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The Dirichlet and Regularity Problems for Some Second Order Linear Elliptic Systems on Bounded Lipschitz Domains

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Abstract In this paper, we investigate divergence-form linear elliptic systems on bounded Lipschitz domains in $\mathbb{R}^{d+1}$, $d \geq 2$, with L^2 boundary data. The coefficients are assumed to be real, bounded, and measurable. We show that when the coefficients are small, in Carleson norm, compared to one that is continuous on the boundary, we obtain solvability for both the Dirichlet and regularity boundary value problems given that the coefficients satisfy a certain "pseudo-symmetry" condition.

Keywords Linear elliptic systems · Second order · Bounded Lipschitz domains · Small Carleson norm condition

Mathematics Subject Classification (2010) 35J57

1 Introduction and Preliminaries

We consider second order elliptic systems of equations $\mathcal{L}u = 0$, where $u = (u^1, \cdots, u^m)$, $m \geq 1$, and

$$\mathcal{L} = \mathcal{L}_A = -\frac{\partial}{\partial x_i}\left[A_{i,j}^{\alpha\beta}(X)\frac{\partial}{\partial x_j}\right] \tag{1.1}$$

is defined in $\mathbb{R}^{d+1}$, $d \geq 2$. Single equations correspond to the case $m = 1$.

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We will assume that the coefficient matrix $A(X) = (A^{\alpha\beta}_{i,j})$, $1 \leq \alpha, \beta \leq m$, $1 \leq i, j \leq d+1$, is real, bounded, and satisfies the following ellipticity condition

$$\lambda^{-1}|\xi|^2 \leq A^{\alpha\beta}_{i,j}(X)\xi^{\beta}_{j}\xi^{\alpha}_{i} \leq \lambda|\xi|^2, \tag{1.2}$$

for all $X \in \mathbb{R}^{d+1}$ and $\xi = (\xi^{\alpha}_{i}) \in \mathbb{R}^{(d+1)m}$, $\xi \neq 0$, where $\lambda > 0$ is called the ellipticity of $\mathcal{L}$ or A.

When A is real, let A^T denote the matrix $(A^T)^{\alpha\beta}_{i,j}$ where $(A^T)^{\alpha\beta}_{i,j} = A^{\alpha\beta}_{j,i}$. In this case, we say that A satisfies the "pseudo-symmetry" condition if

$$A^{\alpha\beta}_{i,j} + A^{\alpha\beta}_{j,i} = A^{\beta\alpha}_{i,j} + A^{\beta\alpha}_{j,i}. \tag{1.3}$$

Notice that this property is satisfied automatically when $m = 1$. Define $G_A = \dfrac{A + A^T}{2}$, and we have $G_A = (G_A)^*$, i.e. G_A is symmetric. Note that if A has ellipticity constant λ, so do A^T and A^*.

As usual, the divergence form equation is interpreted in the weak sense, i.e. we say that $u \in W^{1,2}_{\text{loc}}(V)$ is a solution to $\mathcal{L}u = 0$ in a domain V if

$$\int_V A\nabla u \cdot \nabla\phi = 0, \qquad \forall\phi \in C^{\infty}_{0}(V). \tag{1.4}$$

Here, C^{∞}_{0} denotes the space of smooth functions with compact support. When there is a possibility of confusion, we will specify the domain.

We will use the notations D_j, ∂_{x_j}, $\dfrac{\partial}{\partial x_j}$ interchangeably. For a $d+1$-dimensional vectors $f = (f_i)_{1\leq i\leq d+1}$, let $f_{\perp}$, $f_{\|}$ denote the normal and tangential components of f respectively. We also use $\nabla_{\|}$, $\text{div}_{\|}$, $\text{curl}_{\|}$ to denote the differential operators acting only in the tangential component.

The set $W^{1,p}(E)$ is the usual Sobolev space of functions in $L^p(E)$ whose first derivatives (in the sense of distributions) are also in $L^p(E)$, and the set $W^{1,p}_{\text{loc}}$ consists of functions in $W^{1,p}(E')$ for every compact subset E' of E.

Denote by $X = (x, t)$, $Y = (y, s)$ points in $\mathbb{R}^{d+1}$, with $x, y \in \mathbb{R}^d$, $t, s \in \mathbb{R}$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $\|\nabla\varphi\|_{\infty} = M$. Throughout this paper, let D_{φ} be the domain above φ, i.e.

$$D_{\varphi} = \{X = (x, t) \in \mathbb{R}^{d+1} : t > \varphi(x)\}.$$

When there is no ambiguity, we will drop the subscript φ. Then, for any $r > 0$ and $Q = (z, \varphi(z)) \in \partial D$, define:

$$\Delta_r(Q) = \Delta(Q, r) = B(Q, r) \cap \partial D = \overline{T_r(Q)} \cap \partial D, \text{ where}$$
$$T_r(Q) = T(Q, r) = \{X = (x, t) \in D : |x - z| < r, \varphi(x) < t < \varphi(x) + (1 + M)r\}.$$

We now define a bounded Lipschitz domain following [32].

A bounded open set $\Omega \in \mathbb{R}^{d+1}$ is called a bounded Lipschitz domain if for each $Q \in \partial\Omega$, there exists a rectangular coordinate system (x, t), $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, a neighborhood $U(Q) \equiv U \subset \mathbb{R}^{d+1}$ containing Q, and a function $\varphi_Q \equiv \varphi : \mathbb{R}^d \to \mathbb{R}$ such that

(i) $|\varphi(x) - \varphi(y)| \leq C_Q|x - y|$, for all $x, y \in \mathbb{R}^d$, $C_Q < \infty$,
(ii) $U \cap \Omega = \{(x, t) : t > \varphi(x)\} \cap \Omega$.

The coordinate systems (x, t) may always be taken to be a rotation and translation of the standard rectangular coordinates for $\mathbb{R}^{d+1}$. We will also only consider bounded Lipschitz domains with connected boundaries. We also use V to denote a general domain in $\mathbb{R}^{d+1}$ without having specific properties like D_φ or Ω.

Denote by $Z(X,r)$ an open, right circular, doubly truncated cylinder centered at X with radius r. A coordinate cylinder, $Z = Z(Q,r)$, $Q \in \partial\Omega$ is defined by the following properties.

(i) The bases of Z are some positive distance from $\partial\Omega$.
(ii) There is a rectangular coordinate system for $\mathbb{R}^{d+1}$, (x,t), $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, with t-axis containing the axis of Z.
(iii) There is an associated function $\varphi = \varphi_Z : \mathbb{R}^d \to \mathbb{R}$, that is Lipschitz.
(iv) $Z \cap D = Z \cap \{(x,t) : t > \varphi(x)\}$
(v) $Q = (0, \varphi(0))$.

The pair (Z,φ) is called a coordinate pair. For any positive number R, $RZ(Q,r)$ denotes the dilation of Z by a factor of R.

By compactness, we can cover $\partial\Omega$ with a finite number of coordinate cylinders $Z_1, \cdots, Z_N$. Moreover, it is also possible to do this in such a way that for each Z_j there is a coordinate pair (Z_j^*, φ_j) with $Z_j^* = R_j Z_j$, where R_j is some sufficiently large positive number. For example, $R_j > 10(1 + \|\nabla\varphi_j\|_\infty)^{1/2}$. Whenever we cover $\partial\Omega$ with coordinate cylinders, we assume that Z_j^* exist. Observe also that φ_j can be taken to have compact support.

For a bounded Lipschitz domain Ω, there are numbers $M < \infty$ such that for any covering of coordinate cylinders, the φ_j all have Lipschitz norm at most M. The smallest such number is called the Lipschitz constant for Ω.

We note that $\Delta_r(Q)$, $T_r(Q)$ can then be defined for every $Q \in \partial\Omega$ provided that r is small enough.

For $X \in \Omega$, denote $\delta(X) = \text{dist}\,(X, \partial\Omega)$, the distance from X to the boundary.

For the domain above a Lipschitz graph D_φ, and $Q = (z, \varphi(z)) \in \partial D_\varphi$, a cone at Q with aperture α is defined to be

$$\Gamma_\alpha(Q) = \{X \in D_\varphi : |X - Q| \le (1+\alpha)(t - \varphi(z))\},$$

and, in the special case $\varphi = 0$, i.e. $D = \mathbb{R}^{d+1}_+$,

$$\Gamma(x) = \{(x,t) \in \mathbb{R}^{d+1}_+ : |x - y| < t\}.$$

Note that the largest aperture α is determined by the Lipschitz constant φ.

In the case that Ω is a bounded Lipschitz domain, for $Q \in \partial\Omega$, $\Gamma_\alpha(Q)$ denotes an open, circular, doubly truncated cone with one component in Ω and the other in $\mathbb{R}^{d+1} \setminus \overline{\Omega}$. The component interior to Ω is denoted by $\Gamma_{\alpha,i}$ and the component exterior to $\overline{\Omega}$ will be denoted by $\Gamma_{\alpha,e}$. When the context is clear, we will drop the subscript i, e.

Assigning one cone, $\Gamma(Q)$, to each $Q \in \partial\Omega$, we call the resulting family $\{\Gamma(Q) : Q \in \partial\Omega\}$ regular if there is a finite covering of $\partial\Omega$ by coordinate cylinders, as described above, such that for each $(Z(Q,r), \varphi)$ there are three cones, α, β, and γ, each with vertex at the origin and axis along the axis of Z such that

$$\alpha \subset \overline{\beta} \setminus \{0\} \subset \gamma,$$

and for all $(x, \varphi(x)) = P \in (\frac{4}{5} Z^* \cap \partial\Omega)$,

$$\alpha + P \subset \Gamma(P) \subset \overline{\Gamma(Q)} \setminus \{P\} \subset \beta + P,$$

$$(\gamma + P)_i \subset \Omega \cap Z^*, \qquad \text{and} \qquad (\gamma + P)_e \subset Z^* \setminus \overline{D}.$$

Towards the end of this paper, we will need to approximate a bounded Lipschitz domain using the following result. The reader may consult [27, 28], or [31] for a proof.

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^{d+1}$ be a bounded Lipschitz domain. Then the following hold.*

(1) *There is a regular family of cones $\{\Gamma\}$ for Ω as described above.*
(2) *There is a sequence of C^∞ domains, $\Omega_j \subset \mathbb{R}^{d+1}$, and homeomorphisms, $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\sup_{Q\in\partial\Omega} |Q - \Lambda_j(Q)| \to 0$ as $j \to \infty$ and for all j and all $Q \in \partial\Omega$, $\Lambda_j(Q) \in \Gamma_i(Q)$.*
(3) *There is a covering of $\partial\Omega$ by coordinate cylinders, Z, such that given a coordinate pair, (Z, φ), $Z^* \cap \partial\Omega_j$ is given, for each j, as the graph of a C^∞ function ϕ_j such that $\phi_i \to \varphi$ uniformly, $\|\nabla\phi_j\|_\infty \le \|\nabla\varphi\|_\infty$, and $\nabla\phi_i \to \nabla\varphi$ pointwise a.e. and in every $L^p(Z^* \cap \mathbb{R}^{d+1})$, $1 \le p < \infty$.*
(4) *There are positive functions $\omega_j : \partial\Omega \to \mathbb{R}_+$, which are bounded away from zero and infinity, uniformly in j, such that for any measurable set $E \subset \partial\Omega$, $\int_E \omega_j \, d\sigma = \int_{\Lambda_j(E)} d\sigma_j$, and that $\omega_j \to 1$ pointwise a.e. and in every $L^p(\partial\Omega)$, $i \le p < \infty$.*
(5) *The normal vectors to Ω_j, $\vec{N}(\Lambda_j(Q))$, converges pointwise a.e. and in every $L^p(\partial\Omega)$, $1 \le p < \infty$, to $\vec{N}(Q)$. An analogous statement holds for locally defined tangent vectors.*

Let $\|f\|_{L_1^p(\Omega)}$ be the scale-invariant norm

$$\|f\|_{L_1^p(\Omega)} = |\partial\Omega|^{-\frac{1}{d}} \|f\|_{L^p(\Omega)} + \|\nabla_\| f\|_{L^p(\Omega)}.$$

Let u be a function integrable over a bounded set E, then we denote

$$u_E = \fint_E u \, d\mathrm{vol}_E = \frac{1}{|E|} \int_E u \, d\mathrm{vol}_E.$$

We use the notation $u \to f$ non-tangentially (abbreviated by "n.t.") to mean that for a.e. $Q \in \partial V$, $\lim_{X\to Q} u(X) = f(Q)$, where the limit runs over $X \in \Gamma_i(Q)$.

Definition 1.6 An operator $\mathcal{L} = -\mathrm{div}(A\nabla)$ is said to have the De Giorgi-Nash local Hölder property if for any weak solution u to $\mathcal{L}u = 0$ in V, we have

$$|u(Y) - u(Z)| \le C \left(\frac{r}{R}\right)^{\alpha_0} \left(\fint_{B_R} u^2\right)^{1/2}, \qquad \forall Y, Z \in B_r, \tag{1.7}$$

and for some $0 < \alpha_0 = \alpha_0(\lambda, d, m) < 1$, and $0 < r < R < \delta(X)$.

It has been shown, in [15] for example, that Eq. 1.7 is equivalent to the following gradient estimate

$$\int_{B_\rho(X)} |\nabla u|^2 \le C \left(\frac{\rho}{r}\right)^{d-1+2\alpha_0} \int_{B_r(X)} |\nabla u|^2, \qquad 0 < \rho < r. \tag{1.8}$$

Estimates Eqs. 1.7 and 1.8 combined also imply the following Moser local boundedness estimate

$$\sup_{Y\in B} |u(Y)| \le C \left(\fint_{2B} |u|^2\right)^{1/2}, \tag{1.9}$$

whenever $B_{2r}(X) \subset V$.

Remark 1.10 When $m = 1$, De Giorgi ([13]), and Nash ([26]) independently established that solutions to the equation $\mathcal{L}u = -\text{div}(A\nabla u) = 0$, where A is assumed to be a real, bounded, symmetric and elliptic matrix, automatically satisfy Eq. 1.7. Morrey ([25]) later observed that this property (and many others) still holds even when symmetry is dropped.

Remark 1.11 We note that properties (1.7) and (1.9) are stable under small complex perturbation, as shown in Proposition 2.1 in [14]. Consequently, the same properties are stable with respect to small Carleson norm perturbations, see [3] for example.

For the rest of this paper, we will use the terminology *satisfying the "standard assumptions"* to refer to an operator $\mathcal{L} = -\text{div}(A\nabla)$ whose coefficients are real, bounded, measurable, and strongly elliptic, i.e. satisfying Eq. 1.2, and whose solutions to $\mathcal{L}u = 0$ satisfy the local Hölder condition (1.7).

For any point $(x, t) \in D_\varphi$, its Whitney box is defined to be

$$W(x,t) = \{(y,s) : |y-x| < t-\varphi(x), \frac{1}{2}(t-\varphi(x)) < s-\varphi(y) < \frac{3}{2}(t-\varphi(x))\}.$$

Given a measurable function $f : \mathbb{R}^{d+1}_+ \to \mathbb{R}^m$, consider

$$N_* f(x) = \sup_{(z,t)\in\Gamma(x)} |f(z,t)|$$

$$f_W(x,t) = \left(\iint_{W(x,t)} |f(y,s)|^2 \, dy\, ds\right)^{1/2}$$

and let the operators marked with a *tilde* stand for modifications of the functions above with f_W in lieu of f. For example,

$$\tilde{N}_* f(x) = \sup_{(z,t)\in\Gamma(x)} |f_W(z,t)|.$$

We remark here that the function $\tilde{N}_* f(x)$ and the following usual modified non-tangential maximal function

$$\tilde{N} f(x) = \sup_{\Gamma(x)} \left(\fint_{B((y,t),t/2)} |f|^2\right)^{1/2}$$

have equivalent L^2 norms. Similar definition can be made for a a bounded Lipschitz domain.

We are interested in the Dirichlet and regularity boundary value problems (BVPs) for $\mathcal{L}u = 0$ in a bounded Lipschitz domain Ω whose boundary $\partial\Omega$ is connected, with L^2 data. Specifically, we give the following definitions of solvability.

Definition 1.12 We say that the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u = f \in L^2(\partial\Omega, \mathbb{R}^m) & \text{n.t. on } \partial\Omega \end{cases} \tag{D_2}$$

is solvable, i.e. Eq. D_2 holds, if, whenever $f \in C(\partial\Omega, \mathbb{R}^m)$, there exists a solution u such that Eq. D_2 is satisfied, and $\|N_* u\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}$.

Definition 1.13 We say that the regularity problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u = f & \text{n.t. on } \partial\Omega \end{cases} \tag{R_2}$$

is solvable, i.e. Eq. R_2 holds, if, whenever $f \in C(\partial\Omega, \mathbb{R}^m) \cap L^2_1(\partial\Omega, \mathbb{R}^m)$, there exists a solution u such that Eq. R_2 is satisfied, and the estimate $\|\tilde{N}(\nabla u)\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2_1(\partial\Omega)}$ holds.

To differentiate the case of a bounded Lipschitz domain to that of the upper-half space, and by extension, the domain above a Lipschitz graph, we have the following definitions.

Definition 1.14 We say that the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \mathbb{R}^{d+1}_+ \\ \lim_{t\to 0} u = f \text{ n.t.} \\ \|N_* u\|_{L^2(\mathbb{R}^d)} < \infty \end{cases} \qquad (RD_2)$$

is solvable, i.e. Eq. RD_2 holds, if, whenever $f \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^m)$, there exists a solution $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{d+1}_+, \mathbb{R}^m)$ such that Eq. RD_2 is satisfied, and we have the following estimates

$$\|N_*(u)\|_{L^2(\mathbb{R}^d,\mathbb{R}^m)} \leq C\|f\|_{L^2(\mathbb{R}^d,\mathbb{R}^m)}, \qquad (1.15)$$

and

$$\left(\iint_{\mathbb{R}^{d+1}_+} |\nabla u(x,t)|^2\, t\, dx\, dt\right)^{1/2} \leq C\|f\|_2. \qquad (1.16)$$

Definition 1.17 We say that the regularity problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \mathbb{R}^{d+1}_+ \\ \lim_{t\to 0} u = f \text{ n.t} \\ \|\tilde{N}(\nabla u)\|_{L^2(\mathbb{R}^d,\mathbb{R})} < \infty \end{cases} \qquad (RR_2)$$

is solvable, i.e. Eq. RR_2 holds, if, whenever $f \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^m)$, there exists a solution $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^{d+1}_+, \mathbb{R}^m)$ such that Eq. RR_2 is satisfied, and

$$\|\tilde{N}(\nabla u)\|_{L^2(\mathbb{R}^d,\mathbb{R}^m)} \leq C\|\nabla_\| f\|_{L^2(\mathbb{R}^d,\mathbb{R}^m)}.$$

We now review some history in this area. Most of the results we are going to state are for single equations. Calderón studied BVPs for elliptic partial differential equations in a smooth domain in the late 1950's and early 1960's using symbolic calculus. He also pioneered the use of harmonic analysis techniques in solving these BVPs with the proof of the L^2 boundedness of the Cauchy operator on C^1 and Lipschitz curves with small Lipschitz constant in [1]. Coifman, McIntosh, and Meyer then removed the restriction on the Lipschitz constant in [6], paving the way for many works that follow. For the Laplacian, the solvability of Eqs. D_2 and R_2 was established by Dahlberg in [7], and by Jerison and Kenig in [19] respectively. Solvability of the same problems obtained through harmonic layer potentials using the result in [6] is due to Verchota in [32]. For A real, symmetric, and radially independent, solvability of these problems in the unit ball was established in [18] and [21]. However, the authors did not use layer potentials.

When A is not self-adjoint, solvability of Eq. D_2 was obtained in [9] for small, complex perturbation of constant elliptic matrices. Recently, in [2], Alfonseca et. al. used layer potentials to show that if A_0, A_1 are complex, elliptic, and t-independent, and if the solutions to $\mathcal{L}_0 u = 0$, $\mathcal{L}_0^* v = 0$ satisfy the De Giorgi-Nash estimate, then the solvability of the boundary value problems for $\mathcal{L}_0$ implies that for $\mathcal{L}_1$ on the upper-half space for data in L^2, provided that $\|A_1 - A_0\|_\infty < \varepsilon_0$ for some ε_0 small depending only on the parameters associated to

$\mathcal{L}_0$. Rosén then proved the same result for systems in [30] using functional calculus. Hofmann, Mitrea, and Morris ([17]) extended the perturbation result in [2] for data in other L^p, as well as showed perturbation results for BVPs with data in other spaces such as C^α, BMO.

We also remark that consideration of perturbation in Carleson norm is a natural one, since Caffarelli, Fabes, and Kenig observed in [5] that some regularity is necessary in the transversal direction. They showed that given any positive function $\omega(\tau)$ such that $\int_0^1 (\omega(\tau))^2\, d\tau/\tau = +\infty$, there exists a real, symmetric, elliptic matrix $A(x,t)$, whose modulus of continuity in the t-direction is controlled by ω, and for which the associated elliptic harmonic measure and the surface measure are mutually singular, i.e. the Dirichlet problem with data in L^p, $p > 1$ is not solvable. However, Fabes, Jerison, and Kenig showed in [9] that (D_2) holds, provided that the transverse modulus of continuity

$$\omega(\tau) \equiv \sup_{x\in\mathbb{R}^d, 0<t<\tau} |A(x,t) - A(x,0)| \tag{1.18}$$

satisfies the square Dini condition

$$\int_0^1 \frac{\omega^2(\tau)}{\tau}\, d\tau < +\infty, \tag{1.19}$$

and that $A(x,0)$ is sufficiently close to a constant matrix. Dahlberg ([8]) then introduced a scale-invariant version of the square Dini condition, which was further explored by Fefferman, Kenig, and Piper ([10]), as well as Kenig and Piper in [21, 22]. They proved that for real, symmetric operators $\mathcal{L}_1 = -\mathrm{div}(A_1\nabla)$ and $\mathcal{L}_0 = -\mathrm{div}(A_0\nabla)$, the solvability BVPs for $\mathcal{L}_0$ with data in L^p implies that of BVPs for $\mathcal{L}_1$ with data in (possibly some other) L^p under the assumption that

$$d\mu(x,t) = \left(\sup_{W(x,t)} |A_1 - A_0|\right)^2 \frac{dx\, dt}{t} \tag{1.20}$$

is a Carleson measure.

Definition 1.21 The modified Carleson norm of a function g in $\mathbb{R}^{d+1}_+$ is defined to be

$$\|g\|_C = \left(\sup_Q \frac{1}{|Q|} \iint_{Q\times(0,l(Q))} \sup_{W(x,t)} |g|^2 \frac{dx\, dt}{t}\right)^{1/2},$$

where Q is any cube in $\mathbb{R}^d$ and $l(Q)$ is its length.

Recently, in [16], the authors proved L^p solvability results on the upper half-plane for divergence form elliptic equations with complex, bounded coefficients that are small perturbations of t-independent coefficients, as measured by the Carleson measure norm condition, i.e. $\|A_0 - A_1\|_C$ is sufficiently small and A_0 is t-independent. They also proved the same results for other data spaces. The examples in [10] show that such smallness condition is absolutely essential in preserving solvability for $\mathcal{L}_1$.

In this paper, we will establish the following result.

Theorem 1.22 *Let Ω be a bounded Lipschitz domain with Lipschitz constant M. Consider $A, \bar{A}$ such that $A, \bar{A}$ are real, bounded measurable, and elliptic, and $\bar{A}$ is continuous on $\partial\Omega$, i.e. there exists a $\bar{\delta} > 0$ so that for any $P, Q \in \partial\Omega$, $|P-Q| < \bar{\delta}$ implies $|\bar{A}(P) - \bar{A}(Q)| < \varepsilon_0$, where ε_0 is a small constant depending on d, m, M, and the ellipticity of $\bar{A}$. Assume* *further that $\bar{A}$ satisfies the pseudo-symmetry condition* (1.3). *Let $\partial\Omega$ be covered by a finite number of coordinate pairs $(Z_j(Q_j, R), \varphi_j)$, for $1 \le j \le N$. For each coordinate pair $(Z_j(Q_j, R), \varphi_j)$ where $Q_j = (0, \varphi_j(0))$, define*

$$\varepsilon(x,t) = \sup_{W(x,t)} |A(Y) - \bar{A}(y, \varphi_j(y))|,$$

and

$$h_j(8R, Q_j) = \sup_{\Delta_r(P)\subseteq\Delta_{8R}(Q_j)} \frac{1}{|\Delta_r(P)|} \iint_{T_r(P)} \frac{\varepsilon^2(X)}{\delta(X)}\, dX.$$

Assume that the coordinate pairs satisfy the following conditions

(i) $\{(\frac{1}{8}Z_j(Q_j, R), \varphi_j), 1 \le j \le N\}$ *cover* $\partial\Omega$,

(ii) $R \le \dfrac{1}{2\sqrt{M^2+1}}\bar{\delta}$,

(iii) $h_j(8R, Q_j) < \dfrac{\varepsilon}{C^d}$, *where ε is another small constant depending on d, m, M, and the ellipticity of $\bar{A}$, and C is a constant depending only on the geometry of Ω.*

Then, (D_2) and (R_2) are solvable for A.

We note here that the main difference in our work compared to the aforementioned is that our BVPs are posed for bounded Lipschitz domains. Consequently, the bulk of our work revolves around localization arguments. We also note here that the assumption that $\bar{A}$ is continuous is essential as there are counterexamples for the Dirichlet and regularity problems in [20] and [23] respectively.

In the next sections, we will develop tools needed to prove this theorem. While Theorem 1.22 encompasses the single equation case, we will outline the steps showing the same result using different tools, which are only available when $m = 1$, in the last section.

We end this chapter with the following remark which explains how the results that are stated for the upper-half space $\mathbb{R}^{d+1}_+$, e.g. Theorem 1.1 and 7.1 in [30], can be generalized to the case of a domain above a Lipschitz graph with, of course, the additional dependence of the constants on the Lipschitz constant of the graph.

Remark 1.23 Let D_φ be the domain above the Lipschitz graph φ. Consider the pull back $\rho : \mathbb{R}^{d+1}_+ \to D_\varphi$ defined by $\rho(x,t) = (x, \varphi(x) + t)$. Given a function $\tilde{u} : D_\varphi \to \mathbb{C}$, its pull back $u = \tilde{u} \circ \rho$ is a function on $\mathbb{R}^{d+1}_+$. The chain rule gives $\nabla u = \rho^*(\nabla\tilde{u})$, where $\rho^*(f)(x)^\alpha = \underline{\rho}^t(x) f^\alpha(\rho(x))$, and $\underline{\rho}^t$ denotes the transpose of the Jacobian matrix $\underline{\rho}$. If $\tilde{u}$ satisfies the equation $\mathrm{div}(\tilde{A}\nabla\tilde{u}) = 0$ in D_φ, with coefficient $\tilde{A}$ being bounded, complex, accretive and t-independent, then u is a solution to the equation $\mathrm{div}(A\nabla u) = 0$, where

$$A(X) = |J(\rho)(X)|(\underline{\rho}(X))^{-1}\tilde{A}(\rho(X))(\underline{\rho}^t(X))^{-1},$$

and $J(\rho)$ is the Jacobian determinant of ρ. Observe that A satisfies the same conditions $\tilde{A}$ does. In addition, if the solutions $\tilde{u}$ to $\mathcal{L}_{\tilde{A}}\tilde{u}$ satisfy the estimates Eqs. 1.7 and 1.9 in D_φ then the solutions u to $\mathcal{L}u = 0$ satisfy the same estimates in $\mathbb{R}^{d+1}_+$.

Observe also that the Dirichlet and regularity condition $\tilde{u} \to \tilde{f}$ n.t. on ∂D is equivalent to the Dirichlet and regularity condition $u = f$ on $\mathbb{R}^{d+1}_+$ on $\mathbb{R}^d$, where $f = \tilde{f} \circ \rho$.

2 Constant Coefficients

Consider the domain above a Lipschitz graph $D = D_\varphi = \{(x,t) \in \mathbb{R}^{d+1} : t > \varphi(x)\}$. To our knowledge, the following result and its proof have not been presented in the literature.

Theorem 2.1 *Let $\mathcal{L} = -\mathrm{div}(A\nabla)$ be an operator whose matrix of coefficients A is real, constant, elliptic, i.e. satisfying Eq. 1.2, and pseudo-symmetric as in Eq. 1.3. Then, $\mathcal{S}_A : L^2(\partial D) \to \dot{L}^2_1(\partial D)$ is invertible, and (RD_2) and (RR_2) are solvable for $\mathcal{L}$ on D.*

Recall that $G_A = \dfrac{A+A^T}{2}$ is bounded, elliptic, and symmetric. Let $\vec{N}(Q) = (n_1(Q), \cdots, n_d(Q))$ be the outward unit normal vector on ∂D. Then, the conormal and modified conormal derivatives associated to A are defined to be

$$\left(\frac{\partial u}{\partial \nu_A}\right)^\alpha = (\partial_{\nu_A} u)^\alpha = n_i A^{\alpha\beta}_{i,j} \partial_j u^\beta, \qquad \text{and} \qquad \left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)^\alpha = n_i \frac{A^{\alpha\beta}_{i,j} + A^{\alpha\beta}_{j,i}}{2} \partial_j u^\beta$$

respectively.

Since A is constant, the fundamental matrix solution Γ_A associated to $\mathcal{L}$ exists. Note also that if u is a solution to $\mathcal{L}u = 0$, then for each $\alpha = 1, \cdots, M$,

$$\begin{aligned} 0 = (\mathrm{div}(A\nabla u))^\alpha &= \partial_i (A^{\alpha\beta}_{i,j} \partial_j u^\beta) = A^{\alpha\beta}_{i,j} \partial_{ij} u^\beta = A^{\alpha\beta}_{j,i} \partial_{ji} u^\beta \\ &= (\mathrm{div}(A^T \nabla u))^\alpha = \left[\mathrm{div}\left(\frac{1}{2}(A + A^T)\nabla u\right)\right]^\alpha = (\mathrm{div}(G\nabla u))^\alpha. \end{aligned}$$

This means that $\mathcal{L}_A = -\mathrm{div}(A\nabla)$, $\mathcal{L}_{A^T} = -\mathrm{div}(A^T\nabla)$ and $\mathcal{L}_G = -\mathrm{div}(G\nabla)$ share the same matrix of fundamental solutions $\Gamma_A = \Gamma_{A^T} = \Gamma_G$. Since $G^* = G$, we have $\Gamma_A = \Gamma_{A^T} = \Gamma_G = \Gamma_{G^*} = \Gamma_{A^*} = \Gamma_{(A^*)^T}$. Furthermore, $\Gamma_A(X, 0)$ is even and homogeneous of degree $1 - d$ in X, and we have the following additional properties.

$$\begin{aligned} \Gamma^{\alpha\beta}_A(X,Y) &= \Gamma^{\beta\alpha}_A(X,Y) \\ |\nabla^N_X \Gamma_A(X,Y)| &\le C|X-Y|^{1-d-N} \\ \frac{\partial}{\partial x_i}\Gamma_A(X,Y) &= -\frac{\partial}{\partial y_i}\Gamma_A(X,Y) \end{aligned}$$

for all integers $N \ge 0$, $1 \le \alpha, \beta \le m$, and C depending only on d, m, λ, N.

For $f \in L^p(\partial D, \mathbb{R}^m)$, the single layer potential $\mathcal{S}(f) = \mathcal{S}_A(f) = (u^1, \cdots, u^m)$ is defined by

$$u^\alpha(X) = \int_{\partial D} \Gamma^{\alpha\beta}_A(X,Q) f^\beta(Q)\, d\sigma(Q),$$

and the modified double layer potential $\tilde{\mathcal{D}}(f) = \tilde{\mathcal{D}}_A(f) = (w^1, \cdots, w^m)$, is defined by

$$\begin{aligned} w^\alpha(X) &= \int_{\partial D} \left(\frac{\partial}{\partial \tilde{\nu}_{A^*}} \Gamma^\alpha_{A^*}(Q,X)\right)^\gamma f^\gamma(Q)\, d\sigma(Q) \\ &= \frac{1}{2}\int_{\partial D} n_i(Q)\left(A^{\beta\gamma}_{i,j} + (A^T)^{\beta\gamma}_{i,j}\right) \frac{\partial}{\partial q_j}\Gamma^{\alpha\beta}_A(X,Q) f^\gamma(Q)\, d\sigma(Q). \end{aligned}$$

Clearly, $\mathcal{S}(f)$ and $\tilde{\mathcal{D}}(f)$ are both solutions to $\mathcal{L}u = 0$ in $\mathbb{R}^{d+1} \setminus \partial D$. Furthermore, as A is constant, $\mathcal{S}(f)$, $N_*(\nabla(\mathcal{S}(f)))$, $N_*(\tilde{\mathcal{D}}f)$ also belong to $L^p(\partial D, \mathbb{R}^m)$, and their norms are bounded by $C_p\|f\|_{L^p}$, where C_p depends only on d, m, λ, p, and the Lipschitz constant of D.

Proof of Theorem 2.2 It suffices to show that $\mathcal{S}_A : L^2(\partial D) \to \dot{L}^2_1(\partial D)$ is invertible.

Let $u = \mathcal{S}_A(f)$ for some $f \in L^2(\partial D, \mathbb{R}^m)$. Since A is constant, we get the following trace formula for almost every $P \in \partial D$ (see [24])

$$\left(\frac{\partial u^\alpha}{\partial x_i}\right)_\pm (P) = \pm\frac{1}{2} n_i(P) b^{\alpha\beta}(P) f^\beta(P) + \text{p.v.} \int_{\partial D} \frac{\partial}{\partial p_i} \Gamma_A^{\alpha\beta}(P, Q) f^\beta(Q)\, d\sigma(Q),$$

where $(b^{\alpha\beta}(P))^{m\times m}$ is the inverse matrix of $(A^{\alpha\beta}_{i,j} n_j(P) n_i(P))^{m\times m} = \left\langle A\vec{N}, \vec{N}\right\rangle^{m\times m}$. It follows that $\|\nabla_T(\mathcal{S}_A f)\|_2 \le C\|f\|_2$, where C depends on d, m, λ and the Lipschitz constant of D. Consequently

$$\mathcal{S}_A : L^2(\partial D) \to \dot{L}^2_1(\partial D)$$

is bounded. Furthermore,

$$n_j \left(\frac{\partial u^\alpha}{\partial x_i}\right)_+ - n_i \left(\frac{\partial u^\alpha}{\partial x_j}\right)_+ = n_j \left(\frac{\partial u^\alpha}{\partial x_i}\right)_- - n_i \left(\frac{\partial u^\alpha}{\partial x_j}\right)_-,$$

meaning $(\nabla_T u)_+ = (\nabla_T u)_-$ on ∂D. Moreover,

$$\begin{aligned}\left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)^\alpha_\pm (P) &= \frac{1}{2} n_i (A^{\alpha\beta}_{i,j} + A^{\alpha\beta}_{j,i}) \left(\frac{\partial u^\beta}{\partial x_j}\right)_\pm \\ &= n_i(P) G^{\alpha\beta}_{i,j} \Bigg(\pm\frac{1}{2} n_j(P) b^{\beta\gamma} f^\gamma(P) \\ &\qquad + \text{p.v.} \int_{\partial D} \frac{\partial}{\partial p_j} \Gamma_A^{\beta\gamma}(P, Q) f^\gamma(Q)\, d\sigma(Q)\Bigg) \\ &= \pm\frac{1}{2} f^\alpha(P) + \text{p.v.} \int_{\partial D} \frac{1}{2} n_i(P)(A^{\alpha\beta}_{i,j} + A^{\alpha\beta}_{j,i}) \frac{\partial}{\partial p_j} \Gamma_A^{\beta\gamma}(P, Q) f^\gamma(Q)\, d\sigma(Q)\end{aligned}$$

since $(n_i(P) G^{\alpha\beta}_{i,j} n_j(P))_{m\times m} = (A^{\alpha\beta}_{i,j} n_i(P) n_j(P))_{m\times m}$ by properties of the (real) inner product.

Thus, $\left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)_\pm = (\pm\frac{1}{2} I + \mathcal{K}_A)(f)$, where

$$(\mathcal{K}_A(f)(P))^\alpha = \text{p.v.} \int_{\partial D} K_A^{\alpha\beta}(P, Q) f^\beta(Q)\, d\sigma(Q),$$

and

$$\begin{aligned} K_A^{\alpha\beta}(P, Q) &= \frac{1}{2} n_i(P)(A^{\alpha\gamma}_{i,j} + A^{\alpha\gamma}_{j,i}) \frac{\partial}{\partial p_j} \Gamma_A^{\gamma\beta}(P, Q) \\ &= n_i G^{\alpha\gamma}_{i,j} \frac{\partial}{\partial p_j} \Gamma_A^{\gamma\beta}(P, Q).\end{aligned}$$

From this, we obtain $\|\mathcal{K}_A(f)\|_2 \le C\|f\|_2$, where C depends on d, m, λ, and the Lipschitz constant of D. (See [11], for example.) We also have the following jump relation

$$f = \left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)_+ - \left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)_-.$$

Assume that u is a solution to $\mathcal{L}_A u = 0$ in D. Then, u is also a solution to $\mathcal{L}_G u = 0$ in D. Let e_{d+1} be the unit vector in the x_{d+1} direction, we have

$$\begin{aligned}\text{div}(e_{d+1}\langle G\nabla u(X), \nabla u(X)\rangle) &= 2\langle \partial_t \nabla u(X), G\nabla u(X)\rangle \\ &= 2\text{div}(\partial_t u\, G\nabla u(X)).\end{aligned}$$

Thus, the divergence theorem gives

$$\int_{\partial D} \langle e_{d+1}, \vec{N}\rangle \langle G\nabla u(Q), \nabla u(Q)\rangle \, d\sigma(Q) = 2\int_{\partial D} \langle e_{d+1}, \nabla u(Q)\rangle \langle G\nabla u(Q), \vec{N}\rangle \, d\sigma(Q).$$

By ellipticity of G and the fact that $1 \geq \langle e_{d+1}, \vec{N}\rangle \geq C > 0$, where C depends only on the Lipschitz constant of D, we have

$$\int_{\partial D} |\nabla u|^2 \, d\sigma(Q) \leq C\|\partial_t u\|_2 \left\| \frac{\partial u}{\partial \nu_G} \right\|_2 = C\|\partial_t u\|_2 \left\| \frac{\partial u}{\partial \tilde{\nu}_A} \right\|_2,$$

whence by Cauchy inequality with an ε, we have

$$\|\nabla_T u\|_2 \leq \|\nabla u\|_2 \leq C\|\partial_{\tilde{\nu}_A} u\|_2.$$

Now, note that

$$\begin{aligned}\int_{\partial D} \langle e_{d+1}, \vec{N}\rangle \langle G\nabla u, \nabla u\rangle \, d\sigma &= 2\int_{\partial D} \Big[\langle e_{d+1}, \vec{N}\rangle \langle G\nabla u, \nabla u\rangle - \langle e_{d+1}, \nabla u\rangle \langle G\nabla u, \vec{N}\rangle\Big] \, d\sigma \\ &= 2\int_{\partial D} \Big\langle \nabla u, \langle e_{d+1}, \vec{N}\rangle G\nabla u - \langle G\nabla u, \vec{N}\rangle e_{d+1}\Big\rangle \, d\sigma.\end{aligned}$$

Since $\Big\langle \langle e_{d+1}, \vec{N}\rangle G\nabla u - \langle G\nabla u, \vec{N}\rangle e_{d+1}, \vec{N}\Big\rangle = 0$, $\langle e_{d+1}, \vec{N}\rangle G\nabla u - \langle G\nabla u, \vec{N}\rangle e_{d+1}$ is tangential to ∂D. Again, the fact that $\langle e_{d+1}, \vec{N}\rangle$ together with ellipticity of G imply

$$\|\nabla u\|_2 \leq C\|\nabla_T u\|_2,$$

which means $\|\partial_{\tilde{\nu}_A} u\|_2 \leq C\|\nabla_T u\|_2$. Thus, $\|\partial_{\tilde{\nu}_A} u\|_2 \approx \|\nabla_T u\|_2$. The same comparability holds in $D_- = \{(x,t) \in \mathbb{R}^{d+1} : t < \varphi(x)\}$. We now apply these relationships to $u = \mathcal{S}_A f$ to get

$$\begin{aligned}\|f\|_2 &= \left\| \left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)_+ - \left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)_- \right\|_2 \leq \left\| \left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)_+ \right\|_2 + \left\| \left(\frac{\partial u}{\partial \tilde{\nu}_A}\right)_- \right\|_2 \\ &\approx \|(\nabla_T u)_+\|_2 + \|(\nabla_T u)_-\|_2 \approx \|\nabla_T(\mathcal{S}_A f)\|_2.\end{aligned}$$

Hence, $\mathcal{S}_A : L^2(\partial D, \mathbb{R}^m) \to \dot{L}^2_1(\partial D, \mathbb{R}^m)$ is one-to-one.

For $0 \leq s \leq 1$, consider the operators $\mathcal{L}_s = -\mathrm{div}(A_s\nabla)$, where $A_s = sA + (1-s)I$, and I the identity operator. Then, A_s is constant for each $s \in [0,1]$. Furthermore, the ellipticity constants of A_s is uniformly controlled. Hence, by the preceding argument, $\mathcal{S}_{A_s} : L^2(\partial D) \to \dot{L}^2_1(\partial D)$ is bounded uniformly in s. Also, $\mathcal{S}_{A_s}$ is one-to-one for each $s \in [0,1]$. Note that $\mathcal{L}_0 = \Delta$ so by [12], $\mathcal{S}_{A_0}$ is invertible. Lastly, for each s, $\mathcal{L}_s$ satisfies the standard assumptions, and $\|A_s - A_0\|_\infty = s\|A - I\|_\infty$. Hence, by Theorem 7.1 in [30], $\mathcal{S}_{A_s}$ is invertible for $0 < s < \varepsilon_0/\lambda'$ where λ' is the uniform control for the ellipticity constants of A_s. We then obtain the invertibility of $\mathcal{S}_{A_1} = \mathcal{S}_A$ by iterating this process a finite number of times. □

3 Proof of The Main Result

In this section, we will be proving the results needed in the proof of Theorem 1.22. We remark here that the proof uses some of the results in [30] and [17]. We first remark some of the implications of Theorem 7.1 in [30] that are useful to us here. Note that Rosén uses the notation $\nabla_A u = \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_\| u \end{bmatrix}$, and

$$\mathcal{S}_t^A f = \int_{\mathbb{R}^d} \Gamma_A^{\alpha\beta}(X, (y, 0)) f^\beta(y)\, dy$$

.

Remark 3.1 Since $A \to \nabla_A \mathcal{S}_t^A$ is a holomorphic map and $\nabla_A \mathcal{S}_t^A$ depends locally Lipschitz continuously on A, we have

$$\|\nabla \mathcal{S}_t^A|_{t=0} f - \nabla \mathcal{S}_t^{A'}|_{t=0} f\|_2 \le C\|A - A'\|_\infty \|f\|_2.$$

Secondly,

$$\begin{aligned} |\nabla \mathcal{S}_t^A f|^2 &= |\nabla_\| \mathcal{S}_t^A f|^2 + |\partial_t \mathcal{S}_t^A f|^2 \\ &\le C\left(|\nabla_\| \mathcal{S}_t^A f|^2 + |\partial_{\nu_A} \mathcal{S}_t^A f|^2\right) \end{aligned}$$

where $C = C(\lambda, \|A\|_\infty)$. As a result, the estimate $\|\tilde{N}(\nabla_A \mathcal{S}_t^A f)\|_2 \le C\|f\|_2$ from this theorem gives $\|\tilde{N}(\nabla \mathcal{S}_t^A f)\|_2 \le C\|f\|_2$, uniformly in t. By the same reasoning, $\sup_{t>0} \|\nabla \mathcal{S}_t^A f\|_2 \le C\|f\|_2$.

Next, we state a result that can be obtained from Theorem 7.1 in [30] via a quick argument.

Theorem 3.2 *Suppose that $\mathcal{L}_0 = -\mathrm{div}(A^0\nabla)$ and $\mathcal{L}_1 = -\mathrm{div}(A^1\nabla)$ are operators whose coefficients A^0, A^1 are real, bounded measurable, elliptic, and t-independent. Suppose further that solutions to $\mathcal{L}_0 u = 0$ and $\mathcal{L}_0^* w = 0$ satisfy the local Hölder boundedness condition (1.7). Assume also that $\mathcal{S}_0^{A^0}$ and $\mathcal{S}_0^{A^{0,*}} : L^2(\mathbb{R}^d) \to \dot{L}_1^2(\mathbb{R}^d)$ are invertible. Then, there exists an $\varepsilon_0 > 0$ depending on d, m, λ, and the constants associated to $\mathcal{L}_0$ and $\mathcal{L}_0^*$ such that (RD_2) and (RR_2) hold for $\mathcal{L}_1$ and $\mathcal{L}_1^*$ provided that*

$$\|A^0 - A^1\|_\infty < \varepsilon_0.$$

Proof First, we note that since $\|A^0 - A^1\|_\infty < \varepsilon_0$, solutions to $\mathcal{L}_1 u = 0$ and $\mathcal{L}_1^* w = 0$ also satisfy the estimate (1.7). Furthermore, the real ellipticity condition (1.2) implies the accretiveness condition used in [30].

From Remark 3.1 , we have that $\mathcal{S}_t^{A^1} : L^2(\mathbb{R}^d) \to \dot{L}_1^2(\mathbb{R}^d)$ are bounded, uniformly in t, and

$$\|\nabla \mathcal{S}_t^A|_{t=0} f - \nabla \mathcal{S}_t^{A'}|_{t=0} f\|_2 \le C\|A^0 - A^1\|_\infty \|f\|_2,$$

from which the invertibility of $\mathcal{S}_0^{A^1} = \mathcal{S}_t^{A^1}|_{t=0} : L^2 \to \dot{L}_1^2$ follows by the method of continuity, i.e. an argument similar to that used at the end of the proof of Theorem 2.1.

Similarly, $\mathcal{S}_0^{A^{1,*}} : L^2 \to \dot{L}_1^2$ is invertible.

Now, consider $f \in \dot{L}_1^2(\mathbb{R}^d)$, and set $u(x, t) = \mathcal{S}_t^{A^1}\left[\left((\mathcal{S}_0^{A^1})^{-1} f\right)(x)\right]$. Then, $\mathcal{L}_1 u = 0$. Again, by Remark 3.1, $\|\tilde{N}(\nabla u)\|_2 \le C\left\|(\mathcal{S}_0^{A^1})^{-1} f\right\|_2 \le C\|\nabla_\| f\|_2$. By Theorem 4.3 in

[2], which still holds for systems since the argument is exactly the same, $u \to f$ n.t. Thus, (RR_2) holds for $\mathcal{L}_1$.

Similar argument yields solvability of (RD_2) for $\mathcal{L}_1$. □

Then, given the results in [30], we also observe that Theorem 1.12 in [16] still holds for systems satisfying the same assumptions.

By Remark 1.23, many results, such as Theorem 7.1 in [30], and Theorems 1.12 and 1.35 in [16], still hold under the domain above a Lipschitz graph, D_φ setting.

Before we proceed with the proof of Theorem 1.22, we need the following two results. We first state the result on boundedness of the single layer potential on bounded Lipschitz domains.

Lemma 3.3 *Assume the hypotheses of Theorem 1.22. Let $\mathcal{S}_{A,\Omega}(f)$ be the single layer potential associated to $\mathcal{L}$ on Ω. Then, $\|\tilde{N}(\nabla \mathcal{S}_{A,\Omega} f)\|_2 \lesssim \|f\|_2$.*

Proof Note that from the comment following the definition of a bounded Lipschitz domain, we can choose the coordinate pairs so that $(8Z_j(Q_j, R), \varphi_j)$ is still a coordinate pair for each j, i.e.

$$8Z_j(Q_j, R) \cap \Omega = 8Z_j(Q_j, R) \cap \{(x, t) : t > \varphi_j(x)\}.$$

In other words, $T_{8R}(Q_j) \subset 8Z_j(Q_j, R)$.

Pick one of these Q_j, i.e. in the coordinate pair $(Z_j(Q_j, R), \varphi_j)$, $Q_j = (0, \varphi_j(0))$, and consider the domain

$$D_j = \{(x, t) : t > \varphi_j(x)\}.$$

Let $\theta \in C_0^\infty(\mathbb{R}^d)$ be such that $0 \le \theta \le 1, \theta(y) \equiv 1$ if $y \in \Delta_R(0)$ and $\theta(y) \equiv 0$ if $y \in [\Delta_{2R}(0)]^C$. Define

$$A_1(Y) = A_1(y, s) = \theta(y)\bar{A}(y, \varphi_j(y)) + (1 - \theta(y))\bar{A}(Q_j).$$

It is clear that A_1 is real, bounded, measurable, elliptic, and is independent in the vertical direction. In addition, ellipticity of A_1 implies accretiveness as defined in [30]. Observe also that

$$\|A_1 - \bar{A}(Q_j)\|_\infty = \theta(y)\|\bar{A}(y, \varphi_j(y)) - \bar{A}(Q_j)\|_\infty < \varepsilon_0$$

since in $\operatorname{supp}\theta = \Delta_{2r}$,

$$|(y, \varphi_j(y)) - Q_j| = \sqrt{|y|^2 + (\varphi_j(y) - \varphi_j(0))^2} \le \sqrt{1 + M^2}|y| < 2r\sqrt{1 + M^2} < \bar{\delta}.$$

Since $\bar{A}(Q_j)$ is constant, the solutions to $\mathcal{L}_{\bar{A}(Q_j)}u = 0$ satisfy Eq. 1.7 and Eq. 1.9, and so do solutions to $\mathcal{L}_{A_1}u = 0$. Thus, by Theorem 7.1 in [30], $\|\tilde{N}(\nabla \mathcal{S}_{A_1} f)\|_2 \lesssim \|f\|_2$.

Let $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi(s) \equiv 1$ if $0 < s < R$ and $\psi(s) \equiv 0$ if $s \ge 2R$, and define

$$A_2(Y) = A_2(y, s) = \psi(s - \varphi_j(y))[\theta(y)A(Y) + (1 - \theta(y))\bar{A}(Q_j)] + [1 - \psi(s - \varphi_j(y))]A_1(Y).$$

Then, we have

$$A_2(Y) - A_1(Y) = \psi(s - \varphi_j(y))\theta(y)[A(Y) - \bar{A}(y, \varphi_j(y))].$$

We will proceed to show that $\|A_2 - A_1\|_C$ is small. Define

$$b(r, x_0) = \frac{1}{|\Delta_r(x_0)|} \iint_{T_r(x_0)} \frac{\varepsilon'^2(X)}{\delta(X)}\, dX,$$

where $\varepsilon'(X) = \sup_{W(x,t)} |A_2(Y) - A_1(Y)|$.

Consider any $X = (x, t) \in \mathbb{R}^{d+1}_+$, and any $Y = (y, s) \in W(x, t)$. Then, if $X \in [T_{6R}(0)]^C$, we have

$$s - \varphi_j(y) > \frac{1}{2}(t - \varphi_j(x)) \geq 3R.$$

This means $A_2(Y) - A_1(Y) = 0$, which implies $\varepsilon'(X) = 0$. Thus, for any $x_0 \in \mathbb{R}^d$, and $r > 0$

$$b(r, x_0) = \frac{1}{|\Delta_r(x_0)|} \int_{T_{6R}(0) \cap T_r(x_0)} \frac{\varepsilon'^2(X)}{\delta(X)}\, dX.$$

We now look at the following cases:

- $r \geq 2R$: For any $x_0 \in \mathbb{R}^d$, we have

$$b(r, x_0) \leq \frac{C^d}{|\Delta_{6R}(0)|} \int_{T_{6R}(0)} \frac{\varepsilon^2(X)}{\delta(X)}\, dX \leq C^d h(8R, 0),$$

  where ε and h were defined in the statement of Theorem 1.22.
- $r < 2R$: Here we look at the following possibilities:
  - $|x_0| \geq 6R$: For any $(x, t) \in T_r(x_0)$, if $Y = (y, s) \in W(x, t)$ then $|y| > 2R$, which means $\varepsilon'(x, t) = 0$, and so $b(r, x_0) = 0$.
  - $|x_0| < 6R$: In this case, we see that $T_r(x_0) \subset T_{8R}(0)$ so

$$b(r, x_0) \leq h(8R, 0).$$

Consequently, $\|A_2 - A_1\|_C \leq C^d \sup_{r>0, x_0 \in \mathbb{R}^d} b(r, x_0) \leq C^d h(8R, 0) < \varepsilon$. By Theorem 1.12 in [17], $\|\tilde{N}(\nabla \mathcal{S}_{A_2} f)\|_2 \lesssim \|f\|_2$. Since $A_2 = A$ in T_{2R}, the desired result then follows from a partition of unity and rotation of coordinate systems. □

Next, we state and prove a lemma on localization of the regularity problem (see [24] e.g.)

Lemma 3.4 *Assume the hypotheses and notations of Lemma 3.3. Further assume that* $(RR)_2$ *and* $(RD)_2$ *are solvable for* $\mathcal{L}_2$ *and* $\mathcal{L}_2^*$ *on* D_j*, where* $\mathcal{L}_2 = -\mathrm{div}(A_2 \nabla)$*. Then, for* $r < R/8$, $f \in L^2_1(\Delta_{4r}(Q_j))$*, and* $u = \mathcal{S}_{A,\Omega}(f)(X)$*, we have*

$$\int_{\Delta(Q_j, r)} |\partial_\nu u|^2\, d\sigma \leq \frac{C}{r} \int_{T(Q_j, 2r)} |\nabla u|^2\, dX + C \int_{\Delta(Q_j, 2r)} |\nabla_T f|^2\, d\sigma.$$

Proof For ease of notation, we will denote $\Delta(Q_j, r)$ by Δ_r and $T(Q_j, r)$ by T_r.

From Lemma 3.3, we have $\|\tilde{N}(\nabla u)\|_2 \lesssim \|f\|_2$. Thus, by Theorem 4.3 in [2], $\partial_\nu u$ exists.

Consider two cutoff functions $\varphi, \eta \in C_0^\infty(D_j)$ such that $\varphi \equiv 1$ on $T_{3r/2} \cup \Delta_{3r/2}$, $\varphi \equiv 0$ on $[T_{2r} \cup \Delta_{2r}]^C$, $|\nabla \varphi| \leq C/r$, and $\eta \equiv 1$ on $T_{2r} \cup \Delta_{2r}$ so that $\eta\varphi \equiv \varphi$. Let v be the solution to the regularity problem for $\mathcal{L}_2$ on D_j with data ηf. Let $w = \varphi(u - v)$. Then, on Δ_{2r}, we have

$$w = \varphi(f - f\eta) = f(\varphi - \eta\varphi) = 0,$$

whence $w \equiv 0$ on ∂D_j. Thus, by properties of the Green's function, we have

$$w(X) = \int_{D_j} G_{A_2^*}(Y, X)\mathcal{L}_2 w(Y)\, dY = \int_{D_j} G_{A_2}(X, Y)\mathcal{L}_2 w(Y)\, dY.$$

But

$$\begin{aligned}\mathcal{L}_2 w &= -\mathrm{div}(A_2\nabla(\varphi(u-v))) = -\mathrm{div}(A_2(\nabla\varphi)(u-v)) - \mathrm{div}(\varphi A_2\nabla(u-v))\\ &= -\mathrm{div}(A_2(\nabla\varphi)(u-v)) - A_2\nabla(u-v)\nabla\varphi - \varphi\mathrm{div}(A_2\nabla(u-v)).\end{aligned}$$

Recall that $A_2 = A$ on T_R. Since, $\operatorname{supp}\varphi \subset (T_{2r} \cup \Delta_{2r}) \subsetneq (T_R \cup \Delta_R)$, the third term vanishes to give

$$\mathcal{L}_2 w = -\mathrm{div}(A_2(\nabla\varphi)(u-v)) - A_2\nabla(u-v)\nabla\varphi.$$

Since the Green's function vanishes on the boundary, this means

$$\begin{aligned}w(X) &= -\int_{D_j} G_{A_2}(X, Y)\mathrm{div}(A_2(\nabla\varphi)(u-v))(Y)\, dY - \int_{D_j} G_{A_2}(X, Y)A_2\nabla(u-v)\nabla\varphi\, dY\\ &= \int_{D_j} A_2(Y)(\nabla\varphi)(Y)(u-v)(Y)\nabla_Y G_{A_2}(X, Y)\, dY\\ &\quad - \int_{D_j} G_{A_2}(X, Y)A_2(Y)\nabla(u-v)(Y)\nabla\varphi(Y)\, dY.\end{aligned}$$

Now, consider $h \in L^2(\Delta_{2r})$ with $\operatorname{supp} h \subset \Delta_{2r}$, and let Ψ be the solution to the Dirichlet problem for $\mathcal{L}_2^*$ in D_j with datum h. Then, by changing the order of integration, we get

$$\begin{aligned}\int_{\partial D_j} \frac{\partial w}{\partial \nu_{A_2}}(x)h(x)\, dx &= \int_{D_j} (u-v)A_2(Y)\nabla\varphi(Y)\nabla_Y\left(\int_{\partial D_j}\frac{\partial}{\partial\nu}G_{A_2}(x, Y)h(x)\, dx\right) dY\\ &\quad - \int_{D_j} A_2(Y)\nabla(u-v)(Y)\nabla\varphi(Y)\left(\int_{\partial D_j}\frac{\partial}{\partial\nu}G_{A_2}(x, Y)h(x)\, dx\right) dY\\ &= \int_{D_j} A_2(Y)(u-v)\nabla\varphi(Y)\nabla\Psi(Y)\, dY\\ &\quad - \int_{D_j} A_2(Y)\nabla(u-v)(Y)\nabla\varphi(Y)\Psi(Y)\, dY = I + II.\end{aligned}$$

We now estimate each term. For II, since $\operatorname{supp}\nabla\varphi \subset T_{2r} \setminus T_{3r/2} \cup (\Delta_{2r} \setminus \Delta_{3r/2})$, and $|\nabla\varphi| \le C/r$, we have

$$\begin{aligned}II &\le Cr^{-1}\left(\int_{T_{2r}} |\nabla(u-v)(Y)|^2\, dY\right)^{1/2}\left(\int_{T_{2r}} |\Psi(Y)|^2\, dY\right)^{1/2}\\ &\le Cr^{-1}\left(\int_{T_{2r}} |\nabla(u-v)(Y)|^2\, dY\right)^{1/2}\left(\int_{T_{2r}} |N_*\Psi(x)|^2\, ds\, dy\right)^{1/2}\\ &\le Cr^{-1/2}\left(\int_{T_{2r}} |\nabla(u-v)(Y)|^2\, dY\right)^{1/2}\|h\|_{L^2(\Delta_{2r})},\end{aligned}$$

where we have used pointwise estimate in the second inequality, and the fact that Ψ is a solution to the Dirichlet problem with datum h to get the third.

To estimate I, let $Y = (y, s)$. Again, since $\operatorname{supp} \nabla\varphi \subset T_{2r} \setminus T_{3r/2} \cup (\Delta_{2r} \setminus \Delta_{3r/2})$, Schwarz inequality gives

$$\begin{aligned} I &= \int_{D_j} A_2(Y)\nabla\varphi(Y)\frac{(u-v)(Y)}{\delta(Y)}\nabla\Psi(Y)\delta(Y)\,dY \\ &\leq Cr^{-1}\left(\int_{T_{2r}}\frac{|(u-v)(Y)|^2}{\delta^2(Y)}\,dY\right)^{1/2}\left(\int_{T_{2r}} s^2|\nabla\Psi(y,s)|^2\,dy\,ds\right)^{1/2} \\ &\leq Cr^{-1/2}\left(\int_{T_{2r}}\frac{|(u-v)(Y)|^2}{\delta^2(Y)}\,dY\right)^{1/2}\left(\int_0^\infty\int_{\mathbb{R}^d} s|\nabla\Psi(y,s)|^2\,dy\,ds\right)^{1/2} \\ &\leq Cr^{-1/2}\left(\int_{T_{2r}}|\nabla(u-v)(Y)|^2\,dY\right)^{1/2}\|h\|_{L^2(\Delta_{2r})}, \end{aligned}$$

where we have used the version of Hardy's inequality that was shown in [4] since on Δ_{2r}, $u - v = f - \eta f = 0$, as well as the solvability of $(RD)_2$ for $\mathcal{L}_2^*$. Since the choice for h was arbitrary, we obtain the estimate

$$\left\|\frac{\partial w}{\partial\nu_{A_2}}\right\|_{L^2(\Delta_{2r})} \leq Cr^{-1/2}\left(\int_{T_{2r}}|\nabla(u-v)(Y)|^2\,dY\right)^{1/2}$$

by duality. Observe that

$$\frac{\partial w}{\partial\nu_{A_2}} = \frac{\partial}{\partial\nu_{A_2}}[\varphi(u-v)] = \varphi\frac{\partial(u-v)}{\partial\nu_{A_2}} + (u-v)\frac{\partial\varphi}{\partial\nu_{A_2}} = \varphi\frac{\partial(u-v)}{\partial\nu}$$

since $u - v \equiv 0$ on Δ_{2r}, $\partial_{\nu_{A_2}}\varphi \equiv 0$ on $[\Delta_{2r}]^C$, and $A = A_2$ on T_R. Hence,

$$\begin{aligned} \left\|\frac{\partial u}{\partial\nu}\right\|_{L^2(\Delta_r)} &\leq \left\|\frac{\partial(u-v)}{\partial\nu}\right\|_{L^2(\Delta_r)} + \left\|\frac{\partial v}{\partial\nu}\right\|_{L^2(\Delta_r)} \\ &\leq \left\|\varphi\frac{\partial(u-v)}{\partial\nu}\right\|_{L^2(\Delta_{2r})} + \left\|\frac{\partial v}{\partial\nu}\right\|_{L^2(\Delta_{2r})} \\ &= \left\|\frac{\partial w}{\partial\nu_{A_2}}\right\|_{L^2(\Delta_{2r})} + \left\|\frac{\partial v}{\partial\nu}\right\|_{L^2(\Delta_{2r})} \\ &\leq Cr^{-1/2}\left(\int_{T_{2r}}|\nabla(u-v)(Y)|^2\,dY\right)^{1/2} + C\,\|\nabla_T f\|_{L^2(\Delta_{2r})}, \end{aligned}$$

where the last bound for $\dfrac{\partial v}{\partial\nu}$ comes from the fact that v is a solution to the regularity problem for L_2 on D_j. □

We are ready to present the proof of the main theorem.

Proof of Theorem 1.22 It suffices to show that $\mathcal{S}_{A,\Omega} : L^2(\partial\Omega) \to L^2_1(\partial\Omega)$ is invertible.

Note that if we denote by $\cdot^*$ quantities involving the adjoint operators, then we have $h_j^*(8R, Q_j) < \frac{\varepsilon}{C^d}$.

We continue to use the same notations in the proof of Lemma 3.3 in this proof. Recall from there that for each coordinate pair $(Z_j(Q_j, R), \varphi_j)$, $\|A_1 - \bar{A}(Q_j)\|_\infty < \varepsilon_0$. Since $\bar{A}(Q_j)$ is constant, elliptic, and pseudo-symmetric, $\mathcal{S}_{\bar{A}(Q_j)} : L^2(\partial D_j) \to \dot{L}^2_1(\partial D_j)$ is invertible by Theorem 2.1. The same result holds for $\mathcal{S}_{\bar{A}(Q_j)^*}$. By Theorem 3.2,

(RD_2), (RR_2) are solvable for $\mathcal{L}_1 = -\mathrm{div}(A_1\nabla)$ on D_j. Analogously, (RD_2), (RR_2) are solvable for $\mathcal{L}_1^*$ on D_j.

By Theorem 1.35 in [16], (RD_2) and (RR_2) hold for $\mathcal{L}_2$, $\mathcal{L}_2^*$ in D_j. By Lemma 3.4, we obtain the following.

$$\int_{\Delta_{R/8}(Q_j)} |\partial_\nu u|^2\, d\sigma \le C \int_{\Delta_{3R/8}(Q_j)} |\nabla_\| f|^2\, d\sigma + \frac{C}{R}\int_{T_{3R/8}(Q_j)} |\nabla u|^2\, dX.$$

Since $\{(\frac{1}{8}Z_j(Q_j, R), \varphi_j)\}$ cover $\partial\Omega$, we have

$$\begin{aligned}\int_{\partial\Omega} |\partial_\nu u|^2\, d\sigma &\le \int_{\cup_{j=1}^N \Delta_{R/8}(Q_j)} |\partial_\nu u|^2\, d\sigma \\ &\le C\left(\int_{\partial\Omega} |\nabla_T f|^2\, d\sigma + \iint_{\cup_{j=1}^N T_{3R/8}(Q_j)} |\nabla u|^2\, dX\right) \\ &\le C\left(\int_{\partial\Omega} |\nabla_T f|^2\, d\sigma + \iint_{\Omega} |\nabla u|^2\, dX\right).\end{aligned}$$

Recall that $u_{\partial\Omega}$ denote the average of u on $\partial\Omega$. Since u is a solution to $\mathcal{L}u = 0$, we have

$$\begin{aligned}\int_\Omega |\nabla u|^2\, dX &= \int_\Omega |\nabla(u - u_{\partial\Omega})|^2\, dX \le \int_\Omega A\nabla(u-u_{\partial\Omega})\cdot\nabla(u-u_{\partial\Omega})\, dX \\ &= \int_{\partial\Omega} [u - u_{\partial\Omega}]\partial_\nu u\, d\sigma \\ &\le \frac{C}{\varepsilon}\int_{\partial\Omega} |u - u_{\partial\Omega}|^2\, d\sigma + C\varepsilon\int_{\partial\Omega} |\partial_\nu u|^2\, d\sigma \\ &\le \frac{C}{\varepsilon}\int_{\partial\Omega} |\nabla_T f|^2\, d\sigma + C\varepsilon\int_{\partial\Omega} |\partial_\nu u|^2\, d\sigma,\end{aligned}$$

where we have used Cauchy's inequality with an ε as well as Poincaré's inequality. Choosing ε sufficiently small, we get

$$\int_{\partial\Omega} |\partial_\nu u|^2\, d\sigma \le C\int_{\partial\Omega} |\nabla_T f|^2\, d\sigma.$$

Analogously, we get

$$\int_{\partial\Omega} |(\partial_\nu u)_-|^2\, d\sigma \le C\int_{\partial\Omega} |(\nabla_T f)_-|^2\, d\sigma.$$

Furthermore, for $u = \mathcal{S}_{A,\Omega} f$, we have

$$\|f\|_2 = \left\| \left(\frac{\partial u}{\partial \nu}\right)_+ - \left(\frac{\partial u}{\partial \nu}\right)_- \right\|_2 \le C\|(\nabla_T u)_+\|_2 + \|(\nabla_T u)_-\|_2 = C\|\nabla_T u\|_2.$$

So $\mathcal{S}_A : L^2(\partial\Omega) \to L_1^2(\partial\Omega)$ is one-to-one. An argument similar to that at the end of the proof of Theorem 2.1 gives the invertibility of $\mathcal{S}_A : L^2(\partial\Omega) \to L_1^2(\partial\Omega)$. □

We end this section with a uniqueness result.

Lemma 3.5 *Let A be real, bounded, and elliptic and u be a solution to the equation $\mathcal{L}^A u = 0$ in Ω, a bounded Lipschitz domain with connected boundary, such that $N_*(\nabla u) \in L^2(\partial\Omega)$. Assume further that $u = 0$ nontangentially on $\partial\Omega$. Then, $u = 0$ in Ω.*

Proof First, observe that if $N_*(\nabla u) \in L^2(\partial\Omega)$ then $N_*(u) \in L^2(\partial\Omega)$. This is because for a fixed $Q \in \partial\Omega$ and any $X \in \Gamma(Q)$, fundamental theorem of calculus and the fact that $u = 0$ nontangentially on $\partial\Omega$ give:

$$u(X) = \int_{\gamma[Q,X]} \nabla u(Y)\, dY \le \mathrm{diam}(\Omega) N_*(\nabla u)(Q),$$

where $\gamma[Q, X]$ is the straight line path connecting Q and X.

Now, let $\Omega_k \uparrow \Omega$ be a sequence of smooth domains approximating Ω from the inside. We will use the notations from Theorem (1.5) in the remainder of this proof. Denote by $\vec{N}^k$ the outer unit normal vector on $\partial\Omega_k$, $\left(\frac{\partial u}{\partial \nu_k}\right)^\alpha (P) = \vec{N}_i^k(P) A_{i,j}^{\alpha\beta} \frac{\partial u^\beta}{\partial P_j}$, and $d\sigma_k$ the surface measure on $\partial\Omega_k$. By the uniform boundedness away from 0 of the Jacobians ω_k corresponding to the homeomorphisms Λ_k,

$$\begin{aligned}\int_{\partial\Omega} u^\alpha(Q) \frac{\partial u^\alpha}{\partial \nu}(Q)\, d\sigma &\ge C \int_{\partial\Omega} u^\alpha(Q) \frac{\partial u}{\partial \nu}(Q) \omega_k^{-1}(Q)\, d\sigma \\ &= \int_{\partial\Omega_k} u^\alpha(\Lambda_k(Q)) \frac{\partial u}{\partial \nu_k}(\Lambda_k(Q))\, d\sigma_k \\ &= \int_{\Omega_k} A_{i,j}^{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} \frac{\partial u^\alpha}{\partial x_i}\, dX,\end{aligned}$$

where we have used the fact that $\mathcal{L}u = 0$ in Ω_k in the last step. Thus,

$$\begin{aligned}\int_\Omega A_{i,j}^{\alpha\beta}(X) \frac{\partial u^\beta}{\partial x_j} \frac{\partial u^\alpha}{\partial x_i}\, dX &= \lim_{k\to\infty} \int_{\Omega_k} A_{i,j}^{\alpha\beta}(X) \frac{\partial u^\beta}{\partial x_j} \frac{\partial u^\alpha}{\partial x_i}\, dX \\ &\le \int_{\partial\Omega} u^\alpha(Q) \frac{\partial u^\alpha}{\partial \nu}(Q)\, d\sigma = 0.\end{aligned}$$

This implies $\int_\Omega A_{i,j}^{\alpha\beta}(X) \frac{\partial u^\beta}{\partial x_j} \frac{\partial u^\alpha}{\partial x_i}\, dX = 0$, whence, by ellipticity of A, $\|\nabla u\|_{L^2(\Omega)} = 0$. Consequently, u is constant in Ω. By connectedness of Ω, and $u = 0$ n.t. on $\partial\Omega$, it must follow that $u = 0$ in Ω. □

This lemma gives unique solvability of (D_2) and (R_2) in Theorem 1.22. For the latter, uniqueness is obtained modulo constants.

4 Single Equations, I.E. $m = 1$

For single equations, we have more tools such as harmonic measure associated to $\mathcal{L}_A$ as well as its estimates in terms of the Green's function associated to $\mathcal{L}_A$ available to us. Consequently, the proof of Theorem 1.22 can be obtained via a localization argument pioneered by Kenig and Pipher in [21]. We give a sketch of the proof for when $m = 1$ here. The notations are the same as in Lemma 3.3 and 3.4, as well as in the proof of Theorem 1.22.

Step 1: $\mathcal{S}_{\bar{Q}} : L^2(\partial D_j) \to \dot{L}_1^2(\partial D_j)$ is invertible by Theorem 2.1. By Theorem 3.2, (RD_2) and (RR_2) are solvable for $\mathcal{L}_1$ and $\mathcal{L}_1^*$ on D_j.

Step 2: By Theorem 1.35 in [16], (RD_2) and (RR_2) are solvable for $\mathcal{L}_2$ and $\mathcal{L}_2^*$ on D_j.

Step 3: Prove the following localization argument for the regularity problem.

Theorem 3.2 *Let $\mathcal{L} = -\mathrm{div}(A\nabla)$ where A is real, bounded, and elliptic. Suppose that (RR_2) is solvable for $\mathcal{L}$, and (RD_2) is solvable for $\mathcal{L}^*$ in $\mathbb{R}^{d+1}_+$. Let $u \in W^{1,2}(T_{8r}(x_0))$ be a weak solution to $\mathcal{L}u = 0$ in $T_{8r}(x_0)$ such that $\tilde{N}_r(\nabla u) \in L^2(\Delta_{4r}(x_0))$ and $u = f \in L^2_1(\Delta_{4r}(x_0)) \cap C(\Delta_{4r}(x_0))$ on $\Delta_{4r}(x_0)$. Then*

$$\int_{\Delta_r(x_0)} |\tilde{N}_{r/2}(\nabla u)|^2 \, dx \le C \left\{ \int_{\Delta_{3r}(x_0)} |\nabla_T f|^2 \, dx + \frac{1}{r} \int_{T_{3r}(x_0)} |\nabla u|^2 \, dX \right\}. \tag{1.1}$$

Step 4: Use estimate (1.1) to obtain $\int_{\partial\Omega} |\tilde{N}(\nabla u)|^2 \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\|} f|^2 \, d\sigma$, thus proving solvability of (R_2).
Step 5: Show that solvability of (R_2) for $\mathcal{L}_{A^*}$ implies solvability of (D_2) for $\mathcal{L}_A$ on Ω.
Step 6: Prove uniqueness.

For detailed arguments for Steps 3-5 as well as an alternative argument for Step 6, refer to [29].

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