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Dirichlet Boundary Conditions for Degenerate and Singular Nonlinear Parabolic Equations

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Abstract We study existence and uniqueness of solutions to a class of quasilinear degenerate parabolic equations, in bounded domains. We show that there exists a unique solution which satisfies possibly inhomogeneous Dirichlet boundary conditions. To this purpose some barrier functions are properly introduced and used.

Keywords Parabolic equations · Dirichlet boundary conditions · Barrier functions · Sub– and supersolutions · Comparison principle

Mathematics Subject Classification (2010) 35K15 · 35K20 · 35K55 · 35K65 · 35K67

1 Introduction

We are concerned with bounded solutions to the following nonlinear parabolic equation:

$$\rho\, \partial_t u = \Delta[G(u)] \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ ($N \geq 1$) with boundary $\partial\Omega = \mathcal{S}$ and ρ is a positive function of the space variables. Throughout the paper, we shall make the following assumption:

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H0. $\mathcal{S}$ is an $(N-1)-$dimensional compact submanifold of $\mathbb{R}^N$ of class C^3.

Moreover, we require the functions ρ and G to satisfy the following hypotheses:

H1. $\rho \in C(\Omega),\ \rho > 0$ in Ω;
H2. $G \in C^1(\mathbb{R}),\ G(0) = 0,\ G'(s) > 0$ for any $s \in \mathbb{R} \setminus \{0\}$. Moreover, if $G'(0) = 0$,

then G' is decreasing in $(-\delta, 0)$ and increasing in $(0, \delta)$ for some $\delta > 0$.

Clearly, the character of Eq. 1.1 is determined by G and ρ as one can see by looking at Eq. 1.1 as

$$\partial_t u = \frac{1}{\rho}\Delta[G(u)] \quad \text{in } \Omega \times (0, T], \tag{1.2}$$

In fact, in view of the nonlinear function $G(u)$ and hypothesis **H2**, the Eq. 1.1 can be *degenerate*; however, we also consider the case where such degeneracy does not occur (see **H5** below). Moreover, setting

$$d(x) := \text{dist}(x, \mathcal{S}) \quad (x \in \bar{\Omega}),$$

if $\rho(x) \to 0$ as $d(x) \to 0$, the coefficient $\frac{1}{\rho}$ of the operator $\frac{1}{\rho}\Delta$ is unbounded at $\mathcal{S}$, so the operator is *singular*; whereas, if $\rho(x) \to \infty$ as $d(x) \to 0$, the operator $\frac{1}{\rho}\Delta$ is degenerate at $\mathcal{S}$.

Problem (1.1) appears in a wide number of physical applications (see, e.g., [20]); note that, by choosing $G(u) = |u|^{m-1}u$ for some $m > 1$, we obtain the well known porous medium equation with a variable density $\rho = \rho(x)$ (see [4, 5]).

Previous Results on the Cauchy Problem In the literature, a particular attention has been devoted to the following companion Cauchy problem

$$\begin{cases} \rho\partial_t u = \Delta[G(u)] & \text{in } \mathbb{R}^N \times (0, T], \\ u = u_0 & \text{in} \mathbb{R}^N \times \{0\}. \end{cases} \tag{1.3}$$

In particular, existence and uniqueness of solutions to Eq. 1.3 have been extensively studied; note that here and hereafter we always consider *very weak* solutions (see Section 2.1 for the precise definition). To be specific, if one makes the following assumptions:

(i) $\rho \in C(\mathbb{R}^N),\ \rho > 0$,
(ii) $u_0 \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$,

it is well known (see [5, 15, 20, 29]) that there exists a bounded solution to Eq. 1.3; moreover, for $N = 1$ and $N = 2$ such a solution is unique. When $N \geq 3$, the uniqueness of the solution in the class of bounded functions is no longer guaranteed, and it is strictly related to the behavior at infinity of the density ρ. Indeed, it is possible to prove that if ρ does not decay too fast at infinity, then problem (1.3) admits at most one bounded solution (see [29]). On the contrary, if one suppose that ρ decays sufficiently fast at infinity, then the non uniqueness appears (see [4, 14, 17, 29]).

Following this direction, in [14] the authors prove the existence and uniqueness of the solution to Eq. 1.3 which satisfies the following additional condition at infinity

$$\lim_{|x|\to\infty} u(x, t) = a(t) \quad \text{uniformly for } t \in [0, T], \tag{1.4}$$

supposing $a \in C([0, T]), a > 0$ and $\lim_{|x|\to\infty} u_0(x) = a(0)$. Note that Eq. 1.4 is a pointwise condition at infinity for the solution u. Also, the results of [14] have been generalized in [18, 19] to the case of more general operators.

Previous Results in bounded domains When considering (1.1) in a bounded subset $\Omega \subset \mathbb{R}^N$, in view of **H1**, since ρ is allowed either to vanish or to diverge at $\mathcal{S}$, it is natural to consider the following initial value problem associated with Eq. 1.1:

$$\begin{cases} \rho \partial_t u = \Delta[G(u)] & \text{in } \Omega \times (0, T], \\ u = u_0 & \text{in} \Omega \times \{0\}, \end{cases} \tag{1.5}$$

where no boundary conditions are specified at $\mathcal{S}$. We require $\mathcal{S}$, ρ and G to satisfy hypotheses **H0-2**; furthermore, for the initial datum u_0 we assume that

H3. $u_0 \in L^\infty(\Omega) \cap C(\Omega)$.

Concerning the existence and uniqueness of the solutions to Eq. 1.5, the case $G(u) = u$ has been largely investigated, using both analytical and stochastic methods (see, e.g., [22, 27, 28, 31]). Also analogous elliptic or elliptic-parabolic equations have attracted much attention in the literature (see, e.g., [6–11, 25, 26]); in particular, the question of prescribing continuous data at $\mathcal{S}$ has been addressed (see, e.g., [22, 26–28]).

For general nonlinear function G, the well-posedness of problem (1.5) has been studied in [16] in the case $N = 1$ and subsequently addressed for $N \geq 1$ in [30]. Precisely, in [30] is proven that, if ρ diverges sufficiently fast as $d(x) \to 0$, then one has uniqueness of bounded solutions not satisfying any additional condition at $\mathcal{S}$.

Indeed, if one requires that there exist $\hat{\varepsilon} > 0$ and $\underline{\rho} \in C((0, \hat{\varepsilon}])$ such that

- $\rho(x) \geq \underline{\rho}(d(x)) > 0$, for any $x \in \mathcal{S}^{\hat{\varepsilon}} := \{x \in \Omega \mid d(x) < \hat{\varepsilon}\}$,
- $\int_0^{\hat{\varepsilon}} \eta \, \underline{\rho}(\eta) \, d\eta = +\infty$,

then there exists at most one bounded solution to Eq. 1.5.

Conversely, if either $\rho(x) \to \infty$ sufficiently slow or ρ does not diverge when $d(x) \to 0$, then nonuniqueness prevails in the class of bounded solutions. Precisely, in [30] it is supposed that the function ρ satisfies the next condtion: there exist $\hat{\varepsilon} > 0$ and $\overline{\rho} \in C((0, \hat{\varepsilon}])$ such that

- $\rho(x) \leq \overline{\rho}(d(x))$, for any $x \in \mathcal{S}^{\hat{\varepsilon}}$,
- $\int_0^{\hat{\varepsilon}} \eta \, \overline{\rho}(\eta) \, d\eta < +\infty$.

A natural choice for $\overline{\rho}$ is given by

$$\overline{\rho}(\eta) = \eta^{-\alpha}, \quad \text{for some } \alpha \in (-\infty, 2), \text{ and } \eta \in (0, \hat{\varepsilon}]. \tag{1.6}$$

It is proven that for any $A \in \text{Lip}([0, T])$, $A(0) = 0$, there exists a solution to Eq. 1.5 satisfying

$$\lim_{d(x) \to 0} |U(x, t) - A(t)| = 0, \tag{1.7}$$

uniformly with respect to $t \in [0, T]$, where U is defined as

$$U(x, t) := \int_0^t G(u(x, \tau)) \, d\tau.$$

In particular, the previous result implies non-uniqueness of bounded solutions to Eq. 1.5. Moreover, the solution to problem (1.5) which satisfies (1.7) is unique, provided $A \equiv 0$ or $G(u) = u$.

Outline of the Main Results Formally, the boundary $\mathcal{S}$ for problem (1.5) plays the same role played by *infinity* for the Cauchy problem (1.3); hence, the well-posedness for Eq. 1.5

depends on the behavior of ρ in the limit $d(x) \to 0$, in analogy with the previous results for the Cauchy problem (1.3), where it depends on the behavior of ρ for large $|x|$.

Thus, a natural question that arises is if it is possible to impose at $\mathcal{S}$ Dirichlet boundary conditions, instead of the integral one (1.7). Moreover, on can ask if such a Dirichlet conditions restores uniqueness in more general situations than the ones considered in connection with Eq. 1.7. Observe that, as recalled above, the same question has already been investigated for the linear case $G(u) = u$ (see, e.g., [22, 26–28]), and for the case where $\rho \equiv 1$ and G is general (see [2, 3]). The case where both ρ and G are general, which is a quite natural situation also for various applications (see, e.g., [21]), has not been treated in the literature and is the object of our investigation.

In fact, the main novelty of our paper relies in the following result: we prove existence and uniqueness of a bounded solution to problem (1.5) satisfying Dirichlet possibly non-homogeneous boundary conditions. This is of course a much stronger condition with respect to Eq. 1.7. We require the function ρ to satisfy

H4. **i.** $\rho \in L^\infty(\Omega)$,

or

ii. $\inf_\Omega \rho > 0$ and there exist $\hat{\varepsilon} > 0, \overline{\rho} \in C((0, \hat{\varepsilon}])$ such that

- $\rho(x) \le \overline{\rho}(d(x))$ for any $x \in \mathcal{S}^{\hat{\varepsilon}}$,
- $\int_0^{\hat{\varepsilon}} \eta\, \overline{\rho}(\eta)\, d\eta < +\infty$.

Under the hypothesis **H4**, we show that, for any $\varphi \in C(\mathcal{S} \times [0, T])$, if either G is non degenerate, i.e. there holds

H5. $G \in C^1(\mathbb{R}),\; G'(s) \ge \alpha_0 > 0 \quad \text{for any } \; s \in \mathbb{R}$,

or φ and u_0 satisfy

$$\varphi > 0 \quad \text{in } \; \mathcal{S} \times [0, T], \quad \liminf_{x \to x_0} u_0(x) \ge \alpha_1 > 0 \quad \text{for every } \; x_0 \in \mathcal{S}, \tag{1.8}$$

then there exists a unique bounded solution to Eq. 1.5 such that, for each $\tau \in (0, T)$,

$$\lim_{\substack{x \to x_0 \\ t \to t_0}} u(x, t) = \varphi(x_0, t_0) \quad \text{uniformly with respect to } \; t_0 \in [\tau, T] \text{ and } \; x_0 \in \mathcal{S}. \tag{1.9}$$

If we drop either the assumption of non-degeneracy on G or the assumption (1.8), we need to restrict our analysis to the special class of data φ which only depend on x; in fact, for any $\varphi \in C(\mathcal{S})$ we prove that there exists a unique bounded solution to Eq. 1.5 satisfying

$$\lim_{x \to x_0} u(x, t) = \varphi(x_0) \quad \text{uniformly with respect to } \; t \in [0, T] \text{ and } \; x_0 \in \mathcal{S}, \tag{1.10}$$

provided

$$\lim_{x \to x_0} u_0(x) = \varphi(x_0) \quad \text{for every } x_0 \in \mathcal{S}. \tag{1.11}$$

To prove the existence results we introduce and use suitable barrier functions (see Eqs. 3.18, 3.25, 3.31, below). We should note that the definitions of such barriers seem to be new. Let us observe that in constructing such barrier functions, the cases **H4-i** and **H4-ii** will be treated separately (for more details, see Section 3).

In constructing our barrier functions, besides taking into account the behavior at $\mathcal{S}$ of the density $\rho(x)$ as described above, we have to overcome some difficulties due to the nonlinear function $G(u)$. In this respect, we should note that on the one hand, barrier functions similar to those we construct were used in [14] and in [18], where problem (1.3) was addressed and conditions were prescribed at infinity. However, such barriers cannot be trivially adapted to our case. Indeed, by an easy variation of them we could only consider $\mathcal{S}$ in place of *infinity*, prescribing $u(x,t) \to a(t)$ as $d(x) \to 0$ ($t \in (0,T]$), but we cannot distinguish different points $x_0 \in \mathcal{S}$ and impose conditions (1.9) and (1.10). On the other hand, other similar barriers were used in the literature (see, e.g., [12, 13, 24]) to prescribe Dirichlet boundary conditions to solutions to *linear* parabolic or elliptic equations in bounded domains; however, they cannot be used in our situation, in view of the presence of the nonlinear function $G(u)$.

Let us finally mention that our results have some connections with regularity results up to the boundary. In fact, as a consequence of our results, there exists a unique solution to problem (1.5) which is continuous in $\overline{\Omega} \times [0,T]$. General regularity results could be deduced from results in [2] and in [3], where more general equations are treated, only when

$$C_1 \le \rho(x) \le C_2 \quad \text{for all} \;\; x \in \Omega\,, \tag{1.12}$$

for some $0 < C_1 < C_2$. However, we suppose hypotheses **H1** and **H4**, that are weaker than Eq. 1.12.

We close this introduction with a brief overview of the paper. In Section 2 we present a description of the main contributions of the paper; in particular, we state Theorem 2.3, Theorem 2.4 and Theorem 2.5, that assure, under suitable hypotheses, the existence of a bounded solution to Eq. 1.5 satisfying a proper Dirichlet boundary condition. Subsequently, we show that such a solution is unique (see Theorem 2.7). Section 3 is devoted to the proofs of the existence results, while in Section 4 the proof of the uniqueness result is given.

2 Existence and Uniqueness Results

In this section we present existence and uniqueness results for the solutions to

$$\begin{cases} \rho\partial_t u = \Delta[G(u)] & \text{in } \;\Omega \times (0,T], \\ \quad u = u_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ satisfies hypothesis **H0**, and ρ, G and u_0 satisfy hypotheses **H1-H4**.

Throughout the paper, we will extensively use the following notations:

- $Q_T := \Omega \times (0,T]$;
- $\mathcal{S}^\varepsilon := \{x \in \Omega \,:\, d(x) < \varepsilon\}$ $(\varepsilon > 0)$;
- $\mathcal{A}^\varepsilon := \partial\mathcal{S}^\varepsilon \cap \Omega$;
- $\Omega^\varepsilon := \Omega \setminus \overline{\mathcal{S}^\varepsilon}$.

2.1 Mathematical Background

Before stating our results, let us define the tools we shall use in the following.

Definition 2.1 A function $u \in C(\Omega \times [0,T]) \cap L^\infty(\Omega \times (0,T))$ is a solution to Eq. 2.1 if

$$\begin{aligned}\int_0^\tau \int_{\Omega_1} [u\,\rho\,\partial_t\psi + G(u)\Delta\psi]\,dx\,dt = &\int_{\Omega_1} [u(x,T)\psi(x,T) - u_0(x)\psi(x,0)]\,\rho(x)\,dx \\ &+ \int_0^\tau \int_{\partial\Omega_1} G(u)\langle\nabla\psi, \nu\rangle\,dS\,dt,\end{aligned} \tag{2.2}$$

for any open set Ω_1 with smooth boundary $\partial\Omega_1$ such that $\overline{\Omega}_1 \subset \Omega$, for any $\tau \in (0, T]$ and for any $\psi \in C^{2,1}_{x,t}(\overline{\Omega}_1 \times [0,\tau])$, $\psi \geq 0$, $\psi = 0$ in $\partial\Omega_1 \times [0,\tau]$, where ν denotes the outer normal to Ω_1.

Moreover, we say that u is a supersolution (subsolution respectively) to Eq. 2.1 if Eq. 2.2 holds with $\leq$ ($\geq respectively$).

Given $\varepsilon > 0$, we also consider the following auxiliary problem

$$\begin{cases} \rho\partial_t u = \Delta[G(u)] & \text{in } \Omega^\varepsilon \times (0,T] := Q_T^\varepsilon, \\ u = \phi & \text{in} \mathcal{A}^\varepsilon \times (0,T), \\ u = u_0 & \text{in} \Omega^\varepsilon \times \{0\}; \end{cases} \tag{2.3}$$

where $\phi \in C(\mathcal{A}^\varepsilon \times [0,T])$, $\phi(x,0) = u_0(x)$ for all $x \in \mathcal{A}^\varepsilon$.

Definition 2.2 A function $u \in C(\overline{\Omega^\varepsilon} \times [0,T])$ is a solution to Eq. 2.3 if

$$\begin{aligned} \int_0^\tau \int_{\Omega_1} [u\,\rho\,\partial_t\psi + G(u)\Delta\psi]\,dx\,dt \;=\; & \int_{\Omega_1} [u(x,T)\psi(x,T) - u_0(x)\psi(x,0)]\,\rho(x)\,dx \\ & + \int_0^\tau \int_{\partial\Omega_1\setminus\mathcal{A}^\varepsilon} G(u)\langle\nabla\psi,\nu\rangle\,dS\,dt \\ & + \int_0^\tau \int_{\partial\Omega_1\cap\mathcal{A}^\varepsilon} G(\phi)\langle\nabla\psi,\nu\rangle\,dS\,dt, \end{aligned} \tag{2.4}$$

for any open set $\Omega_1 \subset \Omega^\varepsilon$ with smooth boundary $\partial\Omega_1$, for any $\tau \in (0,T]$ and for any $\psi \in C^{2,1}_{x,t}(\overline{\Omega}_1 \times [0,\tau])$, $\psi \geq 0$, $\psi = 0$ in $\partial\Omega_1 \times [0,\tau]$, where ν denotes the outer normal to Ω_1. Supersolution and subsolution are defined accordingly.

2.2 Existence Results

At first, we consider the case of nondegenerate nonlinearities G satisfying hypothesis **H5**.

Theorem 2.3 *Let hypotheses **H0–H1**, **H3–H5** be satisfied and let* $\varphi \in C(\mathcal{S} \times [0,T])$. *Then there exists the maximal solution to* Eq. 2.1 *such that, for each* $\tau \in (0,T)$,

$$\lim_{\substack{x\to x_0\\ t\to t_0}} u(x,t) = \varphi(x_0,t_0), \tag{2.5}$$

uniformly with respect to $t_0 \in [\tau, T]$ *and* $x_0 \in \mathcal{S}$.

Note that we say that u is the maximal solution to problem (2.1), if $u \geq v$ for any solution v of the same problem satisfying (2.5).

We can also prove similar results to Theorem 2.3 in the case of a general nonlinearity G satisfying hypothesis **H2**.

Theorem 2.4 *Let hypotheses **H0–H4** be satisfied, and let* $\varphi \in C(\mathcal{S})$. *Suppose that condition* (1.11) *holds. Then there exists the maximal solution to* Eq. 2.1 *such that*

$$\lim_{x\to x_0} u(x,t) = \varphi(x_0), \tag{2.6}$$

uniformly with respect to $t \in [0,T]$ *and* $x_0 \in \mathcal{S}$.

Finally, we can also consider data φ and u_0 satisfying

$$\varphi > 0 \quad \text{in } \mathcal{S} \times [0, T] \quad \text{and} \quad \liminf_{x \to x_0} u_0(x) \geq \alpha_1 > 0 \quad \text{for every } x_0 \in \mathcal{S}. \tag{2.7}$$

Theorem 2.5 *Let hypotheses **H0-H4** be satisfied, and let $\varphi \in C(\mathcal{S} \times [0, T])$. Suppose that* Eq. 2.7 *holds. Then there exists the maximal solution to* Eq. 2.1 *such that* Eq. 2.5 *holds.*

Remark 2.6 If we further suppose that

$$\lim_{x \to x_0} u_0(x) = \varphi(x_0, 0) \quad \text{for every } x_0 \in \mathcal{S}, \tag{2.8}$$

then in Theorems 2.3 and 2.5 we can take $\tau = 0$.

2.3 Uniqueness Results

Theorem 2.7 *Let hypotheses **H0–H3** be satisfied, and let $\varphi \in C(\mathcal{S} \times [0, T])$. Suppose that there exists a bounded maximal solution $\bar{u}$ of problem* (2.1) *such that* Eq. 2.5 *holds. Then there exists at most one bounded solution to* Eq. 2.1 *such that* Eq. 2.5 *holds.*

Remark 2.8 Sufficient conditions for existence of the maximal solution $\bar{u}$ in Theorem 2.7 can be found in Theorems 2.3, 2.4, 2.5.

3 Existence Results: Proofs

3.1 Preliminaries

In the proofs of our existence results, in order to show that the solution we construct is *maximal*, we will make use of the following lemma.

Lemma 3.1 *Let hypotheses **H0–H3** be satisfied. Let u be a subsolution to problem* (2.1) *and let $\hat{u}$ be a supersolution to problem* (2.1)*. Suppose that for each $\tau \in (0, T)$ there exists $\varepsilon_\tau > 0$ such that, for all $0 < \varepsilon < \varepsilon_\tau$,*

$$u \leq \hat{u} \quad \text{in} \quad \mathcal{A}^\varepsilon \times (\tau, T]. \tag{3.1}$$

Then

$$u \leq \hat{u} \quad \text{in} \quad Q_T.$$

Lemma 3.2 *Let hypotheses **H0–H3** be satisfied. Let $\varepsilon > 0$. Let*

$$a := \begin{cases} [G(u) - G(\hat{u})]/(u - \hat{u}) & \text{for } u \neq \hat{u}, \\ 0 & \text{elsewhere}, \end{cases} \tag{3.2}$$

with u and $\hat{u}$ as in Lemma 3.1. Then there exists a sequence $\{a_n\} \in C^\infty(\overline{Q_T^\varepsilon})$ such that

$$\frac{1}{n^{N+1}} \leq a_n \leq \|a\|_{L^\infty(Q_T^\varepsilon)} + \frac{1}{n^{N+1}} \qquad \text{and} \qquad \frac{(a_n - a)}{\sqrt{a_n}} \to 0 \ \text{in } L^2(Q_T^\varepsilon).$$

Furthermore, let $\chi \in C^\infty(\Omega^\varepsilon)$ *with* supp $\chi \subset \Omega^{\varepsilon_0}$ *for some* $\varepsilon_0 > \varepsilon$, $0 \le \chi \le 1$. *Then there exists a unique solution* $\psi_n \in C^{2,1}_{x,t}(\overline{Q^\varepsilon_T})$ *to problem*

$$\begin{cases} \rho\partial_t\psi_n + a_n\Delta\psi_n = 0 & \text{in } Q^\varepsilon_T, \\ \psi_n = 0 & \text{in } \mathcal{A}^\varepsilon \times (0,T), \\ \psi_n(x,T) = \chi(x) & \text{in } \Omega^\varepsilon. \end{cases} \tag{3.3}$$

Moreover, ψ_n *has the following properties:*

- **i.** $0 \le \psi_n \le 1$ *on* $\overline{Q}^\varepsilon_T$ *;*
- **ii.** $\int\int_{Q^\varepsilon_T} a_n|\Delta\psi_n|^2 < C$ *, for some* $C > 0$ *independent of* n*;*
- **iii.** $\sup_{0\le t\le T}\int_{\Omega^\varepsilon}|\nabla\psi_n|^2 < C$ *, for some* $C > 0$ *independent of* n*;*
- **iv.** *there exists* $\tilde{C} = \tilde{C}_\varepsilon > 0$ *such that* $\left|\frac{\partial\psi_n}{\partial\nu}\right| \le \tilde{C}$ *on* $\mathcal{A}^\varepsilon \times (0,T)$ *for any* $n \in \mathbb{N}$*, where* ν *is the outer normal at* $\mathcal{A}^\varepsilon$.

Proof Note that **i., ii., iii.** follow by the same arguments as in [1, Lemma 10]. We should note that in [1, Lemma 10] $\rho \equiv 1$; however, since in our case $\partial_t\rho = 0$, we get the conclusion exactly by the same arguments. It remains to prove **iv**. To this aim observe that for any $n \in \mathbb{N}$

$$\frac{\partial\psi_n}{\partial\nu} \le 0 \quad \text{in } \mathcal{A}^\varepsilon \times (0,T). \tag{3.4}$$

Furthermore, the function ψ_n is a subsolution of problem

$$\begin{cases} \rho\partial_t\psi_n + a_n\Delta\psi_n = 0 & \text{in } [\Omega^\varepsilon \setminus \bar{\Omega}^{\varepsilon_0}] \times (0,T], \\ \varphi_n = 0 & \text{in } \mathcal{A}^\varepsilon \times (0,T), \\ \varphi_n = 1 & \text{in } \mathcal{A}^{\varepsilon_0} \times (0,T), \\ \varphi_n(x,T) = 0 & \text{in } [\Omega^\varepsilon \setminus \bar{\Omega}^{\varepsilon_0}] \times \{T\}; \end{cases} \tag{3.5}$$

here we have used **i.** and the fact that supp $\chi \subset \Omega^{\varepsilon_0}$. Now, let ζ be the solution of the elliptic problem

$$\begin{cases} \Delta\zeta = 0 & \text{in } [\Omega^\varepsilon \setminus \bar{\Omega}^{\varepsilon_0}], \\ \zeta = 0 & \text{in } \mathcal{A}^\varepsilon, \\ \zeta = 1 & \text{in } \mathcal{A}^{\varepsilon_0}. \end{cases} \tag{3.6}$$

By the maximum principle,

$$\zeta \ge 0 \quad \text{in } \Omega^\varepsilon \setminus \Omega^{\varepsilon_0}.$$

Observe that the function ζ is a supersolution of problem (3.5). So, by the comparison principle,

$$\zeta \ge \psi_n \quad \text{in } [\Omega^\varepsilon \setminus \Omega^{\varepsilon_0}] \times (0,T).$$

Moreover,

$$\zeta = \psi_n = 0 \quad \text{in } \mathcal{A}^\varepsilon \times (0,T).$$

Hence,

$$\frac{\partial\psi_n}{\partial\nu} \ge \frac{\partial\zeta}{\partial\nu} \quad \text{in } \mathcal{A}^\varepsilon \times (0,T). \tag{3.7}$$

From Eqs. 3.4 and 3.7 we obtain for all $n \in \mathbb{N}$

$$\left|\frac{\partial\psi_n}{\partial\nu}\right| \le \tilde{C}_\varepsilon := \max_{\mathcal{A}^\varepsilon}\frac{\partial\zeta}{\partial\nu} \quad \text{in } \mathcal{A}^\varepsilon \times (0,T).$$

This completes the proof. □

Proof of Lemma 3.1 The proof of this lemma is an adaptation of the arguments used in [1, Proposition 9]. Let a be as in Eq. 3.2; since u and $\hat{u}$ are respectively subsolution and supersolution to Eq. 2.1, in view of the Definition 2.1, with Ω_1 and ψ as in Definition 2.2, by Eq. 2.4 with $\tau = T$, we get

$$\int_{\Omega^\varepsilon} \rho(x)[u(x,T) - \hat{u}(x,T)]\psi(x,T)\,dx - \int_0^T \int_{\Omega^\varepsilon} (u - \hat{u})\,\{\partial_t \psi + a\,\Delta\psi\}\,dt\,dx$$
$$\leq -\int_0^\tau \int_{\mathcal{A}^\varepsilon} [G(u) - G(\hat{u})]\langle \nabla\psi, \nu\rangle dS dt - \int_\tau^T \int_{\mathcal{A}^\varepsilon} [G(u) - G(\hat{u})]\langle \nabla\psi, \nu\rangle dS dt \quad (3.8)$$

Now, let $\{a_n\}$ and ψ_n as in Lemma 3.2. Since, for every $n \in \mathbb{N}$, there holds $\langle \nabla\psi_n, \nu\rangle \leq 0$ on $\mathcal{A}^\varepsilon$, if we set $\psi = \psi_n$ in Eq. 3.8, using Eq. 3.1, we obtain

$$\int_{\Omega^\varepsilon} \rho[u(x,T) - \hat{u}(x,T)]\chi(x)\,dx - \int_0^T \int_{\Omega^\varepsilon} (u - \hat{u})(a - a_n)\,\Delta\psi_n dt\,dx$$
$$\leq -\int_0^\tau \int_{\mathcal{A}^\varepsilon} [G(u) - G(\hat{u})]\langle \nabla\psi_n, \nu\rangle dS\,dt - \int_\tau^T \int_{\mathcal{A}^\varepsilon} [G(u) - G(\hat{u})]\langle \nabla\psi_n, \nu\rangle dS\,dt$$
$$\leq -\int_0^\tau \int_{\mathcal{A}^\varepsilon} [G(u) - G(\hat{u})]\langle \nabla\psi_n, \nu\rangle\,dS\,dt. \quad (3.9)$$

In view of Lemma 3.2, we get

$$\left|\int_0^T \int_{\Omega^\varepsilon} (u - \hat{u})(a - a_n)\,\Delta\psi_n dt\,dx\right| \leq C_1 \left\|\frac{a - a_n}{\sqrt{a_n}}\right\|_{L^2(Q_T)} \left\|\sqrt{a_n}\Delta\psi_n\right\|_{L^2(Q_T)}$$
$$\leq C_1\sqrt{C}\left\|\frac{a - a_n}{\sqrt{a_n}}\right\|_{L^2(Q_T)} \to 0 \quad \text{as } n \to \infty, \quad (3.10)$$

where the constant $C_1 > 0$ depends only on $\|u\|_{L^\infty}$ and $\|\hat{u}\|_{L^\infty}$. Furthermore,

$$\left|\int_0^\tau \int_{\mathcal{A}^\varepsilon} [G(u) - G(\hat{u})]\langle \nabla\psi_n, \nu\rangle dS\,dt\right| \leq C_1 \tilde{C}\,\text{meas}(\mathcal{A}^\varepsilon)\tau \quad (3.11)$$

where we used Lemma 3.2-**iv**. Hence, in view of Eqs. 3.10 and 3.11, letting $n \to \infty$ in Eq. 3.9 and then $\tau \to 0$, we end up with

$$\int_{\Omega^\varepsilon} \rho(x)[u(x,T) - \hat{u}(x,T)]\chi(x)\,dx \leq 0. \quad (3.12)$$

Since Eq. 3.12 holds for every $\chi \in C_0^\infty(\Omega^\varepsilon)$, by approximation it also holds with $\chi(x) = \text{sign}(u(x,T) - \hat{u}(x,T))^+$, $x \in \Omega^\varepsilon$. This implies $u \leq \hat{u}$ in Q_T^ε, from which the thesis immediately follows, letting $\varepsilon \to 0^+$. □

3.2 Proofs of the Theorems

In view of the assumption on $\rho(x)$ given in **H4**, there holds the following lemma (see [30]).

Lemma 3.3 *Let hypotheses **H0–H3** be satisfied. Let there exist $\hat{\varepsilon} > 0$, $\overline{\rho} \in C((0,\hat{\varepsilon}])$ such that $\rho(x) \leq \overline{\rho}(d(x))$ for any $x \in \mathcal{S}^{\hat{\varepsilon}}$, and $\int_0^{\hat{\varepsilon}} \eta\,\overline{\rho}(\eta)\,d\eta < +\infty$.*

Then there exists a function $V(x) \in C^2(\overline{\mathcal{S}^\varepsilon})$ such that

$$\begin{cases} \Delta V(x) \leq -\rho(x), & \text{for all } x \in \mathcal{S}^\varepsilon, \\ V(x) > 0, & \text{for all } x \in \mathcal{S}^\varepsilon, \\ V(x) \to 0 & \text{as } d(x) \to 0. \end{cases}$$

In this section we use the fact that for any $\varphi \in C(\mathcal{S} \times [0,T])$, there exists

$$\tilde{\varphi} \in C(\overline{Q}_T) \quad \text{such that } \tilde{\varphi} = \varphi \quad \text{in } \mathcal{S} \times [0,T]. \tag{3.13}$$

We shall write $\tilde{\varphi} \equiv \varphi$.

Proof of Theorem 2.3 The proof is divided into two main parts. At first, we consider that case of a density ρ satisfying hypothesis **H4-ii**.

Let $\eta_0 > 0$. Since **H5** holds, for any $0 < \eta < \eta_0$, we define $u_\varepsilon^\eta \in C(\overline{\Omega^\varepsilon} \times [0,T])$ as the unique solution (see [23, Chapter 5]; see also [32, Chapter 3]) to

$$\begin{cases} \rho\, \partial_t u = \Delta\,[G(u)] & \text{in } \Omega^\varepsilon \times (0,T)\,, \\ u = \varphi + \eta & \text{on } \mathcal{A}^\varepsilon \times (0,T)\,, \\ u = u_{0,\varepsilon} + \eta & \text{in } \Omega^\varepsilon \times \{0\}\,, \end{cases} \tag{3.14}$$

where

$$u_{0,\varepsilon}(x) := \zeta_\varepsilon\, u_0(x) + (1-\zeta_\varepsilon)\,\varphi(x,0) \quad \text{in } \overline{\Omega}^\varepsilon\,,$$

and $\{\zeta_\varepsilon\} \subset C_c^\infty(\Omega^\varepsilon)$ is a sequence of functions such that, for any $\varepsilon > 0$, $0 \le \zeta_\varepsilon \le 1$ and $\zeta_\varepsilon \equiv 1$ in $\Omega^{2\varepsilon}$. By the comparison principle, there holds

$$|u_\varepsilon^\eta| \le K := \max\{\|u_0\|_\infty, \|\varphi\|_\infty\} + \eta_0 \quad \text{in } \Omega^\varepsilon \times (0,T). \tag{3.15}$$

Moreover, by usual compactness arguments (see, e.g., [23, Chapter 5]), there exists a subsequence $\{u_{\varepsilon_k}^\eta\} \subseteq \{u_\varepsilon^\eta\}$ which converges, as $\varepsilon_k \to 0$, locally uniformly in $\Omega \times [0,T]$, to a solution u^η to the following problem

$$\begin{cases} \rho\, \partial_t u = \Delta\,[G(u)] & \text{in } \Omega \times (0,T]\,, \\ u = u_0 + \eta & \text{in } \Omega \times \{0\}. \end{cases} \tag{3.16}$$

We want to prove that, for each $\tau \in (0,T)$,

$$\lim_{\substack{x \to x_0 \\ t \to t_0}} u^\eta(x,t) = \varphi(x_0,t_0),$$

uniformly with respect to $t_0 \in (\tau, T]$, $x_0 \in \mathcal{S}$ and $\eta \in (0,\eta_0)$.

Take any $\tau \in (0, T/2)$. Let $(x_0,t_0) \in \mathcal{S} \times [2\tau, T]$. Set $N_\delta^\varepsilon(x_0) := B_\delta(x_0) \cap \Omega^\varepsilon$ for any $\delta > 0$ and $\varepsilon > 0$ small enough. From the continuity of the function φ and since $G \in C^1(\mathbb{R})$ is increasing, there follows that, for any $\sigma > 0$, there exists $\delta(\sigma) > 0$, independent of (x_0,t_0), such that

$$G^{-1}\,[G(\varphi(x_0,t_0)+\eta) - \sigma] \le \varphi(x,t) + \eta \le G^{-1}\,[G(\varphi(x_0,t_0)+\eta) + \sigma]\,, \tag{3.17}$$

for all $(x,t) \in \overline{N}_\delta(x_0) \times (\underline{t}_\delta, \overline{t}_\delta)$, where

$$\underline{t}_\delta := t_0 - \delta\,, \quad \text{and} \quad \overline{t}_\delta := \min\{t_0+\delta, T\},$$

and

$$N_\delta(x_0) := B_\delta(x_0) \cap \Omega.$$

Clearly, $\underline{t}_\delta > \tau$. Now, for any $(x,t) \in \overline{N}_\delta(x_0) \times (\underline{t}_\delta, \overline{t}_\delta)$, we define

$$\underline{w}(x,t) := G^{-1}\left[-\underline{M}V(x) - \sigma + G(\varphi(x_0,t_0)+\eta) - \underline{\lambda}(t-t_0)^2 - \underline{\beta}|x-x_0|^2\right], \tag{3.18}$$

with $V(x)$ as in Lemma 3.3 and $\underline{M}$, $\underline{\lambda}$ and $\underline{\beta}$ positive constants to be fixed conveniently in the sequel.

First of all we want to prove that

$$\rho \partial_t \underline{w} \le \Delta G(\underline{w}) \quad \text{in } N^{\varepsilon}_{\delta}(x_0) \times (\underline{t}_{\delta}, \overline{t}_{\delta}). \tag{3.19}$$

To his purpose, we note that

$$\rho \partial_t \underline{w} \le \rho \frac{2\underline{\lambda}\delta}{\alpha_0}, \qquad \text{and} \qquad \Delta G(\underline{w}) \ge \underline{M}\rho - 2\underline{\beta} N.$$

Hence, the function $\underline{w}$ solves (3.19), if

$$\underline{M} \ge \frac{2\underline{\beta} N}{\inf_{\Omega} \rho} + \frac{2\underline{\lambda}\delta}{\alpha_0}. \tag{3.20}$$

Going further, for any $(x, t) \in [B_{\delta}(x_0) \cap \mathcal{A}^{\varepsilon}] \times (\underline{t}_{\delta}, \overline{t}_{\delta})$, we have

$$\underline{w}(x, t) \le G^{-1}[G(\varphi(x_0, t_0) + \eta) - \sigma]. \tag{3.21}$$

Moreover, for $(x, t) \in [\partial B_{\delta}(x_0) \cap \Omega^{\varepsilon}] \times (\underline{t}_{\delta}, \overline{t}_{\delta})$, there holds

$$\underline{w}(x, t) \le -K, \tag{3.22}$$

provided

$$\underline{\beta} \ge \frac{G(||\varphi||_{L^\infty} + \eta_0) - G(-K)}{\delta^2}.$$

Finally, for all $(x, t) \in N^{\varepsilon}_{\delta}(x_0) \times \{\underline{t}_{\delta}\}$, there holds

$$\underline{w}(x, t) \le G^{-1}[G(\varphi(x_0, t_0) + \eta) - \underline{\lambda}\delta^2] \le -K, \tag{3.23}$$

assuming

$$\underline{\lambda} \ge \frac{G(||\varphi||_{L^\infty} + \eta_0) - G(-K)}{\delta^2}.$$

From Eqs. 3.21, 3.22 and 3.23 we obtain that $\underline{w}$ is a subsolution to the following problem

$$\begin{cases} \rho\, \partial_t u = \Delta\, [G(u)] & \text{in } N^{\varepsilon}_{\delta}(x_0) \times (\underline{t}_{\delta}, \overline{t}_{\delta})\,, \\ u = -K & \text{in } [\partial B_{\delta}(x_0) \cap \Omega^{\varepsilon}] \times (\underline{t}_{\delta}, \overline{t}_{\delta})\,, \\ u = G^{-1}[G(\varphi + \eta) - \sigma] & \text{in } [B_{\delta}(x_0) \cap \mathcal{A}^{\varepsilon}] \times (\underline{t}_{\delta}, \overline{t}_{\delta})\,, \\ u = -K & \text{in } N^{\varepsilon}_{\delta}(x_0) \times \{\underline{t}_{\delta}\}. \end{cases} \tag{3.24}$$

Recalling the definition of u^{η}_{ε} given in Eq. 3.14, and by using Eq. 3.15, it follows that u^{η} is a supersolution to problem (3.24). Note that sub– and supersolutions to problem (3.24) are meant similarly to Definition 2.2, considering that $N^{\varepsilon}_{\delta}(x_0)$ is piece-wise smooth; the same holds for problems of the same form we mention in the sequel.

By proceeding with the same methods, for all $(x, t) \in \overline{N}_{\delta}(x_0) \times (\underline{t}_{\delta}, \overline{t}_{\delta})$ we define

$$\overline{w}(x, t) := G^{-1}\left[\overline{M} V(x) + \sigma + G(\varphi(x_0, t_0) + \eta) + \overline{\lambda}(t - t_0)^2 + \overline{\beta}|x - x_0|^2\right], \tag{3.25}$$

proving that, with an appropriate choice for the coefficients $\overline{M}$, $\overline{\lambda}$ and $\overline{\beta}$, $\overline{w}$ is a supersolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\,[G(u)] & \text{in } N_\delta^\varepsilon(x_0)\times(\underline{t}_\delta,\overline{t}_\delta)\,,\\ u\, = \,K & \text{in } [\partial B_\delta(x_0)\cap\Omega^\varepsilon]\times(\underline{t}_\delta,\overline{t}_\delta)\,,\\ u\, = G^{-1}[G(\varphi+\eta)+\sigma] & \text{in } [B_\delta(x_0)\cap\partial\Omega^\varepsilon]\times(\underline{t}_\delta,\overline{t}_\delta)\,,\\ u\, = K & \text{in } N_\delta^\varepsilon(x_0)\times\{\underline{t}_\delta\}. \end{cases} \tag{3.26}$$

Precisely, we require $\overline{M}$ to be such that

$$\overline{M}\geq\frac{2\overline{\beta}N}{\inf_\Omega\rho}+\frac{2\overline{\lambda}\delta}{\alpha_0},$$

while $\overline{\beta}$ and $\overline{\lambda}$ are chosen so that

$$\overline{\beta}\geq\frac{G(K)-G(||\varphi||_{L^\infty}+\eta_0)}{\delta^2},\qquad \underline{\lambda}\geq\frac{G(K)-G(||\varphi||_{L^\infty}+\eta_0)}{\delta^2}.$$

On the other hand, u^η is a subsolution to problem (3.26). Hence, by the comparison principle, and by letting $\varepsilon_k\to 0$, we get

$$\underline{w}\leq u^\eta\leq\overline{w}\quad\text{in } N_\delta(x_0)\times(\underline{t}_\delta,\overline{t}_\delta). \tag{3.27}$$

Take any $\tau\in(0,T/2)$ and $(x_0,t_0)\in\mathcal{S}\times[2\tau,T]$. Due to Eq. 3.27, recalling the definition of $\underline{w}$ and $\overline{w}$ and by letting $x\to x_0$, $t\to t_0$, one has

$$G^{-1}\left[G(\varphi(x_0,t_0)+\eta)-2\sigma\right]\leq u^\eta(x_0,t_0)\leq G^{-1}\left[G(\varphi(x_0,t_0)+\eta)+2\sigma\right].$$

Letting $\sigma\to 0^+$, we end up with

$$\lim_{\substack{x\to x_0\\ t\to t_0}} u^\eta(x,t)=\varphi(x_0,t_0),$$

uniformly with respect to $t_0\in(2\tau,T)$, $x_0\in\mathcal{S}$ and $\eta\in(0,\eta_0)$, for each $\tau\in(0,T/2)$. Moreover, by usual compactness arguments, there exists a subsequence $\{u^{\eta_k}\}\subset\{u^\eta\}$ which converges, as $\eta_k\to 0$, to a solution u to Eq. 2.1, locally uniformly in $\Omega\times[0,T]$. Hence, by using Eq. 3.27, we have, in the limit $\sigma\to 0^+$ and $\eta\to 0^+$,

$$\lim_{\substack{x\to x_0\\ t\to t_0}} u(x,t)=\varphi(x_0,t_0),$$

uniformly with respect to $t_0\in(2\tau,T)$ and $x_0\in\mathcal{S}$, for each $\tau\in(0,T/2)$.

It remains to show that u is the maximal solution. To this end, let v be any solution to problem (2.1) satisfying (2.5). From Eq. 3.27 it follows that for any $\alpha\in(0,\eta_0/4)$ and for any $\tau\in(0,T)$, there exist $\tilde{\varepsilon}>0$, $\eta(\alpha)>0$, with $\eta(\alpha)\to 0$ as $\alpha\to 0$, such that for any $0<\varepsilon<\tilde{\varepsilon}$ and $\eta\in(\eta(\alpha),\eta_0)$

$$v(x,t)\leq\varphi(x,t)+\alpha\leq u^\eta(x,t)\quad\text{for all }\ (x,t)\in\mathcal{A}^\varepsilon\times(\tau,T]. \tag{3.28}$$

Moreover

$$v(x,0)=u_0(x)<u_0(x)+\eta=u^\eta(x,0)\quad\text{for all }\ x\in\Omega. \tag{3.29}$$

Since $v(x,t)$ and $u^\eta(x,t)$ are solutions to the same equations in $\Omega\times(0,T]$, in view of Eqs. 3.28, 3.29 and Lemma 3.1 there holds

$$v(x,t)\leq u^\eta(x,t)\quad\text{for all }\ (x,t)\in Q_T.$$

Passing to the limit $\eta \to 0^+$ we obtain

$$v \le u \quad \text{in } \; Q_T,$$

and the proof is complete, in this case.

In the second part of the proof, we consider a density ρ such that **H4-i** holds. Now, we need to slightly modify the arguments used above. Since $\mathcal{S} \in C^1$, by [13] the uniform exterior sphere condition is satisfied, i.e. there exists $R > 0$ such that for any $x_0 \in \mathcal{S}$ we can find $x_1 \in \mathbb{R}^N \setminus \bar{\Omega}$ such that $B_R(x_1) \subset \mathbb{R}^N \setminus \bar{\Omega}$ and $\overline{B_R(x_1)} \cap \mathcal{S} = \{x_0\}$. Thus, by standard arguments (see [24]), it is proven that the following function

$$h(x) := C[e^{-a\,R^2} - e^{-a\,|x-x_1|^2}] \tag{3.30}$$

satisfies

- $\Delta h \le -1$ in $B_R(x_0) \cap \Omega$;
- $h > 0$ for all $x \in \left[\bar{B}_R(x_0) \cap \bar{\Omega}\right] \setminus \{x_0\}$;
- $h(x_0) = 0$,

for a suitable choice of the constants $C > 0$ and $a > 0$, independent of $x_0 \in \mathcal{S}$.

The function $h(x)$ can be used in order to built suitable barrier functions $\underline{w}(x,t)$ and $\overline{w}(x,t)$. To this end, for $(x,t) \in \overline{N}_\delta(x_0) \times (\underline{t}_\delta, \overline{t}_\delta)$, we define

$$\underline{w}(x,t) := G^{-1}\left[-\underline{M}h(x) - \sigma + G(\varphi(x_0,t_0) + \eta) - \underline{\lambda}(t-t_0)^2\right], \tag{3.31}$$

being $h(x)$ as in Eq. 3.30.

First of all, because of the properties of $h(x)$, there holds $\rho\partial_t\underline{w} \le \Delta G(\underline{w})$, if

$$\underline{M} \ge \frac{2\,\rho(x)\,\underline{\lambda}\,\delta}{\alpha_0},$$

Hence, we require that

$$\underline{M} \ge \frac{2\,\lambda\,\delta}{\alpha_0}\,\|\rho\|_{L^\infty}.$$

Next, let $(x,t) \in [B_\delta(x_0) \cap \mathcal{A}^\varepsilon] \times (\underline{t}_\delta, \overline{t}_\delta)$; we have

$$\underline{w} \le G^{-1}[G(\varphi(x_0,t_0) + \eta) - \sigma]. \tag{3.32}$$

Moreover, for $(x,t) \in [\partial B_\delta(x_0) \cap \Omega^\varepsilon] \times (\underline{t}_\delta, \overline{t}_\delta)$ we have

$$\underline{w}(x,t) \le -K, \tag{3.33}$$

provided

$$\underline{M} \ge \frac{G(||\varphi||_{L^\infty} + \eta_0) - G(-K)}{\inf_{\partial B_\delta(x_0)\cap\Omega} h}.$$

Finally, for $(x,t) \in N_\delta^\varepsilon(x_0) \times \{\underline{t}_\delta\}$

$$\underline{w}(x,t) \le G^{-1}[G(\varphi(x_0,t_0) + \eta) - \underline{\lambda}\delta^2] \le -K \tag{3.34}$$

imposing

$$\underline{\lambda} \ge \frac{G(||\varphi||_{L^\infty} + \eta_0) - G(-K)}{\delta^2}.$$

From Eqs. 3.32, 3.33 and 3.34 we can state that $\underline{w}$ is a subsolution to the following problem

$$\begin{cases} \rho\,\partial_t u = \Delta\,[G(u)] & \text{in } N_\delta^\varepsilon \times (\underline{t}_\delta, \overline{t}_\delta)\,, \\ u\,=-K & \text{on } [\partial B_\delta(x_0)\cap\Omega^\varepsilon]\times(\underline{t}_\delta,\overline{t}_\delta)\,, \\ u\,=G^{-1}[G(\varphi+\eta)-\sigma] & \text{in } [B_\delta(x_0)\cap\partial\Omega^\varepsilon]\times(\underline{t}_\delta,\overline{t}_\delta)\,, \\ u\,=-K & \text{in } N_\delta^\varepsilon(x_0)\times\{\underline{t}_\delta\}\,, \end{cases} \tag{3.35}$$

while u^η is a supersolution to the same problem. By proceeding with the same methods, for all $(x,t)\in\overline{N}_\delta(x_0)\times(\underline{t}_\delta,\overline{t}_\delta)$ we define

$$\overline{w}(x,t) := G^{-1}\left[\overline{M}\,h(x)+\sigma+G(\varphi(x_0,t_0)+\eta)+\overline{\lambda}(t-t_0)^2\right], \tag{3.36}$$

proving that, with the appropriate choices for the coefficients $\overline{M}$, $\overline{\lambda}$ and $\overline{\beta}$, $\overline{w}$ is a supersolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\,[G(u)] & \text{in } N_\delta^\varepsilon \times (\underline{t}_\delta, \overline{t}_\delta)\,, \\ u\,=K & \text{on } [\partial B_\delta(x_0)\cap\Omega^\varepsilon]\times(\underline{t}_\delta,\overline{t}_\delta)\,, \\ u\,=G^{-1}[G(\varphi+\eta)+\sigma] & \text{in } [B_\delta(x_0)\cap\partial\Omega^\varepsilon]\times(\underline{t}_\delta,\overline{t}_\delta)\,, \\ u\,=K & \text{in } N_\delta^\varepsilon(x_0)\times\{\underline{t}_\delta\}\,, \end{cases} \tag{3.37}$$

while u^η is a subsolution to the same problem. Hence, by the comparison principle, and by letting $\varepsilon_k \to 0$, we get

$$\underline{w} \le u^\eta \le \overline{w} \quad \text{in } N_\delta(x_0)\times(\underline{t}_\delta,\overline{t}_\delta). \tag{3.38}$$

Take any $\tau \in (0, T/2)$. Let $(x_0,t_0)\in\mathcal{S}\times[2\tau,T]$. In view of Eq. 3.38, recalling the definition of $\underline{w}$ and $\overline{w}$ and by letting $x\to x_0$ and choosing $t=t_0$, one has

$$G^{-1}\left[G(\varphi(x_0,t_0)+\eta)-2\sigma\right] \le u^\eta(x,t_0) \le G^{-1}\left[G(\varphi(x_0,t_0)+\eta)+2\sigma\right].$$

So, the thesis follows for $\sigma \to 0^+$ as in the previous case, as well as the maximality of u. □

Proof of Theorem 2.4 The conclusion follows arguing as in the proof of Theorem 2.3, choosing $\underline{\lambda}=0$ in Eq. 3.18 and in Eq. 3.31, and $\overline{\lambda}=0$ in Eq. 3.25 and in Eq. 3.36. We only mention that in this case, since we are assuming **H2** instead of **H5**, existence and uniqueness of the solutions u_ε^η follow by results in [32, Chapter 5]; moreover, in view of [2, Lemma 5.2] we can find a subsequence $\{u_{\varepsilon_k}^\eta\}\subseteq\{u_\varepsilon^\eta\}$ which converges, as $\varepsilon_k\to 0$, locally uniformly in $\Omega\times[0,T]$, to a solution u^η to problem (3.16). Furthermore, the fact that Eq. 2.6 holds uniformly for $t\in[0,T]$ is due to assumption (1.11). □

Proof of Theorem 2.5 Let

$$\alpha_2 := \min\left\{\min_{\bar{\Omega}\times[0,T]}\varphi,\ \alpha_1\right\},$$

with $\alpha_1 > 0$ as in Eq. 2.7. Since $\varphi \in C(\mathcal{S} \times [0,T])$ and $\varphi > 0$ in $\mathcal{S} \times [0,T]$, we can select $\tilde{\varphi} \equiv \varphi$ as in Eq. 3.13, such that $\tilde{\varphi} > 0$ in $\bar{Q}_T$. So, $\alpha_2 > 0$. Take $\underline{u}_0 \in C(\bar{\Omega})$ such that

$$\underline{u}_0 \le u_0 \quad \text{in} \quad \Omega \,, \quad \lim_{x \to x_0} \underline{u}_0(x) = \frac{\alpha_2}{2}. \tag{3.39}$$

By Theorem 2.4, there exists a solution $\underline{u}(x,t)$ to the following problem

$$\begin{cases} \rho \partial_t u = \Delta[G(u)] & \text{in} \ \ \Omega \times (0,T], \\ u = \underline{u}_0 & \text{in} \ \Omega \times \{0\}, \end{cases} \tag{3.40}$$

such that

$$\lim_{x \to x_0} \underline{u}(x,t) = \frac{\alpha_2}{2} \quad \text{uniformly for} \ \ x_0 \in \mathcal{S}, t \in [0,T]. \tag{3.41}$$

We construct the approximating sequence $\{u_\varepsilon^\eta\}$ as in the proof of Theorem 2.4. Due to Eqs. 3.40 and 3.41, by the comparison principle, we have that for some $\varepsilon_0 > 0$, for every $0 < \varepsilon < \varepsilon_0$

$$\underline{u}(x,t) \le u_\varepsilon^\eta(x,t) \quad \text{for all } x \in \Omega^\varepsilon \,, \ \ t \in (0,T]. \tag{3.42}$$

Then, by usual compactness arguments (see [2, Lemma 5.2]) there exists a subsequence $\{u_{\varepsilon_k}^\eta\} \subset \{u_\varepsilon^\eta\}$ which converges, as $\varepsilon_k \to 0$, to a solution u^η to Eq. 3.16. From Eq. 3.42 it follows that

$$u^\eta(x,t) \ge \underline{u}(x,t) \quad \text{for all} \ \ x \in \Omega \,, \ \ t \in (0,T].$$

Therefore, for some $0 < \varepsilon_1 < \varepsilon_0$, for all $0 < \eta < \eta_0$ there holds

$$u^\eta(x,t) \ge \frac{\alpha_2}{4} \quad \text{for all } x \in \mathcal{S}^{\varepsilon_1}, \ \ t \in (0,T]. \tag{3.43}$$

Hence, in $\mathcal{S}^{\varepsilon_1} \times (0,T]$ the equation does not degenerate, i.e., for some $\alpha_0 > 0$,

$$G'(u) \ge \alpha_0 \quad \text{in} \ \ \mathcal{S}^{\varepsilon_1} \times (0,T].$$

Select a function G_1 such that hypothesis **H2** is satisfied; moreover, $G_1(u) = G(u)$ for $u \ge \frac{\alpha_2}{4}$ and $G_1'(u) \ge \frac{\alpha_0}{2} > 0$ for all $u \in \mathbb{R}$. From Eq. 3.43, $u^\eta(x,t)$ is a solution to the non-degenerate equation

$$\rho \partial_t u = [G_1(u)] \quad \text{in} \ \ \mathcal{S}^{\varepsilon_1} \times (0,T].$$

Thus we get the conclusion as in the proof of Theorem 2.3. □

4 Uniqueness Results: Proofs

The proof of Theorem 2.7 makes use of the following lemma.

Lemma 4.1 *Let $\varepsilon_0 > 0$ and $F \in C^\infty(\Omega)$ such that $F \ge 0$, $\operatorname{supp} F \subset \Omega^{\varepsilon_0}$. Then, for any $0 < \varepsilon < \varepsilon_0$, there exists a unique classical solution ψ^ε to the problem*

$$\begin{cases} \Delta \psi^\varepsilon = -F \ \text{in} \ \Omega^\varepsilon \\ \\ \psi^\varepsilon = 0 \qquad on \ \mathcal{A}^\varepsilon. \end{cases} \tag{4.1}$$

Moreover, for any $0 < \varepsilon < \varepsilon_0$ there holds:

$$\psi^\varepsilon > 0 \quad \text{in} \ \ \Omega^\varepsilon \,; \tag{4.2}$$

$$\langle \nabla \psi^\varepsilon(x), \nu^\varepsilon(x) \rangle < 0 \quad \textit{for all} \ \ x \in \mathcal{A}^\varepsilon \,; \tag{4.3}$$

$$\int_{\mathcal{A}^\varepsilon} \big| \langle \nabla \psi^\varepsilon, \nu^\varepsilon \rangle \big| dS \le \bar{C} \,, \tag{4.4}$$

for some constant $\bar{C} > 0$ independent of ε; here ν^ε denotes the outer unit normal vector to $\partial\Omega^\varepsilon$.

Proof For any $0 < \varepsilon < \varepsilon_0$, the existence and the uniqueness of the solution ψ_ε to Eq. 4.1 follow immediately. Moreover, since $F \geq 0$, by the strong maximum principle we get Eqs. 4.2 and 4.3. Observe that, since supp $F \subset \Omega^{\varepsilon_0}$, then for any $0 < \varepsilon < \varepsilon_0$ we have

$$\int_{\Omega^\varepsilon} F(x)\,dx = \int_{\Omega^{\varepsilon_0}} F(x)\,dx =: \bar{C}. \tag{4.5}$$

On the other hand, from Eq. 4.1 by integrating by parts,

$$\int_{\Omega^\varepsilon} F(x)dx = -\int_{\Omega^\varepsilon} \Delta\psi^\varepsilon dx = -\int_{\mathcal{A}^\varepsilon} \langle \nabla\psi^\varepsilon, \nu^\varepsilon\rangle dS. \tag{4.6}$$

From Eqs. 4.5, 4.6, and 4.3 we get Eq. 4.4. □

Proof of Theorem 2.7 In view of the hypotheses we made, we can apply Theorem 2.3 to infer that there exists a maximal solution $\bar{u}$ to Eq. 2.1. Let u be any solution to Eq. 2.1, and let $F \in C_c^\infty(\Omega)$.

Without loss of generality, we suppose supp $F \subset \Omega^{\varepsilon_0}$, for some $\varepsilon_0 > 0$, $F \not\equiv 0$ and $F \geq 0$. Since both $\bar{u}$ and u solves (2.1), we apply the equality (2.2) with $\Omega = \Omega^\varepsilon$, $0 < \varepsilon < 2\varepsilon_0$ and $\psi(x,t) = \psi^\varepsilon(x)$, with ψ^ε given by Lemma 4.1. We get

$$\begin{aligned}\int_0^T \int_{\Omega^\varepsilon} [G(\bar{u}) - G(u)]\, F(x)\,dx\,dt \\ = -\int_{\Omega^\varepsilon} [\bar{u}(x,T) - u(x,T)]\rho(x)\,\psi^\varepsilon(x)dx \\ -\int_0^T \int_{\mathcal{A}^\varepsilon} [G(\bar{u}) - G(u)]\langle\nabla\psi^\varepsilon, \nu^\varepsilon\rangle dS\,dt\end{aligned}$$

Since $F \geq 0$, $\psi^\varepsilon \geq 0$, $\bar{u} \geq u$ in Ω^ε and $\langle\nabla\psi^\varepsilon, \nu^\varepsilon\rangle \leq 0$ on $\mathcal{A}^\varepsilon$, the previous equality gives:

$$\begin{aligned}0 \leq& \int_0^T \int_{\Omega^\varepsilon} [G(\bar{u}) - G(u)]F(x)\,dx\,dt \leq -\int_0^T \int_{\mathcal{A}^\varepsilon} [G(\bar{u}) - G(u)]\langle\nabla\psi^\varepsilon, \nu^\varepsilon\rangle dS\,dt \\ =& -\int_0^\tau \int_{\mathcal{A}^\varepsilon} [G(\bar{u}) - G(u)]\langle\nabla\psi^\varepsilon, \nu^\varepsilon\rangle dS\,dt - \int_\tau^T \int_{\mathcal{A}^\varepsilon} [G(\bar{u}) - G(u)]\langle\nabla\psi^\varepsilon, \nu^\varepsilon\rangle dS\,dt\end{aligned} \tag{4.7}$$

Going further, by Eq. 4.4, we get

$$\begin{aligned}\int_\tau^T \int_{\mathcal{A}^\varepsilon} [G(\bar{u}) - G(u)]\langle\nabla\psi^\varepsilon, \nu^\varepsilon\rangle\,dS\,dt &\leq T \sup_{\mathcal{A}^\varepsilon\times(\tau,T)} [G(\bar{u}) - G(u)] \int_{\mathcal{A}^\varepsilon} \left|\langle\nabla\psi^\varepsilon, \nu^\varepsilon\rangle\right| dS \\ &\leq \bar{C}T \sup_{\mathcal{A}^\varepsilon\times(\tau,T)} [G(\bar{u}) - G(u)].\end{aligned} \tag{4.8}$$

Furthermore,

$$\int_0^\tau \int_{\mathcal{A}^\varepsilon} [G(\bar{u}) - G(u)]\langle\nabla\psi^\varepsilon, \nu^\varepsilon\rangle\,dS\,dt \leq \bar{C}\,\tau\,C, \tag{4.9}$$

where the constant C only depends on $\|u\|_{L^\infty}$ and $\|\bar{u}\|_{L^\infty}$. Since any solution to Eq. 2.1 satisfies condition (2.5) uniformly for $t \in [\tau, T]$, for each $\tau \in (0, T)$, we get

$$\sup_{\mathcal{A}^\varepsilon\times(\tau,T)} [G(\bar{u}) - G(u)] \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.10}$$

Hence, in view of Eqs. 4.7, 4.8, 4.9 and 4.10, if we let $\varepsilon \to 0$ in Eq. 4.7 and then $\tau \to 0$, we obtain

$$\int_0^T \int_\Omega [G(\bar{u}) - G(u)]\ F(x)\, dx\, dt\ = 0. \tag{4.11}$$

In view of the hypothesis **H2**, and because of the arbitrariness of F, Eq. 4.11 implies

$$\bar{u}\ = u \quad \text{in}\ \ \Omega \times (0, T]\,,$$

and the proof is completed. □

As outlined in Remark 2.8, Theorem 2.7 holds true either if we consider a non degenerate nonlinearity G satisfying hypothesis **H5** or if we suppose

$$\varphi(x_0, t) \equiv \varphi(x_0), \quad \text{for all}\ \ t \in [0, T].$$

Infact, in both cases, Theorem 2.3 and Theorem 2.5 assure the existence of the maximal solution satisfying (2.5) and (2.6) respectively. Hence, the uniqueness follows as in the proof of Theorem 2.7.

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