

Heat Kernel Bounds on Metric Measure Spaces and Some Applications

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Abstract Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. We derive the upper and lower bounds of the heat kernel on (X, d, μ) by applying the parabolic Harnack inequality and the comparison principle, and then sharp bounds for its gradient, which are also sharp in time. For applications, we study the large time behavior of the heat kernel, the stability of solutions to the heat equation, and show the L^p boundedness of (local) Riesz transforms.

Keywords Metric measure space · Ricci curvature · Heat kernel · Heat equation · Riesz transform

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1 Introduction and Main Results

Let M be a complete (smooth) Riemannian manifold with dimension $n \geq 2$, and let p_t be the heat kernel. Denote the length in the tangent space by $|\cdot|$. It is well-known from Li and Yau [25] that, if M has nonnegative Ricci curvature, then for any $\epsilon > 0$, there exist positive constants $C(\epsilon)$ and $C_1(\epsilon)$, such that

$$\frac{1}{C(\epsilon)\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 - \epsilon)t}\right) \leq p_t(x, y) \leq \frac{C(\epsilon)}{\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 + \epsilon)t}\right),$$

and

$$|\nabla_x p_t(x, y)| \leq \frac{C_1(\epsilon)}{\sqrt{t}\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 + \epsilon)t}\right),$$

for all $t > 0$ and all $x, y \in M$.

It is known that the upper and lower bounds of $p_t(x, y)$ has been extended by Sturm [34, 35] to the Dirichlet space (X, \mathcal{E}, μ) supporting a (weak) local Poincaré inequality and the doubling measure μ , where X is a locally compact separable Hausdorff space, \mathcal{E} is a strongly local, regular symmetric Dirichlet form and μ is a positive Radon measure. Note that here the distance is the so-called intrinsic distance induced by the Dirichlet form \mathcal{E} . For the case of the non-symmetric and time-dependent Dirichlet form in the similar framework, see the recent paper [26].

Throughout this work, let (X, d, μ) be a metric measure space, such that (X, d) is a complete and separable metric space and μ is a locally finite (i.e., finite on bounded sets) Borel regular measure with support the whole space X .

Recently, in the metric measure space (X, d, μ) , Erbar et al. [17] and Ambrosio et al. [8] introduced the Riemannian curvature-dimension condition, denoted by $RCD^*(K, N)$, which is a generalization of the ‘‘Ricci curvature lower bound’’ for the non-smooth setting and a strengthening of the curvature-dimension condition introduced by Lott and Villani [27] and Sturm [36, 37]. The $RCD^*(K, N)$ space resembles more a Riemannian structure in the sense that Cheeger energy and Sobolev spaces are Hilbertian. We refer the reader to [1, 3–5, 7, 17] for more details (see also Section 2 below). In this work, we study the sharp heat kernel bounds on $RCD^*(K, N)$ spaces with $K \in \mathbb{R}$ and $N \in [1, \infty)$, and then give some applications.

Notice that Sturm’s results in [33–35] are valid for a metric measure space (X, d, μ) satisfying the $RCD^*(0, N)$ condition with $N \in [1, \infty)$, since in these cases doubling property and Poincaré inequality hold; see e.g. [17, 29]. However, for example, the constant in the exponential term in the Gaussian lower bound in [35] is not sharp. The first main result below gives a sharper lower bound of $p_t(x, y)$. The approach is adapted from Sturm [32] in the Riemannian setting, by applying the Laplacian comparison principle established by Gigli in [19] and the parabolic Harnack inequality established by Garofalo and Mondino [18] and the first named author [22].

Theorem 1.1 *Let (X, d, μ) be a $RCD^*(0, N)$ space with $N \in [1, \infty)$. Given any $\epsilon > 0$, there exists a positive constant $C_1(\epsilon)$ such that*

$$\frac{1}{C_1(\epsilon)\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 - \epsilon)t}\right) \leq p_t(x, y) \leq \frac{C_1(\epsilon)}{\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 + \epsilon)t}\right) \tag{1.1}$$

for all $t > 0$ and all $x, y \in X$.

Combining this, and the Li–Yau inequality from [18, 22], we can immediately derive a sharp bound for the gradient of the heat kernel. Here, we denote the minimum weak upper gradient of a function $f : X \rightarrow \mathbb{R}$ by $|\nabla f|$ (assume its existence at present).

Corollary 1.1 *Let (X, d, μ) be a $RCD^*(0, N)$ space with $N \in [1, \infty)$. Given any $\epsilon > 0$, there exists a positive constant $C_1(\epsilon)$ such that*

$$|\nabla p_t(x, \cdot)|(y) \leq \frac{C_1(\epsilon)}{\sqrt{t}\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 + \epsilon)t}\right). \quad (1.2)$$

for all $t > 0$ and μ -a.e. $x, y \in X$.

Notice that in [22], the author used similar forms of Eqs. 1.1 and 1.2 but with implicit constants in the exponential terms to obtain the Li–Yau inequality. The strength of Theorem 1.1 and Corollary 1.1 is that we obtain the sharp constants in the exponential terms.

We shall then establish the following heat kernel bounds on general $RCD^*(K, N)$ spaces with $K < 0$ and $N \in [1, \infty)$.

Theorem 1.2 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K < 0$ and $N \in [1, \infty)$. Given any $\epsilon > 0$, there exist positive constants $C_1(\epsilon), C_2(\epsilon)$, depending also on K, N , such that*

$$\begin{aligned} \frac{1}{C_1(\epsilon)\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 - \epsilon)t} - C_2(\epsilon)t\right) &\leq p_t(x, y) \\ &\leq \frac{C_1(\epsilon)}{\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 + \epsilon)t} + C_2(\epsilon)t\right) \end{aligned} \quad (1.3)$$

for all $t > 0$ and all $x, y \in X$.

Corollary 1.2 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K < 0$ and $N \in [1, \infty)$. Given any $\epsilon > 0$, there exist positive constants $C_1(\epsilon), C_2(\epsilon)$ such that*

$$|\nabla p_t(x, \cdot)|(y) \leq \frac{C_1(\epsilon)}{\sqrt{t}\mu(B(y, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4 + \epsilon)t} + C_2(\epsilon)t\right). \quad (1.4)$$

for all $t > 0$ and μ -a.e. $x, y \in X$.

Notice that in [28, Lemma 3.3], by using an elementary argument, Mondino and Naber obtained sharp lower and upper bounds of heat kernels $p_t(x, y)$ for $x, y \in X$ for x, y satisfying $d(x, y) < 10\sqrt{t}$, and an upper bound for the quantity $\int_{X \setminus B(x, r)} p_t(x, y) d\mu(y)$.

In a forthcoming paper, the second named author will use a dimensional free Harnack inequality (cf. [23, 38]) to investigate heat kernel bounds on $RCD(K, \infty)$ spaces (cf. [1, 3, 5]).

The following result generalizes the large time behavior of heat kernels in the Riemannian manifold established by Li [24, Theorem 1] to the present setting with also sharp form.

Theorem 1.3 *Let (X, d, μ) be a $RCD^*(0, N)$ space with $N \in \mathbb{N}$ and $N \geq 2$. Let $x_0 \in X$. If there exists $\theta \in (0, \infty)$ such that $\liminf_{R \rightarrow \infty} \frac{\mu(B(x_0, R))}{R^N} = \theta$, then for any $x, y \in X$, it holds that*

$$\lim_{t \rightarrow \infty} \mu(B(x_0, \sqrt{t})) p_t(x, y) = \omega(N)(4\pi)^{-N/2},$$

where $\omega(N)$ is the volume of the unit ball in \mathbb{R}^N .

According to Xu [39], on an N -dimensional Riemannian manifold with non-negative Ricci curvature, if the volume is not maximal growth (i.e., $\liminf_{R \rightarrow \infty} \frac{\mu(B(x_0, R))}{R^N} = 0$), then the limit of $\mu(B(x_0, \sqrt{t})) p_t(x, y)$ as $t \rightarrow \infty$ does not necessarily exist.

For applications of the bounds on the heat kernel and its gradient, we shall consider stability of solutions to the heat equation, and the boundedness of (local) Riesz transforms.

The paper is organized as follows. In Section 2, we give some basic notations and notions for Sobolev spaces, differential structures, curvature-dimension conditions and heat kernels, and recall some known results. In Section 3, we will provide the proofs of heat kernel and its gradient estimates.

In Section 4, we will prove Theorem 1.3. Stability of solutions to the heat equation will be studied there as well. Notice that, since the approach depends on the comparison results between a $RCD^*(0, N)$ space and the Euclidean space \mathbb{R}^N , we can only obtain the result when $N \in \mathbb{N}$ and $N \geq 2$. The arguments essentially follows from Li [24] with some necessary modifications to our non-smooth context, since e.g. there seems no effective Gauss–Green type formula available so far.

In Section 5, we will establish the boundedness of the Riesz transform $|\nabla(-\Delta)^{-1/2}|$ and its local version on $L^p(X)$ for all $p \in (1, \infty)$, where Δ is the Laplacian (see Definition 2.5 below). The approach follows from the known one in Riemannian manifolds. In the smooth setting, since the Riesz transform of smooth functions with compact supports is well defined, and this class of functions is dense in $L^p(X)$ for each $p \in (1, \infty)$, one only needs to deal with smooth functions with compact support. However, in our non-smooth setting, by applying the results from [9, 14], the main issue left should be to find a suitable acting class for the Riesz transform.

We should mention that, all the results we get generalize the known ones in the Riemannian manifold with Ricci curvature bounded below, and hold in the Alexandrov space with Ricci curvature bounded from below; see [41].

Finally, we make some conventions on notation. Throughout the work, we denote by C, c positive constants which are independent of the main parameters, but which may vary from line to line. The symbol $B(x, R)$ denotes an open ball with center x and radius R with respect to the distance d , and $CB(x, R) = B(x, CR)$. The space $\text{Lip}(X)$ denotes the set of all Lipschitz functions on X .

2 Preliminaries

In this section, we recall some basic notions and several auxiliary results.

2.1 Sobolev Spaces and Differential Structures

Let (X, d) be a complete and separable metric space and let $C([0, 1], X)$ be the space of continuous curves on $[0, 1]$ with values in X equipped with the sup norm. For

$t \in [0, 1]$, the map $e_t : C([0, 1], X) \rightarrow X$ is the evaluation at time t defined by

$$e_t(\gamma) := \gamma_t.$$

A curve $\gamma : [0, 1] \rightarrow X$ is in the absolutely continuous class $AC^q([0, 1], X)$ for some $q \in [1, \infty]$, if there exists $g \in L^q([0, 1])$ such that,

$$d(\gamma_s, \gamma_t) \leq \int_s^t g(r) dr, \quad \text{for any } s, t \in [0, 1] \text{ satisfying } s < t. \tag{2.1}$$

It is true that, if $\gamma \in AC^p([0, 1]; X)$, then the metric slope

$$\lim_{\delta \rightarrow 0} \frac{d(\gamma_{r+\delta}, \gamma_r)}{|\delta|},$$

denoted by $|\dot{\gamma}_r|$, exists for L^1 -a.e. $r \in [0, 1]$, belongs to $L^p([0, 1])$, and it is the minimal function g such that Eq. 2.1 holds (see Theorem 1.1.2 in [2]). The length of the absolutely continuous curve $\gamma : [0, 1] \rightarrow X$ is defined by $\int_0^1 |\dot{\gamma}_r| dr$. We call that (X, d) is a length space if

$$d(x_0, x_1) = \inf \left\{ \int_0^1 |\dot{\gamma}_r| dr : \gamma \in AC^1([0, 1], X), \gamma_i = x_i, i = 0, 1 \right\}, \quad \forall x_0, x_1 \in X.$$

Let μ be a locally finite (i.e., finite on bounded sets) Borel regular measure on (X, d) with support the whole space X . Throughout the work, we call the triple (X, d, μ) the metric measure space.

Definition 2.1 (Test Plan). Let $\pi \in \mathcal{P}(C([0, 1], X))$. We say that π is a test plan if there exists a constant $C > 0$ such that

$$(e_t)_\# \pi \leq C\mu, \quad \text{for all } t \in [0, 1],$$

and

$$\int \int_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma) < \infty.$$

Definition 2.2 (Sobolev Space). The Sobolev class $S^2(X)$ (resp. $S^2_{loc}(X)$) is the space of all Borel functions $f : X \rightarrow \mathbb{R}$, for which there exists a non-negative function $G \in L^2(X)$ (resp. $G \in L^2_{loc}(X)$) such that, for each test plan π , it holds

$$\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) \leq \int \int_0^1 G(\gamma_t) |\dot{\gamma}_t| dt d\pi(\gamma). \tag{2.2}$$

It then follows from a compactness argument that, for each $f \in S^2(X)$ there exists a unique minimal G in the μ -a.e. sense such that Eq. 2.2 holds. We then denote the minimal G by $|\nabla f|$ and call it the minimal weak upper gradient following [6].

The inhomogeneous Sobolev space $W^{1,2}(X)$ is defined as $S^2(X) \cap L^2(X)$ equipped with the norm

$$\|f\|_{W^{1,2}(X)} := \left(\|f\|_{L^2}^2 + \|\nabla f\|_{L^2(X)}^2 \right)^{1/2}.$$

The local Sobolev space $W^{1,2}_{loc}(\Omega)$ for an open set $\Omega \subset X$, and the Sobolev space with compact support $W^{1,2}_c(X)$ can be defined in an obvious manner. The relevant Sobolev spaces have been studied in [6, 12, 30].

The following terminologies and results are mainly taken from [5, 19].

Definition 2.3 (Infinitesimally Hilbertian Space). Let (X, d, μ) be a metric measure space. We say that it is infinitesimally Hilbertian, provided $W^{1,2}(X)$ is a Hilbert space.

Notice that, from the definition, it follows that (X, d, μ) is infinitesimally Hilbertian if and only if, for any $f, g \in S^2(X)$, it holds

$$\|\nabla(f + g)\|_{L^2(X)}^2 + \|\nabla(f - g)\|_{L^2(X)}^2 = 2 \left(\|\nabla f\|_{L^2(X)}^2 + \|\nabla g\|_{L^2(X)}^2 \right).$$

Definition 2.4 Let (X, d, μ) be an infinitesimally Hilbertian space, $\Omega \subset X$ an open set and $f, g \in S_{loc}^2(\Omega)$. The map $\langle \nabla f, \nabla g \rangle : \Omega \rightarrow \mathbb{R}$ is μ -a.e. defined as

$$\langle \nabla f, \nabla g \rangle := \inf_{\epsilon > 0} \frac{|\nabla(g + \epsilon f)|^2 - |\nabla g|^2}{2\epsilon},$$

with the infimum being intended as μ -essential infimum.

We shall sometimes write $\langle \nabla f, \nabla g \rangle$ as $\nabla f \cdot \nabla g$ for convenience. The inner product $\langle \nabla f, \nabla g \rangle$ is linear, and satisfies the Cauchy–Schwarz inequality, the chain rule and the Leibniz rule (see e.g. [19]).

With the aid of the inner product, we can define the Laplacian operator as below. Notice that the Laplacian operator is linear due to (X, d, μ) being infinitesimally Hilbertian.

Definition 2.5 (Laplacian). Let (X, d, μ) be an infinitesimally Hilbertian space. Let $f \in W_{loc}^{1,2}(X)$. We call $f \in \mathcal{D}_{loc}(\Delta)$, if there exists $h \in L_{loc}^1(X)$ such that, for each $\psi \in W_c^{1,2}(X)$, it holds

$$\int_X \langle \nabla f, \nabla \psi \rangle d\mu = - \int_X h \psi d\mu.$$

We will write $\Delta f = h$. If $f \in W^{1,2}(X)$ and $h \in L^2(X)$, we then call $f \in \mathcal{D}(\Delta)$.

From the Leibniz rule, it follows that if $f, g \in \mathcal{D}_{loc}(\Delta) \cap L_{loc}^\infty(X)$ (resp. $f, g \in \mathcal{D}(\Delta) \cap L_{loc}^\infty(X) \cap \text{Lip}(X)$), then $fg \in \mathcal{D}_{loc}(\Delta)$ (resp. $fg \in \mathcal{D}(\Delta)$) satisfies $\Delta(fg) = g\Delta f + f\Delta g + 2\nabla f \cdot \nabla g$.

2.2 Curvature-dimension Conditions and Consequences

Let (X, d, μ) be an infinitesimally Hilbertian space. Denote by H_t the heat flow $e^{t\Delta}$ corresponding to the Dirichlet form $(\mathcal{E}, W^{1,2}(X))$, defined by

$$\mathcal{E}(f, g) = \int_X \langle \nabla f, \nabla g \rangle d\mu, \quad f, g \in W^{1,2}(X).$$

From (X, d, μ) being infinitesimally Hilbertian, it follows that H_t is linear and the Dirichlet form $(\mathcal{E}, W^{1,2}(X))$ is strongly local.

Now we recall the definition of $RCD^*(K, N)$ spaces. See e.g. [17, Sections 3 and 4] for other equivalent characterizations. Here and in what follows, for $K = 0$, $K/(e^{2Kt} - 1) := \lim_{K \rightarrow 0} K/(e^{2Kt} - 1) = 1/(2t)$.

Definition 2.6 Let $K \in \mathbb{R}$ and $N \in [1, \infty)$, and let (X, d, μ) be a length, infinitesimally Hilbertian space satisfying:

- (a) for some constants $C, c > 0$ and some point $o \in X$,

$$\mu(B(o, r)) \leq C e^{cr^2}, \quad \text{for every } r > 0,$$

- (b) each function $f \in W^{1,2}(X)$ with $|\nabla f| \leq 1$ admits a continuous representative,
- (c) for every $f \in W^{1,2}(X)$ and every $t > 0$,

$$|\nabla H_t f|^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)} |\Delta H_t f|^2 \leq e^{-2Kt} H_t(|\nabla f|^2), \quad \mu\text{-a.e. in } X. \quad (2.3)$$

Then we call (X, d, μ) a $RCD^*(K, N)$ space.

On the $RCD^*(K, N)$ space (X, d, μ) , the measure μ satisfies the local doubling (global doubling, provided $K \geq 0$) property, which we present in the next lemma (see e.g. [17, Section 3]).

Lemma 2.1 Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \leq 0$ and $N \in [1, \infty)$, and let $x \in X$ and $0 < r \leq R < \infty$.

- (i) If $K = 0$, then

$$\mu(B(x, R)) \leq \left(\frac{R}{r}\right)^N \mu(B(x, r)).$$

- (ii) If $K < 0$, then

$$\mu(B(x, R)) \leq \frac{l_{K,N}(R)}{l_{K,N}(r)} \mu(B(x, r)),$$

where $(0, \infty) \ni t \mapsto l_{K,N}(t)$ is a continuous function depending on K and N , and $l_{K,N}(t) = O(e^{tC(K,N)})$ as t tends to ∞ for some constant $C(K, N)$ depending on K and N .

From the definition of the $RCD^*(K, N)$ space, we know that (X, d) is a length space. The (local) doubling property immediately implies that every bounded closed ball in (X, d) is totally bounded. Since (X, d) is also complete, it is then proper and geodesic. Recall that a metric space (X, d) is proper if every bounded closed subset is compact. The properness also implies that the Dirichlet form $(\mathcal{E}, W^{1,2}(X))$ is indeed regular.

By [5, Theorem 3.9], we see that the intrinsic metric induced by the Dirichlet form $(\mathcal{E}, W^{1,2}(X))$, defined as

$$d_{\mathcal{E}}(x, y) = \sup\{\psi(x) - \psi(y) : \psi \in W^{1,2}(X) \cap C(X), |\nabla \psi| \leq 1 \mu\text{-a.e. in } X\},$$

for every $x, y \in X$, coincides with the original one, i.e.,

$$d_{\mathcal{E}}(x, y) = d(x, y), \quad \forall x, y \in X.$$

Hence, we can work indifferently with either the distance d or $d_{\mathcal{E}}$.

Recently, Rajala [29] proved that an L^1 weak local Poincaré inequality holds on $RCD^*(K, N)$ spaces, and hence also an L^1 strong local Poincaré inequality holds by the doubling and geodesic properties and by applying [21, Theorem 1]. By using the Hölder inequality, we know that the L^p weak local Poincaré inequality holds for all $p \in (1, \infty)$.

Lemma 2.2 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \leq 0$ and $N \in [1, \infty)$. Then for every $x \in X$ and every $R > 0$, there exists a positive constant $C := C(K, N, R)$ such that for any $r \in (0, R)$,*

$$\int_{B(x,r)} |f - f_B|^2 d\mu \leq Cr^2 \int_{B(x,r)} |\nabla f|^2 d\mu, \quad \text{for all } f \in W^{1,2}(X), \quad (2.4)$$

where $f_B = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu$. In particular, if $K = 0$, then Eq. 2.4 holds with constant $C := C(K, N)$ independent of R .

Now we can apply the results from Sturm [34, Proposition 2.3] to immediately deduce that there exists a heat kernel, i.e., a measurable map $(0, \infty) \times X \times X \ni (t, x, y) \mapsto p_t(x, y) \in [0, \infty)$ such that, for any $t > 0$, $f \in L^1(X) + L^\infty(X)$ and each $x \in X$,

$$H_t f(x) = \int_X f(y) p_t(x, y) d\mu(y);$$

for all $s, t > 0$ and all $x, y \in X$,

$$p_{t+s}(x, y) = \int_X p_t(x, z) p_s(z, y) d\mu(z);$$

the function $u : (t, y) \mapsto p_t(x, y)$ is a solution of the equation $\Delta u = \frac{\partial}{\partial t} u$ on $(0, \infty) \times X$ in the weak sense. By the symmetry of the semi-group, p_t is also symmetric, i.e., for every $t > 0$, $p_t(x, y) = p_t(y, x)$ for all $(x, y) \in X \times X$. The doubling property and the local L^2 Poincaré inequality imply that the function $x \mapsto p_t(x, y)$ is Hölder continuous for every $(t, y) \in (0, \infty) \times X$, by a standard argument; see e.g. [35, Section 3]. Moreover, H_t is stochastically complete (see e.g. [33, Theorem 4]), i.e.,

$$\int_X p_t(x, y) d\mu(y) = 1, \quad \forall t > 0 \text{ and } \forall x \in X.$$

3 Heat Kernel Bounds

In this section, we shall prove the main results on the heat kernel bounds. We shall follow the approach of Sturm [32], by applying the Laplacian comparison principle established by Gigli [19] (see Lemma 3.1 below) and the parabolic Harnack inequality in [18, 22] as our main tools.

In what follows, let $d_{x_0}(x) := d(x_0, x)$ be the distance function for each x_0 .

Lemma 3.1 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \leq 0$ and $N \in [1, \infty)$. Then the distance function $d_{x_0} \in \mathcal{D}_{\text{loc}}(\Delta, X \setminus x_0)$, and*

$$\Delta d_{x_0}|_{X \setminus x_0} \leq \frac{N \tau_{K,N}(d_{x_0}) - 1}{d_{x_0}}.$$

Above, $\tau_{K,N} \equiv 1$ if $K = 0$, and $\tau_{K,N}(\theta) = \theta \sqrt{-K/N} \operatorname{coth}(\theta \sqrt{-K/N})$ for $\theta \in [0, \infty)$.

The following parabolic Harnack inequalities are established in [18, 22].

Lemma 3.2 (*Parabolic Harnack Inequality*). *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Then for each $0 \leq f \in \cup_{1 \leq q < \infty} L^q(X)$, all $0 < s < t < \infty$ and $x, y \in X$, it holds that*

(i) if $K = 0$,

$$H_s f(x) \leq H_t f(y) \exp \left\{ \frac{d(x, y)^2}{4(t-s)} \right\} \left(\frac{t}{s} \right)^{N/2};$$

(ii) if $K < 0$,

$$H_s f(x) \leq H_t f(y) \exp \left\{ \frac{d(x, y)^2}{4(t-s)e^{2Kt/3}} \right\} \left(\frac{1 - e^{2Kt/3}}{1 - e^{2Ks/3}} \right)^{N/2}.$$

Now we begin to prove Theorem 1.1.

Proof of Theorem 1.1 (i) The inequality

$$p_t(x, y) \leq \frac{C_1(\epsilon)}{\mu(B(y, \sqrt{t}))} \cdot \exp \left(-\frac{d^2(x, y)}{(4 + \epsilon)t} \right)$$

follows from Sturm [34, Corollary 2.5], by using the doubling property in Lemma 2.1 and the L^2 Poincaré inequality in Lemma 2.2.

To prove the reverse inequality, we set $N_1 := \min\{m \in \mathbb{N} \mid m \geq N\}$. Since $N \leq N_1$, the space (X, d, μ) satisfies also $RCD^*(0, N_1)$. By applying the same argument in [32, p158-p159] with Lemma 3.1, we conclude that there exists a constant $C(N_1)$ such that

$$\int_{B(y, \sqrt{t})} p_t(x, z) d\mu(z) \geq C(N_1) \cdot \exp \left(-\frac{d^2(x, y)}{4(1 - \epsilon)t} - \frac{1 + \epsilon^{-1}}{2} \right), \tag{3.1}$$

for all $t > 0$ and all $x, y \in X$. Indeed, the argument only used the Laplacian comparison principle for X in Lemma 3.1 and an explicit calculation for the heat kernel on Euclidean space \mathbb{R}^{N_1} of dimension N_1 (see [32, (3.1)]).

Fix any $\epsilon > 0$. According to Lemma 3.2 (i), we have

$$\begin{aligned} p_{(1+\epsilon)t}(x, y) &\geq \mu^{-1} \left(B(y, \sqrt{t}) \right) \cdot \int_{B(y, \sqrt{t})} p_t(x, z) d\mu(z) \cdot \exp \left(-\frac{1}{4\epsilon} \right) \cdot (1 + \epsilon)^{-N_1/2} \\ &\stackrel{(3.1)}{\geq} C(\epsilon, N_1) \cdot \mu^{-1} \left(B(y, \sqrt{t}) \right) \cdot \exp \left(-\frac{d^2(x, y)}{4(1 - \epsilon)t} \right), \end{aligned}$$

for all $t > 0$ and all $x, y \in X$. This implies the desired estimate. □

Proof of Corollary 1.1 It follows from the Li–Yau inequality

$$|\nabla \log p_t(x, \cdot)|^2(y) - \frac{\partial}{\partial t} \log p_t(x, y) \leq \frac{N}{2t}, \quad \text{for } \mu\text{-a.e. } x, y \in X,$$

in [18, 22] and Theorem 1.1 that, for each $\epsilon > 0$,

$$\begin{aligned} |\nabla p_t(x, \cdot)|^2(y) &\leq \frac{N}{2t} p_t(x, y)^2 + p_t(x, y) |\Delta p_t(x, \cdot)(y)| \\ &\leq \frac{C_1(\epsilon)}{t\mu(B(y, \sqrt{t}))^2} \cdot \exp\left(-2\frac{d^2(x, y)}{(4 + \epsilon)t}\right), \end{aligned}$$

as desired. □

We now turn to prove Theorem 1.2. To this end, in particular, we derive a parabolic Harnack inequality for the heat kernel p_t from Lemma 3.2 (ii).

Lemma 3.3 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K < 0$ and $N \in [1, \infty)$. For any $0 < s < s + 1 \leq t < \infty$ and $x, y, z \in X$, it holds*

$$p_t(x, y) \geq p_s(x, z) \exp\left\{-\frac{d(y, z)^2}{2e^{2K/3}}\right\} \left(\frac{1 - e^{K/3}}{1 - e^{2K/3}}\right)^{N/2} \left(\frac{1 - e^{2Ks/3}}{1 - e^{2K(t-1/2)/3}}\right)^{N/2}.$$

Proof Write $p_t(x, y) = H_1(p_{t-1}(x, \cdot))(y)$. Then it follows, from Lemma 3.2 (ii) that

$$\begin{aligned} p_t(x, y) &= H_1(p_{t-1}(x, \cdot))(y) \\ &\geq H_{1/2}(p_{t-1}(x, \cdot))(z) \exp\left\{-\frac{d(y, z)^2}{2e^{2K/3}}\right\} \left(\frac{1 - e^{K/3}}{1 - e^{2K/3}}\right)^{N/2} \\ &= p_{t-1/2}(x, z) \exp\left\{-\frac{d(y, z)^2}{2e^{2K/3}}\right\} \left(\frac{1 - e^{K/3}}{1 - e^{2K/3}}\right)^{N/2} \\ &\geq p_s(x, z) \exp\left\{-\frac{d(y, z)^2}{2e^{2K/3}}\right\} \left(\frac{1 - e^{K/3}}{1 - e^{2K/3}}\right)^{N/2} \left(\frac{1 - e^{2Ks/3}}{1 - e^{2K(t-1/2)/3}}\right)^{N/2}, \end{aligned}$$

as desired. □

The following lemma is a particular case of Sturm [34, Lemma 1.7].

Lemma 3.4 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Suppose that $\psi \in W^{1,2}(X) \cap L^\infty(X)$ with $|\nabla \psi|^2 \leq \gamma^2$, and u is a solution to the heat equation $\frac{\partial}{\partial t} u = \Delta u$ on $X \times [0, \infty)$. Then for all $0 \leq s < t < \infty$, it holds*

$$\|e^\psi u(\cdot, t)\|_{L^2(X)} \leq e^{\gamma^2(t-s)} \|e^\psi u(\cdot, s)\|_{L^2(X)}.$$

We can prove Theorem 1.2 now. We should mention that the idea of the establishment of the upper bound comes from [34, Theorem 2.4] and the lower bound from [32].

Proof of Theorem 1.2

- (i) **The upper bounds.** Let $\epsilon \in (0, \frac{1}{2})$, $t > 0$ and $x, y \in X$. By the parabolic Harnack inequality (Lemma 3.2 (ii)), it follows that, for each $z \in B(y, \sqrt{t})$, it holds

$$p_t(x, y) \leq p_{(1+\epsilon)t}(x, z) \exp\left\{\frac{d(y, z)^2}{4\epsilon t e^{2K(1+\epsilon)t/3}}\right\} \left(\frac{1 - e^{2K(1+\epsilon)t/3}}{1 - e^{2Kt/3}}\right)^{N/2}, \tag{3.2}$$

and, if $t \geq 1/\epsilon$, then, by Lemma 3.3,

$$p_t(x, y) \leq p_{(1+\epsilon)t}(x, z) \exp \left\{ \frac{d(y, z)^2}{2e^{2K/3}} \right\} \left(\frac{1 - e^{2K/3}}{1 - e^{K/3}} \right)^{N/2} \left(\frac{1 - e^{2Kt(1+\epsilon)/3}}{1 - e^{2Kt/3}} \right)^{N/2}. \tag{3.3}$$

Let $\beta \in \mathbb{R}$, $0 \leq f \in L^2(X)$ and let ψ be a Lipschitz cut-off function with $|\nabla \psi|^2 \leq 1$ μ -a.e. in X . Set $u(x, t) := H_t(e^{-\beta\psi} f)$.

Suppose at first that $t < 1/\epsilon$. By Lemma 3.2 (ii), we find that, for all $z \in B(y, \sqrt{t})$, it holds

$$H_{t+t\epsilon}(e^{-\beta\psi} f)(y)^2 \leq H_{t+2\epsilon t}(e^{-\beta\psi} f)(z)^2 \exp \left\{ \frac{d(y, z)^2}{4\epsilon t e^{2K(1+2\epsilon)t/3}} \right\} \left(\frac{1 - e^{2K(1+2\epsilon)t/3}}{1 - e^{2K(1+\epsilon)t/3}} \right)^{N/2}.$$

This implies that

$$\begin{aligned} H_{t+t\epsilon}(e^{-\beta\psi} f)(y)^2 &\leq C(K, \epsilon) e^{-KN\epsilon t} \int_{B(y, \sqrt{t})} H_{t+2\epsilon t}(e^{-\beta\psi} f)(z)^2 d\mu(z) \\ &\leq C(K, \epsilon) e^{-KN\epsilon t} e^{-2\beta\psi(y)+2|\beta|\sqrt{t}} \int_{B(y, \sqrt{t})} e^{2\beta\psi(z)} H_{t+2\epsilon t}(e^{-\beta\psi} f)(z)^2 d\mu(z) \\ &\leq C(K, \epsilon) e^{-KN\epsilon t} e^{-2\beta\psi(y)+2|\beta|\sqrt{t}} e^{2\beta^2 t} \frac{1}{\mu(B(y, \sqrt{t}))} \|f\|_{L^2(X)}^2, \end{aligned} \tag{3.4}$$

where in the last estimate we used Lemma 3.4.

On the other hand, since

$$\begin{aligned} H_{t+t\epsilon}(e^{-\beta\psi} f)(y)^2 &= \left[\int_X p_{t+t\epsilon}(y, z) e^{-\beta\psi(z)} f(z) d\mu(z) \right]^2 \\ &\geq \left[\int_{B(x, \sqrt{t})} p_{t+t\epsilon}(y, z) e^{-\beta\psi(z)} f(z) d\mu(z) \right]^2 \\ &\geq e^{-2\beta\psi(x)-2|\beta|\sqrt{t}} \left[\int_{B(x, \sqrt{t})} p_{t+t\epsilon}(y, z) f(z) d\mu(z) \right]^2, \end{aligned}$$

which, together with Eq. 3.4, and taking supremum with respect to $\|f\|_{L^2(B(x, \sqrt{t}))} \leq 1$, yields that

$$\begin{aligned} e^{-2\beta\psi(x)-2|\beta|\sqrt{t}} \|p_{t+t\epsilon}(y, \cdot)\|_{L^2(B(x, \sqrt{t}))}^2 &\leq C(K, \epsilon) e^{-KN\epsilon t} e^{-2\beta\psi(y)+2|\beta|\sqrt{t}} e^{2\beta^2 t} \mu(B(y, \sqrt{t}))^{-1}. \end{aligned} \tag{3.5}$$

A similar argument as in Eq. 3.4, by applying the parabolic Harnack inequality (3.2), yields,

$$\begin{aligned} p_t(x, y)^2 &\leq C(K, \epsilon) \int_{B(x, \sqrt{t})} p_{t+\epsilon t}(z, y)^2 d\mu(z) e^{-KN\epsilon t} \\ &\leq C(K, \epsilon) \frac{e^{-KN\epsilon t}}{\mu(B(x, \sqrt{t}))} \|p_{t+\epsilon t}(y, \cdot)\|_{L^2(B(x, \sqrt{t}))}^2. \end{aligned} \tag{3.6}$$

Let $\beta = \frac{d(x,y)}{2t}$, and choose ψ such that $\psi(x) - \psi(y)$ is sufficiently close to $-d(x, y)$. Combining the estimates (3.5) and (3.6), we find

$$\begin{aligned}
 p_t(x, y)^2 &\leq C(K, \epsilon) \frac{\exp \left\{ -2KN\epsilon t + 4\beta\sqrt{t} + 2\beta(\psi(x) - \psi(y)) + 2\beta^2 t \right\}}{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))} \\
 &\leq C(K, \epsilon) \frac{\exp \left\{ -2KN\epsilon t + \frac{2d(x,y)}{\sqrt{t}} - \frac{d(x,y)^2}{2t} \right\}}{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}.
 \end{aligned}$$

On the other hand, by the local doubling property (Lemma 2.1 (ii)) and $t < 1/\epsilon$, we have

$$\begin{aligned}
 \mu(B(y, \sqrt{t})) &\leq \mu(B(x, \sqrt{t} + d(x, y))) \leq C \exp \left\{ C(K, N)(d(x, y) + \sqrt{t}) \right\} \mu(B(x, \sqrt{t})) \\
 &\leq C(\epsilon) \exp \left\{ C(K, N, \epsilon) \frac{d(x, y)}{\sqrt{t}} \right\} \mu(B(x, \sqrt{t})).
 \end{aligned}$$

Combining the above two estimates and the Young inequality, we conclude that

$$\begin{aligned}
 p_t(x, y) &\leq \sqrt{C(K, \epsilon)} \frac{\exp \left\{ -KN\epsilon t + \frac{d(x,y)}{\sqrt{t}} - \frac{d(x,y)^2}{4t} + C(K, N, \epsilon) \frac{d(x,y)}{\sqrt{t}} \right\}}{\mu(B(y, \sqrt{t}))} \\
 &\leq C_1(\epsilon) \frac{1}{\mu(B(y, \sqrt{t}))} \exp \left\{ C_2(\epsilon)t - \frac{d(x, y)^2}{(4 + \epsilon)t} \right\}. \tag{3.7}
 \end{aligned}$$

For the case $t \geq 1/\epsilon$, by using the parabolic Harnack inequality (3.3) instead of Eq. 3.2 in the proof of Eq. 3.6, we can conclude that Eq. 3.7 also holds. This completes the proof of upper bounds.

- (ii) **The lower bounds.** Set $N_1 := \min\{m \in \mathbb{N} \mid m \geq N\}$. Since $N \leq N_1$, the metric measure space (X, d, μ) satisfies also $RC D^*(K, N_1)$. By applying the Laplacian comparison theorem (Lemma 3.1), and the parabolic maximum principle, we conclude that, there exists a constant $C(N_1)$ such that

$$\int_{B(y, \sqrt{t})} p_t(x, z) d\mu(z) \geq C(K, N_1, \epsilon) \cdot \exp \left(-\frac{d^2(x, y)}{4(1 - \epsilon)t} - C_2(\epsilon)t \right) \tag{3.8}$$

for all $t > 0$ and all $x, y \in X$. Notice that, the argument only used the Laplacian comparison theorem (Lemma 3.1), and an explicit calculation for heat kernel on the hyperbolic space E^{k, N_1} of dimension N_1 and constant sectional curvature $k = \frac{K}{N_1 - 1}$; see [32, Corollary 3.6].

For $t < 1/\epsilon$, by the parabolic Harnack inequality (3.2), we have

$$\begin{aligned}
 p_t(x, y) &\geq C(K, \epsilon) \int_{B(y, \sqrt{t})} p_t(x, z) d\mu(z) e^{KN\epsilon t} \\
 &\geq \frac{1}{C_1(\epsilon)\mu(B(y, \sqrt{t}))} \exp \left(-\frac{d^2(x, y)}{(4 - \epsilon)t} - C_2(\epsilon)t \right). \tag{3.9}
 \end{aligned}$$

Suppose now $t \geq \max\{1, 1/\epsilon\}$. By the parabolic Harnack inequality (3.3), we conclude that

$$\begin{aligned} p_{(1+\epsilon)t}(x, y) &\geq C(K, N) \int_{B(y, \sqrt{t})} p_t(x, z) \exp\left\{-\frac{d(y, z)^2}{2e^{2K/3}}\right\} d\mu(z) \left(\frac{1 - e^{2Kt/3}}{1 - e^{2K((1+\epsilon)t)/3}}\right)^{N/2} \\ &\geq C(K, N, \epsilon) \int_{B(y, \sqrt{t})} p_t(x, z) d\mu(z) \exp\left\{-\frac{t}{2e^{2K/3}} - C(\epsilon)t\right\} \\ &\geq C(K, N, \epsilon) \exp\left(-\frac{d^2(x, y)}{4(1 - \epsilon)t} - C_2(\epsilon)t\right). \end{aligned}$$

The proof is completed. □

Proof of Corollary 1.2 Notice that by [22, Theorem 1.2], it holds for each $f \in L^1(X)$ and each $t > 0$ that,

$$|\nabla \log H_t f|^2 \leq e^{-2Kt/3} \frac{\Delta H_t f}{H_t f} + \frac{NK}{3} \frac{e^{-4Kt/3}}{1 - e^{-2Kt/3}}, \quad \mu\text{-a.e.}$$

This gives, when $0 < t \leq 1$, that

$$|\nabla p_t(x, \cdot)|^2(y) \leq C(K)|\Delta p_t(x, \cdot)(y)|p_t(x, y) + C(K)t^{-1}p_t(x, y)^2. \tag{3.10}$$

If $t > 1$, by writing $p_t(x, y) = H_1(p_{t-1}(x, \cdot))(y)$, we can also conclude that

$$|\nabla p_t(x, \cdot)|^2(y) = |\nabla H_1(p_{t-1}(x, \cdot))|^2(y) \leq C(K)|\Delta p_t(x, \cdot)(y)|p_t(x, y) + C(K)p_t(x, y)^2,$$

Notice that, by using Davies [15, Theorem 4] and Theorem 1.2, we see that for each $t > 0$, and almost all $x, y \in X$,

$$\left| \frac{\partial}{\partial t} p_t(x, y) \right| \leq \frac{C(\epsilon)}{t\mu(B(y, \sqrt{t}))} \exp\left\{C_2(\epsilon)t - \frac{d(x, y)^2}{(4 + \epsilon)t}\right\}. \tag{3.11}$$

This, together with Theorem 1.2 again, yields

$$|\nabla p_t(x, \cdot)| \leq \frac{C(\epsilon)}{\sqrt{t}\mu(B(y, \sqrt{t}))} \exp\left\{C_2(\epsilon)t - \frac{d(x, y)^2}{(4 + \epsilon)t}\right\},$$

as desired. □

Gong and Wang [20] established a characterization of compactness of Riemannian manifolds by using heat kernel bounds. Their arguments work also in our $RC D^*(K, N)$ setting. We omit the details of the proof here, but just mention that the heat flow satisfies the semi-group Poincaré inequality (see [5, Corollary 2.3]): for every $f \in W^{1,2}(X)$ and every $t > 0$,

$$H_t(f^2) - (H_t f)^2 \leq \frac{1 - e^{-2Kt}}{K} H_t(|\nabla f|^2), \quad \mu\text{-a.e. in } X,$$

and the mass preserving property (see [4, Section 4]): for every $t > 0$ and every $f \in L^1(X) \cap L^2(X)$,

$$\int_X H_t f d\mu = \int_X f d\mu,$$

and for every ball $B(x, r) \subset X$, the bottom of the spectrum of the operator $-\Delta$ in $L^2(B(x, r))$ is bounded above by a positive constant $C(K, N, R)$ with $R > r$, independent of x , by applying Lemma 2.1 (see [35, Proposition 2.1]).

Theorem 3.1 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Then the following conditions are equivalent.*

- (i) (X, d) is compact;
- (ii) There exist $x \in X$ and $t_0 > 0$ such that $\int_X p_{t_0}(x, y)^{-1} d\mu(y) < \infty$;
- (iii) There exists $t_0 > 0$ such that $\int_X p_{t_0}(x, x) d\mu(x) < \infty$.

4 Stability of Solutions to the Heat Equation

In this section, we apply the heat kernel bounds (Theorem 1.1) to the study of large time behaviors of the heat kernel and stability of solutions to the heat equation. Our arguments will be based on the method of Li [24] with some necessary modifications, due to lacking of the Stokes’ formula (or the Gauss–Green formula).

Definition 4.1 (Boundary measure). For a fixed $x_0 \in X$ and $r \in [0, \infty)$, define

$$s(x_0, r) := \limsup_{\delta \rightarrow 0^+} \frac{1}{\delta} \mu(B(x_0, r + \delta) \setminus B(x_0, r)).$$

We remark here that, in the Riemannian manifold M with μ being the volume measure, it is immediate to see that $s(x_0, r)$ is equal to the $(n - 1)$ -dimensional Hausdorff measure of $\partial B(x_0, r)$, for every $B(x_0, r)$ in M .

The first part in the next lemma is known (see e.g. [37, Theorem 2.3]), and the second part is immediate from the last definition and the local Lipschitz continuity of the function $r \mapsto \mu(B(x_0, r))$ in $(0, \infty)$, for each $x_0 \in X$ (see e.g. [37, p.148]).

Lemma 4.1 *Let (X, d, μ) be a $RCD^*(0, N)$ spaces with $N \in (1, \infty)$. Then for all $x_0 \in X$ and $0 < r < R < \infty$, it holds*

$$\frac{s(x_0, R)}{s(x_0, r)} \leq \left(\frac{R}{r}\right)^{N-1},$$

and

$$\mu(B(x_0, R)) = \int_0^R s(x_0, r) dr.$$

Lemma 4.2 *Let (X, d, μ) be a $RCD^*(0, N)$ spaces with $N \in (1, \infty)$. If there exists $x_0 \in X$, and $\theta \in (0, \infty)$ such that*

$$\liminf_{R \rightarrow \infty} \frac{\mu(B(x_0, R))}{R^N} = \theta.$$

Then for each $R > 0$, it holds

$$s(x_0, R) \geq N\theta R^{N-1}.$$

Proof From Lemma 4.1, it follows the function $R \mapsto \frac{s(x_0, R)}{R^{N-1}}$ is non-increasing on $(0, \infty)$. Hence, if there exists $R_0 \in (0, \infty)$ such that $s(x_0, R_0) < N\theta R_0^{N-1}$, then for all $R > R_0$, it holds $s(x_0, R) < N\theta R^{N-1}$. This together with Lemma 4.1 implies that for each $R > R_0$,

$$\mu(B(x_0, R) \setminus B(x_0, R_0)) = \int_{R_0}^R s(x_0, r) dr < \theta (R^N - R_0^N),$$

and hence

$$\liminf_{R \rightarrow \infty} \frac{\mu(B(x_0, R))}{R^N} < \liminf_{R \rightarrow \infty} \frac{\theta (R^N - R_0^N) + \mu(B(x_0, R_0))}{R^N} = \theta.$$

This contradicts with the assumption. Therefore, for each $R > 0$, we see that $s(x_0, R) \geq N\theta R^{N-1}$, as desired. □

Definition 4.2 (Boundary Integral). Let $x_0 \in X$ and $r \in [0, \infty)$. Suppose $f \in L^\infty_{\text{loc}}(X)$. Define the integral of f on $\partial B(p, r)$ as

$$|f|_{\partial B(x_0, r)} := \limsup_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_{B(x_0, r+\delta) \setminus B(x_0, r)} f(x) d\mu(x).$$

Lemma 4.3 Let (X, d, μ) be a $RC D^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in (1, \infty)$. Let $x_0 \in X$ and $R \in (0, \infty)$. Then for each $f \in L^\infty_{\text{loc}}(X)$, it holds

$$\int_{B(x_0, R)} f(x) d\mu(x) = \int_0^R |f|_{\partial B(x_0, r)} dr.$$

Proof Notice that, for each fixed $x_0 \in X$, the function $r \mapsto \mu(B(x_0, r))$ is locally Lipschitz continuous on $(0, \infty)$. From this, we conclude that, for each $f \in L^\infty_{\text{loc}}(X)$, the function

$$r \mapsto \int_{B(x_0, r)} f d\mu$$

is locally Lipschitz continuous on $(0, \infty)$, and hence, the required equality holds. □

Proof of Theorem 1.3 It is shown in [22, Theorem 1.4] that

$$\lim_{t \rightarrow \infty} t^{N/2} p_t(x, y) = C(\theta), \tag{4.1}$$

for some constant $C(\theta) \in (0, \infty)$. Let us prove that $\theta C(\theta) = \omega(N)(4\pi)^{-N/2}$.

Following [32, Proof of Theorem 4.5], by applying the Laplacian comparison theorem (Lemma 3.1), and the parabolic maximum principle, it follows that, for all $x \in X$ and $t, r > 0$,

$$\int_{B(x, r)} p_t(x, y) d\mu(y) \geq \int_{B(x_r, r)} \frac{1}{(4\pi t)^{N/2}} \exp \left\{ -\frac{|z|^2}{4t} \right\} dz, \tag{4.2}$$

where $x_r = (r, 0, \dots, 0) \in \mathbb{R}^N$. This, together with Eq. 4.1, implies that

$$\mu(B(x, r))C(\theta) = \lim_{t \rightarrow \infty} t^{N/2} \int_{B(x, r)} p_t(x, y) d\mu(y) \geq \frac{\omega(N)r^N}{(4\pi)^{N/2}}.$$

Letting $r \rightarrow \infty$, we find that

$$\theta C(\theta) \geq \frac{\omega(N)}{(4\pi)^{N/2}}. \tag{4.3}$$

Let us prove the upper bound of $\theta C(\theta)$. For any $\delta > 0$ and all $x, y \in X$, the parabolic Harnack inequality in Lemma 3.2 (i) yields

$$p_t(x, x)^2 \leq p_{(1+\delta)t}(x, y)^2 \exp \left\{ \frac{d(x, y)^2}{2\delta t} \right\} (1 + \delta)^N,$$

which implies that, for all $r, \epsilon > 0$,

$$\begin{aligned}
 & p_t(x, x)^2 \frac{\mu(B(x, r + \epsilon) \setminus B(x, r))}{\epsilon} \\
 & \leq \exp \left\{ \frac{(r + \epsilon)^2}{2\delta t} \right\} (1 + \delta)^N \frac{1}{\epsilon} \int_{\mu(B(x, r + \epsilon) \setminus B(x, r))} p_{(1+\delta)t}(x, y)^2 d\mu(y).
 \end{aligned}$$

Letting $\epsilon \rightarrow 0$ and applying Lemma 4.2, we conclude that for a.e. $r > 0$,

$$\begin{aligned}
 (1 + \delta)^N |p_{(1+\delta)t}(x, \cdot)^2|_{\partial B(x, r)} & \geq \exp \left\{ -\frac{r^2}{2\delta t} \right\} s(x, r) p_t(x, x)^2 \\
 & \geq N\theta r^{N-1} \exp \left\{ -\frac{r^2}{2\delta t} \right\} p_t(x, x)^2. \tag{4.4}
 \end{aligned}$$

Integrating over $(0, \infty)$, and applying Lemma 4.1 and Lemma 4.3, we find that

$$\begin{aligned}
 (1 + \delta)^N t^{N/2} p_{2(1+\delta)t}(x, x) & = (1 + \delta)^N t^{N/2} \int_X p_{(1+\delta)t}(x, y)^2 d\mu(y) \\
 & = (1 + \delta)^N t^{N/2} \int_{(0, \infty)} |p_{(1+\delta)t}(x, \cdot)^2|_{\partial B(x, r)} dr \\
 & \geq t^{N/2} N\theta p_t(x, x)^2 \int_0^\infty r^{N-1} \exp \left(-\frac{r^2}{2\delta t} \right) dr \\
 & = \frac{\theta t^N}{\omega(N)} (2\pi\delta)^{N/2} p_t(x, x)^2,
 \end{aligned}$$

where in the last inequality, we used Eq. 4.4. With Eq. 4.1, letting $t \rightarrow \infty$, we derive that

$$\begin{aligned}
 \left(\frac{1 + \delta}{2} \right)^{N/2} C(\theta) & = \lim_{t \rightarrow \infty} \left(\frac{1 + \delta}{2} \right)^{N/2} (2(1 + \delta)t)^{N/2} p_{2(1+\delta)t}(x, x) \\
 & \geq \lim_{t \rightarrow \infty} \frac{(2\pi\delta)^{N/2}}{\omega(N)} \theta t^N p_t(x, x)^2 = \frac{(2\pi\delta)^{N/2}}{\omega(N)} \theta C(\theta)^2,
 \end{aligned}$$

and hence,

$$\theta C(\theta) \leq \frac{\omega(N)}{(2\pi\delta)^{N/2}} \left(\frac{1 + \delta}{2} \right)^{N/2}.$$

Letting $\delta \rightarrow \infty$ and using Eq. 4.3, we have

$$\theta C(\theta) = \frac{\omega(N)}{(4\pi)^{N/2}},$$

which implies that

$$\lim_{t \rightarrow \infty} \mu(B(x_0, \sqrt{t})) p_t(x, y) = \lim_{t \rightarrow \infty} \frac{\mu(B(x_0, \sqrt{t}))}{t^{N/2}} t^{N/2} p_t(x, y) = \theta C(\theta) = \frac{\omega(N)}{(4\pi)^{N/2}},$$

and hence, we complete the proof. □

The existence of the large time limit of heat kernels yields the following stability property of solutions to the heat equation. The proof is a slightly modification of the proof of [24, Theorem 3] to our non-smooth context.

Theorem 4.1 *Let (X, d, μ) be a $RC D^*(0, N)$ space with $N \in \mathbb{N}$ and $N \geq 2$. Let $x_0 \in X$. Suppose that there exists a constant $\theta \in (0, \infty)$ such that $\liminf_{R \rightarrow \infty} \frac{\mu(B(x_0, R))}{R^N} = \theta$. Then, for each $f \in L^\infty(X)$, and for any $x \in X$, the limit $\lim_{t \rightarrow \infty} H_t(f)(x)$ exists, if and only if, the limit*

$$\lim_{r \rightarrow \infty} \int_{B(x,r)} f(y) d\mu(y)$$

exists; moreover, it holds

$$\lim_{t \rightarrow \infty} H_t(f)(x) = \lim_{r \rightarrow \infty} \int_{B(x,r)} f(y) d\mu(y).$$

Proof It follows from Theorem 1.3 that for each $x \in X$,

$$\lim_{t \rightarrow \infty} t^{N/2} p_t(x, x) = \frac{\omega(N)}{\theta(4\pi)^{N/2}} =: C(\theta).$$

This, together with the parabolic Harnack inequality

$$p_s(x, x) \leq p_t(x, y) \exp \left\{ \frac{d(x, y)^2}{4(t-s)} \right\} \left(\frac{t}{s} \right)^{N/2},$$

for all $x, y \in X$ and $0 < s < t$, yields that there exists $\epsilon(s) > 0$ satisfying $\epsilon(s) \rightarrow 0$ as $s \rightarrow \infty$ such that

$$p_t(x, y) \geq (C(\theta) - \epsilon(s))t^{-N/2} \exp \left\{ -\frac{d(x, y)^2}{4(t-s)} \right\}.$$

For any $0 \leq g \in L^\infty(X)$ and each $x \in X$, it holds

$$\begin{aligned} H_t(g)(x) &\geq (C(\theta) - \epsilon(s))t^{-N/2} \int_X \exp \left\{ -\frac{d(x, y)^2}{4(t-s)} \right\} g(y) d\mu(y) \\ &\geq (C(\theta) - \epsilon(s))t^{-N/2} \int_1^\infty \exp \left\{ -\frac{r^2}{4(t-s)} \right\} |g|_{\partial B(x,r)} dr. \end{aligned}$$

Let $0 \leq g \in L^\infty(X)$ and let $z \in \mathbb{R}^N$. We set for each $x \in X$,

$$\bar{g}_x(z) := \frac{|g|_{\partial B(x,|z|)}}{N\omega(N)|z|^{N-1}} \chi_{\{|z| \geq 1\}}(z), \quad z \in \mathbb{R}^N.$$

By Lemma 4.1, we have

$$0 \leq \bar{g}_x(z) \leq \|g\|_{L^\infty(X)} \frac{s(x, |z|)}{N\omega(N)|z|^{N-1}} \chi_{\{|z| \geq 1\}}(z) \leq \|g\|_{L^\infty(X)} \frac{s(x, 1)}{N\omega(N)} < \infty.$$

Hence, $0 \leq \bar{g}_x \in L^\infty(\mathbb{R}^N)$.

Let \mathcal{H}_t be the heat semigroup on \mathbb{R}^N . Then for each $x \in X$, it follows that

$$H_t(g)(x) \geq (C(\theta) - \epsilon(s)) \left(\frac{4\pi(t-s)}{t} \right)^{N/2} \mathcal{H}_{t-s}(\bar{g}_x)(0). \tag{4.5}$$

For a function $f \in L^\infty(X)$, we may assume $|f| \leq 1$. By setting $g_1 := 1 + f$ and $g_2 := 1 - f$, we then conclude from Eq. 4.5 that

$$1 + H_t(f)(x) \geq (C(\theta) - \epsilon(s)) \left(\frac{4\pi(t-s)}{t} \right)^{N/2} [\mathcal{H}_{t-s}(\bar{1}_x)(0) + \mathcal{H}_{t-s}(\bar{f}_x)(0)], \tag{4.6}$$

and

$$1 - H_t(f)(x) \geq (C(\theta) - \epsilon(s)) \left(\frac{4\pi(t-s)}{t} \right)^{N/2} [\mathcal{H}_{t-s}(\bar{I}_x)(0) - \mathcal{H}_{t-s}(\bar{f}_x)(0)]. \tag{4.7}$$

Using Lemma 4.2 and the definition of \bar{I} , we find that

$$\bar{I}_x(z) = \frac{s(x, |z|)}{N\omega(N)|z|^{N-1}} \chi_{\{|z| \geq 1\}}(z) \geq \frac{\theta}{\omega(N)} \chi_{\{|z| \geq 1\}}(z), \quad \forall z \in \mathbb{R}^N$$

which, together with Eqs. 4.6 and 4.7, implies that

$$\begin{aligned} 1 - \frac{(C(\theta) - \epsilon(s))}{C(\theta)} \left(\frac{t-s}{t} \right)^{N/2} \mathcal{H}_{t-s}(\chi_{\{|z| \geq 1\}})(0) \\ \geq (C(\theta) - \epsilon(s)) \left(\frac{4\pi(t-s)}{t} \right)^{N/2} \mathcal{H}_{t-s}(\bar{f}_x)(0) - H_t(f)(x) \\ \geq \frac{(C(\theta) - \epsilon(s))}{C(\theta)} \left(\frac{t-s}{t} \right)^{N/2} \mathcal{H}_{t-s}(\chi_{\{|z| \geq 1\}})(0) - 1. \end{aligned}$$

Letting $s = \sqrt{t}$ and $t \rightarrow \infty$, we find that

$$\lim_{t \rightarrow \infty} \left[H_t(f)(x) - C(\theta)(4\pi)^{N/2} \mathcal{H}_{t-\sqrt{t}}(\bar{f}_x)(0) \right] = 0,$$

since $\epsilon(\sqrt{t}) \rightarrow 0$ and $\mathcal{H}_{t-\sqrt{t}}(\chi_{\{|z| \geq 1\}})(0) \rightarrow 1$, as $t \rightarrow \infty$. Hence, $\lim_{t \rightarrow \infty} H_t(f)(x)$ exists, if and only if, $\lim_{t \rightarrow \infty} \mathcal{H}_{t-\sqrt{t}}(\bar{f}_x)(0)$ exists. From the proof of [24, Theorem 3], it follows that the limit $\lim_{t \rightarrow \infty} \mathcal{H}_{t-\sqrt{t}}(\bar{f}_x)(0)$ exists, if and only if,

$$\lim_{R \rightarrow \infty} \int_{B(0,R) \subset \mathbb{R}^N} \bar{f}_x(z) dz$$

exists. Notice that, for $R > 1$,

$$\begin{aligned} \int_{B(0,R) \subset \mathbb{R}^N} \bar{f}_x(z) dz &= \frac{1}{\omega(N)R^N} \int_1^R |f|_{\partial B(x,r)} dr \\ &= \frac{\mu(B(x,R))}{\omega(N)R^N} \left[\int_{B(x,R)} f(y) d\mu(y) - \frac{1}{\mu(B(x,R))} \int_{B(x,1)} f(y) d\mu(y) \right]. \end{aligned}$$

Since $\lim_{R \rightarrow \infty} \frac{\mu(B(x,R))}{R^N} = \theta$, we find that, the limit $\lim_{R \rightarrow \infty} \int_{B(0,R) \subset \mathbb{R}^N} \bar{f}_x(z) dz$ exists, if and only if, $\lim_{R \rightarrow \infty} \int_{B(x,R)} f(y) d\mu(y)$ exists.

Hence, we see that the limit $\lim_{t \rightarrow \infty} H_t(f)(x)$ exists, if and only if, $\lim_{r \rightarrow \infty} \int_{B(x,r)} f(y) d\mu(y)$ exists. Similar arguments as in the proof of [24, Theorem 3] imply that the two limits must equal, if they exist. The proof is therefore completed. \square

5 Riesz Transforms

In this section, we consider the boundedness of the Riesz transform $|\nabla(-\Delta)^{-1/2}|$ and the local version $|\nabla(-\Delta + a)^{-1/2}|$ for some $a > 0$. For simplicity, we only consider the case $\mu(X) = \infty$, the case $\mu(X) < \infty$ follows from minor modifications.

In what follows, we shall let $a_K = 0$ if $K = 0$, and let $a_K > 0$ large enough for $K < 0$. Since $-\Delta + a_K$ is a non-negative self-adjoint operator, we can define fractional powers of $-\Delta + a_K$ as

$$(-\Delta + a_K)^{-1/2} = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-a_K s} e^{s\Delta} \frac{ds}{\sqrt{s}},$$

and

$$(-\Delta + a_K)^{1/2} = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-a_K s} (\Delta + a_K) e^{s\Delta} \frac{ds}{\sqrt{s}}.$$

We refer the readers to Yosida [40, Chapter 9.11] for more details.

In smooth settings, it is easy to see that $(-\Delta + a_K)^{-1/2} f$ and $(-\Delta + a_K)^{1/2} f$ are well defined if f is a smooth function with compact support; see, e.g., [9, 14]. In the non-smooth cases, we need a bit work to show that these two operators are well-defined on some dense spaces.

Definition 5.1 (Acting Class). The acting class $\mathbb{V}(X)$ of the Riesz transform $|\nabla(-\Delta + a_K)^{-1/2}|$ is defined as

$$\mathbb{V}(X) := \{f \in L^1(X) \cap L^\infty(X) : (-\Delta + a_K)^{-1/2} f \in \mathcal{D}(\Delta)\}.$$

Remark 5.1 If $\mu(X) < \infty$, we need to consider the subclass $L^p_0(X)$ of $L^p(X)$, where the elements f are required to satisfy $f \in L^p(X)$ and $\int_X f d\mu = 0$. In this case, we need to redefine the acting class $\mathbb{V}(X)$ as

$$\mathbb{V}(X) := \left\{ f \in L^\infty(X) : \int_X f d\mu = 0, (-\Delta + a_K)^{-1/2} f \in \mathcal{D}(\Delta) \right\}.$$

Lemma 5.1 Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. For each $p \in [1, \infty)$, $\mathbb{V}(X)$ is dense in $L^p(X)$.

To prove this lemma, let us recall the following mapping property established in [22].

Proposition 5.1 Let (X, d, μ) be a $RCD^*(K, N)$ space, where $K \in \mathbb{R}$ and $N \in [1, \infty)$.

- (i) For each $t > 0$ and $p \in [1, \infty]$, the operator H_t is bounded on $L^p(X)$ with $\|H_t\|_{p,p} \leq 1$.
- (ii) If $K \geq 0$, then, for each $t > 0$, the operators $\sqrt{t}|\nabla H_t|$ and $t\Delta H_t$ are bounded on $L^p(X)$ for all $p \in [1, \infty]$. Moreover, there exists $C > 0$, such that, for all $t > 0$ and all $p \in [1, \infty]$,

$$\max \left\{ \|\sqrt{t}|\nabla H_t|\|_{p,p}, \|t\Delta H_t\|_{p,p} \right\} \leq C.$$

- (iii) If $K < 0$, then, for each $t > 0$, the operators $\sqrt{t}|\nabla H_t|$ and $t\Delta H_t$ are bounded on $L^p(X)$ for all $p \in [1, \infty]$. Moreover, there exists $C > 0$, such that, for all $t > 0$ and all $p \in [1, \infty]$,

$$\max \left\{ \|\sqrt{(t \wedge 1)}|\nabla H_t|\|_{p,p}, \|(t \wedge 1)\Delta H_t\|_{p,p} \right\} \leq C.$$

We can now prove Lemma 5.1.

Proof of Lemma 5.1 We only need to show that, for each $f \in L^1(X) \cap L^\infty(X)$, there exists $\{f_k\}_{k \in \mathbb{N}} \subset \mathbb{V}(X)$ such that $f_k \rightarrow f$ in $L^p(X)$ as $k \rightarrow \infty$.

For each $\epsilon > 0$, define

$$f_\epsilon := e^{-a_K \epsilon} H_\epsilon(f) - e^{-a_K/\epsilon} H_{1/\epsilon}(f).$$

From the property of the heat semi-group, it follows that $f_\epsilon \in L^1(X) \cap L^\infty(X)$, and $e^{-a_K \epsilon} H_\epsilon(f) \rightarrow f$ in $L^p(X)$ as $\epsilon \rightarrow 0$. From the heat kernel bounds in Theorems 1.1 and 1.2, it follows that for each $x \in X$,

$$\begin{aligned} |e^{-a_K/\epsilon} H_{1/\epsilon}(f)|(x) &\leq \frac{C_1 e^{-a_K/\epsilon}}{\mu(B(x, \sqrt{1/\epsilon}))} \int_X \exp\left(-\frac{d^2(x, y)}{5/\epsilon} + C_2/\epsilon\right) |f(y)| d\mu(y) \\ &\leq \frac{C_1 e^{-a_K/\epsilon + C_2/\epsilon}}{\mu(B(x, \sqrt{1/\epsilon}))} \|f\|_{L^1(X)}, \end{aligned}$$

which tends to 0 as $\epsilon \rightarrow 0$, where we choose $a_K > C_2 > 0$ if $K < 0$. This implies that $e^{-a_K/\epsilon} H_{1/\epsilon}(f) \rightarrow 0$ in $L^p(X)$, and hence, $f_\epsilon \rightarrow f$ in $L^p(X)$, as $\epsilon \rightarrow 0$.

Let us show that $f_\epsilon \in \mathbb{V}(X)$. By the analytic property of the heat semi-group, and the fact

$$(-\Delta + a_K)^{-1/2} f_\epsilon = \int_\epsilon^{1/\epsilon} \sqrt{s} (-\Delta + a_K)^{1/2} e^{s(\Delta - a_K)}(f) \frac{ds}{\sqrt{s}},$$

we conclude that

$$\|(-\Delta + a_K)^{-1/2} f_\epsilon\|_{L^2(X)} \leq C(\epsilon) \|f\|_{L^2(X)},$$

for some constant $C(\epsilon) > 0$.

To show that $(-\Delta + a_K)^{-1/2} f_\epsilon \in \mathcal{D}(\Delta)$, we write

$$(-\Delta + a_K)^{-1/2} f_\epsilon = \frac{\sqrt{\pi}}{2} \int_0^\infty \int_\epsilon^{1/\epsilon} (-\Delta + a_K) e^{(s+t)(\Delta - a_K)}(f) dt \frac{ds}{\sqrt{s}}.$$

If $K = 0$, then by using Theorem 1.1, the Minkowski inequality and Proposition 5.1, we find

$$\begin{aligned} \|\nabla(-\Delta)^{-1/2} f_\epsilon\|_{L^2(X)} &\leq \frac{\sqrt{\pi}}{2} \int_0^\infty \int_\epsilon^{1/\epsilon} \|\nabla(-\Delta) e^{(s+t)\Delta}(f)\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq C \int_0^\infty \int_\epsilon^{1/\epsilon} \|\nabla e^{\frac{(s+t)}{2}\Delta}(-\Delta) e^{\frac{(s+t)}{2}\Delta}(f)\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq C \int_0^\infty \int_\epsilon^{1/\epsilon} \frac{1}{(s+t)^{3/2}} \|f\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq C(\epsilon) \|f\|_{L^2(X)}, \end{aligned}$$

and

$$\begin{aligned} \|\Delta(-\Delta)^{-1/2} f_\epsilon\|_{L^2(X)} &\leq \frac{\sqrt{\pi}}{2} \int_0^\infty \int_\epsilon^{1/\epsilon} \|\Delta(-\Delta) e^{(s+t)\Delta}(f)\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq \frac{\sqrt{\pi}}{2} \int_0^\infty \int_\epsilon^{1/\epsilon} \frac{1}{(s+t)^2} \|f\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq C(\epsilon) \|f\|_{L^2(X)}, \end{aligned}$$

for some constant $C(\epsilon) > 0$. These show that $f_\epsilon \in \mathbb{V}(X)$ when $K = 0$.

Suppose now $K < 0$. Using Theorem 1.2, the Minkowski inequality and Proposition 5.1, we find

$$\begin{aligned} \|\nabla(-\Delta + a_K)^{-1/2} f_\epsilon\|_{L^2(X)} &\leq \frac{\sqrt{\pi}}{2} \int_0^\infty \int_\epsilon^{1/\epsilon} \|\nabla(-\Delta + a_K)e^{(s+t)(\Delta - a_K)}(f)\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq \frac{\sqrt{\pi}}{2} \int_0^\infty \int_\epsilon^{1/\epsilon} \frac{e^{-a_K(t+s)}}{[(s+t) \wedge 1]^{3/2}} \|f\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq C(\epsilon, a_K) \|f\|_{L^2(X)}, \end{aligned}$$

and

$$\begin{aligned} \|\Delta(-\Delta + a_K)^{-1/2} f_\epsilon\|_{L^2(X)} &\leq \frac{\sqrt{\pi}}{2} \int_0^\infty \int_\epsilon^{1/\epsilon} \|\Delta(-\Delta + a_K)e^{(s+t)(\Delta - a_K)}(f)\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq \frac{\sqrt{\pi}}{2} \int_0^\infty \int_\epsilon^{1/\epsilon} \frac{e^{-a_K(t+s)}}{[(s+t) \wedge 1]^2} \|f\|_{L^2(X)} dt \frac{ds}{\sqrt{s}} \\ &\leq C(\epsilon, a_K) \|f\|_{L^2(X)}, \end{aligned}$$

for some constant $C(\epsilon, a_K) > 0$. Hence, we can conclude now that $f_\epsilon \in \mathbb{V}(X)$ for each $\epsilon > 0$.

Therefore, $\mathbb{V}(X)$ is dense in $L^p(X)$ for each $p \in [1, \infty)$. The proof is completed. □

Now we present the main results of this section. The first theorem is on the L^p boundedness of the Riesz transform $|\nabla(-\Delta)^{-1/2}|$. We recall that a sub-linear operator T is of weak type $(1, 1)$ if, for any $f \in L^1(X)$, there exists a constant $C > 0$ such that

$$\sup_{\lambda > 0} \lambda \mu(\{x \in X : |Tf(x)| > \lambda\}) \leq C \|f\|_{L^1(X)}.$$

Theorem 5.1 *Let (X, d, μ) be a $RCD^*(0, N)$ space with $N \in [1, \infty)$. Then, the Riesz transform $|\nabla(-\Delta)^{-1/2}|$, initially defined on $\mathbb{V}(X)$, extends to a sub-linear operator of weak type $(1, 1)$ and it is also bounded on $L^p(X)$, for each $p \in (1, \infty)$.*

The next one is the local version of Theorem 5.1.

Theorem 5.2 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K < 0$ and $N \in [1, \infty)$. Then, for large enough $a_K > 0$, the local Riesz transform $|\nabla(-\Delta + a_K)^{-1/2}|$, initially defined on $\mathbb{V}(X)$, extends to a bounded sub-linear operator on $L^p(X)$, for each $p \in (1, \infty)$.*

Notice that, as pointed out by [11, p.6], although the results from [9] were stated in manifolds, their method of proofs indeeds work on metric measure spaces. Therefore, Theorem 5.1 and Theorem 5.2 follow directly from [9, Theorem 1.4] and [9, Theorem 1.7] by using Corollary 1.1 and Corollary 1.2, respectively.

In what follows, we outline two different proofs of these results, since the proofs are basically identical to [9, 14].

5.1 Proofs by Using Heat Kernel Gradient Estimates

Now we begin the proof of the main results by combing the heat kernel gradient estimate in Corollaries 1.1 and 1.2, and applying the original method of [14] and [9].

Proof of Theorem 5.1 For each $f \in \mathbb{V}(X)$, it follows from the definition of $\mathbb{V}(X)$ that,

$$\|\nabla(-\Delta)^{-1/2}(f)\|_{L^2(X)} = \left(\int_X (-\Delta)^{-1/2}(f)(x)(-\Delta)^{1/2}(f)(x) \, d\mu(x) \right)^{1/2} = \|f\|_{L^2(X)}.$$

Then by Lemma 5.1, we conclude that $|\nabla(-\Delta)^{-1/2}|$ is bounded on $L^2(X)$. Following the method in [14] and [9], we divide the proof into two cases, namely, $p \in (1, 2)$ and $p \in (2, \infty)$.

Following the proofs in [14, Sections 2 and 3], by applying the Calderón–Zygmund decomposition (see, e.g., [14, pp.1154–1155]) for functions and the Marcinkiewicz interpolation theorem, in order to show the boundedness of $|\nabla(-\Delta)^{-1/2}|$ on $L^p(X)$ for all $p \in (1, 2)$, we only need to prove that $|\nabla(-\Delta)^{-1/2}|$ is of weak type $(1, 1)$. It will follow from by using the heat kernel estimate in Theorem 1.1 and by showing that, there exist constants $C, c > 0$ such that for all $s, t > 0$ and μ -a.e. $x \in X$,

$$\int_{d(x,y) \geq t^{1/2}} |\nabla p_s(x, \cdot)|(y) \, d\mu(y) \leq C e^{-ct/s} s^{-1/2}.$$

Indeed, by using the gradient estimate (1.2) and the doubling property in Lemma 2.1 (i), we see that

$$\begin{aligned} \int_{d(x,y) \geq t^{1/2}} |\nabla p_s(x, \cdot)|(y) \, d\mu(y) &\leq \sum_{k=1}^{\infty} \frac{C\mu(B(x, 2^k\sqrt{t}) \setminus B(x, 2^{k-1}\sqrt{t}))}{\sqrt{s}\mu(B(x, \sqrt{s}))} \exp\left\{-\frac{2^{2k}t}{20s}\right\} \\ &\leq C \exp\left\{-\frac{t}{10s}\right\} s^{-1/2}. \end{aligned}$$

Hence, $|\nabla(-\Delta)^{-1/2}|$ is bounded on $L^p(X)$, for all $1 < p < 2$.

For the remaining case that $p \in (2, \infty)$, combining the estimate of heat kernel in Theorem 1.1 and its gradient estimate (1.2) and the real variable result on singular integrals in [9, Theorem 2.1], and following the proof in [9, pp.936-938], we conclude that $|\nabla(-\Delta)^{-1/2}|$ is bounded on $L^p(X)$ for all $p \in (2, \infty)$, which completes the proof. \square

Now we begin to prove Theorem 5.2.

Proof of Theorem 5.2 For each $f \in \mathbb{V}(X)$, it follows from the definition of $\mathbb{V}(X)$ that,

$$\begin{aligned} &\|\nabla(-\Delta + a_K)^{-1/2}(f)\|_{L^2(X)}^2 + a_K \|(-\Delta + a_K)^{-1/2}(f)\|_{L^2(X)}^2 \\ &= \int_X \langle (-\Delta + a_K)(-\Delta + a_K)^{-1/2}(f), (-\Delta + a_K)^{-1/2}(f) \rangle \, d\mu \\ &= \|f\|_{L^2(X)}^2. \end{aligned}$$

Hence, $|\nabla(-\Delta + a_K)^{-1/2}|$ is bounded on $L^2(X)$ with

$$\|\nabla(-\Delta + a_K)^{-1/2}(f)\|_{L^2(X)} \leq \|f\|_{L^2(X)}.$$

The rest of the proof follows from similar proofs in the cases $p \in (1, 2)$ and $p \in (2, \infty)$ as in [9, 14], by noticing that

$$|\nabla(-\Delta + a_K)^{-1/2} f| \leq \frac{\sqrt{\pi}}{2} \int_0^\infty |\nabla e^{t(\Delta - a_K)}(f)| \frac{dt}{\sqrt{t}},$$

and combining the estimates of the heat kernel and its gradient in Theorem 1.2 and in Corollary 1.2, respectively. The proof is then finished. \square

5.2 Proofs Without Explicit Heat Kernel Gradient Estimates

In fact, also by the method in [9, 14], the L^p boundedness of the Riesz transform $|\nabla(-\Delta)^{-1/2}|$ and the local Riesz transform $|\nabla(a_K - \Delta)^{-1/2}|$ for every $p \in (1, \infty)$ can be shown without using the explicit heat kernel gradient estimates (1.2) and (1.4). The approach demands the following main ingredients: (1) doubling property, (2) stochastic completeness, (3) L^2 version of the local weak Poincaré inequality, (4) Caccioppoli type inequalities, (5) L^p boundedness of $\sqrt{t}|\nabla H_t|$, (6) Davies–Gaffney type estimates, and (7) L^2 boundedness of the Riesz transform.

Now we are going to establish the missing parts (4)–(6) one by one and then give another proof of Theorems 5.1 and 5.2.

The Caccioppoli type estimate for the semi-group H_t can be deduced from the following inequality (see [5, Corollary 2.3]), for every $t > 0$ and $f \in L^2(X)$,

$$\frac{e^{2Kt} - 1}{K} |\nabla H_t f|^2 + \frac{e^{2Kt} - 2Kt - 1}{NK^2} (\Delta H_t f)^2 \leq H_t(f^2) - (H_t f)^2, \quad \mu\text{-a.e. in } X \tag{5.1}$$

Here and in the next proposition, $N = \infty$ is allowable.

Proposition 5.2 *Let (X, d, μ) be a $RD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$ and let $p \in [2, \infty]$. For any $t > 0$, it holds*

$$\|\nabla H_t f\|_{L^p(X)} \leq \sqrt{\frac{K}{e^{2Kt} - 1}} \|f\|_{L^p(X)}, \quad \text{for every } f \in L^2 \cap L^p(X); \tag{5.2}$$

in particular, for $K \geq 0$, it holds

$$\|\nabla H_t f\|_{L^p(X)} \leq \frac{1}{\sqrt{2t}} \|f\|_{L^p(X)}, \quad \text{for every } f \in L^2 \cap L^p(X), \tag{5.3}$$

and for $K < 0$, it holds

$$\|\nabla H_t f\|_{L^p(X)} \leq \frac{1}{\sqrt{2(t \wedge 1)}} \|f\|_{L^p(X)}, \quad \text{for every } f \in L^2 \cap L^p(X). \tag{5.4}$$

Proof By Eq. 5.1, it is immediate to get, for every $t > 0$ and $f \in L^2(X)$,

$$|\nabla H_t f|^2 \leq \frac{K}{e^{2Kt} - 1} H_t(f^2), \quad \mu\text{-a.e. in } X. \tag{5.5}$$

Note that for $t > 0$, H_t is a contraction in all $L^p(X)$ with $p \in [1, \infty]$. For $p = \infty$, the proof of Eq. 5.2 is obvious by Eq. 5.5. Now take $p \in [2, \infty)$ and $f \in L^2 \cap L^p(X)$. Then, by Eq. 5.5, we have

$$\|\nabla H_t f\|_{L^p(X)} \leq \sqrt{\frac{K}{e^{2Kt} - 1}} \left(\int_X (H_t f^2)^{\frac{p}{2}} d\mu \right)^{\frac{1}{p}} \leq \sqrt{\frac{K}{e^{2Kt} - 1}} \|f\|_{L^p(X)},$$

which is just Eq. 5.2. For $K \geq 0$, (5.3) is immediately implied by Eq. 5.2 and the elementary inequality $e^{2Kt} \geq 1 + 2Kt$. For $K < 0$, (5.4) follows from the fact that the function $t \mapsto \frac{e^{2Kt} - 1}{K}$ is decreasing in $[0, \infty)$ with maximum value 2 at $t = 0$. □

The first Davies–Gaffney type estimate can be established by the same proof of [10, Lemma 3.6] with minor modifications. In fact, if we let

$$\Gamma(f, g) = \langle \nabla f, \nabla g \rangle, \quad \text{for every } f, g \in W^{1,2}(X),$$

then $\Gamma : W^{1,2}(X) \times W^{1,2}(X) \rightarrow L^1(X)$ is a *carré du champ*. The second one can be established by the general method since $(H_t)_{t \geq 0}$ is an analytic semi-group in $L^p(X)$ for every $p \in (1, \infty)$; see [10, Lemma 3.7]. We present them in the next lemma and omit the details here.

Lemma 5.2 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. For every closed subsets E, F of X , every $t > 0$ and every function $f \in L^2(X)$ with support in E , it holds*

$$\|H_t f\|_{L^2(F)} \leq e^{-\frac{d^2(E,F)}{4t}} \|f\|_{L^2(E)}, \tag{5.6}$$

and there exists a constant $C > 0$, such that

$$\|t \Delta H_t f\|_{L^2(F)} \leq C e^{-\frac{d^2(E,F)}{6t}} \|f\|_{L^2(E)}.$$

Remark 5.2 One may also use two side heat kernel bounds, Theorem 1.1 and Theorem 1.2, and Phragmén-Lindelöf theorem (cf. [13, Proposition 2.1]) to show that

$$\|H_t f\|_{L^2(F)} \leq e^{-\frac{d^2(E,F)}{5t}} \|f\|_{L^2(E)}.$$

Notice that in the above estimate the constant in the exponential term is larger than in Eq. 5.6.

Applying the similar argument as the proof of Eq. 5.6, we can prove the following corollary, which is in fact equivalent to Eq. 5.6.

Corollary 5.1 *Let (X, d, μ) be a $RCD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. For every $t > 0$ and $f_i \in L^2(B_i)$ with support in two balls $B_i := B(x_i, r_i)$ in X , $i = 1, 2$, it holds*

$$\int_X (H_t f_1) f_2 d\mu \leq e^{-\frac{d^2(B_1, B_2)}{4t}} \|f_1\|_{L^2(B_1)} \|f_2\|_{L^2(B_2)}. \tag{5.7}$$

In order to establish the third Davies–Gaffney type estimates, we need to construct the Lipschitz cut-off function with quantitative estimate on its gradient in X , which is possible since for fixed $x \in X$, the function $y \mapsto d(x, y)$ is Lipschitz continuous in X with respect to the distance d itself.

Lemma 5.3 *Let $\epsilon > 0$. For a closed subset F of X , there exists a function $\chi \in \text{Lip}(X)$ such that $\chi \in [0, 1]$, $\chi \equiv 1$ on $F^{(\epsilon/2)}$ and $\chi \equiv 0$ in $X \setminus F^{(\epsilon)}$, and satisfying*

$$|\nabla \chi| \leq \frac{2}{\epsilon}, \quad \mu\text{-a.e. in } X,$$

where $F^{(\epsilon)} := \{x \in X : d(x, F) < \epsilon\}$ is the ϵ -neighborhood of F with respect to d .

Proof Choose

$$\chi(x) = \frac{\left(\frac{\epsilon}{2} - d(x, F^{(\epsilon/2)})\right)^+}{\frac{\epsilon}{2}} \wedge 1, \quad x \in X,$$

where $d(x, A) := \inf_{y \in A} d(x, y)$ for a subset A in X . We complete the proof. □

The third Davies–Gaffney type estimate is presented in the next lemma. The key ingredient in the proof is the following inequality: for every $t > 0$,

$$t \int_X \chi^2 |\nabla H_t f|^2 d\mu \leq \|t \Delta H_t f\|_{L^2(\bar{F}^{(\epsilon)})} \|H_t f\|_{L^2(X)} \\ + 2 \left(t \int_X |\nabla \chi|^2 (H_t f)^2 d\mu \right)^{\frac{1}{2}} \left(t \int_X \chi^2 |\nabla H_t f|^2 d\mu \right)^{\frac{1}{2}},$$

where $\bar{F}^{(\epsilon)}$ is the closure of $F^{(\epsilon)}$ and χ is the cut-off function constructed in Lemma 5.3, and then apply the first two Davies–Gaffney type estimates in Lemma 5.2 (see [10, Lemma 3.10] for the smooth manifold case). We also omit the details here.

Lemma 5.4 *Let (X, d, μ) be a $RD^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. For every closed sets $E, F \subset X$, every $t > 0$ and every function $f \in L^2(X)$ with support in E , there exist constants $C > 0$ and $\beta > 0$ such that*

$$\sqrt{t} \|\nabla H_t f\|_{L^2(F)} \leq C e^{-\beta \frac{d^2(E,F)}{t}} \|f\|_{L^2(E)}.$$

Now we are in position to present another proof of Theorem 5.1.

Proof of Theorem 5.1 for $p \in (1, 2)$ As the proof given in Section 5.1, we only need to show that $|\nabla(-\Delta)^{-1/2}|$ is of weak type $(1, 1)$, which is implied by combining Theorem 1.1, the doubling property in Lemma 2.1(i) and the third Davies–Gaffney type estimate in Lemma 5.4. \square

Remark 5.3 The result that the Riesz transform $|\nabla(-\Delta)^{-1/2}|$ is bounded in $L^p(X)$ for all $p \in (1, 2)$ is in fact can also be shown following the proof of [31, Theorem 5], once we let $\alpha = \frac{1}{2}$, $L = \Delta$ and $A = |\nabla \cdot|$ in the aforementioned theorem and combine with Theorem 1.1 and Corollary 5.1.

Proof of Theorem 5.1 for $p \in (2, \infty)$ Following the method in [9, Sections 2 and 3], we only need to combine Lemma 2.1(i), the first part of Lemma 2.2, Proposition 5.2, Lemma 5.2 and Lemma 5.4, and then complete the proof. \square

As for Theorem 5.2, by the localization technique in [16, Section 5] (see also [9, Section 4] in the Riemannian setting), the local case can be proved as in the global case.

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