappa \mathcal{L}^\kappa f\right)(x)\,ds = P_t\mathcal{L}^\kappa f(x) = \partial_t u(t,x).
\end{aligned}$$

Therefore $u(t,x)$ satisfies the assumptions of Theorem 5.1. Since $u(0,x) = f(x)$, it follows from the maximum principle that

$$P_t^\kappa f(x) = u(t,x) = f(x) + \int_0^t P_s^\kappa \mathcal{L}^\kappa f(x)\,ds. \tag{5.27}$$

Since $\mathcal{L}^\kappa f$ is bounded and uniformly continuous, we can use Eq. 1.13 to get

$$\lim_{t\downarrow 0}\frac{1}{t}\left(P_t^\kappa f(x) - f(x)\right) = \lim_{t\downarrow 0}\frac{1}{t}\int_0^t P_s^\kappa \mathcal{L}^\kappa f(x)ds = \mathcal{L}^\kappa f(x)$$

and the convergence is uniform.
(2) Using our Theorem 1.1(iii), Theorem 1.2(1) and Lemma 5.4, the proof of this part is the same as in [6]. □

5.4 Lower Bound Estimate of $p^\kappa(t, x, y)$

By Theorem 1.3, we have that $(P_t^\kappa)_{t\geq 0}$ is a Feller semigroup and there exists a Feller process $X = (X_t, \mathbb{P}_x)$ corresponding to $(P_t^\kappa)_{t\geq 0}$. Moreover, by Eq. 5.27 for $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$,

$$f(X_t) - f(x) - \int_0^t \mathcal{L}^\kappa f(X_s)\, ds \tag{5.28}$$

is a martingale with respect to the filtration $\sigma(X_s, s \leq t)$. Therefore by the same argument as that in [6, Section 4.4], we have the following Lévy system formula: for every function $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ vanishing on the diagonal and every stopping time S,

$$\mathbb{E}_x \sum_{0<s\leq S} f(X_{s-}, X_s) = \mathbb{E}_x \int_0^S f(X_s, y) J_X(X_s, dy) ds\,, \tag{5.29}$$

where $J_X(x, y) := \kappa(x, y - x) J(x - y)$.

For $A \in \mathcal{B}(\mathbb{R}^d)$ we define $\tau_A := \inf\{t \geq 0 : X_t \notin A\}$.

The following result is the counterpart of [6, Lemma 4.6].

Lemma 5.7 *For each $\gamma \in (0, 1)$ there exists $A = A(\gamma) > 0$ such that for every $r > 0$,*

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}_x \left(\tau_{B(x,r)} \leq (A\Phi(1/(4r)))^{-1}\right) \leq \gamma\,. \tag{5.30}$$

Proof Without loss of generality, we take $x = 0$. The constant A will be chosen later. Let $f \in C_b^\infty(\mathbb{R}^d)$ with $f(0) = 0$ and $f(y) = 1$ for $|y| \geq 1$. For any $r > 0$ set $f_r(y) = f(y/r)$. By the definition of f_r and the martingale property in Eq. 5.28 we have

$$\begin{aligned}\mathbb{P}_0 \left(\tau_{B(0,r)} \leq (A\Phi(1/(4r)))^{-1}\right) &\leq \mathbb{E}_0 \left[f_r \left(X_{\tau_{B(0,r)}\wedge(A\Phi(1/(4r)))^{-1}}\right)\right] \\ &= \mathbb{E}_0 \left(\int_0^{\tau_{B(0,r)}\wedge(A\Phi(1/(4r)))^{-1}} \mathcal{L}^\kappa f_r(X_s)\, ds\right).\end{aligned} \tag{5.31}$$

By the definition of $\mathcal{L}^\kappa$, Eqs. 1.1 and 1.7 we have

$$\begin{aligned}|\mathcal{L}^\kappa f_r(y)| &= \frac{1}{2}\left|\int_{\mathbb{R}^d} (f_r(y+z) + f_r(y-z) - 2f_r(y))\, \kappa(y, z) J(z)\, dz\right| \\ &\leq \frac{\kappa_1\gamma_0\|\nabla^2 f_r\|_\infty}{2} \int_{|z|\leq r} |z|^2 j(|z|)\, dz + 2\kappa_1\gamma_0\|f_r\|_\infty \int_{|z|>r} j(|z|)\, dz \\ &\leq c_1 \left(\frac{\|\nabla^2 f\|_\infty}{r^2} r^2 \mathcal{P}(r) + \|f\|_\infty \mathcal{P}(r)\right) \leq c_2\, \Phi(r^{-1})\,,\end{aligned}$$

where $c_2 = c_2(\kappa_1, \gamma_0, f)$. Here the last inequality is a consequence of Eq. 3.7. Substituting in Eq. 5.31 we get that

$$\mathbb{P}_0 \left(\tau_{B(0,r)} \leq (A\Phi(1/(4r)))^{-1}\right) \leq c_2\Phi(r^{-1})(A\Phi(1/(4r)))^{-1} \leq 4c_2 A^{-1}\,.$$

With $A = 4c_2/\gamma$ the lemma is proved. □

Proof of Theorem 1.4 Throughout the proof, we fix $T, M \geq 1$ and, without loss of generality, we assume that $\Phi^{-1}(T^{-1})^{-1} = M$.

By [4, Theorem 2.4] and the same argument as the one in [5, Proposition 2.2] (see also [7, Proposition 6.4(1)] or [3, Proposition 6.2]), Eqs. 1.4, 1.20, 1.1 and 1.7 imply that there exists a constant $c_0 > 0$ such that

$$p_y(t,x) \geq c_0 \left(\Phi^{-1}(t^{-1})^d \wedge t j(|x|) \right) \quad (t,x,y) \in (0,T] \times B(0,4M) \times \mathbb{R}^d . \tag{5.32}$$

Since by [11, Lemma 3.2(a)],

$$j(|x|) \geq c_1 |x|^{-d} \Phi(|x|^{-1}), \quad |x| \leq 4M \tag{5.33}$$

for some $c_1 \in (0,1)$, by Proposition 2.1 we have

$$p_y(t,x) \geq c_0 c_1 t \rho(t,x) \quad (t,x,y) \in (0,T] \times B(0,4M) \times \mathbb{R}^d . \tag{5.34}$$

(1) Let $\lambda = 1/A$ where A is the constant from Lemma 5.7 for $\gamma = 1/2$. Then for every $t > 0$,

$$\sup_{z \in \mathbb{R}^d} \mathbb{P}_z(\tau_{B(z, 2^{-2}\Phi^{-1}(t^{-1})^{-1})} \leq \lambda t) \leq \frac{1}{2} . \tag{5.35}$$

Let $t \in (0,T]$ and $|x-y| \leq 3\Phi^{-1}(t^{-1})^{-1}$(so that $|x-y| \leq 3M$). By Eq. 4.47 we have that there exists a constant $c_2 > 0$ such that

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z) q(s,z,y)\, dz\, ds &\geq -c_2 t \left(\rho_\beta^0 + \rho_0^\beta \right)(t, x-y) \\
&= -c_2 t \left(\Phi^{-1}(t^{-1})^{-\beta} + |x-y|^\beta \wedge 1 \right) \rho(t, x-y) \\
&\geq -c_2 t \left(\Phi^{-1}(t^{-1})^{-\beta} + 3^\beta \Phi^{-1}(t^{-1})^{-\beta} \right) \rho(t, x-y) .
\end{aligned}$$

We choose $t_0 \in (0,1)$ so that for all $t \in (0,t_0)$, $c_2(1+3^\beta)\Phi^{-1}(t^{-1})^{-\beta} \leq c_1/2$. Together with Eqs. 5.34 and 4.46 we conclude that for all $t \in (0,t_0)$ and all $x, y \in \mathbb{R}^d$ satisfying $|x-y| \leq 3\Phi^{-1}(t^{-1})^{-1}$ we have

$$p^\kappa(t,x,y) \geq \frac{c_1}{2} t \rho(t, x-y) \geq c_3 t \frac{\Phi\left(\frac{1}{\Phi^{-1}(t^{-1})} + \frac{3}{\Phi^{-1}(t^{-1})} \right)}{\left(\frac{1}{\Phi^{-1}(t^{-1})} + \frac{3}{\Phi^{-1}(t^{-1})} \right)^d} \geq c_4 \Phi^{-1}(t^{-1})^d .$$

By Eq. 1.15 and iterating $\lfloor T/t_0 \rfloor + 1$ times, we obtain the following near-diagonal lower bound

$$p^\kappa(t,x,y) \geq c_5 \Phi^{-1}(t^{-1})^d \text{ for all } t \in (0,T] \text{ and } |x-y| \leq 3\Phi^{-1}(t^{-1})^{-1} . \tag{5.36}$$

Now we assume $|x-y| > 3\Phi^{-1}(t^{-1})^{-1}$ and let $\sigma = \inf\{t \geq 0 : X_t \in B(y, 2^{-1}\Phi^{-1}(t^{-1})^{-1})\}$. By the strong Markov property and Eq. 5.35 we have

$$\begin{aligned}
&\mathbb{P}_x\left(X_{\lambda t} \in B(y, \Phi^{-1}(t^{-1})^{-1}) \right) \geq \mathbb{P}_x\left(\sigma \leq \lambda t, \sup_{s \in [\sigma, \sigma+\lambda t]} |X_s - X_\sigma| < 2^{-1}\Phi^{-1}(t^{-1})^{-1} \right) \\
&= \mathbb{E}_x\left(\mathbb{P}_{X_\sigma}\left(\sup_{s \in [0,\lambda t]} |X_s - X_0| < 2^{-1}\Phi^{-1}(t^{-1})^{-1} \right); \sigma \leq \lambda t \right) \\
&\geq \inf_{z \in B(y, 2^{-1}\Phi^{-1}(t^{-1})^{-1})} \mathbb{P}_z\left(\tau_{B(z, 2^{-1}\Phi^{-1}(t^{-1})^{-1})} > \lambda t \right) \mathbb{P}_x(\sigma \leq \lambda t) \\
&\geq \frac{1}{2}\mathbb{P}_x(\sigma \leq \lambda t) \geq \frac{1}{2}\mathbb{P}_x\left(X_{\lambda t \wedge \tau_{B(x, \Phi^{-1}(t^{-1})^{-1})}} \in B(y, 2^{-1}\Phi^{-1}(t^{-1})^{-1}) \right) .
\end{aligned} \tag{5.37}$$

Since

$$X_s \notin B\left(y, 2^{-1}\Phi^{-1}(t^{-1})^{-1}\right) \subset B\left(x, \Phi^{-1}(t^{-1})^{-1}\right)^c, \quad s < \lambda t \wedge \tau_{B(x,\Phi^{-1}(t^{-1})^{-1})},$$

we have

$$\mathbf{1}_{X_{\lambda t \wedge \tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \in B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} = \sum_{s \le \lambda t \wedge \tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \mathbf{1}_{X_s \in B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1}}.$$

Thus, by the Lévy system formula in Eq. 5.29 we have

$$\begin{aligned}
&\mathbb{P}_x\left(X_{\lambda t \wedge \tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \in B(y, 2^{-1}\Phi^{-1}(t^{-1})^{-1})\right) \\
&= \mathbb{E}_x\left[\int_0^{\lambda t \wedge \tau_{B(x,\Phi^{-1}(t^{-1})^{-1})}} \int_{B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} J_X(X_s, u)\, du\, ds\right] \\
&\ge \mathbb{E}_x\left[\int_0^{\lambda t \wedge \tau_{B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})}} \int_{B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} \kappa_0 j(|X_s - u|)\mathbf{1}_{\{u:|X_s - u| < |x-y|\}}\, du\, ds\right].
\end{aligned} \tag{5.38}$$

Let w be the point on the line connecting x and y (i.e., $|x-y| = |x-w| + |w-y|$) such that $|w-y| = 7\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1}$. Then $B(w, 2^{-4}\Phi^{-1}(t^{-1})^{-1}) \subset B(y,\ 2^{-1}\Phi^{-1}(t^{-1})^{-1})$. Moreover, for every $(z, u) \in B(x, 6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1}) \times B(w, 2^{-4}\Phi^{-1}(t^{-1})^{-1})$, we have

$$\begin{aligned}
|z-u| &\le |z-x| + |w-u| + |x-w| = |z-x| + |w-u| + |x-y| - |w-y| \\
&< (6\cdot 2^{-4} + 2^{-4})\Phi^{-1}(t^{-1})^{-1} + |x-y| - 7\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1} = |x-y|.
\end{aligned}$$

Thus

$$B(w, 2^{-4}\Phi^{-1}(t^{-1})^{-1}) \subset \{u : |z-u| < |x-y|\} \quad \text{for } z \in B(x, 6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1}). \tag{5.39}$$

Equations 5.39 and 5.35 imply that

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\lambda t \wedge \tau_{B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})}} \int_{B(y,2^{-1}\Phi^{-1}(t^{-1})^{-1})} j(|X_s - u|)\mathbf{1}_{\{u:|X_s-u|<|x-y|\}}\, du\, ds\right] \\
&\ge \mathbb{E}_x\left[\lambda t \wedge \tau_{B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})}\right] \int_{B(w,2^{-4}\Phi^{-1}(t^{-1})^{-1})} j(|x-y|)\, du \\
&\ge \lambda t \mathbb{P}_x\left(\tau_{B(x,6\cdot 2^{-4}\Phi^{-1}(t^{-1})^{-1})} \ge \lambda t\right) \left|B(w, 2^{-4}\Phi^{-1}(t^{-1})^{-1})\right| j(|x-y|) \\
&\ge c_6 t \Phi^{-1}(t^{-1})^{-d} j(|x-y|).
\end{aligned} \tag{5.40}$$

By combining Eq. 5.37, 5.38 and 5.40 we get that

$$\mathbb{P}_x\left(X_{\lambda t} \in B(y, \Phi^{-1}(t^{-1})^{-1})\right) \ge \frac{1}{2} c_6 t \Phi^{-1}(t^{-1})^{-d} j(|x-y|) \tag{5.41}$$

By Eqs. 1.15, 5.36 and 5.41 we have

$$\begin{aligned}
p^\kappa(t,x,y) &\ge \int_{B(y,\Phi^{-1}(t^{-1})^{-1})} p^\kappa(\lambda t, x, z) p^\kappa((1-\lambda)t, z, y)\, dz \\
&\ge \inf_{z \in B(y,\Phi^{-1}(t^{-1})^{-1})} p^\kappa((1-\lambda)t, z, y) \int_{B(y,\Phi^{-1}(t^{-1})^{-1})} p^\kappa(\lambda t, x, z)\, dz \\
&\ge c_7 \Phi^{-1}(t^{-1})^d t \Phi^{-1}(t^{-1})^{-d} j(|x-y|) = c_7 t j(|x-y|).
\end{aligned}$$

Combining this estimate with Eq. 5.36 we obtain Eq. 1.21. Inequality Eq. 1.22 follows from Eq. 1.21, Proposition 2.1 and Eq. 5.33. □

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