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Absence of Non-Constant Harmonic Functions with ℓ^p-gradient in some Semi-Direct Products

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Abstract To obtain groups with bounded harmonic functions (which are amenable), one of the most frequent way is to look at some semi-direct products (*e.g.* lamplighter groups). The aim here is to show that many of these semi-direct products do not admit harmonic functions with gradient in ℓ^p, for $p \in [1, \infty[$.

Keywords Harmonic functions on groups; Energy; ℓ^p cohomology · Wreath products

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In [5] and [6], the author showed that many groups do not have non-constant harmonic functions with gradient in ℓ^p (for $p \in [1, \infty[$): *e.g.* Liouville groups, lamplighters on $\mathbb{Z}^d$ with amenable lamp states, groups with infinitely many finite conjugacy classes, ... The aim of this short paper is to show that many semi-direct products (including lamplighter groups on bigger spaces) also have this property. This contrasts with the fact that all groups admit non-constant harmonic functions with gradient in ℓ^∞ (*i.e.* Lipschitz) and that the groups under consideration have many non-constant bounded harmonic functions.

The graphs $\Gamma = (X, E)$ considered here will always be the Cayley graphs of finitely generated groups. The gradient of $f : X \to \mathbb{R}$ is $\nabla f : E \to \mathbb{R}$ defined by $\nabla f(x, y) = f(y) - f(x)$. The space of p-Dirichlet functions is $\mathrm{D}^p(\Gamma) = \{f : X \to \mathbb{R} \mid \nabla f \in \ell^p(E)\}$ and the space of harmonic functions is $\mathcal{H}(\Gamma) = \ker(\nabla^* \nabla)$. Harmonic functions with gradient in $\ell^p(E)$ are denoted $\mathcal{H}\mathrm{D}^p(\Gamma) = \mathcal{H}(\Gamma) \cap \mathrm{D}^p(\Gamma)$, and $\mathrm{B}\mathcal{H}\mathrm{D}^p(\Gamma) = \mathcal{H}\mathrm{D}^p(\Gamma \cap \ell^\infty(X)$ is the subspace of bounded such functions.

Although these spaces depend *a priori* on the generating set, an abuse of notation will be made by replacing the Cayley graph Γ by the finitely generated group it represents: for

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a group H and a finite generating set, $\mathrm{D}^p(H)$ is to be understood as the D^p space on the associated Cayley graph.

Theorem 1 *Let H be of growth at least polynomial of degree $d \geq 2p$ and, if H is of superpolynomial growth, assume that only constant functions belong to $\mathrm{B}\mathcal{H}\mathrm{D}^q(H)$ for some $q > p$. Let C be a group which is not finitely generated, assume $G = C \rtimes_\phi H$ is finitely generated and assume the hypothesis* (TC) *hold. Then $\mathcal{H}\mathrm{D}^p(G)$ contains only constant functions.*

The results actually holds for any graph quasi-isometric to a Cayley graph of G.

The hypothesis (TC) (see Section 1.3) means that elements in C which are "far away" commute where "far away" can intuitively be thought of as that in order to make some specific subgroup which contain them intersect, one needs to apply a ϕ_h with $h \in H$ large. This hypothesis always holds for Abelian C and in some other interesting examples (see Example 10). The author is inclined to believe that other methods could weaken this hypothesis, however D. Osin pointed out to the author it may not be removed.

Particular examples are lamplighter groups $L \wr H$, as long as H has no bounded harmonic functions (and $d \geq 2p$). Using [6, §4], one can check that iterated wreath products (*e.g.* $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$) have a trivial $\mathcal{H}\mathrm{D}^p$. This gives a partial answers to a problem of Georgakopoulos [4, Problem 3.1], see Section 2.1. See also Section 2.2 below for more details and [7] for more results on lamplighter groups.

Together with [5, Theorem 1.4], this results indicates that some extension operations should be avoided to construct groups with harmonic functions with gradient in ℓ^p out of groups which do not have them. In fact, using [5, Theorem 1.2], an open question of Gromov can be stated as: is it true that any finitely generated amenable group G has only constant functions in $\mathcal{H}\mathrm{D}^p(G)$ (for any $p \in]2, \infty[$. For further questions and comments see Section 2.2 below.

Lemma 5 shows there are no harmonic functions with gradient in c_0 in groups of polynomial growth. It would be nice to have an amenable groups where this fails. Recall that for $p = 2$, this result can be interpreted in terms of [reduced] ℓ^2-cohomology in degree one (or first ℓ^2-Betti numbers). For links between the current results and [reduced] ℓ^p-cohomology [in degree 1], the reader is directed to [5].

1 Proof

1.1 Boundary Values

The following lemma, taken from [5], will come in handy. Let $\mathrm{BD}^p(\Gamma) = \mathrm{D}^p(\Gamma) \cap \ell^\infty(X)$ be the space of bounded functions in D^p.

Lemma 2 *Let $g \in \mathrm{D}^p(H)$ and H be a group of growth at least polynomial of degree $d > 2p$. Let P_H be the random walk operator on H (for some finite generating set). Then $\tilde{g} = \lim_{n\to\infty} P_H^n g$ exists and there is a constant K_1 depending on the isoperimetric profile (in particular, possibly on d) such that*

$$\|g - \tilde{g}\|_{\ell^\infty} \leq K_1 \|g\|_{\mathrm{D}^p}.$$

Furthermore $\tilde{g} \in \mathrm{D}^q(H)$ for all $q \in \left[\frac{pd}{d-2p}, \infty\right]$. If $g \in \mathrm{BD}^p(H)$ then $\tilde{g} \in \mathrm{BD}^q(X)$.

Proof Let $g_n = P^n g$, then

$$g - g_n = g - P^n g = \sum_{i=0}^{n-1} P^i g - P^{i+1} g = \sum_{i=0}^{n-1} P^i (I-P) g = \sum_{i=0}^{n-1} P^i (-\Delta g).$$

But if $g \in D^p(H)$ then $\Delta g \in \ell^p(H)$.

Let $p^{(i)} = P^i \delta_e$ where δ_e is the Dirac mass at the identity element of H. Note that the above expression reads $g - g_n = (-\Delta g) * \big(\sum_{i=0}^{n-1} p^{(i)}\big)$.

On the other hand if H has polynomial growth of degree at least d then $\|p^{(i)}\|_{\ell^r} \leq K n^{-d/2r'}$ where r' is the Hölder conjugate of r. Indeed, use Varopoulos to have a bound on the ℓ^∞ norm: $\|p^{(n)}\|_{\ell^\infty} \leq K_2 n^{-d/2}$. The ℓ^1-norm is always 1. Interpolate to get the ℓ^r norm:

$$\|p^{(n)}\|_{\ell^r}^r = \sum p^{(n)}(\gamma)^r \leq \|p^{(n)}\|_\infty^{r-1} \|p^{(n)}\|_1 \leq K_2^{r-1} n^{\frac{-d}{2}(r-1)}$$

Recall that for $r, r_1, r_2 \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ satisfying $1 + \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, Young's inequality (see [13, Theorem 0.3.1]) gives $\|f * g\|_r \leq \|f\|_{r_1} \|g\|_{r_2}$. Applying this inequality to $g - g_n = p^{(n)} * (-\Delta g)$, one deduces convergence (of $g - g_n$ in ℓ^q-norm) and

$$\|g - \tilde{g}\|_{\ell^q} \leq K_1(q) \|g\|_{\mathrm{D}^p}$$

for $q \in \big[\frac{pd}{d-2p}, \infty\big]$ and $K_1(q)$ is the product of the norm of ∇^* (from ℓ^p of the edges ℓ^p of the vertices) and of Green's kernel (from $\ell^p(H) \to \ell^q(H)$). Thus

$$\|g - \tilde{g}\|_{\ell^\infty} \leq K_1 \|g\|_{\mathrm{D}^p},$$

where $K_1 = K_1(\infty)$. For the last assertion, note that $\tilde{g}$ is harmonic bounded (given g is bounded) and has gradient in ℓ^q (being the sum of a function in $\mathrm{D}^p(H)$ and a function in $\ell^q(H)$). □

1.2 Some Slicing and a Reduction

Take $G = C \rtimes_\phi H$. Assume H is finitely generated (by S_H) and there is a finite set S_C such that $\phi_H(S_C) := \{\phi_z(c) \mid z \in H, c \in S_C\}$ generates C. For G, consider the generating set $\{e_C\} \times S_H \cup S_C \times \{e_H\}$. This will turn out to be unimportant but makes the proof much simpler.

Recall c_0 is the closure of finitely supported functions in ℓ^∞: for some countable set Y,

$$c_0(Y) := \{f : X \to \mathbb{R} \mid \forall \epsilon > 0, \exists F \subset Y \text{ finite such that } \|f\|_{\ell^\infty(X \setminus F} < \epsilon\}.$$

For $z \in H$ let $|z|$ be the word length of z (for S_H). Let $\phi_F(S_C) := \{\phi_z(c) \mid z \in F, c \in S_C\}$. For $c \in C$, let $|c|$ be the word length for [the infinite alphabet] $\phi_H(S_C)$. Let

$$\lfloor c \rfloor := \min\{r \in \mathbb{Z}_{\geq 0} \mid c \text{ belongs to the subgroup generated by } \phi_{S_H^r}(S_C)\}.$$

Let $\operatorname{supp} c$ be union of the sets $F \subset H$ such that c belongs to the group generated by $\phi_F(S_C)$, F is minimal with respect to inclusion and $\max_{f \in F} |f| \leq \lfloor c \rfloor$.

Lemma 3 *Let $f \in \mathrm{D}^p(\Gamma)$. There exists a sequence ϵ'_n of positive real numbers tending to 0*

$$\forall c \in C, \forall s \in S_C \quad |f(c,z) - f(c \cdot \phi_z(s), z)| < \epsilon'_{|z|}.$$

Proof Writing the terms in ∇f gives

$$\|f\|^p_{\mathrm{D}^p(G)} = \sum_{z \in Z} \sum_{c \in C} \left(\sum_{s \in S_C} |f(c \cdot \phi_z(s), z) - f(c,z)|^p + \sum_{s \in S_H} |f(c, z \cdot h) - f(c,z)|^p \right)$$

This implies that $\sum_{c \in C, s \in S_C} |f(c \cdot \phi_z(s), z) - f(c,z)|^p$ tends to 0 in z. Since $\ell^p \subset c_0$, one has that $|f(c \cdot \phi_z(s), z) - f(c,z)|$ tends to 0 uniformly in c and $s \in S_C$. □

Similarly:

Lemma 4 *Let $f \in \mathrm{D}^p(\Gamma)$. There exists a sequence ϵ_n of positive real numbers tending to 0 so that*

$$\|f(c,\cdot)\|_{\mathrm{D}^p(H)} \leq \epsilon_{\lfloor c \rfloor}.$$

Proof $\forall c \in C$, $f(c,\cdot) \in \mathrm{D}^p(H)$, and $\|f\|^p_{\mathrm{D}^p(G)} \geq \sum_{c \in C} \|f(c,\cdot)\|^p_{\mathrm{D}^p(H)}$. Again, the terms in this sum tend to 0 (formally, we use again that $\ell^p \subset c_0$). □

The following lemma is probably well-known.

Lemma 5 *Assume H has polynomial growth. Then there are no harmonic functions with gradient in c_0.*

Proof It is known by the works of Colding & Minicozzi [3] (see also Kleiner [10, Theorem 1.4]) that groups of polynomial growth have a finite dimensional space of harmonic functions with gradient in ℓ^∞. Recall that $\lambda_\gamma f(x) := f(\gamma x)$. Since left-multiplication is an isometry of the Cayley graph, $\lambda_\gamma f$ is harmonic if and only if f is. Furthermore, their gradients are the same up to this shift. Given f non-constant with gradient in c_0, the aim is to show that the $\lambda_\gamma f$ span a vector space of arbitrarily large dimension.

To do this, note that there is some edge $e \in E$ so that, up to multiplying f by a constant, $\nabla f(e) = 1$ (this is possible since f is non-constant). For any ϵ, let γ_1 be so that $|\nabla(\lambda_{\gamma_1} f)(e)| < \epsilon/2$, $|\nabla f(\gamma_1^{-1} e)| < \epsilon/2$. This is possible since $\nabla f \in c_0(E)$. Pick γ_2 so that $\lambda_{\gamma_2} f$ has gradient $< \epsilon/4$ at e and $\gamma_1^{-1} e$ while f and $\lambda_{\gamma_1} f$ have gradient $< \epsilon/4$ at $\gamma_2^{-1} e$. Continue this similarly to get a sequence γ_i with "errors" $\epsilon/2^i$. Restricting to the edges $\gamma_i^{-1} e$ one sees vectors of the form:

$$\begin{matrix} 1 & \epsilon/2 & \epsilon/4 & \epsilon/8 & \dots \\ \epsilon/2 & 1 & \epsilon/4 & \epsilon/8 & \dots \\ \epsilon/4 & \epsilon/4 & 1 & \epsilon/8 & \dots \\ \epsilon/8 & \epsilon/8 & \epsilon/8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Now if $L : \mathbb{R}^N \to \mathbb{R}^N$ is a linear map such that $\|L\vec{e}_i - \vec{e}_i\|_{\ell^2(N)} \leq \epsilon$ for $\{\vec{e}_i\}_{1 \leq i \leq N}$ the usual basis of $\mathbb{R}^N$. Then a standard exercises shows $\dim \ker L \leq \epsilon^2 N$.

Indeed, let $\{\vec{v}_i\}_{1 \leq i \leq N}$ be an orthogonal basis of $\mathbb{R}^N$, and $M : \mathbb{R}^N \to \mathbb{R}^N$ be a linear map such that $M\vec{v}_i = \vec{v}_i$ for $1 \leq i \leq k$ then $k \leq \|M\|^2_{\ell^2(N^2)}$. This is because the $\ell^2(N^2)$ norm for these matrices can also be expressed by $\mathrm{Tr} M^\mathsf{T} M$ and is consequently independent of the choice of orthogonal basis. As $M\vec{v}_i = \vec{v}_i$ for $1 \leq i \leq k$, a simple computation yields $k \leq \mathrm{Tr} M^t M = \|M\|^2_{\ell^2(N^2)}$.

Let $\dim \ker L = k$ and $M = L - \mathrm{Id}$. Since there is an orthogonal basis of $\mathbb{R}^N$ such that the first k elements actually form a basis of $\ker L$, this implies $\|M\|^2_{\ell^2(N^2)} \geq k$. On the other hand, $\|M\|^2_{\ell^2(N^2)} = \sum_i \|M\vec{e}_i\|^2_{\ell^2(N)} \leq N\epsilon^2$. It follows that $k \leq \epsilon^2 N$ as claimed.

This means that the space spanned by $\{\lambda_{\gamma_i} f\}_{i=1}^N$ is of dimension at least $(1-\epsilon^2)N$. In particular there is an infinite dimensional space of Lipschitz harmonic functions. This implies the group is not of polynomial growth. □

The preceding lemma does not *a priori* exclude the existence of non-constant harmonic functions with sublinear growth (see either Hebisch & Saloff-Coste [9, Theorem 6.1] or Meyerovitch, Perl, Tointon & Yadin [12, Theorem 1.3]).

Remark 6 Before moving on, it is necessary to note that groups of polynomial growth are Liouville, *i.e.* they have no non-constant bounded harmonic functions. Thus for such groups H, $\mathrm{B}\mathcal{H}\mathrm{D}^q(H) \simeq \mathbb{R}$ for any $q \in [1, \infty]$. In fact, if $q < \infty$, then $\mathcal{H}\mathrm{D}^q(H)$ contains only constants, by Lemma 5.

If H has growth at least polynomial of degree $d > 2p$ and $q \in \left]\frac{dp}{d-2p}, \infty\right]$, [5, Theorem 1.2] shows that $\mathrm{B}\mathcal{H}\mathrm{D}^q(H) \simeq \mathbb{R}$ implies $\mathcal{H}\mathrm{D}^p(H) \simeq \mathbb{R}$. In particular, if H has superpolynomial growth, $q > p$ and $\mathrm{B}\mathcal{H}\mathrm{D}^q(H) \simeq \mathbb{R}$ implies $\mathcal{H}\mathrm{D}^p(H) \simeq \mathbb{R}$.

1.3 Constant at Infinity

Let us state the hypothesis (TC) that is required on ϕ and C. It states that for any finite $F \subset H$, there are infinitely many z so that there exists a s satisfying $\phi_z(s)$ is not in the subgroup generated by $\phi_F(S_C)$ and

$$\text{for all } w \in F \text{ and } s' \in S_C, \quad [\phi_z(s), \phi_w(s')] = 1. \tag{TC}$$

The easiest case where this hypothesis hold is when C is Abelian. For another common example see Example 10.

Assume H is either of polynomial growth or has superpolynomial growth and $\mathrm{B}\mathcal{H}\mathrm{D}^q(H) \simeq \mathbb{R}$ for some $q > p$. For $f \in \mathrm{D}^p(G)$ and $c \in C$, let $\tilde{f}(c, \cdot) = \lim_{n\to\infty} P^n_H f(c, \cdot)$, where P^n_H is the random walk operator restricted to H. Thus, by Lemma 4, Lemma 2 and Remark 6, $\|f(c, \cdot) - \mathrm{cst}_c\|_{\ell^\infty(H)} \leq K_1 \epsilon_{\lfloor c \rfloor}$, where cst_c is the constant function $\tilde{f}(c, \cdot)$.

Given a set $F \subset H$, let $\overline{F} = \{z \in H \mid \phi_z(S_C) \subset \phi_F(S_C)\}$. If C is not finitely generated, then $H \setminus \overline{F}$ is infinite for any finite set F.

Theorem 7 *Let H be as above (i.e. either of polynomial growth $d > 2p$ or of superpolynomial growth and $\mathrm{B}\mathcal{H}\mathrm{D}^q(H) \simeq \mathbb{R}$ for some $q > p$). Assume C is not finitely generated. Let $f \in \mathrm{D}^p(\Gamma)$, and define $\bar{f} : C \to \mathbb{R}$ by $\bar{f}(c)$ is equal to the constant of the constant function $\tilde{f}(c, \cdot)$. Then*

$$|\bar{f}(c_1) - \bar{f}(c_2)| \leq K_1(\epsilon_{\lfloor c_1 \rfloor} + \epsilon_{\lfloor c_2 \rfloor}).$$

In particular, $\lim_{\lfloor c \rfloor \to \infty} \bar{f}(c)$ *exists.*

Proof We need to show that the constants cst_{c_i} corresponding to c_1 and $c_2 \in C$ are close. Let $|c_2^{-1}c_1|$ be the distance from c_1 to c_2 (in the infinitely generated Cayley graph of C for the generating set $\phi_H(S_C)$).

Fix some $\epsilon > 0$ and assume for simplicity that $K_1 \geq 1$.

Let z be so that $\nu := \max(\epsilon'_{|z|}, \epsilon_{|z|}) < \epsilon/K_1(3|c_2^{-1}c_1| + 5)$ and z lies outside $\overline{\Sigma}$ where $\Sigma = \mathrm{supp}\, c_1 \cup \mathrm{supp}\, c_2$. Since, for some $s \in S_C$ and any c with $\overline{\mathrm{supp}\, c} \subset \overline{\Sigma}$, $\lfloor c \cdot \phi_z(s) \rfloor \geq |z|$, by Lemmas 4 and 2,

$$|f(c \cdot \phi_z(s), w) - \mathrm{cst}_{c \cdot \phi_z(s)}| < K_1 \nu \quad \text{for any } w \in H \text{ and any } c \text{ with } \overline{\mathrm{supp}\, c} \subset \overline{\Sigma}.$$

Also, by Lemma 3, for any $c \in C$, $|f(c, z) - f(c \cdot \phi_z(s), z)| < \nu$. Thus, for any $s' \in S_C$, for any $w \in \overline{\Sigma}$ and any c with $\overline{\mathrm{supp}\, c} \subset \overline{\Sigma}$,

$$\begin{aligned}
&|\mathrm{cst}_{c \cdot \phi_w(s') \cdot \phi_z(s)} - \mathrm{cst}_{c \cdot \phi_z(s)}| \\
&\quad \leq |\mathrm{cst}_{c \cdot \phi_w(s') \cdot \phi_z(s)} - f(c \cdot \phi_w(s') \cdot \phi_z(s), w)| \\
&\qquad + |f(c \cdot \phi_w(s') \cdot \phi_z(s), w) - f(c \cdot \phi_z(s), w)| \\
&\qquad + |f(c \cdot \phi_z(s), w) - \mathrm{cst}_{c \cdot \phi_z(s)}| \\
&\quad \leq K_1 \nu + |f(c \cdot \phi_z(s) \cdot \phi_w(s'), w) - f(c \cdot \phi_z(s), w)| + K_1 \nu. \\
&\quad \leq 2K_1 \nu + \nu.
\end{aligned}$$

where the above estimate as well as hypothesis (TC) was used in the second inequality and Lemma 4 was used in the third inequality. Hence, one has

$$|\mathrm{cst}_{c_1 \cdot \phi_z(s)} - \mathrm{cst}_{c_2 \cdot \phi_z(s)}| \leq 3K_1 |c_2^{-1}c_1| \nu.$$

Finally,

$$\begin{aligned}
|\mathrm{cst}_{c_i \cdot \phi_z(s)} - \mathrm{cst}_{c_i}| &\leq |\mathrm{cst}_{c_i \cdot \phi_z(s)} - f(c_i \cdot \phi_z(s), z)| \\
&\quad + |f(c_i \cdot \phi_z(s), z) - f(c_i, z)| \\
&\quad + |f(c_i, z) - \mathrm{cst}_{c_i}| \\
&\leq (K_1 + 1)\nu + K_1 \epsilon_{\lfloor c_i \rfloor}.
\end{aligned}$$

So

$$\begin{aligned}
|\mathrm{cst}_{c_1} - \mathrm{cst}_{c_2}| &\leq K_1 \big[(3|c_2^{-1}c_1| + 4)\nu + \epsilon_{\lfloor c_1 \rfloor} + \epsilon_{\lfloor c_2 \rfloor}\big] \\
&< K_1 \big(\epsilon + \epsilon_{\lfloor c_1 \rfloor} + \epsilon_{\lfloor c_2 \rfloor}\big).
\end{aligned}$$

Since the above holds for any $\epsilon > 0$, the conclusion follows.

To see the limit exists, note that the sequence is Cauchy. □

Proof of theorem 1 First, the proof is done for the generating set build with S_C and S_H. Let B_n be a sequence of balls centred at the identity in G. Say a function has only one value at infinity if, up to changing f by a constant function, $f(B_n^{\mathsf{c}}) \subset]-\sigma_n, \sigma_n[$ for some sequence of positive numbers σ_n tending to 0.

Take $f \in \mathcal{H}\mathrm{D}^p(G)$. If f takes only one value at infinity, then f is constant (by the maximum principle).

Thus, the theorem follows if we show any $f \in \mathrm{D}^p(G)$ has one value at infinity (it is not even required that f be harmonic). Change f by a constant so that the function $\bar{f}$ from Theorem 7 tends to 0 as $\lfloor c \rfloor \to \infty$. This implies

$$|f(c, z)| = |f(c, z) - \mathrm{cst}_c + \mathrm{cst}_c| \leq 3K_1 \epsilon_{\lfloor c \rfloor},$$

by bounding the first term as in the proof of Theorem 7 and the second by the result of Theorem 7. It remains to check that $f(c, z)$ also tends to 0 as $|z| \to \infty$. Assume $z \notin S_H^{\lfloor c \rfloor}$ (*i.e.* $|z| > \lfloor c \rfloor$), then, using the same bounds and Lemma 3,

$$\begin{aligned} |f(c,z)| &\leq |\mathrm{cst}_{c\cdot\phi_z(s)}| + |\mathrm{cst}_{c\cdot\phi_z(s)} - f(c\cdot\phi_z(s), z)| \\ &\quad + |f(c\cdot\phi_z(s), z) - f(c,z)| \\ &\leq 2K_1\epsilon_{|z|} + K_1\epsilon_{|z|} + \epsilon'_{|z|} \\ &\leq 3K_1\epsilon_{|z|} + \epsilon'_{|z|}. \end{aligned}$$

Thus f has only one value at infinity.

If one considered another generating set, then a simple way is to do as follows. Note that G is not virtually nilpotent, hence satisfies a d-dimensional isoperimetric profile for any d. By [5, Theorem 1.4], for any Cayley graph Γ of G, one has: the reduced ℓ^p-cohomology in degree one of Γ is non-trivial for some $p \in [1, \infty[$ if and only if $\mathcal{H}\mathrm{D}^q(\Gamma) \not\simeq \mathbb{R}$ for some $q \in [1, \infty[$. Since the reduced ℓ^p-cohomology in degree one is an invariant of quasi-isometry (in particular, of the choice of generating set) the result follows for other generating sets. □

2 Some Examples and Questions

2.1 Examples

Le us rewrite Theorem 1.

Corollary 8 *Let $G = C \rtimes_\phi H$, assume C is not finitely generated but G is and that hypothesis* (TC) *holds.*

- *If H has polynomial growth of degree d, then, for all $p \in [1, d/2[$, $\mathcal{H}\mathrm{D}^p(G)$ contains only constant functions.*
- *If H has intermediate growth, then, for all $p \in [1, \infty[$, $\mathcal{H}\mathrm{D}^p(G)$ contains only constant functions.*
- *If H has exponential growth and $p \in [1, \infty[$ and let $1 \leq p < q \leq \infty$. Then $\mathrm{B}\mathcal{H}\mathrm{D}^q(H) \simeq \mathbb{R}$ implies $\mathcal{H}\mathrm{D}^p(G) \simeq \mathbb{R}$. Also $\mathcal{H}\mathrm{D}^p(H) \simeq \mathbb{R}$ implies $\mathcal{H}\mathrm{D}^p(G) \simeq \mathbb{R}$.*

Example 9 The classical example is to take L finitely generated group ("lamp state") and $C = \oplus_H L$ with H acting by shifting the index. C is the "lamp configuration group", and the semi-direct product is called a lamplighter group. If H and L are finitely generated, then S_C can be picked to be the generating set of L (at the index e_H).

Georgakopoulos [4] showed lamplighter *graphs* do not have harmonic functions with gradient in ℓ^2. His methods extends to harmonic functions with gradient in ℓ^p. However, the lamp groups must be finite.

Using Theorem 1, [6, §4], [11, Theorem.(iv)] and work of Georgakopoulos [4], one may readily check that the only lamplighter groups for which it is not proven that $\mathcal{H}\mathrm{D}^p(\Gamma) \simeq \mathbb{R}$ for any $p \in [1, \infty[$ are those where L is infinite amenable and either 1- H is of polynomial growth and not virtually Abelian or 2- H has $\mathcal{H}\mathrm{D}^p(H) \not\simeq \mathbb{R}$.

Though the proof was not done in this generality, Theorem 1 extends almost *verbatim* to the case of lamplighter graphs. The correct hypothesis is that the graph H must have IS_d for

$d > 2p$ and either: 1- $\mathcal{H}\mathrm{D}^p(H) \simeq \mathbb{R}$ or 2- $\mathrm{B}\mathcal{H}\mathrm{D}^q(H) \simeq \mathbb{R}$ for some $q \in \left]\frac{dp}{d-2p}, \infty\right]$. This gives a partial answer to a problem raised by Georgakopoulos [4, Problem 3.1].

Example 10 Another classical example is to take Sym_H to be the permutations $H \to H$ which are not the identity only on a finite set. There is a natural action (say, on the right) of H on itself by permutation. This gives $G = \mathrm{Sym}_H \rtimes_\phi H$ which is finitely generated (although Sym_H is not finitely generated).

2.2 Further Comments and Questions

A simple way to show that the gradient of a harmonic function is not in ℓ^p is to think in terms of electric currents. This is essentially the method used by Georgakopoulos [4] to show lamplighter *graph* do not have harmonic functions with gradient in ℓ^2. Indeed, if one exhibits "many" paths which are "not too long" between points where the potential is $\geq 5/8$ and points where its $\leq 3/8$, then one gets a lower bound on the gradient.

Note that, for groups, using [5, Theorem 1.2], it is sufficient to consider the case of bounded harmonic functions. Still, assume for simplicity that f is a bounded harmonic function. Then, up to normalisation, its values are between 0 and 1. Let $N = f^{-1}[0, 3/8]$ and $P = f^{-1}[5/8, 1]$ (both are infinite sets). Let k_n be the maximal number of edge-disjoint paths of length $\leq n$ between N and P. Then the ℓ^p norm of the current is at least

$$\frac{1}{4} \sum_{\text{paths}} \frac{1}{\text{length of the paths}} \geq \frac{k_n}{n^p}$$

Let $p_c := \inf\{p \mid \mathcal{H}\mathrm{D}^p(\Gamma) \not\simeq \mathbb{R}\}$. Then, $\forall \epsilon > 0$, $\lim\limits_{n \to \infty} \frac{k_n}{n^{p_c+\epsilon}} = 0$. This gives an amusing view of the critical exponent from ℓ^p-cohomology, see Bourdon, Martin & Valette [1] or Bourdon & Pajot [2].

Question 11 Given a group of exponential growth and divergence rate $n \mapsto n^d$, is it possible to show that k_n grows exponentially?

Indeed, there are always two geodesic rays $\{x_n\}$ and $\{y_n\}$ with $f(x_n) \to 1$ and $f(y_n) \to 0$. If the divergence does not grow too quickly, then there should be [exponentially] many paths of distance roughly $Kn^{1/d}$ between those (for some $K > 0$).

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