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An Equivalence Between the Dirichlet and the Neumann Problem for the Laplace Operator

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Abstract We give a representation of the solution of the Neumann problem for the Laplace operator on the n-dimensional unit ball in terms of the solution of an associated Dirichlet problem. The representation is extended to other operators besides the Laplacian, to smooth simply connected planar domains, and to the infinite-dimensional Laplacian on the unit ball of an abstract Wiener space, providing in particular an explicit solution for the Neumann problem in this case. As an application, we derive an explicit formula for the Dirichlet-to-Neumann operator, which may be of independent interest.

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For a smooth bounded domain $D \subset \mathbb{R}^n$ ($n \geq 1$), consider the corresponding Dirichlet and Neumann problems for the Laplacian in D

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = \varphi \text{ on } \partial D \end{cases} \tag{1}$$

and

$$\begin{cases} \Delta U = 0 \text{ in } D \\ \frac{\partial U}{\partial \nu} = \phi \text{ on } \partial D \end{cases}, \tag{2}$$

where ν is the outward unit normal to the boundary of D.

As it is known, for continuous boundary data, the Dirichlet problem (1) has a unique solution and the Neumann problem (2) has a solution, unique up to additive constants, if we require in addition the condition $\int_{\partial D} \phi(z)\,\sigma(dz) = 0$. Note that this is a necessary condition for the existence of a solution, since by Green's first identity we have

$$\int_{\partial D} \phi(z)\,\sigma(dz) = \int_{\partial D} 1 \frac{\partial U}{\partial \nu}(z)\,\sigma(dz) = \int_D 1\Delta U(z) + \nabla 1 \cdot \nabla U(z)\,dz = 0.$$

There are several representations of the solutions of the Dirichlet and Neumann problems above in the literature: by single / double layer potentials (see for example [7], Theorem 3.40), by spherical harmonics ([7], Theorem 2.60), or even by probabilistic methods (see Theorem 2.1 in [2] for the Dirichlet problem, and [5, 11], or Theorem 5.3 in [3] for the Neumann problem).

It is usually agreed that the Neumann problem is in general "harder" than the Dirichlet problem. In the present paper, we derive explicit relations between the solutions of (1) and (2), which do not seem to appear in the literature. This shows that the Dirichlet and Neumann problems are equivalent in this case (equally "hard"), in the sense that solving one of them leads to the solution of the other one.

Remark 1 Although there are various connections between the Neumann and the Dirichlet problems in the literature, they are not explicit. For example, it is known (see e.g. [13, Chap. 7, Sect. 11]) that the Neumann problem (2) can be reduced to the Dirichlet problem (1) for the choice of φ given by $\phi = \Lambda_n \varphi$, where Λ_n denotes the Dirichlet-to-Neumann operator for $D \subset \mathbb{R}^n$ (see also Section 1.3). In turn, this requires to invert the operator Λ_n, without an explicit form being given. Alternately, following the same reference, the previous condition is shown to be equivalent to $(I - N)\varphi = -2S\phi$, where the operators S, N are related to single and double layer potentials on D (see [13, Proposition 7.11.1]). Again, in order to relate the Neumann problem (2) to the Dirichlet problem (1), this requires to invert the operator $I - N$, without an explicit form being given.

Our main result in Theorem 1 (and its extensions given in Theorems 2, 5, and 6) provide an explicit relation between the solution(s) of (2) and (1), in the sense that the normalized solution of (2) can be found as a weighted average of the solution of (1). As noted above, this is equivalent to inverting the Dirichlet-to-Neumann operator, and as an application we provide an explicit form of the inverse in the case of the unit ball in $\mathbb{R}^n$ (see Section 1.3).

The results are interesting in their own respect. The result in Theorem 1 shows that in the case of the unit ball in $\mathbb{R}^n$ ($n \geq 1$) we can find the solution of the Neumann problem by solving the Dirichlet problem with the same boundary values. Probabilistically, this result has an interesting consequence, since the solution of the Neumann problem can be represented by Brossamler's formula (see [3, 5]) in terms of the reflecting Brownian motion, while the solution of the Dirichlet problem (see e.g. [2, Sect. 2.2]) is given in terms of the killed Brownian motion. The result in Theorem 1 shows therefore that the expected value of certain functionals of reflected Brownian motion can be computed alternately in terms of the expected value of the killed Brownian motion in the same domain, a result which may be of independent interest. From a different prospective, in terms of extensions of Dirichlet spaces, the connection between the killed and the reflecting Brownian motion (and more general processes) have been pointed out in the recent monograph [6, Chap. 6,7].

The structure of the paper is as follows. In Section 1, we consider the case of the unit ball in $\mathbb{R}^n$ ($n \geq 1$). We begin with the heuristics which led us to the result (Section 1.1), and then we give the main theorem in this case, Theorem 1. The result can be extended to other operators besides the Laplacian, and in Theorem 2 we present such an extension.

As an application, in Section 1.3 we derive an explicit representation of the inverse of the Dirichlet-to-Neumann operator (a particular case of the Poincaré-Steklov operator, which encapsulates the boundary response of a system modelled by a certain partial differential equation). The result could be relevant to people working in this area of research (Calderon's problem, domain decomposition methods, a.s.o.).

In Section 2, we use the method of conformal maps to extend the result obtained in the case of the unit disk to the general case of smooth bounded simply connected domains. We point out that the linear operators which appear in Theorem 1 and Theorem 5 (the operator $T(u) = U$ which establishes the correspondence between the solution u of the Dirichlet problem and the solution U of the Neumann problem, given by Eq. 4, respectively by Eq. 20) transform the Dirichlet boundary condition of the input into a Neumann boundary condition for the output, and thus may be of further interest besides the case of Laplace operator.

We conclude with the a connection between the Dirichlet and the Neumann problem for the Laplacian in the case of the infinite-dimensional ball on an abstract Wiener space. Under the appropriate conditions, in Theorem 6 we establish the same connection between the two problems as in the finite dimensional case. An interesting byproduct of this result is that it provides an (explicit) solution for the Neumann problem for the infinite-dimensional Laplacian.

In what follows, we will identify as usual the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, that is we identify the vector $(x, y) \in \mathbb{R}^2$ with the complex number $z = x + iy \in \mathbb{C}$. In particular, the dot product of two vectors $a, b \in \mathbb{R}^2$ will be written in terms of multiplication of complex numbers as $a \cdot b = \operatorname{Re}\left(a\bar{b}\right)$, and for a complex number $z \in \mathbb{C}$ we denote the real part and the imaginary part of z by $\operatorname{Re}(z)$, respectively $\operatorname{Im}(z)$. Also, for a function u defined on a subset D of $\mathbb{R}^2$ (or $\mathbb{C}$), we will write equivalently $u(x, y)$ or $u(z)$, where $z = x + iy \in D$.

For a smooth bounded domain we will be denote by $\sigma(\cdot)$ and $\sigma_0(\cdot)$ the surface measure on its boundary, respectively the surface measure normalized to have total mass 1.

1 The Case of the Unit Ball $\mathbb{U} \subset \mathbb{R}^n$

In this section we consider the Dirichlet and Neumann problems for the Laplace operator in the particular case $D = \mathbb{U} = \{z \in \mathbb{R}^n : |z| < 1\}$ of the unit ball in $\mathbb{R}^n$, $n \geq 1$.

1.1 Heuristics

Assume for the moment that the dimension of the space is $n = 2$. It is known that if u is a solution to the Dirichlet problem (1), then u is the real part of an analytic function $G = u + iv$ in $\mathbb{U}$ (if u is known, the function v can be determined from the Cauchy-Riemann equations for the function G). So we may interpret the Dirichlet problem (1) as the problem of finding an analytic function in $\mathbb{U}$ (continuous up to the boundary), with prescribed values φ on the boundary for its real part.

Similarly, if U is a solution of the Neumann problem (2), then U is the real part of an analytic function $F = U + iV$. Since in the case of the unit disk the outward unit normal to $\partial\mathbb{U}$ is $\nu(z) = z$, $z = x + iy \in \partial\mathbb{U}$, we can write the Neumann boundary condition for U using the Cauchy-Riemann equations for F, as follows

$$\phi(z) = \frac{\partial U}{\partial \nu}(z) = xU_x(z) + yU_y(z) = xU_x(z) - yV_x(z) = \mathrm{Re}\left(zF'(z)\right).$$

If we set $G(z) = zF'(z)$, $z \in \mathbb{U}$, it follows that G is analytic in $\mathbb{U}$ and has boundary values ϕ for its real part on $\partial\mathbb{U}$. The Neumann problem (2) is therefore equivalent to finding the analytic function G in $\mathbb{U}$ (continuous up to the boundary) with prescribed values ϕ for its real part on $\partial\mathbb{U}$. Once G is determined, we can find F by complex integration as follows

$$F(z) = F(0) + \int_0^z \frac{G(\xi)}{\xi} d\xi, \qquad z \in \mathbb{U}, \tag{3}$$

and we can then determine the solution of the Neumann problem (2) as $U(z) = \mathrm{Re}F(z)$, $z \in \mathbb{U}$.

The above heuristics show that both the Dirichlet and Neumann problems (at least in the case of the unit disk) are equivalent to finding an analytic function in $\mathbb{U}$, continuous up to the boundary, with prescribed values on the boundary for its real part. This shows that the Dirichlet and Neumann problems are "equally hard" in this case.

Formula (3) also suggests a direct way of finding the solution $U = \mathrm{Re}F$ of the Neumann problem from the solution $u = \mathrm{Re}G$ of the Dirichlet problem, by circumventing the problem of finding the corresponding analytic functions F and G. Integrating in (3) along the line segment from 0 to $z \in \mathbb{U}$, we have $\xi(\rho) = \rho z$ with $0 \le \rho \le 1$, and we obtain

$$F(z) = F(0) + \int_0^1 \frac{G(\rho z)}{\rho z} z d\rho = F(0) + \int_0^1 \frac{G(\rho z)}{\rho} d\rho,$$

and taking real parts we get $U(z) = U(0) + \int_0^1 \frac{u(\rho z)}{\rho} d\rho$, $z \in \mathbb{U}$.

1.2 The results

The results in the previous section lead us to the following.

Theorem 1 *Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is continuous and satisfies $\int_{\partial\mathbb{U}} \phi(z)\, \sigma_0(dz) = 0$. If u is the solution of the Dirichlet problem* (1) *with boundary condition $\varphi = \phi$ on $\partial\mathbb{U}$, then*

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}}, \tag{4}$$

is the solution to the Neumann problem (2) *with $U(0) = 0$.*

Proof The heuristics above could serve as a proof in the particular case $n = 2$, provided we show that the analytic functions F, G and their derivatives can be extended continuously to the boundary of $\mathbb{U}$. We will not do this, and instead we will provide a direct proof of (4) in the general case $n \geq 1$.

First note that since u is harmonic in $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$, by bounded convergence theorem we have

$$u(0) = \lim_{r \nearrow 1} \int_{\partial(r\mathbb{U})} u(z)\, d\sigma_0(z) = \int_{\partial\mathbb{U}} u(z)\, d\sigma_0(z) = \int_{\partial\mathbb{U}} \phi(z)\, d\sigma_0(z) = 0,$$

which in particular shows that the integrand in (4) is continuous at the origin: $\lim_{\rho\to 0} \frac{u(\rho z)}{\rho} = \lim_{\rho \to 0} \frac{u(\rho z) - u(0)}{\rho} = \nabla u(0) \cdot z$.

Since u is harmonic in $\mathbb{U}$, it is $C^\infty(\mathbb{U})$, and therefore the integrand in (4) and its second order partial derivatives are continuous functions of $\rho \in [0, 1]$. Differentiating under the integral sign in (4), we obtain

$$\Delta_z U(z) = \int_0^1 \Delta_z \frac{u(\rho z)}{\rho} d\rho = \int_0^1 \rho \Delta u(\rho x, \rho y)\, d\rho = 0,$$

for any $z \in \mathbb{U}$, where we denoted by Δ_z the Laplace operator $\sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$.

To see that U has the prescribed normal derivative on $\partial\mathbb{U}$, fix $z_0 \in \partial\mathbb{U}$ and recall that we are using the outward normal $\nu(z_0) = z_0$ to the boundary of $\partial\mathbb{U}$. We have

$$\begin{aligned}\frac{\partial U}{\partial \nu}(z_0) &= \lim_{\varepsilon \nearrow 0} \frac{U(z_0 + \varepsilon\nu(z_0)) - U(z_0)}{\varepsilon} = \lim_{\varepsilon\nearrow 0} \frac{1}{\varepsilon}\left(\int_0^{1+\varepsilon} \frac{u(\rho z_0)}{\rho} d\rho - \int_0^1 \frac{u(\rho z_0)}{\rho} d\rho\right) \\ &= \lim_{\varepsilon \nearrow 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} \frac{u(\rho z_0)}{\rho} d\rho \qquad = \lim_{\varepsilon\nearrow 0} \frac{u(\rho^* z_0)}{\rho^*} \\ &= u(z_0),\end{aligned}$$

by the continuity of u in $\overline{\mathbb{U}}$, where $\rho^* \in (1 + \varepsilon, 1)$ denotes the intermediate point given by the mean value theorem. This shows that the values of the normal derivative $\frac{\partial U}{\partial \nu}$ on $\partial\mathbb{U}$ coincide with the boundary values of u on $\partial\mathbb{U}$, concluding the proof. □

With only cosmetic changes, the proof of the previous theorem can also be applied to other operators besides the Laplacian. For example, if $\mathcal{L}$ is the operator

$$\mathcal{L}f(z) = \sum_{i,j=1}^n a_{ij}(z) \frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_{i=1}^n a_i(z) \frac{\partial f}{\partial z_i}(z), \tag{5}$$

where the coefficients a_{ij} are smooth and homogeneous of degree $k \in [0, 1]$, i.e.

$$a_{ij}(\rho z) = \rho^k a_{ij}(z), \qquad 0 \leq \rho \leq 1, z \in \mathbb{U}, 1 \leq i, j \leq n, \tag{6}$$

and the coefficients a_i are also smooth and homogeneous of degree $k - 1$, i.e.

$$a_i(\rho z) = \rho^{k-1} a_{ij}(z), \qquad 0 \leq \rho \leq 1, z \in \mathbb{U}, 1 \leq i \leq n, \tag{7}$$

and if u and U are related by (4), then

$$\mathcal{L}U(z) = \int_0^1 \rho^{1-k} \mathcal{L}u(\rho z)\, d\rho, \qquad z \in \mathbb{U},$$

and

$$\frac{\partial U}{\partial \nu}(z) = u(z), \qquad z \in \mathbb{U}.$$

The previous observation leads us to the following more general result.

Theorem 2 *Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is continuous and satisfies $\int_{\partial\mathbb{U}} \phi(z)\,\sigma_0(dz) = 0$. If u is the solution of the Dirichlet problem*

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \mathbb{U} \\ u = \phi \text{ on } \partial\mathbb{U} \end{cases} \tag{8}$$

where $\mathcal{L}$ is the operator given by (5) which satisfies (6) and (7), then

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}}, \tag{9}$$

is the solution to the Neumann problem

$$\begin{cases} \mathcal{L}U = 0 & \text{in } \mathbb{U} \\ \frac{\partial U}{\partial \nu} = \phi & \text{on } \partial\mathbb{U} \end{cases}, \tag{10}$$

with $U(0) = 0$.

A result similar to the one in Theorem 1 can be given for the other direction, from the Neumann to the Dirichlet problem. As it is known, for a harmonic function the Laplacian and the partial derivatives commute. This observation allows to write the solution of the Dirichlet problem in terms of the solution of the Neumann problem, as follows.

Theorem 3 *Assume $\varphi : \partial\mathbb{U} \to \mathbb{R}$ is continuous and let U be the solution of the Neumann problem (2) with boundary condition $\phi = \varphi - \int_{\partial\mathbb{U}} \varphi(\xi)\,\sigma_0(d\xi)$. If we define*

$$u(z) = z \cdot \nabla U(z) + \int_{\partial\mathbb{U}} \varphi(\xi)\,\sigma_0(d\xi), \qquad z \in \overline{\mathbb{U}}, \tag{11}$$

then u is the solution to the Dirichlet problem (1).

Proof The function ϕ satisfies the necessary condition for the existence of a solution of the Neumann problem. Using the previous observation it is not difficult to see that

$$\Delta u(z) = 2\Delta U(z) + z \cdot \nabla(\Delta U(z)) = 0, \qquad z \in \mathbb{U},$$

so u is harmonic on $\mathbb{U}$. Since u also assumes the correct boundary values φ, it is the solution of the Dirichlet problem (1), concluding the proof. □

1.3 A Representation of the Dirichlet-to-Neumann Operator in the Case of the Unit Ball in $\mathbb{R}^n$

As an application of Theorem 1, we derive an explicit representation of the inverse of the Dirichlet-to-Neumann operator Λ_n (see e.g. [10, Sect. 5.0]) in the case of the unit ball $\mathbb{U} \subset \mathbb{R}^n$ ($n \geq 2$), defined formally as follows.

For $\varphi \in C(\partial\mathbb{U})$, let u^φ be the solution to the Dirichlet problem for the Laplacian on $\mathbb{U}$ with boundary values φ on $\partial\mathbb{U}$. The Dirichlet-to-Neumann operator $\Lambda_n : C(\partial\mathbb{U}) \to C(\partial\mathbb{U})$ is a particular case of a Poincaré-Steklov operator, which maps the Dirichlet boundary values φ of a harmonic function u^φ in $\mathbb{U}$ to the corresponding Neumann boundary values $\phi = \frac{\partial u^\varphi}{\partial \nu}$ on $\partial\mathbb{U}$, more precisely

$$\Lambda_n(\varphi) = \left.\frac{\partial u^\varphi}{\partial \nu}\right|_{\partial\mathbb{U}}, \tag{12}$$

where $\nu(z) = z$ denotes the outward unit pointing normal to the boundary of $\mathbb{U}$.

The operator Λ_n is not injective since $\Lambda_n(\varphi + c) = \Lambda_n(\varphi)$, for any constant $c \in \mathbb{R}$. Conversely, if $\Lambda_n(\varphi_1) = \Lambda_n(\varphi_2)$ then $\varphi_1 - \varphi_2$ is constant in $\mathbb{U}$, by the uniqueness up to additive constant of the solution of the Neumann problem. Identifying the class $\{\varphi \in C(\partial\mathbb{U}) : \Lambda_n(\varphi) = \phi\}$ with the representant φ of it satisfying the normalization condition $\int_{\partial\mathbb{U}} \varphi(\xi)\sigma_0(\xi) = 0$ (or equivalent $u^{\varphi}(0) = 0$), the Dirichlet-to-Neumann operator Λ_n becomes an injective operator. It follows that if we consider the restriction of the Dirichlet-to-Neumann operator

$$\Lambda_n : \left\{\varphi \in C(\partial\mathbb{U}) : \int_{\partial\mathbb{U}} \varphi(\xi)\sigma_0(\xi) = 0\right\} \to \left\{\phi \in C(\partial\mathbb{U}) : \int_{\partial\mathbb{U}} \phi(\xi)\sigma_0(\xi) = 0\right\}, \quad (13)$$

then Λ_n is a bijective operator, and we will use this definition in the sequel.

Using the connection between the solutions of the Dirichlet and Neumann problems for the Laplacian given in Theorem 1, we can derive an explicit representation of the inverse of the operator Λ_n as follows.

Theorem 4 *Assume $\phi : \partial\mathbb{U} \to \mathbb{R}$ is continuous and satisfies $\int_{\partial\mathbb{U}} \phi(\xi)\,\sigma(d\xi) = 0$. We have*

$$\Lambda_n^{-1}(\phi)(z) = \int_{\partial\mathbb{U}} \phi(\xi)\, k_n(z,\xi)\sigma_0(d\xi), \qquad z \in \partial\mathbb{U}, \quad (14)$$

where $k_n(z,\zeta) = \int_0^1 \frac{1}{\rho}\left(\frac{1-\rho^2}{|\rho z - \xi|^n} - 1\right) d\rho$, $z, \xi \in \partial\mathbb{U}$.

Explicitly, $k_2(z,\xi) = -2\ln|z-\xi|$, $k_3(z,\xi) = \frac{2}{|z-\xi|} - 2 + \ln 2 - \ln\left(\frac{|z-\xi|^2}{2} + |z-\xi|\right)$, and for $n > 4$ the kernel $k_n(z,\xi)$ can be computed using the recurrence formulae (17) – (18) below.

Proof Let $u = u^{\phi}$ be the solution of the Dirichlet problem (1) with boundary condition $\varphi = \phi$ on $\partial\mathbb{U}$. By Theorem 1 it follows that the function U defined by

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho} d\rho, \qquad z \in \overline{\mathbb{U}}, \quad (15)$$

is the solution to the Neumann problem (2) with boundary values ϕ on $\partial\mathbb{U}$ and $U(0) = 0$.

Consider now $z \in \partial\mathbb{U}$ arbitrarily fixed and $r \in (0,1)$. Combining the above with the Poisson integral formula for the ball, and using that $u(0) = \int_{\partial\mathbb{U}} \phi(\xi)\,\sigma_0(d\xi) = 0$, we obtain

$$U(rz) = \int_0^1 \frac{u(\rho r z) - u(0)}{\rho} d\rho = \int_0^1 \frac{1}{\rho} \int_{\partial\mathbb{U}} \left(\frac{1-\rho^2 r^2}{|\rho r z - \xi|^n} - 1\right) \varphi(\xi)\,\sigma_0(d\xi) d\rho. \quad (16)$$

For arbitrary $\xi \in \mathbb{U}\setminus\{\pm z\}$ and $\rho \in (0,1]$, denoting by $\theta_1 \in (0,\pi)$ the angle between z and ξ, we have

$$\begin{aligned}
&\left|\varphi(\xi)\frac{1}{\rho}\left(\frac{1-\rho^2 r^2}{|\rho r z - \xi|^n} - 1\right)\right| \le \frac{\|\varphi\|_\infty}{\rho\,|\rho r z - \xi|^{n-2}}\left(\left|\frac{1-\rho^2 r^2}{|\rho r z - \xi|^2} - 1\right| + \left|1 - |\rho r z - \xi|^{n-2}\right|\right)\\
&\le \frac{\|\varphi\|_\infty}{\rho\sqrt{(\rho r - \cos\theta_1)^2 + \sin^2\theta_1}^{\,n-2}}\left(\frac{2\rho r\,|\rho r - \cos\theta_1|}{\rho^2 r^2 - 2\rho r\cos\theta_1 + 1} + 2\rho r\left(2^{n-2} - 1\right)\right)\\
&\le \frac{2r\,\|\varphi\|_\infty}{|\sin\theta_1|^{n-2}}\left(\frac{2}{(1-\rho r\cos\theta_1)^2} + 2^{n-2} - 1\right) \le \frac{2^n r\,\|\varphi\|_\infty}{|\sin\theta_1|^{n-2}(1-r)^2}.
\end{aligned}$$

The last expression is integrable on $\partial\mathbb{U}\times[0,1]$. To see this, consider spherical coordinates $(\theta_1,\ldots,\theta_{n-1})\in[0,\pi]^{n-2}\times[0,2\pi]$, with θ_1 chosen to denote the angle between z and ξ. Since the expression above depends only on θ_1, integrating in the remaining $(n-2)$ variables we obtain the area $\omega_{n-1}\left|\sin\theta_1\right|^{n-2}$ of the $(n-1)$-dimensional sphere of radius $\left|\sin\theta_1\right|$, and therefore

$$\begin{aligned}\int_{\partial\mathbb{U}}\int_0^1\frac{2^n r\,\|\varphi\|_\infty}{\left|\sin\theta_1\right|^{n-2}(1-r)^2}d\rho\sigma_0\left(d\xi\right) &= \frac{2^n r\,\|\varphi\|_\infty}{(1-r)^2}\int_0^\pi\frac{1}{\left|\sin\theta_1\right|^{n-2}}\frac{\omega_{n-1}\left|\sin\theta_1\right|^{n-2}}{\omega_n}d\theta_1\\ &= \frac{2^n\pi r\omega_{n-1}\,\|\varphi\|_\infty}{\omega_n\,(1-r)^2}<\infty.\end{aligned}$$

The argument above shows that the integrand in (16) is absolutely integrable for any $r\in(0,1)$, and using Tonelli-Fubini theorem we obtain the representation

$$U\left(rz\right)=\int_{\partial\mathbb{U}}\varphi\left(\xi\right)k_n\left(rz,\xi\right)\sigma_0(d\xi),\qquad z\in\partial\mathbb{U}, r\in(0,1)\,.$$

The claim (14) of the theorem follows from the above, once we show that we can take limits with $r\nearrow 1$.

Note that for any $r\in(0,1)$ we have

$$k_n\left(rz,\xi\right)=\int_0^1\frac{1}{\rho}\left(\frac{1-\rho^2r^2}{\left|\rho rz-\xi\right|^n}-1\right)d\rho=\int_0^r\frac{1}{\rho}\left(\frac{1-\rho^2}{\left|\rho z-\xi\right|^n}-1\right)d\rho,$$

and therefore using the inequality $\left|\rho z-\xi\right|\geq\max\left\{1-\rho,\sin\theta_1\right\}$, we obtain

$$\begin{aligned}\left|k_n\left(rz,\xi\right)-k_n\left(z,\xi\right)\right| &= \left|\int_r^1\frac{1}{\rho}\left(\frac{1-\rho^2}{\left|\rho z-\xi\right|^n}-1\right)d\rho\right|\leq\int_r^1\frac{1}{\rho}\frac{1-\rho^2}{\left|\rho z-\xi\right|^n}d\rho-\ln r\\ &\leq -\ln r+\frac{1}{r\left|\sin\theta_1\right|^{n-3/2}}\int_r^1\frac{2\left(1-\rho\right)}{\left(1-\rho\right)^{3/2}}d\rho=-\ln r+\frac{4\sqrt{1-r}}{r\left|\sin\theta_1\right|^{n-3/2}}.\end{aligned}$$

Using the above inequality and passing to spherical coordinates as in the proof above, we conclude

$$\begin{aligned}\left|U\left(rz\right)-\int_{\partial\mathbb{U}}\varphi\left(\xi\right)k_n\left(z,\xi\right)\sigma_0\left(d\xi\right)\right| &= \left|\int_{\partial\mathbb{U}}\varphi\left(\xi\right)\left(k_n\left(rz,\xi\right)-k_n\left(z,\xi\right)\right)\sigma_0\left(d\xi\right)\right|\\ &\leq -\|\varphi\|_\infty\ln r+\frac{4\sqrt{1-r}\,\|\varphi\|_\infty}{r}\int_{\partial\mathbb{U}}\frac{1}{\left|\sin\theta_1\right|^{n-3/2}}\sigma_0\left(d\xi\right)\\ &= -\|\varphi\|_\infty\ln r+\frac{4\sqrt{1-r}\,\|\varphi\|_\infty}{r}\int_0^\pi\frac{1}{\sqrt{\left|\sin\theta_1\right|}}\frac{\omega_{n-1}}{\omega_n}d\theta_1\\ &= -\|\varphi\|_\infty\ln r+\frac{4\sqrt{1-r}\,\|\varphi\|_\infty}{r}2\sqrt{2}K\left(\tfrac{1}{2}\right)\frac{\omega_{n-1}}{\omega_n},\end{aligned}$$

where $K\left(\frac{1}{2}\right)\approx 1.854$ is the complete integral of the first kind. Passing to the limit with $r\nearrow 1$ and using the fact that $\lim_{r\to 1}U\left(rz\right)=U\left(z\right)=\Lambda^{-1}\left(\phi\right)\left(z\right)$ proves the first claim of the theorem.

To prove the second claim, denoting by $a = z \cdot \xi$, we have

$$k_2(z,\xi) = -\int_0^1 \frac{2\rho - 2a}{\rho^2 - 2a\rho + 1} d\rho = -\ln\left(\rho^2 - 2a\rho + 1\right)\Big|_{\rho=0}^{\rho=1} = -\ln(2-2a) = -2\ln|z-\xi|,$$

and

$$\begin{aligned} k_3(z,\xi) &= \int_0^1 \frac{-2\rho + 2a}{\left(\rho^2 - 2a\rho + 1\right)^{3/2}} d\rho + \int_0^1 \frac{1}{\rho^2}\left(\frac{1}{\left(1 - 2a/\rho + 1/\rho^2\right)^{1/2}} - \rho\right) d\rho \\ &= \frac{2}{\left(\rho^2 - 2a\rho + 1\right)^{1/2}}\Bigg|_0^1 - \int_\infty^1 \frac{1}{\left(t^2 - 2at + 1\right)^{1/2}} - \frac{1}{t} dt \\ &= \frac{2}{\sqrt{2-2a}} - 2 + \left(\ln \frac{\left(t - a + \sqrt{t^2 - 2at + 1}\right)}{t}\right)\Bigg|_{t=1}^{t=\infty} \\ &= \frac{2}{|z-\xi|} - 2 + \ln 2 - \ln\left(\frac{|z-\xi|^2}{2} + |z-\xi|\right), \end{aligned}$$

where in the second integral we have used the substitution $\frac{1}{\rho} = t$.

Finally, using algebraic manipulations and integration by parts, for $n > 4$ we obtain the recurrence formula

$$\begin{aligned} k_n(z,\xi) &= k_{n-2}(z,\xi) + \frac{2\left(1 - |z-\xi|^{n-2}\right)}{(n-2)|z-\xi|^{n-2}} - \frac{1 - |z-\xi|^{n-4}}{(n-4)|z-\xi|^{n-4}} + \left(1 - \frac{|z-\xi|^2}{2}\right) \\ &\quad \times J_{n-2}(z,\xi), \end{aligned} \tag{17}$$

where $J_n(z,\xi) = \int_0^1 \frac{1}{|\rho z - \xi|^n} d\rho$ satisfies the recurrence formula

$$J_n(z,\xi) = \frac{4(n-3) J_{n-2}(z,\xi)}{(n-2)\left(4 - |z-\xi|^2\right)|z-\xi|^2} + \frac{2\left(1 + 4|z-\xi|^{n-4} - |z-\xi|^{n-2}\right)}{(n-2)\left(4 - |z-\xi|^2\right)|z-\xi|^{n-2}}. \tag{18}$$

□

2 The Case of a Smooth Planar Domain D

Using conformal mapping arguments, we can extend the results in the previous section to the general case of a smooth bounded simply connected domain $D \subset \mathbb{C}$ ($C^{1,\alpha}$ boundary with $0 < \alpha < 1$ will suffice), as follows.

Fix $w_0 \in D$ and let $f : \mathbb{U} \to D$ be the conformal map of the unit disk $\mathbb{U}$ onto D with $f(0) = w_0$, $\arg f'(0) = 0$, and let $g = f^{-1}$ denote its inverse. It is known (see e.g. [12, Chap. 3]) that if D is smooth ($C^{1,\alpha}$ boundary with $0 < \alpha < 1$, or Dini-smooth boundary suffices), then f, f' have continuous extensions to $\overline{\mathbb{U}}$, and moreover $f' \neq 0$ in $\overline{\mathbb{U}}$. The geometric interpretation of the complex derivative $f'(z)$ (local rotation by $\arg f'(z)$ and local scaling by $\left|f'(z)\right|$ at z under the map f) shows that f preserves the angles between curves at any point $z \in \overline{\mathbb{U}}$. In particular, this allows us to reduce the Neumann problem in D to an equivalent Neumann problem in the disk. Combining this with the results of the preceding section we are led to the following.

Theorem 5 *Let $D \subset \mathbb{C}$ be a smooth bounded simply connected domain ($C^{1,\alpha}$ boundary with $0 < \alpha < 1$ will suffice), and for an arbitrarily fixed $w_0 \in D$ let $f : \mathbb{U} \to D$ be the conformal map of the unit disk $\mathbb{U}$ onto D with $f(0) = w_0$, $\arg f'(0) = 0$, and let $g = f^{-1} : D \to \mathbb{U}$ be its inverse.*

Assume $\phi : \partial D \to \mathbb{R}$ is continuous and satisfies $\int_{\partial D} \phi(w)\,\sigma(dw) = 0$. If u is the solution of the Dirichlet problem (1) with boundary condition

$$\varphi(w) = \frac{1}{|g'(w)|}\phi(w), \qquad w \in \partial D, \tag{19}$$

then

$$U(w) = \int_0^1 \frac{u(f(\rho g(w)))}{\rho}\,d\rho, \qquad w \in D, \tag{20}$$

is the solution to the Neumann problem (2) with $U(w_0) = 0$.

Proof Under the assumption on the smoothness of D, f has a conformal extension to $\overline{\mathbb{U}}$, and therefore $u \circ f$ is harmonic in $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$. We have

$$u(w_0) = u(f(0)) = \frac{1}{2\pi}\int_0^{2\pi} u\left(f\left(re^{i\theta}\right)\right)d\theta,$$

for any $0 < r < 1$. Passing to the limit with $r \nearrow 1$ and using the dominated convergence theorem we obtain

$$\begin{aligned} u(w_0) &= \frac{1}{2\pi}\int_0^{2\pi} u\left(f\left(e^{i\theta}\right)\right)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\left|g'\left(f\left(e^{i\theta}\right)\right)\right|}\phi\left(f\left(e^{i\theta}\right)\right)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \phi\left(f\left(e^{i\theta}\right)\right)\left|f'\left(e^{i\theta}\right)\right|d\theta = \int_{\partial D}\phi(w)\,\sigma(dw) \\ &= 0, \end{aligned}$$

by hypothesis. In particular, this shows that the integrand in (20) is continuous at the origin: $\lim_{\rho\to 0}\frac{u(f(\rho g(w)))}{\rho} = \lim_{\rho\to 0}\frac{u(f(\rho g(w)))-u(f(0))}{\rho} = (\nabla u \circ f)(g(w)) \cdot g(w)$.

The functions f, g, and u are infinitely differentiable, and therefore the integrand in (20) has continuous second order partial derivatives. Differentiating inside the integral, for any $w \in D$ we have

$$\begin{aligned} \Delta_w U(w) &= \int_0^1 \Delta_w \frac{u(f(\rho g(w)))}{\rho}\,d\rho = \int_0^1 \frac{1}{\rho}\Delta_w\left(u\left(h_\rho(w)\right)\right)d\rho \\ &= \int_0^1 \frac{1}{\rho}\left|h'_\rho(w)\right|^2 (\Delta u)\left(h_\rho(w)\right)d\rho = 0, \end{aligned}$$

since u is assumed harmonic in D, where Δ_w represents the Laplace operator with respect to the real variables $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$, and h_ρ denotes the analytic function $h_\rho = f \circ (\rho g) : D \to D$.

Similarly, using the Cauchy-Riemann equations for $h_\rho = f \circ (\rho g)$, we obtain

$$\begin{aligned} \nabla_w U(w) &= \nabla_w \int_0^1 \frac{u(f(\rho g(w)))}{\rho}\,d\rho = \int_0^1 \frac{1}{\rho}\nabla_w\left(u\left(h_\rho(w)\right)\right)d\rho \\ &= \int_0^1 \frac{1}{\rho}(\nabla u)\left(h_\rho(w)\right)\overline{h'_\rho(w)}\,d\rho = \int_0^1 (\nabla u)\left(h_\rho(w)\right)\overline{f'(\rho g(w))\,g'(w)}\,d\rho, \end{aligned}$$

for $w \in \overline{D}$. Since the outward unit normal to $\partial\mathbb{U}$ at $z \in \partial\mathbb{U}$ is $\nu_{\mathbb{U}}(z) = z$, the geometric interpretation of the argument of $f'(z)$ shows that the unitary outward pointing normal to ∂D at $f(z)$ is $\nu(f(z)) = z\frac{f'(z)}{|f'(z)|}$, $z \in \partial\mathbb{U}$.

From the last two relations, for $w = f(z) \in \partial D$ we obtain

$$\begin{aligned}
\frac{\partial U}{\partial \nu}(w) &= \nabla_w U(f(z)) \cdot z\frac{f'(z)}{|f'(z)|} = \operatorname{Re}\left(\nabla_w U(f(z))\,\overline{\left(z\frac{f'(z)}{|f'(z)|}\right)}\right) \\
&= \operatorname{Re}\int_0^1 (\nabla u)\left(h_\rho(f(z))\right)\overline{f'(\rho g(f(z)))\,g'(f(z))}\left(z\frac{f'(z)}{|f'(z)|}\right)d\rho \\
&= \frac{1}{|f'(z)|}\int_0^1 \operatorname{Re}\left((\nabla u)(f(\rho z))\,\overline{zf'(\rho z)}\right)d\rho = \frac{1}{|f'(z)|}\int_0^1 \frac{d}{d\rho}(u(f(\rho z)))\,d\rho \\
&= \frac{1}{|f'(z)|}u(f(\rho z))\Big|_{\rho=0}^{\rho=1} = \frac{1}{|f'(z)|}(u(f(z)) - u(f(0))) \\
&= \frac{1}{|f'(z)|}\left(\frac{1}{|g'(f(z))|}\phi(f(z))\right) = \phi(f(z)) = \phi(w),
\end{aligned}$$

which shows that U has the prescribed boundary values ϕ on ∂D, concluding the proof. □

3 The Case of the Infinite-dimensional Ball

The result in Section 1.2 can be extended to the case of Dirichlet and Neumann problems for the infinite-dimensional ball on an abstract Wiener space. In order to give the result, we begin with some preliminaries.

Following [9] and [8], let (H, B) be an abstract Wiener space, i.e. $(H, \langle\cdot\rangle)$ is a separable real Hilbert space with corresponding norm $|\cdot|$, which is continuously and densely embedded into a Banach space $(B, \|\cdot\|)$, and let $(W_t)_{t\geq 0}$ be the standard Wiener process with state space B. For an open set $V \subset B$ we will denote by $\tau_x^V = \inf\{t \geq 0 : x + W_t \notin V\}$ the first exit time from V of the Brownian motion $(x + W_t)_{t\geq 0}$ starting at x. In particular, if $V = \mathbb{U}_r(0) = \{x \in B : \|x\| < r\}$ we will write $\tau_x^{(r)}$ for $\tau_x^{\mathbb{U}_r(0)}$, and if $r = 1$ we will omit the upper index and write τ_x for $\tau_x^{(1)}$. Also, if $x = 0$ we will omit the lower index 0 in the preceding notations.

For a Borel measurable function $f : \mathbb{S}_r(x) \to \mathbb{R}$, consider the average $(A_r f)(x)$ of f over $\mathbb{S}_r(x)$ (see [8]) defined by

$$(A_r f)(x) = \int_{\mathbb{S}_r(0)} f(x+y)\pi_r(dy), \tag{21}$$

whenever the integral exists, where $\mathbb{S}_r(x) = \partial\mathbb{U}_r(x)$ denotes the boundary of the ball $\mathbb{U}_r(x)$ of radius $r > 0$ centered at x in B, and π_r is the *central hitting measure* defined on the Borel subsets of $\mathbb{S}_r(0)$ by $\pi_r(E) = P(W_{\tau^{(r)}} \in E)$.

Also recall ([8]) that a function $f : V \subset B \to \mathbb{R}$ defined on an open subset V of the Wiener space (H, B) is called *harmonic* if it is locally bounded, Borel measurable, finely continuous, and there exists $\rho > 0$ such that $(A_r f)(x) = f(x)$, for every $0 < r < \rho$ for which $\overline{\mathbb{U}_r(x)} \subset V$.

The *generalized Laplacian* ([9, Definition 4]) of a Borel measurable function $f : V \subset B \to \mathbb{R}$ defined on (H, B) at the point $x \in V$ is defined by

$$\Delta f(x) = 2 \lim_{r \searrow 0} \frac{(A_r f)(x) - f(x)}{E\tau^{(r)}}, \tag{22}$$

if this limit exists.

Remark 2 The strong Markov property of the infinite-dimensional Brownian motion shows (see for example Corollary 1.2 and Remark 3.4 in [9]) that if V is strongly regular and if $f : \partial V \to \mathbb{R}$ is bounded and continuous, the *stochastic solution* of the Dirichlet problem for V with boundary values f given by

$$u(x) = E\left(f(x + W_{\tau_x^V}) 1_{\tau_x^V < \infty}\right), \qquad x \in V, \tag{23}$$

is a continuous, harmonic function in V, and has limiting boundary values f on ∂V.

There are known sufficient conditions under which the generalized Laplace operator $\Delta f(x)$ defined above coincides with the usual trace $D^2 f(x)$ operator for a smooth function f (see for example [9, Corollary 8.1]).

We consider the corresponding generalized Dirichlet and Neumann problems for the generalized Laplace operator Δ on a smooth open set $V \subset B$ as the problem of finding a continuous function $u : \overline{V} \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta u = 0 \text{ in } V \\ u = \varphi \text{ on } \partial V \end{cases}, \tag{24}$$

respectively

$$\begin{cases} \Delta u = 0 \text{ in } V \\ \frac{\partial u}{\partial \nu} = \phi \text{ on } \partial V \end{cases}, \tag{25}$$

where $\nu(x)$ denotes the outward unit normal to the boundary of V at $x \in \partial V$.

Remark 3 Remark 2 above shows that in the case $V = \mathbb{U}_1(0)$ of the unit sphere, the infinite-dimensional Dirichlet problem (24) above has at least a solution, namely the stochastic solution. In turn, as we will see in the next theorem, this will allow us to construct a solution for the infinite-dimensional Neumann problem (25).

In order to carry out the infinite-dimensional analogue of the correspondence between Dirichlet and Neumann problem presented in the previous sections, in the sequel we will restrict to the particular case of Hilbert spaces. More precisely, for the remaining part of this section we will assume that the Banach space $(B, \|\cdot\|)$ is also a Hilbert space, with corresponding inner product denoted by $\langle \cdot, \cdot \rangle_B$, and we will also assume that the inclusion $H \subset B$ is Hilbert-Schmidt.

Although the above additional assumptions may not be necessary for the validity of the main theorem below, we could not find a simple proof for the Hölder continuity of the stochastic solution of the Dirichlet problem in the general setup of abstract Wiener spaces (see Lemma 1 and Corollary 1 below), and, in order not to obscure the general idea, we have chosen the case of Hilbert spaces.

The additional hypotheses on the Wiener space (H, B) show (as in [4], Section 3, or [1], Proposition 3.5) that there exists an orthonormal basis $\{e_n : n \geq 1\}$ of H contained in B' and a sequence of positive numbers $(\lambda_n)_{n\geq 1}$ with $\sum_{n=1}^{\infty}\lambda_n < \infty$, such that $\{\overline{e_n} = \frac{e_n}{\sqrt{\lambda_n}} : n \geq 1\}$ is an orthonormal basis of B satisfying

$$\lambda_n\, {}_{B'}\langle e_n, z\rangle_B = \langle e_n, z\rangle_B, \qquad n \geq 1,\ z \in B. \tag{26}$$

With the above preparation we can now state the extension of Theorem 1 to this infinite-dimensional setting, as follows.

Theorem 6 *Assume* $\phi : \mathbb{S}_1(0) \to \mathbb{R}$ *satisfies* $\int_{\mathbb{S}_1(0)} \phi(z)\,\pi_1(dz) = 0$ *and is Lipschitz continuous. If* u *is the stochastic solution of the Dirichlet problem* (24) *for* $V = \mathbb{U}_1(0)$ *with boundary condition* $\varphi = \phi$ *on* $\mathbb{S}_1(0)$*, then*

$$U(z) = \int_0^1 \frac{u(\rho z)}{\rho}\,d\rho, \qquad z \in \mathbb{U}_1(0), \tag{27}$$

is a solution to the Neumann problem (25) *for* $V = \mathbb{U}_1(0)$ *with* $U(0) = 0$.

Before proceeding with the proof, we will first show that under the hypotheses of the theorem, the function U given by (27) is well-defined. We begin with the following lemma which shows that with high probability, an infinite-dimensional Brownian motion starting near the boundary of the unit sphere will exit from it near its starting point.

Lemma 1 *Let* $(W_t)_{t\geq 0}$ *be an infinite-dimensional Brownian motion starting at* $W_0 = x \in \mathbb{U}_1(0) \setminus \{0\}$ *and let* $\tau = \inf\{t \geq 0 : W_t \notin \mathbb{U}_1(0)\}$*. Setting* $\varepsilon = 1 - \|x\|$*, we have*

$$P\left(\|W_\tau - W_0\| \leq 5\sqrt[4]{\varepsilon}\right) \geq 1 - \sqrt{\varepsilon}. \tag{28}$$

Proof For $z \in B'\setminus\{0\}$, denote by $\mathcal{H}_z = \{y \in B : \langle z, y - z\rangle_B = 0\}$, by $\mathcal{H}z^+ = \{y \in B :< z, y - z\rangle_B \geq 0\}$ the closed half-space delimited by $\mathcal{H}_z$ which contains $2z$, and by $\mathcal{S}_z = \partial\mathbb{S}_1(0) \cap \mathcal{H}_z^+$ the hyperspherical cap with base centered z (see Fig. 1).

For $x \in \mathbb{U}_1(0) \setminus \{0\}$ arbitrarily fixed, consider $x_n = \sum_{k=1}^{n} {}_{B'}\langle e_k, x\rangle_B e_k$, $n \geq 1$. Since $x \neq 0$, without loss of generality we may assume that ${}_{B'}\langle x_n, x\rangle_B > 0$ for all $n \geq 1$.

The projection $B_t^n = {}_{B'}\langle \frac{1}{|x_n|}x_n, W_t\rangle_B$, $t \geq 0$, of the Brownian motion $(W_t)_{t\geq 0}$ on the direction of the unitary vector $x_n^* = \frac{1}{|x_n|}x_n$ of H is a 1-dimensional Brownian motion starting $B_0^n = \frac{1}{|x_n|}\, {}_{B'}\langle x_n, x\rangle_B$.

Using that $\{e_n : n = 1, 2, \ldots\} \subset B' \subset H$ is an orthonormal basis in H and ${}_{B'}\langle b, h\rangle_B = \langle b, h\rangle$ for $b \in B'$ and $h \in H$, we obtain

$$0 < {}_{B'}\langle x_n, x\rangle_B = \sum_{k=1}^{n} {}_{B'}\langle e_k, x\rangle_B^2 = \sum_{k=1}^{n} {}_{B'}\langle e_k, x_n\rangle_B^2 = \sum_{k=1}^{n}\langle e_k, x_n\rangle^2 \leq |x_n|^2,$$

by Bessel's inequality.

Also,

$$0 < {}_{B'}\langle x_n, x\rangle_B = \sum_{k=1}^{n} {}_{B'}\langle e_k, x\rangle_B^2 = \sum_{k=1}^{n}\left\langle \frac{e_k}{\sqrt{\lambda_k}}, x_n\right\rangle_B^2 \leq \|x\|^2,$$

again by Bessel's inequality, since $\left\{\overline{e_k} = \frac{e_k}{\sqrt{\lambda_k}} : k = 1, 2, \ldots\right\}$ is an orthonormal basis of B.

Combining the last two inequalities above we conclude $0 < {}_{B'}\langle x_n, x\rangle_B \leq |x_n|\,\|x\|$, and therefore $\varepsilon_n = 1 - B_0^n = 1 - {}_{B'}\langle \frac{1}{|x_n|}x_n, x\rangle_B \in (1 - \|x\|, 1) \subset (0, 1)$ for all $n \geq 1$.

The continuity of the paths of Brownian motion show that if the Brownian motion B^n exits the interval $\left(1 - \varepsilon_n - \sqrt{\varepsilon_n}, 1\right)$ through the endpoint 1, the Brownian motion W will

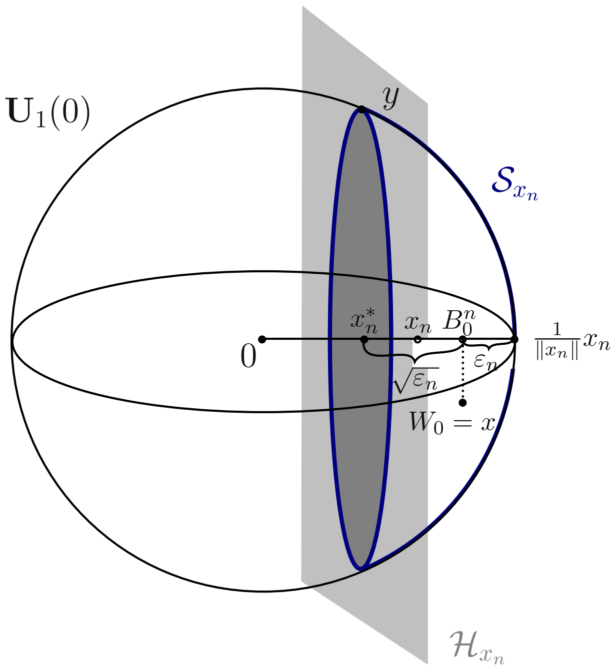

Fig. 1 The half-space $\mathcal{H}_{x_n}$ and the hyperspherical cap $\mathcal{S}_{x_n}$ in the proof of Lemma 1

exit from the sphere $\mathbb{U}_1(0)$ through the hyperspherical cap $\mathcal{S}_{x_n^*} \subset \mathbb{S}_1(0)$ with base vertex at $x_n^* = \frac{1-\varepsilon_n-\sqrt{\varepsilon_n}}{\|x_n\|}x_n$ and height $\|x_n^* - \frac{1}{\|x_n\|}x_n\| = \varepsilon_n + \sqrt{\varepsilon_n}$, and therefore we obtain

$$P\left(W_\tau \in \mathcal{S}_{x_n^*}\right) \geq P\left(B_{\tau'}^n = 1\right). \tag{29}$$

where $\tau' = \inf\left\{t \geq 0 : B_t^n \notin \left(1 - \varepsilon_n - \sqrt{\varepsilon_n}, 1\right)\right\}$ is the exit time of the 1-dimensional Brownian motion B^n from the interval $\left(1 - \varepsilon_n - \sqrt{\varepsilon_n}, 1\right)$.

Since $\{\overline{e_n} = \frac{e_n}{\sqrt{\lambda_n}} : n \geq 1\}$ is an orthonormal basis of B and using (26), we obtain

$$\left\langle \frac{1}{\|x_n\|}x_n, x\right\rangle_B = \frac{1}{\sqrt{\sum_{k=1}^n \langle \overline{e}_k, x\rangle_B^2}} \sum_{k=1}^n \langle \overline{e}_k, x\rangle_B^2 = \sqrt{\sum_{k=1}^n \langle \overline{e}_k, x\rangle_B^2} \nearrow \|x\|$$

as $n \to \infty$, and therefore for n large enough we have

$$\left\| \frac{1}{\|x_n\|}x_n - x\right\| = \sqrt{\|x\|^2 + 1 - 2\left\langle \frac{1}{\|x_n\|}x_n, x\right\rangle_B} \leq 2(1 - \|x\|) = 2\varepsilon.$$

It follows that for n large enough and all $y \in \mathcal{S}_{x_n^*}$ we have

$$\begin{aligned}\|y - x\| &\leq \left\| y - B_0^n \frac{1}{\|x_n\|}x_n\right\| + \left\| B_0^n \frac{1}{\|x_n\|}x_n - \frac{1}{\|x_n\|}x_n\right\| + \left\| \frac{1}{\|x_n\|}x_n - x\right\| \\ &\leq \sqrt{1 + \varepsilon_n - \left(1 - \varepsilon_n - \sqrt{\varepsilon_n}\right)^2} + \varepsilon_n + 2\varepsilon,\end{aligned}$$

and combining with (29) we obtain

$$P\left(\|W_\tau - W_0\| \le \sqrt{1+\varepsilon_n - \left(1-\varepsilon_n-\sqrt{\varepsilon_n}\right)^2} + \varepsilon_n + 2\varepsilon\right) \ge P\left(B^n_{\tau'} = 1\right).$$

Since $(B_{t\wedge\tau'})_{t\ge 0}$ is a bounded martingale, using Doob's optional stopping and dominated convergence theorems we obtain

$$1-\varepsilon_n = EB^n_0 = EB^n_{\tau'} = \left(1-\varepsilon_n-\sqrt{\varepsilon_n}\right) P\left(B^n_{\tau'} = 1-\varepsilon_n-\sqrt{\varepsilon_n}\right) + P\left(B^n_{\tau'}=1\right),$$

and therefore

$$P\left(B^n_{\tau'}=1\right) = \frac{1}{1+\sqrt{\varepsilon_n}} \ge 1-\sqrt{\varepsilon_n}.$$

Together with the previous inequality this shows that

$$P\left(\|W_\tau - W_0\| \le \sqrt{1+\varepsilon_n - \left(1-\varepsilon_n-\sqrt{\varepsilon_n}\right)^2} + \varepsilon_n + 2\varepsilon\right) \ge 1-\sqrt{\varepsilon_n}. \tag{30}$$

Using the inequality

$$\sqrt{1+x-\left(1-x-\sqrt{x}\right)^2}+x \le \sqrt{2x+2\sqrt{x}}+x \le 2\sqrt[4]{x}+x \le 3\sqrt[4]{x}, \tag{31}$$

valid for $x \in [0, 1]$, we obtain

$$P\left(\|W_\tau - W_0\| \le 3\sqrt[4]{\varepsilon_n}+2\varepsilon\right) \ge 1-\sqrt{\varepsilon_n}, \qquad n \ge 1. \tag{32}$$

Note that

$$\begin{aligned}
\lim_{n\to\infty} |x_n|^2 &= \lim_{n\to\infty} \langle x_n, x_n\rangle = \lim_{n\to\infty} {}_{B'}\langle x_n, x_n\rangle_B \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} {}_{B'}\langle e_k, x\rangle_B^2 = \lim_{n\to\infty} \sum_{k=1}^{n} \langle \overline{e_k}, x\rangle_B^2 = \sum_{k=1}^{\infty} \langle \overline{e_k}, x\rangle_B^2 \\
&= \|x\|^2
\end{aligned}$$

and

$$\lim_{n\to\infty} {}_{B'}\langle x_n, x\rangle_B = \lim_{n\to\infty} \sum_{k=1}^{n} {}_{B'}\langle e_k, x\rangle_B^2 = \lim_{n\to\infty} \sum_{k=1}^{n} \langle \overline{e_k}, x\rangle_B^2 = \sum_{k=1}^{\infty} \langle \overline{e_k}, x\rangle_B^2 = \|x\|^2,$$

by Parseval's identity, and therefore

$$\lim_{n\to\infty} \varepsilon_n = 1 - \lim_{n\to\infty} {}_{B'}\left\langle \frac{1}{|x_n|} x_n, x\right\rangle_B = 1-\|x\| = \varepsilon.$$

Passing to the limit in (32) with $n \to \infty$ we obtain

$$P\left(\|W_\tau - W_0\| \le 5\sqrt[4]{\varepsilon}\right) \ge P\left(\|W_\tau - W_0\| \le 3\sqrt[4]{\varepsilon}+2\varepsilon\right) \ge 1-\sqrt{\varepsilon},$$

concluding the proof of the lemma. □

Assume now that $\phi : \mathbb{S}_1(0) \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant K, i.e.

$$|\phi(x) - \phi(y)| \le K\|x-y\|, \qquad x, y \in \mathbb{S}_1(0), \tag{33}$$

and satisfies the normalization condition $\int_{\mathbb{S}_1(0)} \phi(z)\,\pi_1(dz) = 0$. Let u be the stochastic solution of the Dirichlet problem (24) for $V = \mathbb{U}_1(0)$ with boundary values ϕ, that is

$$u(x) = E\phi(x+W_\tau), \qquad x \in \mathbb{U}_1(0),$$

where E denotes the expectation with respect to the infinite-dimensional Brownian motion $(W_t)_{t\geq 0}$ starting at the origin and $\tau_x = \inf\{t \geq 0 : x + W_t \notin \mathbb{U}_1(0)\}$ is the exit time of $x + W$ from $\mathbb{U}_1(0)$.

Since by hypothesis $u(0) = \int_{\mathbb{S}_1(0)} \phi(z)\, \pi_1(dz)$, for $x \in \mathbb{U}_1(0) - \{0\}$ arbitrarily fixed, we have

$$|u(x)| = |u(x) - u(0)| = \left| E\left(\phi\left(W_{\tau_x}\right)\right) - \phi\left(W_{\tau_0}\right) \right|,$$

Considering the event $A = \left\{ \left\| (x + W_{\tau_x} - W_{\tau_0}) \right\| \geq 5\sqrt[4]{\|x\|} \right\}$, and using the hypothesis (33) on ϕ, we obtain

$$\begin{aligned} |u(x)| &\leq K E\left(\left\| x + W_{\tau_x} - W_{\tau_0} \right\|\right) \\ &\leq K E\left(\left\| x + W_{\tau_x} - W_{\tau_0} \right\| 1_{A^c}\right) + K E\left(\left\| x + W_{\tau_x} - W_{\tau_0} \right\| 1_A\right) \\ &\leq 5K\sqrt[4]{\|x\|} + 2K P(A). \end{aligned} \tag{34}$$

To estimate the probability of the event A, we use the strong Markov property of the infinite-dimensional Brownian motion (see [9, Remark 3.4]) at the a.s. finite stopping time $\tau_0 \wedge \tau_x$ and Lemma 1 above, as follows

$$P(A) = E\left(E\left(1_A | \mathcal{F}_{\tau_0 \wedge \tau_x}\right)\right) = E\left(1_{\tau_0 < \tau_x} \psi\left(x + W_{\tau_0}\right) + 1_{\tau_x < \tau_0} \psi\left(W_{\tau_x}\right)\right), \tag{35}$$

where $\psi(y) = P^y\left(\left\| W'_{\tau'_y} - W'_0 \right\| > 5\sqrt[4]{1 - \|y\|}\right)$ denotes the probability that an infinite-dimensional Brownian motion $\left(W'_t\right)_{t\geq 0}$ starting at $y \in \mathbb{U}_1(0)$ will exit $\mathbb{U}_1(0)$ at a distance greater than $5\sqrt[4]{1 - \|y\|}$ from its starting point.

Note that on the event $\{\tau_0 < \tau_x\}$ we have $\operatorname{dist}\left(x + W_{\tau_0}, \partial\mathbb{U}_1(0)\right) \leq \left\| (x + W_{\tau_0}) - W_{\tau_0} \right\| = \|x\|$, so using Lemma 1 we obtain

$$\begin{aligned} \psi\left(x + W_{\tau_0}\right) &\leq P^{x + W_{\tau_0}}\left(\left\| W'_{\tau_{x+W_{\tau_0}}} - W'_0 \right\| > 5\sqrt[4]{\operatorname{dist}\left(W'_0, \partial\mathbb{U}_1(0)\right)}\right) \\ &\leq \sqrt{\operatorname{dist}\left(W'_0, \partial\mathbb{U}_1(0)\right)} = \sqrt{\operatorname{dist}\left(x + W_{\tau_0}, \partial\mathbb{U}_1(0)\right)} \\ &\leq \sqrt{\|x\|}. \end{aligned} \tag{36}$$

Similarly, on the event $\{\tau_0 < \tau_x\}$ we obtain $\psi\left(W_{\tau_x}\right) \leq \sqrt{\|x\|}$, and using (34) – (36) we conclude

$$|u(x)| \leq 5K\sqrt[4]{\|x\|} + 4K\sqrt{\|x\|}, \qquad x \in \mathbb{U}_1(0). \tag{37}$$

The inequality above is sufficient to prove that the function U given by (27) is well defined, since under the hypothesis on ϕ the function u is continuous on $\mathbb{U}_1(0)$, and

$$\begin{aligned} \lim_{r \searrow 0} \left| \int_0^r \frac{u(\rho x)}{\rho} d\rho \right| &\leq \lim_{r \searrow 0} \int_0^r \frac{5K\sqrt[4]{\rho\|x\|} + 4K\sqrt{\rho\|x\|}}{\rho} d\rho \\ &= \lim_{r \searrow 0} \left(20K\sqrt[4]{r\|x\|} + 8K\sqrt{r\|x\|}\right) \\ &= 0, \end{aligned}$$

so that the integral in (27) is well defined for all $z \in \mathbb{U}_1(0)$.

Remark 4 We believe that the function U given by (27) is also well-defined under the weaker hypotheses of just continuity and boundedness of ϕ on $\mathbb{S}_1(0)$ (rather than the Lipschitz continuity hypothesis), but we were unable to find a proof of this fact.

The proof above shows that $|u(x)-u(0)| \leq 9K\sqrt[4]{\|x\|}$ for $x \in \mathbb{U}_1(0)$. The same proof, but with Brownian motions starting at x and y instead of x and 0, shows that

$$|u(x)-u(y)| \leq 9K\|x-y\|^{1/4}, \qquad x, y \in \mathbb{U}_1(0),$$

and we obtain the following.

Corollary 1 *The stochastic solution of the Dirichlet problem* (24) *for* $V = \mathbb{U}_1(0)$, *with Lipschitz continuous Dirichlet boundary condition* $\varphi : \mathbb{S}_1(0) \to \mathbb{R}$ *satisfying* $\int_{\mathbb{S}_1(0)} \varphi(z)\,\pi_1(dz) = 0$ *is Hölder continuous of order* $\frac{1}{4}$ *in* $\mathbb{U}_1(0)$.

Proof of Theorem 6. For arbitrarily fixed $x \in \mathbb{U}_1(0)$ and $0 < r < 1 - \|x\|$, the inequality (37) shows that for any $\rho \in (0, 1]$ and $y \in \mathbb{S}_r(0)$ we have

$$\frac{u(\rho(x+y))}{\rho} \leq \frac{1}{\rho}\left(5K\sqrt[4]{\|\rho(x+y)\|} + 4K\sqrt{\|\rho(x+y)\|}\right) \leq 5K\rho^{-3/4} + 4K\rho^{-1/2},$$

and therefore the function $\frac{u(\rho(x+y))}{\rho}$ is integrable on $\mathbb{S}_r(0) \times [0, 1]$ with respect to the product measure $\pi_r(dy) \times d\rho$. Note that the scaling invariance of the infinite-dimensional Brownian motion (see e.g. [9, Remark 3.3]) implies that $\pi_r(dy) = \pi_{r\rho}(d(\rho y))$. Using this, Fubini's theorem and the substitution $y' = \rho y$, we obtain

$$\begin{aligned} A_r U(x) &= \int_{\mathbb{S}_r(0)} \int_0^1 \frac{u(\rho(x+y))}{\rho} d\rho \pi_r(dy) = \int_0^1 \frac{1}{\rho} \int_{\mathbb{S}_r(0)} u(\rho x + \rho y)\pi_r(dy) d\rho \\ &= \int_0^1 \frac{1}{\rho} \int_{\mathbb{S}_{r\rho}(0)} u(\rho x + y')\pi_{r\rho}(dy') d\rho = \int_0^1 \frac{1}{\rho} A_{r\rho} u(\rho x) d\rho = \int_0^1 \frac{u(\rho x)}{\rho} d\rho \\ &= U(x), \end{aligned}$$

since u is a harmonic function in $\mathbb{U}_1(0)$ (see Remark 2). It follows that U is also harmonic in $\mathbb{U}_1(0)$, so in particular $\Delta U = 0$ in $\mathbb{U}_1(0)$.

The outward unit normal to the boundary of $\mathbb{U}_1(0)$ at $x \in \mathbb{S}_1(0)$ is $\nu(x) = x$. For an arbitrary $x \in \mathbb{U}_1(0)$ we have

$$\begin{aligned} \frac{\partial U}{\partial \nu}(x) &= \lim_{\varepsilon \nearrow 0} \frac{U(x + \varepsilon\nu(x)) - U(x)}{\varepsilon} = \lim_{\varepsilon \nearrow 0} \frac{1}{\varepsilon} \left(\int_0^1 \frac{u(\rho(1+\varepsilon)x)}{\rho} d\rho - \int_0^1 \frac{u(\rho x)}{\rho} d\rho \right) \\ &= \lim_{\varepsilon \nearrow 0} \frac{1}{\varepsilon} \left(\int_0^{1+\varepsilon} \frac{u(\rho' x)}{\rho'} d\rho' - \int_0^1 \frac{u(\rho x)}{\rho} d\rho \right) = \lim_{\varepsilon \nearrow 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} \frac{u(\rho x)}{\rho} d\rho \\ &= \lim_{\varepsilon \nearrow 0} \frac{u(\rho^* x)}{\rho^*} \\ &= u(x), \end{aligned}$$

by the continuity of u in $\overline{\mathbb{U}_1(0)}$, where $\rho^* \in (1+\varepsilon, 1)$ denotes the intermediate point given by the mean value theorem. This shows that the values of the normal derivative $\frac{\partial U}{\partial \nu}$ on $\partial\mathbb{U}_1(0)$ coincide with the boundary values of u on $\partial\mathbb{U}_1(0)$, concluding the proof. □

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