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On Harnack's Inequality for the Linearized Parabolic Monge-Ampère Equation

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Abstract It is shown that the parabolic Harnack property stands as an intrinsic feature of the Monge-Ampère quasi-metric structure by proving Harnack's inequality for non-negative solutions to the linearized parabolic Monge-Ampère equation under minimal geometric assumptions.

Keywords Linearized parabolic Monge-Ampère equation · Monge-Ampère measure · Harnack's inequality

Mathematics Subject Classification (2010) Primary 35K96 · 35K10 · Secondary 35K65 · 35A15

1 Introduction and Main Result

The results in this article constitute the parabolic part of a program started in [7, 9] (where the elliptic case was treated) with the dominant theme of establishing Harnack's property within the Monge-Ampère quasi-metric structure under minimal geometric assumptions. As a consequence, the Harnack property is shown to hold as an intrinsic feature of the Monge-Ampère quasi-metric space (see more on this below). The rest of this section will be devoted to a detailed description of our main result, the plan of the proof, and the distinguishing features of the techniques involved.

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Throughout the article $\varphi : \mathbb{R}^n \to \mathbb{R}$ will be a strictly convex, twice continuously differentiable function with $D^2\varphi(x) > 0$ for every $x \in \mathbb{R}^n$. We will be concerned with the typically degenerate and singular parabolic operator

$$L_\varphi(u) := u_t - \operatorname{trace}((D^2\varphi)^{-1} D^2 u). \tag{1.1}$$

The natural measure-theoretic and geometric objects for understanding L_φ are dictated by φ as well. The Monge-Ampère measure associated to φ is $d\mu_\varphi(x) := \det D^2\varphi(x)\, dx$ and, given $x \in \mathbb{R}^n$ and $r > 0$, a section of φ centered at x with height r is the open bounded convex set

$$S_\varphi(x, r) := \{y \in \mathbb{R}^n : \varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y - x\rangle < r\}.$$

Our main result is the following:

Theorem 1 (Parabolic Harnack inequality) *Assume $\mu_\varphi \in (DC)_\varphi$. There exist geometric constants $C_H, C_K \geq 1$ such that for every $(x_0, t_0) \in \mathbb{R}^{n+1}$, every $R > 0$, and every non-negative solution u to*

$$L_\varphi(u) = 0 \quad \text{in } S_\varphi(x_0, C_K R) \times (t_0 - 3R/2, t_0 + 2R],$$

we have

$$\sup_{Q^-} u \leq C_H \inf_{Q^+} u,$$

where $Q^+ := S_\varphi(x_0, R) \times (t_0 + R, t_0 + 2R]$ and $Q^- := S_\varphi(x_0, R) \times (t_0 - R, t_0]$.

Based on the pioneering work of L. Caffarelli and C. Gutiérrez [1, 2] in the elliptic context, Q. Huang successfully implemented a program to prove Theorem 1 in [6] under the stronger hypothesis $\mu_\varphi \in (\mu_\infty)$ (see [6, Theorem 1.1]). Thus, as the primary purpose of this article, we establish Theorem 1 under the weaker (and minimal) assumption $\mu_\varphi \in (\text{DC})_\varphi$. A brief comment on the hypotheses $\mu_\varphi \in (\mu_\infty)$ versus $\mu_\varphi \in (\text{DC})_\varphi$ follows.

The hypothesis $\mu_\varphi \in (\mu_\infty)$ corresponds to a Coifman-Fefferman-type property for μ_φ. That is, $\mu_\varphi \in (\mu_\infty)$ if and only if there exist constants $0 < \alpha \leq 1 \leq C$ such that for every section $S := S_\varphi(x_0, r)$ and every Borel set $E \subset S$ it holds true that

$$\frac{\mu_\varphi(E)}{\mu_\varphi(S)} \leq C \left(\frac{|E|}{|S|}\right)^\alpha, \tag{1.2}$$

where $|F|$ denotes the Lebesgue measure of a set $F \subset \mathbb{R}^n$. Property (1.2) has been extensively used by Caffarelli-Gutiérrez [2] and Huang [6] in order to obtain Harnack's inequalities in the degenerate elliptic and parabolic settings, respectively. Following [2, 5] we write $\mu_\varphi \in (\text{DC})_\varphi$ if there exist constants $\alpha_0 \in (0, 1)$ and $C_0 \geq 1$ such that

$$\mu_\varphi(S_\varphi(x, r)) \leq C_0\ \mu_\varphi(\alpha_0 S_\varphi(x, r)) \quad \forall x \in \mathbb{R}^n, \forall r > 0, \tag{1.3}$$

where $\alpha_0 S_\varphi(x, r)$ is the α_0-contraction of $S_\varphi(x, r)$ with respect to its center of mass (the center of mass being computed with respect to Lebesgue measure). Constants depending only on α_0 and C_0 in Eq. 1.3, as well as on dimension n, will be called *geometric constants*.

It turns out that $\mu_\varphi \in (\mu_\infty)$ implies $\mu_\varphi \in (\text{DC})_\varphi$ (with the reverse implication not being true in general) and that the difference between properties (1.2) and (1.3) can be regarded as the one between Muckenhoupt's A_∞ weights and doubling weights. In particular, notice how, as opposed to Eq. 1.2, condition (1.3) requires no a priori regularity with respect to Lebesgue measure. (For a more detailed comparison between $(\mathrm{DC})_\varphi$ and (μ_∞), see [3, Section 3].) More importantly, the condition $\mu_\varphi \in (\mathrm{DC})_\varphi$ turns out to be equivalent to a quasi-metric structure associated to φ (see Section 2) that requires no a priori intervention of Lebesgue measure or Euclidean balls. Thus, the relevance of Theorem 1 stems from placing the parabolic Harnack inequality as an *intrinsic* feature of the Monge-Ampère quasi-metric structure.

In both the elliptic and parabolic settings treated in [2, 6], the method for obtaining Harnack's inequality was modeled after the work of N. Krylov and M. Safonov on uniformly elliptic and parabolic operators in non-divergence form which consists of three basic steps: a mean-value property for positive sub-solutions, a double-ball property for positive super-solutions, and a suitable covering lemma to obtain a power-like decay of the distribution function of a positive solution. Despite the degeneracy of the linearized elliptic and parabolic Monge-Ampère operators, this method was carried out in [2, 6] with the hypothesis $\mu_\varphi \in (\mu_\infty)$ playing a key role in overcoming such degeneracy. In this work we use the same approach, but with different techniques that overcome the degeneracy under $\mu_\varphi \in (\mathrm{DC})_\varphi$ only.

1.1 Plan of the Proof of Theorem 1

In Q. Huang's implementation of the method above in the parabolic case, the mean-value property for positive sub-solutions (a combination of Lemma 4.2 and Theorem 4.2 in [6]) is obtained just under $\mu_\varphi \in (\mathrm{DC})_\varphi$. More precisely, the computations from [6, pp. 2051–53] imply the following mean-value inequality for sub-solutions under the assumption $\mu_\varphi \in (\mathrm{DC})_\varphi$ only.

Theorem 2 ([6, pp. 2051–53]) *Assume $\mu_\varphi \in (DC)_\varphi$. For every $q > 0$ and $0 < \tau' < \tau$, there exists a constant $K_9 > 0$, depending only on geometric constants as well as on q and τ'/τ, such that for every $(x_0, t_0) \in \mathbb{R}^{n+1}$, every $R > 0$, and every $u \geq 0$ satisfying $L_\varphi(u) \leq 0$ in $Q^\tau(K)$, where*

$$Q^\tau(K) := S_\varphi(x_0, KR) \times (t_0 - \tau R, t_0]$$

(and K is as in Eq. 2.12*), we have*

$$\sup_{Q'} u \leq K_9 \left(\frac{1}{\mathcal{M}(Q^\tau(K))} \iint_{Q^\tau(K)} u^q \, d\mathcal{M} \right)^{\frac{1}{q}}, \tag{1.4}$$

where $Q' := S(x_0, \tau' R/\tau) \times (t_0 - \tau' R, t_0]$.

On the other hand, the hypothesis $\mu_\varphi \in (\mu_\infty)$ was used in [6] twice: once to build a Calderón-Zygmund covering lemma based on parabolic cylinders (see [6, Theorem 2.1]) and then to prove the so-called double-ball property for positive super-solutions (see [6, Lemma 3.3]). In turn, those results led to a power-like decay property for solutions, see Theorem 4.1 and Corollary 4.1 in [6]. As mentioned, in this work the hypothesis $\mu_\varphi \in (\mu_\infty)$ is bypassed and, after suitable preparations, in Section 6 we prove.

Theorem 3 *Assume $\mu_\varphi \in (DC)_\varphi$. There exist geometric constants $K_{10} > 0$ and $0 < \delta_1 < 1$ such that every positive solution u of*

$$L_\varphi(u) = 0 \quad \text{in } S_\varphi(x_0, 8K^2R) \times (t_0 - 3R/2, t_0 + 2R]$$

satisfies

$$\left(\frac{1}{\mathcal{M}(Q^-)} \iint_{Q^-} u^{\delta_1} \, d\mathcal{M} \right)^{\frac{1}{\delta_1}} \leq K_{10} \inf_{Q^+} u. \tag{1.5}$$

Thus, Theorem 3 encompasses Theorem 4.1 and Corollary 4.1 in [6] and Theorem 1 follows from Theorem 2 and Theorem 3.

The proof of Theorem 3 relies on taking advantage of the variational side of L_φ. The connection between the divergence and non-divergence forms for L_φ comes from the fact that $A_\varphi(x)$, the matrix of co-factors of $D^2\varphi(x)$

$$A_\varphi(x) := D^2\varphi(x)^{-1} \det D^2\varphi(x) \quad \forall x \in \mathbb{R}^n,$$

possesses the null-Lagrangian property; namely,

$$\operatorname{div}(A_\varphi \nabla h)(x) = \operatorname{trace}(A_\varphi(x) D^2 h(x)), \tag{1.6}$$

for every function h that is twice-differentiable function at a point $x \in \mathbb{R}^n$. The identity (1.6) follows from fact that the columns of A_φ are divergence-free.

Indeed, Eq. 1.6 will allow us to deal with L_φ as a divergence-form operator whose degeneracy will be addressed by Poincaré-type inequalities adapted to φ (see Section 2.1). That is, as opposed to the techniques in [2, 6] based on the ABP maximum principle, our techniques will hinge upon Poincaré-type inequalities and integration by parts. This approach was introduced in [7, 9] in the elliptic Monge-Ampère setting.

The rest of the article is organized as follows: In Section 2 the notation and basic properties for the Monge-Ampère quasi-metric structure are recorded. In Section 3, and always under the hypothesis $\mu_\varphi \in (\mathrm{DC})_\varphi$ only, we prove a uniform "size-transfer", from parabolic cylinders to sections, for positive super-solutions. Meaning that whenever a positive super-solution is large in a portion of a parabolic cylinder, then, in a shorter time range and as functions of the space variable only, they remain uniformly above zero in a portion of an inner section (see Theorem 8). Then, by means of Poincaré-type inequalities and integration by parts, in Section 4 we prove an arbitrarily sensitive critical-density property (see Theorem 10).

In Section 5, it is shown how Theorem 10, along with a suitable calibration of some constants, implies the double-ball property for positive super-solutions (see Theorem 11). Also, a new proof for Lemma 3.3 in [6], but under $\mu_\varphi \in (\mathrm{DC})_\varphi$ only, is included in the form of Corollary 12.

Theorem 3 is proved in Section 6 where the role of the Calderón-Zygmund covering lemma in [6] is played by the so-called *crawling ink spots theorem*. To achieve this, the approach implemented by Schwab-Silvestre [10, Section 6] in the context of parabolic integro-differential equations with very irregular kernels has been adapted to the Monge-Ampère setting.

Throughout the article one can assume that the sub- and super-solutions under consideration are classical ones, keeping in mind that all constants involved will always be geometric constants, in particular, they depend on neither the smoothness of the sub- and super-solutions nor the largest or smallest eigenvalues of the Hessian $D^2\varphi$.

2 Preliminaries and Notation

For $x, y \in \mathbb{R}^n$ set

$$\delta_\varphi(x, y) := \varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y - x\rangle. \tag{2.7}$$

Then, for $x \in \mathbb{R}^n$ and $r > 0$, the section $S_\varphi(x, r) = \{y \in \mathbb{R}^n : \delta_\varphi(x, y) < r\}$. Notice that, due the strict convexity of φ, $\delta_\varphi(x, y) = 0$ if and only if $x = y$.

The parabolic Monge-Ampère measure in $\mathbb{R}^{n+1}$ is defined by

$$d\mathcal{M}(x, t) := \det D^2\varphi(x)\, dx\, dt \quad \forall (x, t) \in \mathbb{R}^{n+1}. \tag{2.8}$$

The condition $\mu_\varphi \in (\mathrm{DC})_\varphi$ implies the existence of a geometric constant $K_d \geq 1$ such that

$$\mu_\varphi(S_\varphi(x, 2r)) \leq K_d\ \mu_\varphi(S_\varphi(x, r)) \quad \forall x \in \mathbb{R}^n, \forall r > 0. \tag{2.9}$$

By setting $\nu := \log_2 K_d$, Eq. 2.9 yields

$$\mu_\varphi(S_\varphi(x, R_2)) \leq K_d \left(\frac{R_2}{R_1}\right)^\nu \mu_\varphi(S_\varphi(x, R_1)) \quad \forall x \in \mathbb{R}^n, \forall\, 0 < R_1 < R_2. \tag{2.10}$$

Also, for a geometric constant $K \geq 1$, we have

$$\delta_\varphi(x, y) \leq K\delta_\varphi(y, x) \quad \forall x, y \in \mathbb{R}^n \tag{2.11}$$

as well as the following symmetrized K-quasi-triangle inequality which holds true for every $x, y, z \in \mathbb{R}^n$

$$\delta_\varphi(x, y) \leq K\left(\min\{\delta_\varphi(z, x), \delta_\varphi(x, z)\} + \min\{\delta_\varphi(z, y), \delta_\varphi(y, z)\}\right). \tag{2.12}$$

Conversely, if Eqs. 2.11 and 2.12 hold true for some $K \geq 1$, then so does Eq. 1.3 with some constants α_0 and C_0 depending only on K and dimension n. That is, the condition $\mu_\varphi \in (\mathrm{DC})_\varphi$ exactly determines when the pair $(\mathbb{R}^n, \delta_\varphi)$ becomes a quasi-metric space (in the sense of Eqs. 2.11 and 2.12). Hence, the condition $\mu_\varphi \in (\mathrm{DC})_\varphi$ is referred to as a minimal geometric hypothesis, in the sense that a quasi-metric space represents a minimal platform on which real-analysis techniques can be carried out. See [2, 3, 5], [4, Chapter 3], and references therein, for more on the Monge-Ampère quasi-metric structure and its related real analysis.

In Sections 3 and 4 it will be necessary to deal with measures of "thin annuli" of the form $S_\varphi(x_0, R) \setminus S_\varphi(x_0, \beta R)$ with $0 < \beta < 1$ and β close to 1. To this end we resort to the following lemma by Caffarelli-Gutiérrez in [1].

Lemma 4 ([1, Lemma 2]) *Suppose that μ_φ satisfies the doubling condition (2.9). Then, given $R > 0$ and $\epsilon > 0$ there exists $\xi \in (1, 2]$, depending only on R and ϵ, such that $\xi - \epsilon \geq 1$ and*

$$\frac{\mu_\varphi(S_\varphi(x_0, \xi R) \setminus S_\varphi(x_0, (\xi - \epsilon)R))}{\mu_\varphi(S_\varphi(x_0, \xi R))} \leq \epsilon \log K_d. \tag{2.13}$$

Remark 5 Lemma 4 will be used in the following way: for every $x_0 \in \mathbb{R}^n$, $R > 0$, and $\beta \in (0, 1)$ we have that

$$\mu_\varphi(S_\varphi(x_0, R')) \le \frac{\mu_\varphi(S_\varphi(x_0, \beta R'))}{1 - 2(1-\beta)\log K_d}, \tag{2.14}$$

with $R' := \xi R$ and $\xi = \xi(R, \beta) \in (1, 2]$. Indeed, given $x_0 \in \mathbb{R}^n$, $R > 0$, and $\beta \in (0, 1)$, set $\epsilon := 2(1-\beta)$. Then, let $\xi = \xi(R, \epsilon)$ be as in Lemma 4 so that putting $R' := \xi R$ and $\epsilon' := \epsilon/\xi$, Eq. 2.13 means

$$\frac{\mu_\varphi(S_\varphi(x_0, R') \setminus S_\varphi(x_0, (1-\epsilon')R'))}{\mu_\varphi(S_\varphi(x_0, R'))} \le \xi\epsilon' \log K_d. \tag{2.15}$$

Next, by setting $\beta' := 1 - \epsilon' = 1 - \epsilon/\xi$ we have

$$1 - \epsilon < 1 - \epsilon/\xi = \beta' \le 1 - \epsilon/2 = \beta$$

and Eq. 2.15 gives

$$\begin{aligned} \mu_\varphi(S_\varphi(x_0, R')) &= \mu_\varphi(S_\varphi(x_0, R') \setminus S_\varphi(x_0, \beta' R')) + \mu_\varphi(S_\varphi(x_0, \beta' R')) \\ &\le \epsilon \log K_d\, \mu_\varphi(S_\varphi(x_0, R')) + \mu_\varphi(S_\varphi(x_0, \beta' R')), \end{aligned}$$

so that Eq. 2.14 follows from

$$\mu_\varphi(S_\varphi(x_0, R')) \le \frac{\mu_\varphi(S_\varphi(x_0, \beta' R'))}{1 - \epsilon \log K_d} \le \frac{\mu_\varphi(S_\varphi(x_0, \beta R'))}{1 - 2(1-\beta)\log K_d}.$$

2.1 Poincaré-Type Inequalities

The appropriate notion of gradient is also adapted to the convex function φ. Given a function v differentiable at a point $x \in \mathbb{R}^n$ we define

$$\nabla^\varphi v(x) := D^2\varphi(x)^{-\frac{1}{2}} \nabla v(x). \tag{2.16}$$

Poincaré and Sobolev inequalities for the Monge-Ampère quasi-metric structure (that is, under the assumption $\mu_\varphi \in (\mathrm{DC})_\varphi$ only) with respect to ∇^φ have been proved in [7, 8]. The next theorem extends the Poincaré-type inequalities from [7] to allow for averages of functions on arbitrary measurable subsets of the Monge-Ampère sections.

Theorem 6 *Assume $\mu_\varphi \in (DC)_\varphi$. There exist geometric constants $K_3, K_5, K_0 \ge 1$ such that for every section $S := S_\varphi(x_0, r)$ and every (Lebesgue-measurable) subset $\mathcal{N} \subset S$, the following Poincaré-type inequalities hold true:*

(i) *For every $h \in C^1(S)$ we have*

$$\int_S |h - h_{\mathcal{N}}|\, dx \le K_3 \frac{|S|}{|\mathcal{N}|} \left(r|S| \int_S |\nabla^\varphi h|^2\, dx \right)^{\frac{1}{2}}, \tag{2.17}$$

where $h_{\mathcal{N}} := \frac{1}{|\mathcal{N}|} \int_{\mathcal{N}} h(x)\, dx$.

(ii) *For every $h \in C^1(S_\varphi(x_0, K_0^2 r))$ we have*

$$\int_S |h - h^\varphi_{\mathcal{N}}|\, d\mu_\varphi \le K_5 \frac{\mu_\varphi(S)}{\mu_\varphi(\mathcal{N})} \left(r\, \mu_\varphi(S) \int_{S_\varphi(x_0, K_0^2 r)} |\nabla^\varphi h|^2\, d\mu_\varphi \right)^{\frac{1}{2}}, \tag{2.18}$$

where $h^\varphi_{\mathcal{N}} := \frac{1}{\mu_\varphi(\mathcal{N})} \int_{\mathcal{N}} h(x)\, d\mu_\varphi(x)$.

The proof of Theorem 6 proceeds as the one for Theorem 1.3 in [7] and we briefly sketch it. Starting with an affine transformation T that normalizes S so that $T(\mathcal{N}) \subset T(S) \subset B(0, 1)$ and $T(S)$ is convex. For $h \in C^1(S)$ define $\bar{h} \in C^1(T(S))$ as

$$\bar{h}(y) := h(T^{-1}y) \quad \forall y \in T(S).$$

By the usual Poincaré inequality on convex sets (see, for instance, Lemma 5.2.1 [11, p.146]), there exists a constant $C^* \geq 1$, depending only on dimension n, such that

$$\int_{T(S)} |\bar{h}(y) - \bar{h}_{T(\mathcal{N})}|\, dy \leq \frac{C^* \operatorname{diam}(T(S))^{n+1}}{|T(\mathcal{N})|} \int_{T(S)} |\nabla \bar{h}(y)|\, dy, \tag{2.19}$$

with

$$h_{T(\mathcal{N})} := \frac{1}{|T(\mathcal{N})|} \int_{T(\mathcal{N})} \bar{h}(y)\, dy.$$

Then Eq. 2.17 follows by reasoning along the lines of the proof of Theorem 1.3 in [7, Section 5] and, by means of the convex conjugate of φ, so does Eq. 2.18. See [7, Section 5] and [8, Section 4] for further details.

3 A Uniform Estimate from Parabolic Cylinders to Sections

From this point on we fundamentally depart from the non-variational techniques (based on maximum principles) used in [2] and [6]. Instead, we implement a mix of techniques from the context of divergence-form parabolic operators (see, for instance, [11, Section 5.2] on Harnack's inequality for the heat equation) and from [7, 9] where the degeneracy of the linearized elliptic Monge-Ampère operator has been dealt with by exploring the variational side of the operator.

We start by mentioning the following lemma whose proof can be found, for instance, in [11, p.148].

Lemma 7 ([11, p.148]) *There exists a twice continuously differentiable function* $g : (0, \infty) \to [0, \infty)$ *such that*

(i) $g'(s) \leq 0$ *for every* $s > 0$,
(ii) $g''(s) \geq (g'(s))^2 - g'(s)$ *for every* $s > 0$,
(iii) g *satisfies*

$$g(s) = \log\left(\frac{1 - e^{-1}}{1 - e^{-s}}\right) \quad \forall s \in (0, 1/2),$$

(iv) $g(s) = 0$ *for every* $s \geq 1$.

As mentioned in the introduction, the next theorem quantifies the fact that whenever a non-negative super-solution u is large in a portion of a parabolic cylinder, then, for t in a shorter time range, the functions $u(\cdot, t)$ remain uniformly above zero in a portion of an inner section.

Theorem 8 *Assume* $\mu_\varphi \in (DC)_\varphi$. *Given* $(x_0, t_0) \in \mathbb{R}^{n+1}$ *and* $R > 0$, *suppose that* u *satisfies* $L_\varphi(u) \geq 0$ *and* $u > 0$ *in* Q_{2R} *where*

$$Q_{2R}(x_0, t_0) := S_\varphi(x_0, 2R) \times (t_0 - 2R, t_0].$$

Let α, β, ε, and σ be any numbers satisfying $0 < \sigma < \varepsilon < \beta < 1$ *and*

$$\frac{(1-\varepsilon)}{(1-\sigma)} < (1-\alpha)[1 - 2(1-\beta)\log K_d], \tag{3.20}$$

in particular, β needs to be close enough to 1 so that $1 - 2(1-\beta)\log K_d > 0$. *Let us put*

$$Q_{R'} := S_\varphi(x_0, R') \times (t_0 - R', t_0]$$

with $R' := \xi R \in (R, 2R]$ and $\xi = \xi(R, \beta)$ as in Remark 5.

Then, there is a constant $\lambda \in (0, 1)$, depending only on geometric constants as well as on α, β, ε, and σ, such that the inequality

$$\mathcal{M}(\{(x,t) \in Q_{R'} : u(x,t) \geq 1\}) \geq \varepsilon \mathcal{M}(Q_{R'}) \tag{3.21}$$

implies that

$$\mu_\varphi(\{x \in S_\varphi(x_0, \beta R') : u(x,t) \geq \lambda\}) \geq \alpha\, \mu_\varphi(S_\varphi(x_0, \beta R')) \quad \forall t \in (t_0 - \sigma R', t_0]. \tag{3.22}$$

Proof Set $S := S_\varphi(x_0, R')$. Take $\zeta \in C_0^1(S)$ (independent of time t) to be specified later and any $t_1, t_2 \in \mathbb{R}$ such that $t_0 - R' \leq t_1 < t_2 \leq t_0$. For a function $G : (0, \infty) \to [0, \infty)$ satisfying

$$G \in C^2(0, \infty), \quad G' \leq 0, \quad \text{and} \quad G'' \geq (G')^2, \tag{3.23}$$

also to be fixed later, multiply the inequality $L_\varphi(u) \geq 0$, which means

$$u_t - \operatorname{trace}((D^2\varphi)^{-1} D^2 u) \geq 0,$$

by $\zeta^2 G'(u)\chi_{[t_1,t_2]}(\leq 0)$ and integrate over $S \times [t_1, t_2]$ with respect to $d\mathcal{M}$ to obtain

$$\int_{t_1}^{t_2}\int_S \Big[\zeta^2 G'(u)u_t\, \mu_\varphi - \operatorname{trace}(A_\varphi D^2 u)\zeta^2 G'(u)\Big]\, dxdt \leq 0. \tag{3.24}$$

By applying the null-Lagrangian property (1.6) and integrating by parts, for each fixed $t \in [t_0 - R, t_0]$, Eq. 3.24 now reads as

$$\int_{t_1}^{t_2}\int_S \Big[\zeta^2 G'(u)u_t\, \mu_\varphi + \langle A_\varphi \nabla u, \nabla(\zeta^2 G'(u))\rangle\Big]\, dxdt \leq 0. \tag{3.25}$$

Setting $w := G(u)$ and recalling the definition of ∇^φ in Eqs. 2.16, 3.25 yields

$$\int_{t_1}^{t_2}\int_S \Big[\zeta^2 w_t + G''\zeta^2|\nabla^\varphi u|^2 + \langle \nabla^\varphi w, \nabla^\varphi(\zeta^2)\rangle\Big]\, d\mathcal{M} \leq 0. \tag{3.26}$$

Now, by using Eq. 3.26, we can bound

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_S \Big[\zeta^2 w_t + \langle \nabla^\varphi w, \nabla^\varphi(\zeta^2)\rangle + \zeta^2|\nabla^\varphi w|^2\Big]\, d\mathcal{M} \\
&= \int_{t_1}^{t_2}\int_S \Big[\zeta^2 w_t + G''\zeta^2|\nabla^\varphi u|^2 + \langle \nabla^\varphi w, \nabla^\varphi(\zeta^2)\rangle\Big]\, d\mathcal{M} \\
&+ \int_{t_1}^{t_2}\int_S (-G''\zeta^2|\nabla^\varphi u|^2 + \zeta^2|\nabla^\varphi w|^2)\, d\mathcal{M} \\
&\leq \int_{t_1}^{t_2}\int_S (-G''\zeta^2|\nabla^\varphi u|^2 + \zeta^2|\nabla^\varphi w|^2)\, d\mathcal{M} \\
&= \int_{t_1}^{t_2}\int_S \Big[-G'' + (G')^2\Big]\zeta^2|\nabla^\varphi u|^2\, d\mathcal{M} \leq 0,
\end{aligned}$$

where in the last line we used that $\nabla^\varphi w = G'(u)\nabla^\varphi u$ and Eq. 3.23. Consequently,

$$\int_{t_1}^{t_2}\int_S \left(\zeta^2 w_t + \zeta^2|\nabla^\varphi w|^2\right) d\mathcal{M} \le -\int_{t_1}^{t_2}\int_S \langle \nabla^\varphi w, \nabla^\varphi(\zeta^2)\rangle\, d\mathcal{M}$$

and, by Cauchy-Schwarz and Young's inequalities,

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_S \left(\zeta^2 w_t + \zeta^2|\nabla^\varphi w|^2\right) d\mathcal{M} \le \int_{t_1}^{t_2}\int_S |\langle \nabla^\varphi w, \nabla^\varphi(\zeta^2)\rangle|\, d\mathcal{M}\\
&\le 2\int_{t_1}^{t_2}\int_S \zeta|\nabla^\varphi w||\nabla^\varphi \zeta|\, d\mathcal{M}\\
&\le 2\left(\int_{t_1}^{t_2}\int_S \zeta^2|\nabla^\varphi w|^2\, d\mathcal{M}\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\int_S |\nabla^\varphi \zeta|^2\, d\mathcal{M}\right)^{\frac{1}{2}}\\
&\le \frac{1}{2}\int_{t_1}^{t_2}\int_S \zeta^2|\nabla^\varphi w|^2\, d\mathcal{M} + 2\int_{t_1}^{t_2}\int_S |\nabla^\varphi \zeta|^2\, d\mathcal{M}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\textstyle\int_{t_1}^{t_2}\int_S \left(\zeta^2 w_t + \frac{1}{2}\zeta^2|\nabla^\varphi w|^2\right) d\mu_\varphi\, dt &\le 2\int_{t_1}^{t_2}\int_S |\nabla^\varphi \zeta|^2\, d\mu_\varphi\, dt\\
&= 2(t_2 - t_1)\int_S |\nabla^\varphi \zeta|^2\, d\mu_\varphi \le 2R'\int_S |\nabla^\varphi \zeta|^2\, d\mu_\varphi. \qquad (3.27)
\end{aligned}$$

Next, in order to estimate $\int_S |\nabla^\varphi \zeta|^2\, d\mu_\varphi$, we proceed along the lines of some computations in [9, Section 6], this is the step where the degeneracy of L_φ is circumvented by means of the null Lagrangian property (1.6). Details follow. For $\beta \in (0, 1)$ as in the statement of the theorem (it will be helpful to bear in mind that β is close to 1), let $h : \mathbb{R} \to [0, 1]$ be a differentiable function such that

$$h \equiv 1 \text{ in } [0, \beta R'], \quad h \equiv 0 \text{ in } [R', \infty), \text{ and } \quad \|h'\|_{L^\infty} \le \frac{1}{R'(1-\beta)}. \qquad (3.28)$$

Recalling the definition of δ_φ in Eq. 2.7, for $x \in \mathbb{R}^n$ set

$$\delta_{x_0}(x) := \delta_\varphi(x_0, x) \quad \text{and} \quad \zeta_0(x) := h(\delta_{x_0}(x))$$

so that $\zeta_0 \in C_0^1(S)$ and $\nabla\zeta_0(x) = h'(\delta_{x_0}(x))\nabla\delta_{x_0}(x)$. Integrating by parts again (notice that $\delta_{x_0}(x) = R'$ for every $x \in \partial S$) and using the null-Lagrangian property (1.6), we can write

$$\begin{aligned}
&\int_S |\nabla^\varphi \zeta_0|^2\, d\mu_\varphi = \int_S \langle A_\varphi \nabla\zeta_0, \nabla\zeta_0\rangle\, dx = \int_S h'(\delta_{x_0})^2 \langle A_\varphi \nabla\delta_{x_0}, \nabla\delta_{x_0}\rangle\, dx\\
&\le \|h'\|_{L^\infty}^2 \int_S \langle A_\varphi \nabla\delta_{x_0}, \nabla\delta_{x_0}\rangle\, dx = -\|h'\|_{L^\infty}^2 \int_S \langle A_\varphi \nabla\delta_{x_0}, \nabla(R' - \delta_{x_0})\rangle\, dx\\
&= \|h'\|_{L^\infty}^2 \int_S \operatorname{div}(A_\varphi \nabla\delta_{x_0})(R' - \delta_{x_0})\, dx\\
&= \|h'\|_{L^\infty}^2 \int_S \operatorname{trace}(A_\varphi D^2\delta_{x_0})(R' - \delta_{x_0})\, dx\\
&= \|h'\|_{L^\infty}^2 \int_S \operatorname{trace}(A_\varphi D^2\varphi)(R' - \delta_{x_0})\, dx = n\|h'\|_{L^\infty}^2 \int_S (R' - \delta_{x_0})\, \mu_\varphi\, dx\\
&\le \frac{nR'}{R'^2(1-\beta)^2}\,\mu_\varphi(S) = \frac{n}{R'(1-\beta)^2}\,\mu_\varphi(S).
\end{aligned}$$

Putting this together with Eq. 3.27 implies that for every t_1, t_2 in $[t_0 - R', t_0]$, with $t_1 < t_2$, the following inequality holds true

$$\int_{t_1}^{t_2} \int_S \left(\zeta_0^2 w_t + \frac{1}{2} \zeta_0^2 |\nabla^\varphi w|^2 \right) d\mathcal{M} \le \frac{2n}{(1-\beta)^2} \, \mu_\varphi(S). \tag{3.29}$$

For $\lambda > 0$ to be fixed later, set $w = G(u) := g(u + \lambda) \ge 0$ where g is the function from Lemma 7. For $t \in [t_0 - R', t_0]$ set

$$\mu(t) := \mu_\varphi(\{x \in S_\varphi(x_0, R') : u(x,t) \ge 1\})$$

and

$$N_t := \{x \in S_\varphi(x_0, \beta R') : u(x,t) \ge \lambda\}.$$

From Eq. 3.21 we have

$$\int_{t_0 - R'}^{t_0} \mu(t)\, dt = \mathcal{M}(\{(x,t) \in Q_{R'} : u(x,t) \ge 1\}) \ge \varepsilon \mathcal{M}(Q_{R'}) = \varepsilon\, \mu_\varphi(S) R'.$$

Therefore, since we also have $\mu(t) \le \mu_\varphi(S)$ for every $t \in [t_0 - R', t_0]$,

$$\int_{t_0 - R'}^{t_0 - \sigma R'} \mu(t)\, dt = \int_{t_0 - R'}^{t_0} \mu(t)\, dt - \int_{t_0 - \sigma R'}^{t_0} \mu(t)\, dt \ge (\varepsilon - \sigma)\, \mu_\varphi(S) R'.$$

By the mean-value theorem there is $\tau \in [t_0 - R', t_0 - \sigma R']$ such that

$$\mu(\tau) \ge \frac{\varepsilon - \sigma}{1 - \sigma}\, \mu_\varphi(S). \tag{3.30}$$

Now, using Eq. 3.29 with $t_1 := \tau$ and any $t_2 \in (t_0 - \sigma R', t_0]$ followed by the doubling property (2.10), we can write

$$\begin{aligned}
&\int_S \zeta_0^2(x) w(x, t_2)\, d\mu_\varphi(x) \\
&= \int_\tau^{t_2} \int_S \zeta_0^2(x) w_t(x,t)\, d\mu_\varphi(x)\, dt + \int_S \zeta_0^2(x) w(x,\tau)\, d\mu_\varphi(x) \\
&\le \frac{2n\, \mu_\varphi(S_\varphi(x_0, R'))}{(1-\beta)^2} + \int_S \zeta_0^2(x) w(x,\tau)\, d\mu_\varphi(x) \\
&\le \frac{2n K_d\, \mu_\varphi(S_\varphi(x_0, \varepsilon R'))}{(1-\beta)^2 \varepsilon^\nu} + \int_S \zeta_0^2(x) w(x,\tau)\, d\mu_\varphi(x),
\end{aligned}$$

which, since $\varepsilon < \beta < 1$, implies that

$$\int_S \zeta_0^2(x) w(x, t_2)\, d\mu_\varphi(x) \le \frac{2n K_d\, \mu_\varphi(S_\varphi(x_0, \beta R'))}{(1-\beta)^2 \varepsilon^\nu} + \int_S \zeta_0^2(x) w(x,\tau)\, d\mu_\varphi(x). \tag{3.31}$$

On the other hand, by using that $u(\cdot, t) < \lambda$ in $S_\varphi(x_0, \beta R') \setminus N_t$, that $w = g(u + \lambda)$ with $g' \le 0$, and that $\zeta_0 \equiv 1$ in $S_\varphi(x_0, \beta R')$, we get

$$\begin{aligned}
\int_S \zeta_0^2(x) w(x, t_2)\, d\mu_\varphi(x) &\ge \int_{S_\varphi(x_0, \beta R')} w(x, t_2)\, d\mu_\varphi(x) \\
&\ge \int_{S_\varphi(x_0, \beta R') \setminus N_{t_2}} w(x, t_2)\, d\mu_\varphi(x) \\
&\ge g(2\lambda)\, \mu_\varphi(S_\varphi(x_0, \beta R') \setminus N_{t_2}).
\end{aligned}$$

Also, since $g \equiv 0$ in $[1, \infty)$ and $w = g(u + \lambda) \leq g(\lambda)$, Eq. 3.30 yields

$$\begin{aligned}\int_S \zeta_0^2(x) w(x,\tau)\, d\mu_\varphi(x) &\leq \int_S w(x,\tau)\, d\mu_\varphi(x) = \int_{\{x\in S: u(x,\tau)<1\}} w(x,\tau)\, d\mu_\varphi(x) \\ &\leq g(\lambda)\, \mu_\varphi(\{x \in S : u(x,\tau) < 1\}) \\ &= g(\lambda)(\mu_\varphi(S) - \mu(\tau)) \leq \left(\frac{1-\varepsilon}{1-\sigma}\right) g(\lambda)\, \mu_\varphi(S) \\ &\leq \left(\frac{1-\varepsilon}{1-\sigma}\right) \frac{g(\lambda)\, \mu_\varphi(S_\varphi(x_0, \beta R'))}{1 - 2(1-\beta)\log K_d}, \end{aligned} \tag{3.32}$$

where for the last inequality we used Eq. 2.14. Connecting the above inequalities through Eq. 3.31, for any $t_2 \in (t_0 - \sigma R', t_0]$ it follows that

$$\begin{aligned} g(2\lambda)\, \mu_\varphi(S_\varphi(x_0, \beta R') \setminus N_{t_2}) \leq\ & \frac{2nK_d\, \mu_\varphi(S_\varphi(x_0, \beta R'))}{(1-\beta)^2 \varepsilon^\nu} \\ & + \left(\frac{1-\varepsilon}{1-\sigma}\right) \frac{g(\lambda)\, \mu_\varphi(S_\varphi(x_0, \beta R'))}{[1 - 2(1-\beta)\log K_d]}. \end{aligned}$$

That is, for every $t_2 \in (t_0 - \sigma R', t_0]$, we obtained

$$\begin{aligned} \frac{\mu_\varphi(S_\varphi(x_0, \beta R') \setminus N_{t_2})}{\mu_\varphi(S_\varphi(x_0, \beta R'))} \leq\ & \frac{2nK_d}{(1-\beta)^2 \varepsilon^\nu g(2\lambda)} \\ & + \frac{(1-\varepsilon) g(\lambda)}{(1-\sigma) g(2\lambda)[1 - 2(1-\beta)\log K_d]}. \end{aligned}$$

Therefore, given any $\alpha, \varepsilon, \beta, \sigma$ as in Eq. 3.20, by (iii) in Lemma 7 we can choose $\lambda \in (0, 1)$, close to 0 and depending only on $\alpha, \varepsilon, \beta, \sigma$, dimension n, and the geometric constant K_d, such that

$$\frac{2nK_d}{(1-\beta)^2 \varepsilon^\nu g(2\lambda)} + \frac{(1-\varepsilon) g(\lambda)}{(1-\sigma) g(2\lambda)[1 - 2(1-\beta)\log K_d]} < (1-\alpha).$$

Consequently,

$$\mu_\varphi(N_{t_2}) \geq \alpha\, \mu_\varphi(S_\varphi(x_0, \beta R')) \quad \forall t_2 \in (t_0 - \sigma R', t_0]$$

and this is precisely (3.22). □

4 An Arbitrarily Sensitive Critical-Density Property

Our next lemma, combined with Theorem 8, will imply that every density is critical (Theorem 10 below).

Lemma 9 *Assume $\mu_\varphi \in (DC)_\varphi$. Let $K_0 > 1$ be the geometric constant from Theorem 6. Given $(x_0, t_0) \in \mathbb{R}^{n+1}$ and $R > 0$, set $Q_{2R} := S_\varphi(x_0, 2R) \times (t_0 - 2R, t_0]$ and*

$$Q_R^* := S_\varphi(x_0, 4K_0^2 R) \times (t_0 - 2R, t_0].$$

Suppose that u satisfies $L_\varphi(u) \geq 0$ *and* $u > 0$ *in* Q_R^* *and that there are some constants* α, β, λ, $\sigma \in (0, 1)$ *such that*

$$\mu_\varphi(\{x \in S_\varphi(x_0, \beta R') : u(x,t) \geq \lambda\}) \geq \alpha\, \mu_\varphi(S_\varphi(x_0, \beta R')) \quad \forall t \in (t_0 - \sigma R', t_0] \tag{4.33}$$

where $R' := \xi R \in (R, 2R]$ *and* $\xi = \xi(R, \beta)$ *are always as in Remark 5. Fix an arbitrary* $\sigma' \in (0, \sigma)$*. Then, there exists* $\theta \in (0, 1)$*, depending only on geometric constants as well as on* α, β, λ, σ *and* σ'*, such that*

$$u(x,t) \geq \theta\lambda \quad \forall (x,t) \in Q'_{\beta,\sigma,\sigma'} := S(x_0, \sigma'\beta R'/(K\sigma)) \times (t_0 - \sigma' R', t_0]. \tag{4.34}$$

Proof Let us put $\gamma := \theta\lambda$, where $\theta \in (0, 1)$ will be determined later on. Since we are assuming $L_\varphi(u) \geq 0$ in Q_R^*, by replicating the proof of Theorem 8 up through the point where we obtained (3.29), again with $w := G(u)$ where G satisfies (3.23), but now with the section $S_0 := S_\varphi(x_0, 2K_0^2 R')$ and $\zeta(x) := h(\delta_\varphi(x_0, x))$ with h such that, instead of Eq. 3.28, it verifies $h \equiv 1$ in $[0, K_0^2 R']$, $h \equiv 0$ in $[2K_0^2, \infty)$, and $\|h'\|_{L^\infty} \leq 1/K_0^2$, for every t_1, t_2 in $[t_0 - R', t_0]$, with $t_1 < t_2$, we obtain

$$\int_{t_1}^{t_2} \int_{S_0} \left(\zeta^2 w_t + \frac{1}{2}\zeta^2 |\nabla^\varphi w|^2\right) d\mathcal{M} \leq \frac{2n}{{K_0}^4}\, \mu_\varphi(S_0). \tag{4.35}$$

Now by Eq. 4.35 with $t_1 := t_0 - \sigma R'$ and $t_2 := t_0$

$$\int_{t_0 - \sigma R'}^{t_0} \int_{S_0} \left(\zeta^2 w_t + \frac{1}{2}\zeta^2 |\nabla^\varphi w|^2\right) d\mathcal{M} \leq \frac{2n}{{K_0}^4}\, \mu_\varphi(S_0). \tag{4.36}$$

At this point we choose G as $w := G(u) := g\left(\frac{u+\gamma}{\lambda}\right)$, where g is always as in Lemma 7. In particular, the fact that $g' \leq 0$ implies that $w \leq g(\gamma/\lambda)$ in Q_R^*; therefore,

$$\begin{aligned}\int_{t_0 - \sigma R'}^{t_0} \int_{S_0} \zeta^2 w_t \, d\mathcal{M} &= \int_{S_0} \zeta^2(x)[w(x,t_0) - w(x, t_0 - \sigma R')]\, d\mu_\varphi(x) \\ &\geq -\int_{S_0} \zeta^2(x) w(x, t_0 - \sigma R')\, d\mu_\varphi(x) \geq -g(\gamma/\lambda)\, \mu_\varphi(S_0).\end{aligned}$$

Along with Eq. 4.36, this yields

$$\begin{aligned}\int_{t_0 - \sigma R'}^{t_0} \int_{S_\varphi(x_0, K_0^2 R')} |\nabla^\varphi w|^2 \, d\mathcal{M} &\leq \int_{t_0 - \sigma R'}^{t_0} \int_{S_0} \zeta^2 |\nabla^\varphi w|^2 \, d\mathcal{M} \\ &\leq \left(\frac{4n}{{K_0}^4} + 2g(\gamma/\lambda)\right) \mu_\varphi(S_0).\end{aligned} \tag{4.37}$$

From the facts that $g' \leq 0$ and $g \equiv 0$ in $[1, \infty)$ and the hypothesis (4.33), for every $t \in (t_0 - \sigma R', t_0]$ we have

$$\begin{aligned}\alpha\, \mu_\varphi(S_\varphi(x_0, \beta R')) &\leq \mu_\varphi(\{x \in S_\varphi(x_0, \beta R') : u(x,t) \geq \lambda\}) \\ &\leq \mu_\varphi(\{x \in S_\varphi(x_0, \beta R') : w(x,t) = 0\}).\end{aligned}$$

Now, for each fixed $t \in (t_0 - \sigma R', t_0]$, we use the Poincaré inequality (2.18) from Theorem 6 with the function $w(\cdot, t)$, the section $S_\varphi(x_0, \beta R')$ and the set $\mathcal{N} := \{x \in S_\varphi(x_0, \beta R') : w(x, t) = 0\}$ to obtain

$$\int_{S_\varphi(x_0,\beta R')} w(x,t)\, d\mu_\varphi(x) \le \alpha^{-1} K_5 (\beta R')^{\frac{1}{2}} \mu_\varphi(S_\varphi(x_0, \beta R'))^{\frac{1}{2}} \times \left(\int_{S(x_0, \beta K_0^2 R')} |\nabla^\varphi w(x,t)|^2\, d\mu_\varphi(x) \right)^{\frac{1}{2}}$$

which, after integration in t over $(t_0 - \sigma R', t_0]$ and the Cauchy-Schwarz inequality, yields

$$\int_{t_0 - \sigma R'}^{t_0} \int_{S_\varphi(x_0,\beta R')} w\, d\mathcal{M} \le \alpha^{-1} K_5 (\beta\sigma)^{\frac{1}{2}} R' \mu_\varphi(S_\varphi(x_0, \beta R'))^{\frac{1}{2}} \left(\int_{t_0-\sigma R'}^{t_0} \int_{S(x_0, \beta K_0^2 R')} |\nabla^\varphi w|^2\, d\mathcal{M} \right)^{\frac{1}{2}}.$$

Then, by Eq. 4.37 and the doubling property (2.10), we can further estimate

$$\begin{aligned} &\int_{t_0 - \sigma R'}^{t_0} \int_{S_\varphi(x_0,\beta R')} w\, d\mathcal{M} \\ &\le \alpha^{-1} K_5 (\beta\sigma)^{\frac{1}{2}} R' \mu_\varphi(S_\varphi(x_0, \beta R'))^{\frac{1}{2}} \mu_\varphi(S_0)^{\frac{1}{2}} \left(\frac{4n}{K_0{}^4} + 2g(\gamma/\lambda) \right)^{\frac{1}{2}} \\ &\le K_d^{\frac{1}{2}} K_5 \alpha^{-1} (\beta\sigma)^{\frac{1}{2}} (2K_0^2/\beta)^{\frac{\nu}{2}} R' \mu_\varphi(S_\varphi(x_0, \beta R')) \left(\frac{4n}{K_0{}^4} + 2g(\gamma/\lambda) \right)^{\frac{1}{2}}. \end{aligned}$$

Setting $K(\alpha, \beta, \sigma) := K_d^{\frac{1}{2}} K_5 \alpha^{-1} (\beta/\sigma)^{\frac{1}{2}} (2K_0^2/\beta)^{\frac{\nu}{2}}$ and

$$Q := S_\varphi(x_0, \beta R') \times (t_0 - \sigma R', t_0] \subset Q_{2R} \subset Q_R^*$$

we get

$$\frac{1}{\mathcal{M}(Q)} \iint_Q w\, d\mathcal{M} \le K(\alpha, \beta, \sigma) \left(\frac{4n}{K_0{}^4} + 2g(\gamma/\lambda) \right)^{\frac{1}{2}}. \tag{4.38}$$

On the other hand, since $L_\varphi(u) \ge 0$, $G'(u) \le 0$, and $G''(u) \ge 0$ in all of Q_R^* we have

$$\begin{aligned} L_\varphi(w) &= w_t - \operatorname{trace}((D^2\varphi)^{-1} D^2 w) \\ &= G'(u) u_t - G'(u) \operatorname{trace}((D^2\varphi)^{-1} D^2 u) - G''(u) \operatorname{trace}((D^2\varphi)^{-1} \nabla u \otimes \nabla u) \\ &= G'(u) L_\varphi(u) - G''(u) \operatorname{trace}((D^2\varphi)^{-1} \nabla u \otimes \nabla u) \\ &\le -G''(u) \operatorname{trace}((D^2\varphi)^{-1} \nabla u \otimes \nabla u) = -G''(u) |\nabla^\varphi u|^2 \le 0. \end{aligned}$$

Hence, by applying Theorem 2 (with $q = 1$, $\tau = \sigma$ and $\tau' = \sigma'$) to w in $Q \subset Q_R^*$ it follows that, for $K_9 > 0$ as in Theorem 2,

$$\sup_{Q_{\beta,\sigma,\sigma'}} w \le \frac{K_9}{\mathcal{M}(Q)} \iint_Q w\, d\mathcal{M}, \tag{4.39}$$

with $Q_{\beta,\sigma,\sigma'} := S(x_0, \sigma'\beta R'/(\sigma K)) \times (t_0 - \sigma' R', t_0]$. Recalling that $\theta := \gamma/\lambda$, by property (iii) in Lemma 7 we can choose $\theta \in (0, 1)$ small enough, depending only on geometric constants as well as α, β, and σ'/σ, so that

$$g(2\gamma/\lambda) > K_9 K(\alpha, \beta, \sigma)\left(\frac{4n}{{K_0}^4} + 2g(\gamma/\lambda)\right)^{\frac{1}{2}}. \tag{4.40}$$

Then, we claim that

$$\inf_{Q'_{\beta,\sigma,\sigma'}} u \geq \gamma. \tag{4.41}$$

Indeed, if there were $(x', t') \in Q'_{\beta,\sigma,\sigma'}$ such that $u(x', t') < \gamma$, then the fact that $g' \leq 0$, Eqs. 4.38, and 4.39 would imply

$$\begin{aligned} g(2\gamma/\lambda) &\leq g((u(x', t') + \gamma)/\lambda) = w(x', t') \leq \sup_{Q'_{\beta,\sigma,\sigma'}} w \\ &\leq \frac{K_9}{\mathcal{M}(Q)} \iint_Q w\, d\mathcal{M} \leq K_9 K(\alpha, \beta, \sigma)\left(\frac{4n}{{K_0}^4} + 2g(\gamma/\lambda)\right)^{\frac{1}{2}}, \end{aligned}$$

contradicting (4.40). Hence, Eq. 4.41 holds and the proof is complete. □

Theorem 10 (Every density is critical). *Assume $\mu_\varphi \in (DC)_\varphi$. Then, for every $\varepsilon_c \in (0, 1)$ and every σ, σ' with $0 < \sigma' < \sigma < \varepsilon_c/(2K_d) < 1$ there exist constants $\beta, \gamma \in (0, 1)$, depending only on $\sigma, \sigma', \varepsilon_c$ and geometric constants, such that for every $(x_0, t_0) \in \mathbb{R}^{n+1}$, $R > 0$, and u satisfying $L_\varphi(u) \geq 0$ and $u > 0$ in $Q^*_R := S_\varphi(x_0, 4K_0^2 R) \times (t_0 - 2R, t_0]$ the inequality*

$$\mathcal{M}(\{(x, t) \in Q_R : u(x, t) \geq 1\}) \geq \varepsilon_c\, \mathcal{M}(Q_R), \tag{4.42}$$

where $Q_R := S_\varphi(x_0, R) \times (t_0 - R, t_0]$, implies

$$\inf_{Q_{\beta,\sigma,\sigma'}} u \geq \gamma, \tag{4.43}$$

where $Q_{\beta,\sigma,\sigma'} := S(x_0, \beta\sigma' R/(\sigma K)) \times (t_0 - \sigma' R, t_0]$.

Proof Given $\varepsilon_c \in (0, 1)$ (think ε_c close to 0) put $\varepsilon := \varepsilon_c/(2K_d)$. Given σ, σ' with $0 < \sigma' < \sigma < \varepsilon < 1$, choose $\alpha, \beta \in (0, 1)$ as in Eq. 3.20 of Theorem 8 (that is, β close to 1 and α close to 0). For this choice of β, let $R' = \xi R$, with $\xi = \xi(R, \beta) \in (1, 2]$, be as in Remark 5 and set

$$Q_{R'} := S_\varphi(x_0, R') \times (t_0 - R', t_0]$$

so that $Q_R \subset Q_{R'}$ and, by the doubling property (2.9),

$$\mathcal{M}(Q_R) = \mu_\varphi(S_\varphi(x_0, R))R \geq \frac{1}{2K_d}\, \mu_\varphi(S_\varphi(x_0, 2R))2R \geq \frac{\mathcal{M}(Q_{R'})}{2K_d},$$

which, along with Eq. 4.42, yields

$$\mathcal{M}(\{(x, t) \in Q_{R'} : u(x, t) \geq 1\}) \geq \frac{\varepsilon_c}{2K_d}\mathcal{M}(Q_{R'}) = \varepsilon\mathcal{M}(Q_{R'}). \tag{4.44}$$

By means of Eq. 4.44 and Theorem 8, the values of α, β, σ, and σ' chosen above produce $\lambda \in (0, 1)$ satisfying (3.22) and then Lemma 9 gives $\gamma \in (0, 1)$ with

$$\inf_{Q'_{\beta,\sigma,\sigma'}} u \geq \gamma, \tag{4.45}$$

where $Q'_{\beta,\sigma,\sigma'} := S(x_0, \beta\sigma' R'/(\sigma K)) \times (t_0 - \sigma' R', t_0]$. Now, since $Q_{\beta,\sigma,\sigma'} \subset Q'_{\beta,\sigma,\sigma'}$ (due to the fact that $R < R'$), Eq. 4.43 follows from Eq. 4.45. □

5 The Double-Ball Property

With Theorem 10 in hand, we are now in position to prove the double-ball property (also known as property of *expansion of positivity*) for super-solutions under the hypothesis $\mu_\varphi \in (\mathrm{DC})_\varphi$ only.

Theorem 11 (The double-ball property) *Assume $\mu_\varphi \in (DC)_\varphi$ and introduce the geometric constant*

$$\kappa_0 := \frac{1}{K_d(4K)^{\nu+1}} \in (0, 1). \tag{5.46}$$

Then, for every $\tau \in (0, 2K]$, $(x_0, t_0) \in \mathbb{R}^{n+1}$, $R > 0$, and u satisfying $L_\varphi(u) \geq 0$ and $u > 0$ in $S_\varphi(x_0, 4K_0^2 K R) \times (t_0 - 2KR, t_0]$, the following implication holds true

$$\inf_{Q_1} u \geq 1 \Rightarrow \inf_{Q_2} u \geq \gamma, \tag{5.47}$$

where

$$Q_1 := S_\varphi(x_0, R/2) \times (t_0 - \tau R, t_0] \quad \textit{and} \quad Q_2 := S_\varphi(x_0, R) \times (t_0 - \kappa_0 \tau R, t_0].$$

Proof Given $\tau \in (0, 2K]$, set $\varepsilon_c := 2\kappa_0\tau \in (0, 1)$, $\sigma' := \varepsilon_c/(4K_d)$, and $\sigma \in (\sigma', 1)$ close enough to σ' so that $\sigma'/\sigma > 4/5$, and let β be as in Theorem 10 (notice that β will depend on τ, but we can always assume $\beta > 5/6$, since, by Eq. 3.20, β must be close to 1).

Assume $\inf_{Q_1} u \geq 1$ and, by contradiction, suppose that $\inf_{Q_2} u \geq \gamma$ does not hold. Now, since $\sigma'/\sigma > 4/5$ and $\beta > 5/6$, we get $2\sigma'\beta/\sigma > 4/3$ and consequently

$$Q_2 = S(x_0, R) \times (t_0 - \sigma' R, t_0] \subset S(x_0, 2\sigma'\beta R/\sigma) \times (t_0 - 2\sigma' K R, t_0] =: Q'_2.$$

From the assumption $\inf_{Q_2} u < \gamma$ it then follows that $\inf_{Q'_2} u < \gamma$ and, by Theorem 10 applied with the just chosen values of σ', σ, ε_c (and with R replaced by $2KR$), we get

$$\mathcal{M}(Q_{2KR} \cap \{u \geq 1\}) < \varepsilon_c \mathcal{M}(Q_{2KR}), \tag{5.48}$$

where $Q_{2KR} := S_\varphi(x_0, 2KR) \times (t_0 - 2KR, t_0]$. Since $\tau \leq 2K$ we obtain the inclusion $Q_1 \subset Q_{2KR} \cap \{u \geq 1\}$ and, from Eq. 5.48, the doubling property (2.10), and the definitions of $\varepsilon_c := 2\kappa_0\tau$ and of κ_0 in Eq. 5.46,

$$\begin{aligned}
\mathcal{M}(Q_1) &\leq \mathcal{M}(Q_{2KR} \cap \{u \geq 1\}) < \varepsilon_c \mathcal{M}(Q_{2KR}) = \varepsilon_c\, \mu_\varphi(S_\varphi(x_0, 2KR))2KR \\
&\leq \varepsilon_c K_d (4K)^\nu \mu_\varphi(S_\varphi(x_0, R/2))2KR \\
&= \kappa_0 K_d (4K)^{\nu+1} \mu_\varphi(S_\varphi(x_0, R/2))R\tau \\
&= \mu_\varphi(S_\varphi(x_0, R/2))\tau R = \mathcal{M}(Q_1),
\end{aligned}$$

leading to the contradiction $\mathcal{M}(Q_1) < \mathcal{M}(Q_1)$. Hence, the implication (5.47) must hold true. □

As a consequence of Theorem 11, one can prove the next result which plays the role of Lemma 3.3 in [6] (where $\mu_\varphi \in (\mu_\infty)$ was assumed).

Corollary 12 *Assume $\mu_\varphi \in (DC)_\varphi$. Given $0 < M < N$ with $N - M \leq 4K$, there exists $\gamma \in (0,1)$, depending only on M, N, and geometric constants, such that for every $(x_0, t_0) \in \mathbb{R}^{n+1}$, $R > 0$, and u satisfying $L_\varphi(u) \geq 0$ and $u > 0$ in $S_\varphi(x_0, 4K_0^2KR) \times (t_0 - 2KR, t_0]$, the following implication holds true*

$$\inf_{Q_N} u \geq 1 \Rightarrow \inf_{Q_M} u \geq \gamma, \tag{5.49}$$

where

$$Q_N := S_\varphi(x_0, R/2) \times (t_0 - NR, t_0] \quad \text{and} \quad Q_M := S_\varphi(x_0, R) \times (t_0 - MR, t_0].$$

Proof Given $0 < M < N$ with $N - M \leq 4K$, set $\tau := (N-M)/2$. For every $t' \in (t_0 - MR, t_0]$ it follows that $(t' - \tau R, t'] \subset (t_0 - NR, t_0]$ and then $S_\varphi(x_0, R/2) \times (t' - \tau R, t'] \subset Q_N$. Thus, $\inf_{Q_N} u \geq 1$ implies

$$\inf_{S_\varphi(x_0, R/2) \times (t' - \tau R, t']} u \geq 1.$$

Next, Theorem 11 applied to u on $S_\varphi(x_0, R/2) \times (t' - \tau R, t']$ yields

$$\inf_{S_\varphi(x_0, R) \times (t' - \kappa_0 \tau R, t']} u \geq \gamma. \tag{5.50}$$

Since (5.50) holds true for every $t' \in (t_0 - MR, t_0]$, we get $\inf_{Q_M} u \geq \gamma$. □

6 Proof of Theorem 3

Some further notation and preparations are in order. Given a parabolic cylinder $Q := S_\varphi(x_1, \rho) \times (t_1 - \rho, t_1]$ and $m \in \mathbb{N}$, let Q_m be the parabolic cylinder defined as

$$Q_m := S_\varphi(x_1, \rho) \times (t_1, t_1 + m\rho].$$

In order to prove Theorem 3 we will use the following version of the result known as the *crawling ink spots theorem* (see [10, Section 6]). Namely,

Theorem 13 *Fix $\mu \in (0,1)$ and $m \in \mathbb{N}$. Suppose that for (Lebesgue-measurable) subsets $E \subset F \subset S_\varphi(x_0, R) \times \mathbb{R}$ the following two conditions hold true:*

(i) *for every $(x,t) \in F$ there exists a parabolic cylinder $Q \subset S_\varphi(x_0, R) \times \mathbb{R}$ such that $(x,t) \in Q$ and*
$$\mathcal{M}(E \cap Q) \leq (1 - \mu)\mathcal{M}(Q), \tag{6.51}$$
(ii) *for every parabolic cylinder $Q \subset S_\varphi(x_0, R) \times \mathbb{R}$ with $\mathcal{M}(E \cap Q) \geq (1 - \mu)\mathcal{M}(Q)$ it follows that $Q_m \subset F$.*

Then,

$$\mathcal{M}(E) \leq \frac{(m+1)}{m}(1 - \kappa\mu)\mathcal{M}(F), \tag{6.52}$$

where $\kappa := (5K)^{-\nu}(10K_d)^{-1} \in (0,1)$.

The proof of Theorem 13 follows, just as in Appendix A in [10] (and see also Lemma 2.3 in [6]), from the Vitali covering lemma and the doubling condition (2.9). Another advantage of Theorem 13 (besides not requiring $\mu_\varphi \in (\mu_\infty)$) is that it can be implemented locally (because so can the Vitali covering lemma), thus allowing for the possibility that the domain of the convex function φ be a bounded convex open subset, instead of all of $\mathbb{R}^n$. We mention in passing that the proof of the Calderón-Zygmund decomposition in [6, Theorem 2.1] requires, in the notation from [6, p. 2033], that $\lim_{r\to\infty} a(r) = 0$, which necessitates that the domain of φ be all of $\mathbb{R}^n$.

Another element in the proof of Theorem 3 will be Lemma 4.1 from [6] which prescribes the smallness of radii of parabolic cylinders satisfying a critical density estimate. For fixed $z_0 := (x_0, t_0) \in \mathbb{R}^{n+1}$ and $R > 0$, set

$$\begin{aligned} \widetilde{Q}(z, R) &:= S_\varphi(x_0, R) \times (t_0 - R/2, t_0 + R/2] \\ Q_A &:= S_\varphi(x_0, 2R) \times (t_0 - 3R/2, t_0 + R/2] \\ Q_B &:= S_\varphi(x_0, 8K^2 R) \times (t_0 - 3R/2, t_0 + 2R]. \end{aligned}$$

Lemma 14 ([6, Lemma 4.1]) *There exist geometric constants C_2, M_2, δ_0, $\varepsilon_0 > 0$ such that for every positive solution u of $L_\varphi(u) = 0$ in Q_B with $\inf_{Q^+} u \le 1$, every $z' = (x', t') \in Q_A$, every $\rho > 0$, and every $M > M_2$, the inequality*

$$\mathcal{M}\{z \in \widetilde{Q}(z', \rho) : u(z) \ge M\} \ge (1 - \varepsilon_0)\mathcal{M}(\widetilde{Q}(z', \rho)) \tag{6.53}$$

implies that $\rho \le C_2 M^{-\delta_0} R$.

The proof of this Lemma, that is, of Lemma 4.1 in [6], ultimately relies on Lemma 3.3 in [6] which, as mentioned, corresponds to our Corollary 12. Therefore, all constants involved are geometric constants depending only on dimension n and the constants α_0, C_0 in Eq. 1.3 (or, equivalently, on K in Eqs. 2.11 and 2.12).

The next lemma says that whenever sub-solutions attain critical density in a parabolic cylinder, then they are uniformly bounded away from zero forward in time. More precisely,

Lemma 15 *Assume $\mu_\varphi \in (DC)_\varphi$. Let $(x_1, t_1) \in \mathbb{R}^{n+1}$, $\rho > 0$, and u satisfy $L_\varphi(u) \ge 0$ and $u > 0$ in $Q^*_\rho := S_\varphi(x_1, 4K_0^2\rho) \times (t_1 - 2\rho, t_1]$. Set $Q := S_\varphi(x_1, \rho) \times (t_1 - \rho, t_1]$ and suppose that*

$$\mathcal{M}(\{u \ge 1\} \cap Q) \ge \varepsilon_1 \mathcal{M}(Q) \tag{6.54}$$

for some $\varepsilon_1 \in (0, 1)$. Then, for every $m \in \mathbb{N}$ there exists $\gamma_m \in (0, 1)$, depending only on ε_1, m, and geometric constants, such that

$$\inf_{Q_m} u \ge \gamma_m, \tag{6.55}$$

where $Q_m := S_\varphi(x_1, \rho) \times (t_1, t_1 + m\rho]$.

Proof Given $m \in \mathbb{N}$, let $\tau \in \mathbb{R}$ satisfy $0 \le \tau \le m/(2K)$ and set

$$Q^\tau := S_\varphi(x_1, 2(\tau + 1)K\rho) \times (t_1 - 2K\rho, t_1 + 2K\tau\rho]$$

so that $Q \subset Q^\tau$ and, by Eq. 6.54 and the doubling property (2.9),

$$\begin{aligned} &\mathcal{M}(\{u \ge 1\} \cap Q^\tau) \ge \mathcal{M}(\{u \ge 1\} \cap Q) \ge \varepsilon_1 \mathcal{M}(Q) \\ &\ge \frac{\varepsilon_1 \mathcal{M}(Q^\tau)}{K_d[2K(1+\tau)]^{\nu+1}} \ge \frac{\varepsilon_1 \mathcal{M}(Q^\tau)}{K_d(2K+m)^{\nu+1}} =: \varepsilon(m)\, \mathcal{M}(Q^\tau). \end{aligned}$$

Next, for every $\sigma'_m, \sigma_m \in (0,1)$ with $\sigma'_m < \sigma_m < \varepsilon(m)/(2K_d)$, Theorem 10 (applied with $\varepsilon_c := \varepsilon(m)$, $R := R_\tau := 2(\tau+1)K\rho$ and $t_0 := t_1 + 2K\tau\rho$) yields $\beta_m, \gamma_m \in (0,1)$, depending only on $\sigma_m, \sigma'_m, \varepsilon(m)$ and geometric constants (in particular, they are independent of τ) such that

$$\inf_{Q^\tau_{\beta_m,\sigma_m,\sigma'_m}} u \geq \gamma_m \quad \forall \tau \in [0, m/(2K)], \tag{6.56}$$

where $Q^\tau_{\beta_m,\sigma_m,\sigma'_m} := S(x_1, \beta_m\sigma'_m R_\tau/(\sigma_m K)) \times (t_1 + 2K\tau\rho - \sigma'_m R_\tau, t_1 + 2K\tau\rho]$. As before, by choosing $\sigma'_m, \sigma_m \in (0,1)$ also satisfying $\sigma'_m/\sigma_m > 4/5$ and assuming that $\beta_m > 5/6$, we get $2\sigma'_m\beta_m/\sigma_m > 4/3$ so that

$$\frac{\beta_m\sigma'_m R_\tau}{\sigma_m K} = \frac{2\beta_m\sigma'_m(\tau+1)K\rho}{\sigma_m K} \geq \frac{4(\tau+1)\rho}{3} > \rho \quad \forall \tau \in [0, m/(2K)],$$

and we get the inclusion

$$S_\varphi(x_1, \rho) \subset S_\varphi(x_1, \beta_m\sigma'_m R_\tau/(\sigma_m K)) \quad \forall \tau \in [0, m/(2K)]. \tag{6.57}$$

Also, as τ moves along the interval $[0, m/(2K)]$ the time intervals $(t_1 + 2K\tau\rho - \sigma'_m R_\tau, t_1 + 2K\tau\rho]$ cover all of $(t_1, t_1 + m\rho]$, which, combined with Eq. 6.57, implies that the cylinders $Q^\tau_{\beta_m,\sigma_m,\sigma'_m}$ cover all of Q_m. Hence, the lower-bound estimate (6.56), which is uniform in τ, gives (6.55). □

of Theorem 3 By homogeneity, we can assume that $\inf_{Q^+} u = 1$. Let $\varepsilon_0, \kappa \in (0,1)$ be the geometric constants from Lemma 14 and Theorem 13, respectively, and choose $m \in \mathbb{N}$ large enough so that

$$\varepsilon_2 := \frac{(m+1)}{m}(1 - \kappa\varepsilon_0) < 1. \tag{6.58}$$

Let $\gamma_m \in (0,1)$ be as in Lemma 15 corresponding to m as in Eq. 6.58 and to $\varepsilon_1 := (1 - \varepsilon_0)$. Next, fix $M > 0$ such that

$$M^{\delta_0} > \max\{2C_2 m, \varepsilon_2^{-1}\} \quad \text{and} \quad M > \max\{M_2, \gamma_m^{-1}\} \tag{6.59}$$

where C_2, δ_0, and M_2 are the geometric constants from Lemma 14. The choices above make m and M geometric constants.

Given M as in Eq. 6.59 and $k \in \mathbb{N}$, let us put

$$E := \{u \geq M^{k+1}\} \cap Q^-$$

and

$$F := \{u \geq M^k\} \cap S_\varphi(x_0, R) \times (t_0 - R, t_0 + C_2 m M^{-k\delta_0} R]. \tag{6.60}$$

Notice that by the choice of M in Eq. 6.59 we have $t_0 + C_2 m M^{-k\delta_0} R < t_0 + R/2$; in particular, $F \subset Q_A$. In order to use Theorem 13 with $E \subset F \subset Q_A$ let us check that its two conditions hold true.

First, given $(x,t) \in F$ (hence $(x,t) \in Q_A$) pick any $r > 0$ with

$$\max\{C_2 M^{-\delta_0} R, \delta_\varphi(x_0, x)\} < r < R,$$

so that $\delta_\varphi(x_0, x) < r < R$ gives $x \in S_\varphi(x_0, r)$ and then

$$(x,t) \in Q_r := S_\varphi(x_0, r) \times (t - r/2, t + r/2] \subset S_\varphi(x_0, R) \times \mathbb{R}.$$

Now, the inequality $C_2 M^{-\delta_0} R < r$ implies that $r > C_2 M^{-\delta_0(k+1)} R$ and, by Lemma 14 (used with M^{k+1} instead of M), the cylinder Q_r satisfies

$$\mathcal{M}(\{u \geq M^{k+1}\} \cap Q_r) < (1 - \varepsilon_0)\mathcal{M}(Q_r)$$

which, combined with the trivial inclusion

$$E \cap Q_r = \{u \geq M^{k+1}\} \cap Q^- \cap Q_r \subset \{u \geq M^{k+1}\} \cap Q_r,$$

yields

$$\mathcal{M}(E \cap Q_r) < (1 - \varepsilon_0)\mathcal{M}(Q_r)$$

and condition (i) in Theorem 13 is met with $\mu := \varepsilon_0$ and $Q := Q_r$.

Next, given any cylinder $Q := S_\varphi(x_1, \rho) \times (t_1 - \rho, t_1]$ such that $Q \subset S_\varphi(x_0, R) \times \mathbb{R}$ and $\mathcal{M}(E \cap Q) \geq (1 - \varepsilon_0)\mathcal{M}(Q)$, we will show that

$$Q_m := S_\varphi(x_1, \rho) \times (t_1, t_1 + m\rho] \subset F. \tag{6.61}$$

From $\mathcal{M}(E \cap Q) \geq (1 - \varepsilon_0)\mathcal{M}(Q)$ and the inclusion $Q \supset Q \cap Q^-$ we get

$$\mathcal{M}(\{u \geq M^{k+1}\} \cap Q) \geq \mathcal{M}(E \cap Q) \geq (1 - \varepsilon_0)\mathcal{M}(Q).$$

Then, by Lemma 15 used with $\varepsilon_1 := (1-\varepsilon_0)$ and with $uM^{-(k+1)}$ in place of u we obtain that

$$\inf_{Q_m} uM^{-(k+1)} \geq \gamma_m$$

and then

$$\inf_{Q_m} u \geq \gamma_m M^{k+1} \geq M^k,$$

where for the last inequality we used Eq. 6.59. That is, we have proved that

$$Q_m \subset \{u \geq M^k\}. \tag{6.62}$$

Also, the fact that $\mathcal{M}(E \cap Q) \geq (1 - \varepsilon_0)\mathcal{M}(Q) > 0$ implies the existence of $(x'', t'') \in Q^- \cap Q$. Then $t'' \leq t_0$ and $0 \leq t_1 - t'' \leq \rho$ and, due to Lemma 14 used with M^{k+1} in place of M, it follows that $\rho \leq C_2 M^{-\delta_0(k+1)} R$. Therefore,

$$\begin{aligned} t_1 + m\rho &= t'' + (t_1 - t'') + m\rho \leq t_0 + (m+1)\rho \\ &\leq t_0 + C_2(m+1)M^{-\delta_0(k+1)}R < t_0 + C_2 m M^{-\delta_0 k} R. \end{aligned}$$

On the other hand, since $t_1 - t'' \geq 0$ and $t_0 - R < t''$, we get

$$t_1 = t_1 - t'' + t'' > t_0 - R.$$

Consequently,

$$Q_m \subset S_\varphi(x_0, R) \times (t_0 - R, t_0 + C_2 m M^{-k\delta_0} R). \tag{6.63}$$

Thus, by Eqs. 6.62, 6.63, and the definition of F in Eq. 6.60 the inclusion (6.61) holds true and condition (ii) in Theorem 13 is met as well.

From Theorem 13 we then obtain that, for every $k \in \mathbb{N}$,

$$\begin{aligned} &\mathcal{M}(\{u \geq M^{k+1}\} \cap Q^-) = \mathcal{M}(E) \leq \frac{(m+1)}{m}(1 - \kappa\varepsilon_0)\mathcal{M}(F) =: \varepsilon_2\mathcal{M}(F) \\ &\leq \varepsilon_2\mathcal{M}(\{u \geq M^k\} \cap Q^-) + \varepsilon_2 C_2 m M^{-k\delta_0}\, \mu_\varphi(S_\varphi(x_0, R))R \\ &= \varepsilon_2\mathcal{M}(\{u \geq M^k\} \cap Q^-) + \varepsilon_2 C_2 m M^{-k\delta_0}\mathcal{M}(Q^-), \end{aligned}$$

which yields

$$\frac{\mathcal{M}(\{u \geq M^{k+1}\} \cap Q^-)}{\mathcal{M}(Q^-)} \leq \varepsilon_2 \frac{\mathcal{M}(\{u \geq M^k\} \cap Q^-)}{\mathcal{M}(Q^-)} + \varepsilon_2 C_2 m M^{-k\delta_0} \tag{6.64}$$

for every $k \in \mathbb{N}$. In order to prove (1.5), for $k \in \mathbb{N}$ set

$$m_k := \frac{\mathcal{M}(\{u \geq M^k\} \cap Q^-)}{\mathcal{M}(Q^-)} \quad \text{and} \quad b_k := \varepsilon_2 C_2 m M^{-k\delta_0}$$

so that Eq. 6.64 implies that

$$m_{k+1} \leq \varepsilon_2^k m_1 + \sum_{j=1}^{k} b_j \varepsilon_2^{k-j} \quad \forall k \in \mathbb{N}. \tag{6.65}$$

Then, from the fact that $\varepsilon_2 M^{\delta_0} > 1$, the sum in Eq. 6.65 can be bounded by a convergent geometric series to obtain

$$m_{k+1} \leq \left(1 + \frac{\varepsilon_2^2 C_2 m M^{\delta_0}}{M^{\delta_0}\varepsilon_2 - 1}\right) \varepsilon_2^k \quad \forall k \in \mathbb{N}$$

and Eq. 1.5 follows with δ_1 and K_{10} depending only on the geometric constants ε_2, C_2, m, δ_0, and M. □

Finally, having proved Theorem 3 under $\mu_\varphi \in (\mathrm{DC})_\varphi$ only, the proof of Theorem 1 follows just as in [6, pp. 2051–53].

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