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Variations on Quantum Ergodic Theorems

Michael Taylor[1]

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Abstract We derive some quantum ergodic theorems, related to microlocal behavior of eigenfunctions of a positive, self-adjoint, elliptic pseudodifferential operator Λ on a compact Riemannian manifold M, emphasizing results that hold without the hypothesis that the Hamiltonian flow generated by the symbol of Λ be ergodic. Cases treated include both integrable Hamiltonians and some associated with "soft chaos."

Keywords Quantum ergodic theorem · Spectrum · Eigenfunction · Chaos

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1 Introduction

Let M be a compact Riemannian manifold and $\Lambda \in OPS^1(M)$ a first order, scalar, elliptic, positive self adjoint operator. Say

$$\operatorname{Spec}\Lambda = \{\lambda_k : k \in \mathbb{N}\}, \tag{1.1}$$

counted with multiplicity, with $\lambda_k \nearrow$. Let $\{\varphi_k : k \in \mathbb{N}\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions,

$$\Lambda\varphi_k = \lambda_k\varphi_k,$$

or more generally we can let $\{\varphi_k\}$ be an orthonormal basis of $L^2(M)$ consisting of quasimodes, satisfying (with λ_k as in Eq. 1.1)

$$\sup_{0\le s\le 1} \|e^{-s(\Lambda-\lambda_k)}(\Lambda-\lambda_k)\varphi_k\|_{L^2} = \varepsilon_k \longrightarrow 0, \tag{1.2}$$

as $k \to \infty$, though in examples we generally stick to the setting of actual eigenfunctions.

✉ Michael Taylor
met@math.unc.edu

[1] UNC at Chapel Hill, Chapel Hill, NC 27599, USA

Let $X \approx S^*M \subset T^*M$ denote the set where the principal symbol $\sigma(\Lambda)$ is equal to 1. The symplectic form on T^*M induces a volume form dS on X, which we normalize to have unit volume. The Hamiltonian vector field associated to $\sigma(\Lambda)$ generates a smooth flow G_t on X, preserving the volume form dS. Let P be the orthogonal projection of $L^2(X, dS)$ onto

$$V = \Big\{ b \in L^2(X, dS) : b \circ G_t \equiv b \Big\}. \tag{1.3}$$

We aim to prove the following.

Theorem 1.1 *There is a set $\mathcal{N} \subset \mathbb{N}$, of density zero, with the following property. Let $A \in OPS^0(M)$ and assume $a = \sigma(A)|_X$ satisfies*

$$Pa = \overline{a} := \int_X a \, dS. \tag{1.4}$$

Then

$$\lim_{k \notin \mathcal{N}, k \to \infty} (A\varphi_k, \varphi_k) = \overline{a}. \tag{1.5}$$

If the flow $\{G_t\}$ is ergodic on X, then V consists of constants, and Eq. 1.4 holds for all a. The ergodic case has been studied for some time, in [5, 6, 8, 14, 17], and other works. Theorem 1.1 applies to cases where such ergodicity does not hold. Such a formulation was mentioned in [13]. Here we intend to explore this formulation and its implications more thoroughly.

More generally than taking $A \in OPS^0(M)$, which leads to $a \in C^\infty(X)$, we can take

$$a \in C(X), \tag{1.6}$$

and assign to a an operator A in the C^*-algebra $\Psi(M)$ of operators on $L^2(M)$ generated by $OPS^0(M)$, with principal symbol a. See Section 3 for details. In this more general setting, still (1.4) $\Rightarrow$ (1.5).

Going further, we obtain the following result, which we propose to call the Quantum Ergodic Theorem.

Theorem 1.2 *Take $A \in \Psi(M)$, with principal symbol $a \in C(X)$ and weaken the hypothesis* (1.4) *to*

$$Pa \in C(X). \tag{1.7}$$

If $A_p \in \Psi(M)$ has principal symbol Pa, then, with $\mathcal{N}$ as in Theorem 1.1,

$$\lim_{k \notin \mathcal{N}, k \to \infty} (A\varphi_k, \varphi_k) - (A_p\varphi_k, \varphi_k) = 0. \tag{1.8}$$

As a corollary of Eq. 1.8, we get

$$\limsup_{k \notin \mathcal{N}, k \to \infty} |\overline{a} - (A\varphi_k, \varphi_k)| \le \sup_X |Pa - \overline{a}|. \tag{1.9}$$

The proof of Theorem 1.1 is parallel to previous proofs (cf. [5, 6, 8, 18], Chap. 15, and [9], pp. 313–325), done in the context where $\{G_t\}$ is ergodic. We record the details to verify that the result holds in the more general setting put forward here. In Section 2 we show that a Weyl formula, standard if $\{\varphi_k\}$ are eigenfunctions of Λ, holds under the more general hypothesis (1.2). In Section 3 we discuss a special class of quantization procedures, known as Friedrichs quantization, which gives that, if $A = \mathrm{op}_{\mathrm{F}}(a)$,

$$(A\varphi_k, \varphi_k) = \langle a, \mu_k \rangle \tag{1.10}$$

defines μ_k as a probability measure on X, not just a distribution. In Section 4 we apply Egorov's theorem to show that, given $a \in C(X)$,

$$\int\limits_X (a - a \circ G_t)\, d\mu_k \longrightarrow 0 \ \text{ as } \ k \to \infty, \tag{1.11}$$

for each t (under the hypothesis (1.2)). In Section 5 we use a standard Mean Ergodic Theorem argument to complete the proof of Theorem 1.1. In Section 6 we prove the Quantum Ergodic Theorem stated above.

In Section 7 we give examples of cases where $\{G_t\}$ is not ergodic on X but Eq. 1.4 holds for interesting classes of operators A, and Eq. 1.7 holds for a general class. The examples treated there are the following.

1. $M = \mathbb{T}^n$, $\Lambda = \sqrt{-\Delta}$. Here (1.4) holds for $a = a(x)$, and (1.7) holds for all $a \in C(X)$.
2. $M = S^n$, $\Lambda = \sqrt{-\Delta}$. Here (1.7) holds for all $a \in C(X)$, and an explicit formula reveals many cases where (1.4) holds.
3. $M = \mathcal{T}^2$, a certain non-flat torus in $\mathbb{R}^3$, $\Lambda = \sqrt{-\Delta}$. The analysis of Pa is more complicated here, due partly to the existence of a hyperbolic closed geodesic on $\mathcal{T}^2$. Again, Eq. 1.7 holds for all $a \in C(X)$. However, unlike Cases 1 and 2, in this case we do not have $P : C^\infty(X) \to C^\infty(X)$. In fact, members of the image of P can have less than Hölder regularity. This illustrates the value of having results for $A \in \Psi(M)$, rather than just for $A \in OPS^0(M)$.
4. $M = S^2$, $\Lambda = \Lambda_c = \sqrt{-\left(Y_1^2 + Y_2^2 + cY_3^2\right)}$, Y_j generate rotation about the x_j-axis. We take $c > 0$, $c \neq 1$. There is a uniform description for the dynamics on X and behavior of the projection P, despite the fact that the spectral behavior of Λ_c depends delicately on c. Thus, certain implications of the quantum ergodic theorem that look natural, for irrational c, seem a bit more surprising for rational c $(\neq 1)$.

The curious phenomenon arising in Example 4 is related to an issue that arises when two self-adjoint operators with discrete spectra commute. Namely, must all eigenfunctions of one operator be joint eignfunctions of both? As seen in Example 4, sometimes they do and sometimes they do not. In Section 8, we look into this phenomenon in a more general setting.

The examples arising in Section 7 are all integrable systems. In Section 9 we take a look at situations that might yield "soft chaos," involving a mixture of integrability and chaos. This analysis makes use of some basic results in KAM theory. In Section 10 we discuss the issue of the existence of soft chaos, describing both known results and open problems. We bring in some examples expected to exhibit soft chaos, and describe how Theorem 1.1 applies to some of them. We plan to investigate in future work some problems raised in Sections 9 and 10.

We end with three appendices. In Appendix A, we establish some results on conditional expectation of use in Section 7. In Appendix B we establish a general result on invariance properties of commuting, measure-preserving flows, which contains Proposition 8.2, the "classical" version of the "quantum" result, Proposition 8.1. Appendix C provides a proof of the last proposition in Section 7.

2 Weyl Law

Our first goal is to prove the following. This will play a role in the proof of Theorem 1.1 in Section 5.

Proposition 2.1 *Let* $A \in OPS^0(M)$*, with principal symbol* $\sigma(A) \in C^\infty(T^*M \setminus 0)$*, homogeneous of degree zero. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (A\varphi_k, \varphi_k) = \int_X \sigma(A)\, dS. \tag{2.1}$$

Note that if the limit on the left side of Eq. 2.1 were known to exist for all $A \in OPS^0(M)$, it must depend only on $\sigma(A)$, since if $\widetilde{A} \in OPS^0(M)$ had the same principal symbol, then $K = A - \widetilde{A}$ would be compact on $L^2(M)$. Since $\varphi_k \to 0$ weakly in $L^2(M)$, we then have $K\varphi_k \to 0$ in L^2-norm, so

$$(A\varphi_k, \varphi_k) - (\widetilde{A}\varphi_k, \varphi_k) \longrightarrow 0. \tag{2.2}$$

Proof of Proposition 2.1 First note that

$$\begin{aligned} e^{-t\Lambda}\varphi_k - e^{-t\lambda_k}\varphi_k &= e^{-t\lambda_k} \int_0^t \frac{d}{ds} e^{-s(\Lambda-\lambda_k)} \varphi_k \, ds \\ &= e^{-t\lambda_k} \int_0^t e^{-s(\Lambda-\lambda_k)} (\Lambda - \lambda_k)\varphi_k \, ds, \end{aligned} \tag{2.3}$$

so, for $t \in (0, 1]$,

$$\left\| e^{-t\Lambda}\varphi_k - e^{-t\lambda_k}\varphi_k \right\|_{L^2} \le \varepsilon_k |t|\, e^{-t\lambda_k}, \tag{2.4}$$

with ε_k as in Eq. 1.2. Hence

$$\sum_k e^{-t\lambda_k} (A\varphi_k, \varphi_k) = \sum_k \left(A\varphi_k, e^{-t\Lambda}\varphi_k\right) + r(t), \tag{2.5}$$

with

$$|r(t)| \le |t| \sum_k \varepsilon_k e^{-t\lambda_k} = o\left(\operatorname{Tr} e^{-t\Lambda}\right), \quad \text{as } t \searrow 0. \tag{2.6}$$

Hence, as $t \searrow 0$,

$$\begin{aligned} \sum_k e^{-t\lambda_k} (A\varphi_k, \varphi_k) &\sim \sum_k \left(A\varphi_k, e^{-t\Lambda}\varphi_k\right) \\ &= \operatorname{Tr} A e^{-t\Lambda} \\ &\sim \Big(\int_X \sigma(A)\, dS\Big) \operatorname{Tr} e^{-t\Lambda} \\ &= \Big(\int_X \sigma(A)\, dS\Big) \sum_k e^{-t\lambda_k}. \end{aligned} \tag{2.7}$$

The third line holds via a standard parametrix construction for $e^{-t\Lambda}$. The result (2.1) follows from this, via Karamata's Tauberian theorem. □

3 Friedrichs Quantization

To proceed, we discuss the notion of a “quantization,” which is a continuous linear map

$$\mathrm{op} : C^\infty(X) \longrightarrow OPS^0_{1,0}(M) \tag{3.1}$$

with the property that, given $a \in C^\infty(X)$, $A = \mathrm{op}(a)$ has principal symbol a (mod $OPS^{-1}_{1,0}(M)$). We insist that $\mathrm{op}(1) = I$, the identity map. The existence of quantizations follows via local coordinate charts and partitions of unity from ψDO calculus on Euclidean space. There are many different quantizations. Each one gives rise to a sequence of elements $\mu_k \in \mathcal{D}'(X)$, defined by

$$\langle a, \mu_k\rangle = (\mathrm{op}(a)\varphi_k, \varphi_k). \tag{3.2}$$

It follows from Eq. 2.2 that if $\widetilde{\mathrm{op}}$ is another quantization, yielding $\tilde{\mu}_k \in \mathcal{D}'(X)$, then for each $a \in C^\infty(X)$, $\langle a, \mu_k\rangle - \langle a, \tilde{\mu}_k\rangle \to 0$ as $k \to \infty$. Basic examples are “Kohn-Nirenberg” quantizations:

$$\mathrm{op}_{\mathrm{KN}} : C^\infty(X) \longrightarrow OPS^0(M) \subset OPS^0_{1,0}(M). \tag{3.3}$$

However, for the analysis to follow, it is useful to bring in the existence of a “Friedrichs quantization,”

$$\mathrm{op}_{\mathrm{F}} : C^\infty(X) \longrightarrow OPS^0_{1,0}(M), \tag{3.4}$$

having the property

$$a \ge 0 \Longrightarrow \mathrm{op}_{\mathrm{F}}(a) \ge 0. \tag{3.5}$$

This is constructed on the Euclidean space level from $\mathrm{op}_{\mathrm{KN}}(a)$ via “Friedrichs symmetrization.” See [16], Chapter 7. It is the case that

$$a \in C^\infty(X) \Longrightarrow \mathrm{op}_{\mathrm{F}}(a) - \mathrm{op}_{\mathrm{KN}}(a) \in OPS^{-1}_{1,0}(M). \tag{3.6}$$

This result is not trivial. In fact, it is the main technical result in the Friedrichs approach to the proof of the sharp Gårding inequality. From Eq. 3.5 it follows that

$$\|\mathrm{op}_{\mathrm{F}}(a)\|_{\mathcal{L}(L^2)} \le \sup_X |a|, \tag{3.7}$$

and hence (3.4) has a unique extension to

$$\mathrm{op}_{\mathrm{F}} : C(X) \longrightarrow \mathcal{L}\left(L^2(M)\right), \tag{3.8}$$

with (3.5) holding for all $a \in C(X)$. From here on we will take a Friedrichs quantization, and set $A = \mathrm{op}_{\mathrm{F}}(a)$. In such a case, the distributions $\mu_k \in \mathcal{D}'(X)$ defined by Eq. 3.2 satisfy

$$a \ge 0 \Longrightarrow \langle a, \mu_k\rangle \ge 0. \tag{3.9}$$

Also $\langle 1, \mu_k\rangle = (\varphi_k, \varphi_k) = 1$. Consequently, for each k,

$$\mu_k \text{ is a probability measure on } X. \tag{3.10}$$

We write

$$(A\varphi_k, \varphi_k) = \int_X a\, d\mu_k. \tag{3.11}$$

Remark The image of $C(X)$ in Eq. 3.8 is contained in the C^*-algebra of operators on $L^2(M)$ generated by $OPS^0(M)$, which we denote $\Psi(M)$. If we compose the map

$$\mathrm{op}_F : C(X) \longrightarrow \Psi(M) \tag{3.12}$$

with taking the quotient by $\mathcal{K}\left(L^2(M)\right)$, the space of compact operators on $L^2(M)$, we get an isomorphism of C^*-algebras:

$$C(X) \overset{\approx}{\longrightarrow} \Psi(M)/\mathcal{K}\left(L^2(M)\right). \tag{3.13}$$

4 Application of Egorov's Theorem

The following result will play a useful role in Section 5.

Proposition 4.1 *Given $a \in C(X)$, we have*

$$\int\limits_X (a - a \circ G_t)\, d\mu_k \longrightarrow 0, \quad \text{as} \ \ k \to \infty, \tag{4.1}$$

locally uniformly in t.

Proof It suffices to prove (4.1) for $a \in C^\infty(X)$. Set $A = \mathrm{op}_{\mathrm{F}}(a)$ and

$$A_t = e^{-it\Lambda} A e^{it\Lambda}. \tag{4.2}$$

By Egorov's theorem (and Eq. 3.6)

$$A_t - \mathrm{op}_{\mathrm{F}}(a \circ G_t) \in OPS^{-1}_{1,0}(M), \tag{4.3}$$

and this holds locally uniformly in t, so

$$\int\limits_X a \circ G_t\, d\mu_k - \left(Ae^{it\Lambda}\varphi_k, e^{it\Lambda}\varphi_k\right) \longrightarrow 0, \quad \text{as} \ \ k \to \infty, \tag{4.4}$$

locally uniformly in t. Now

$$\begin{aligned} e^{it\Lambda}\varphi_k - e^{it\lambda_k}\varphi_k &= e^{it\lambda_k} \int_0^t \frac{d}{ds} e^{is(\Lambda - \lambda_k)} \varphi_k\, ds \\ &= e^{it\lambda_k} \int_0^t e^{is(\Lambda-\lambda_k)} (\Lambda - \lambda_k)\varphi_k\, ds, \end{aligned} \tag{4.5}$$

so, with ε_k as in Eq. 1.2,

$$\left\| e^{it\Lambda}\varphi_k - e^{it\lambda_k}\varphi_k \right\|_{L^2} \le \varepsilon_k |t|. \tag{4.6}$$

Hence

$$\left(Ae^{it\Lambda}\varphi_k, e^{it\Lambda}\varphi_k\right) = (A\varphi_k, \varphi_k) + r_k(t), \tag{4.7}$$

with $r_k(t) \to 0$ as $k \to \infty$, locally uniformly in t. This plus (4.4) gives (4.1), for $a \in C^\infty(X)$, and the general result follows, via (3.10). ☐

5 Proof of Theorem 1.1

To proceed, given $a \in C(X)$, set

$$a_T = \frac{1}{T} \int_0^T a \circ G_t\, dt, \quad \overline{a} = \int\limits_X a\, dS. \tag{5.1}$$

The Mean Ergodic Theorem implies that, as $T \to \infty$,

$$a_T \longrightarrow Pa \ \text{ in } \ L^2\text{-norm}, \tag{5.2}$$

where P is the orthogonal projection of $L^2(X)$ onto

$$V = \{b \in L^2(X) : b \circ G_t \equiv b\}. \tag{5.3}$$

If the flow $\{G_t\}$ is *ergodic* on X, then V consists of constants, and then

$$Pa = \overline{a}. \tag{5.4}$$

Rather than assuming $\{G_t\}$ is ergodic, we will make (5.4) an hypothesis. Under this hypothesis, we have

$$\int_X |a_T - \overline{a}|\, dS \longrightarrow 0 \ \text{ as } \ T \to \infty. \tag{5.5}$$

Thus, for $\varepsilon \in (0, 1]$, there exists $T_\varepsilon < \infty$ such that

$$T \geq T_\varepsilon \Longrightarrow \int_X |a_T - \overline{a}|\, dS \leq \varepsilon. \tag{5.6}$$

Now, Proposition 4.1 gives, for all $a \in C(X),\ T < \infty$,

$$\int_X (a_T - a)\, d\mu_k \longrightarrow 0 \tag{5.7}$$

as $k \to \infty$, hence

$$\int_X (a_T - \overline{a})\, d\mu_k - \int_X (a - \overline{a})\, d\mu_k \longrightarrow 0, \tag{5.8}$$

as $k \to \infty$. Furthermore, Proposition 2.1 implies

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_X b\, d\mu_k = \int_X b\, dS, \tag{5.9}$$

for each $b \in C^\infty(X)$, and Eq. 3.10 then gives this result for all $b \in C(X)$. Taking $b = |a_T - \overline{a}|$ gives

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_X |a_T - \overline{a}|\, d\mu_k = \int_X |a_T - \overline{a}|\, dS, \tag{5.10}$$

for each $T < \infty$. Comparison with Eq. 5.6 gives

$$T \geq T_\varepsilon \Longrightarrow \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_X |a_T - \overline{a}|\, d\mu_k \leq \varepsilon, \tag{5.11}$$

if a satisfies (5.4). It follows that there exists a set $\mathcal{N}_\varepsilon(a) \subset \mathbb{N}$, of density zero, such that

$$T = T_\varepsilon \Longrightarrow \limsup_{k \notin \mathcal{N}_\varepsilon(a), k \to \infty} \int_X |a_T - \overline{a}|\, d\mu_k \leq 2\varepsilon. \tag{5.12}$$

Hence, by Eq. 5.8, for all $\varepsilon > 0$,

$$\limsup_{k \notin \mathcal{N}_\varepsilon(a), k \to \infty} \left| \overline{a} - \int_X a\, d\mu_k \right| \leq 2\varepsilon. \tag{5.13}$$

Now we can produce

$$\mathcal{N}(a) \subset \mathbb{N}, \quad \text{of density zero,} \tag{5.14}$$

such that, for all $\ell \in \mathbb{N}$,

$$\mathcal{N}_{2^{-\ell}}(a) \setminus \mathcal{N}(a) \ \text{ is finite.} \tag{5.15}$$

Then Eq. 5.13 gives, for all $\varepsilon \in (0, 1]$,

$$\limsup_{k \notin \mathcal{N}(a), k \to \infty} \left| \overline{a} - \int_X a \, d\mu_k \right| \leq 2\varepsilon, \tag{5.16}$$

so, if $a \in C(X)$ satisfies (5.4),

$$\lim_{k \notin \mathcal{N}(a), k \to \infty} \left| \overline{a} - \int_X a \, d\mu_k \right| = 0. \tag{5.17}$$

To proceed, let

$$\mathcal{I} = \{a \in C(X) : Pa = \overline{a}\}, \tag{5.18}$$

which is a closed, linear subspace of $C(X)$ (equal to $C(X)$ if $\{G_t\}$ is ergodic). We can take a countable set $\{a_\nu\}$, dense in $\mathcal{I}$, and produce

$$\mathcal{N} \subset \mathbb{N}, \quad \text{of density zero,} \tag{5.19}$$

such that, for all ν,

$$\mathcal{N}(a_\nu) \setminus \mathcal{N} \ \text{ is finite.} \tag{5.20}$$

Then

$$\lim_{k \notin \mathcal{N}, k \to \infty} \left| \overline{a} - \int_X a \, d\mu_k \right| = 0, \tag{5.21}$$

whenever $a = a_\nu$, and hence, by a limiting argument, using Eq. 3.10, for all $a \in \mathcal{I}$. We record the conclusion.

Theorem 5.1 *Let $A \in OPS^0(M)$ and assume $a = \sigma(A)|_X$ satisfies (5.4). Then, with $\mathcal{N}$ as in Eqs. 5.19–5.20,*

$$\lim_{k \notin \mathcal{N}, k \to \infty} (A\varphi_k, \varphi_k) = \int_X a \, dS. \tag{5.22}$$

This also holds whenever $a \in C(X)$ satisfies (5.4) and $A = \mathrm{op}_F(a)$.

6 Proof of the Quantum Ergodic Theorem

Here we will weaken the hypothesis (5.4). Instead, we will assume $A \in \Psi(M)$ has principal symbol

$$a \in L \subset C(X), \tag{6.1}$$

where L is a closed linear subspace of $C(X)$ satisfying

$$P : L \longrightarrow C(X). \tag{6.2}$$

Under the hypotheses (6.1–6.2), we can apply Theorem 5.1 to

$$b = a - Pa. \tag{6.3}$$

Note that $Pb = 0 = \overline{b}$, so Theorem 5.1 implies

$$\lim_{k \notin \mathcal{N}, k \to \infty} ((A - A_p)\varphi_k, \varphi_k) = 0, \tag{6.4}$$

where

$$A_p = \mathrm{op}_F(Pa). \tag{6.5}$$

This gives the Quantum Ergodic Theorem, which we restate here.

Theorem 6.1 *Assume* (6.1–6.2) *hold. There is a set* $\mathcal{N} \subset \mathbb{N}$*, of density 0, independent of the choice of such* $a \in L$*, with the property that* (6.4) *holds, with* A_p *given by* Eq. 6.5.

If $Pa = \overline{a}$, then $A_p\varphi = \overline{a}\varphi$, so Eq. 6.4 contains Theorem 5.1. Note also that

$$\overline{a} - (A\varphi_k, \varphi_k) = ((\overline{a} - A_p)\varphi_k, \varphi_k) + ((A_p - A)\varphi_k, \varphi_k), \tag{6.6}$$

and, by Eq. 3.7,

$$\|\overline{a} - A_p\|_{\mathcal{L}(L^2)} \le \sup_X |\overline{a} - Pa|. \tag{6.7}$$

This together with Eq. 6.4 yields the following.

Corollary 6.2 *Assume* (6.1–6.2) *hold. Take* $A \in \Psi(M)$ *with principal symbol* a*. Then, with* $\mathcal{N}$ *as in Theorem 6.1,*

$$\limsup_{k\notin\mathcal{N}, k\to\infty} |\overline{a} - (A\varphi_k, \varphi_k)| \le \sup_X |Pa - \overline{a}|. \tag{6.8}$$

7 Examples

To begin, let $M = \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$, the flat torus. Take $\Lambda = \sqrt{-\Delta}$. In such a case, the geodesic flow is integrable. However, it is elementary that

$$a(x,\xi) = a(x) \Longrightarrow Pa = \overline{a}. \tag{7.1}$$

More generally,

$$Pa(x,\xi) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{T}^n} a(y,\xi)\, dy. \tag{7.2}$$

Hence, if $\{\varphi_k\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$, satisfying (1.2), then there exists a sparse set $\mathcal{N} \subset \mathbb{N}$ such that, for all $a \in C^\infty(\mathbb{T}^n)$,

$$\lim_{k\notin\mathcal{N}, k\to\infty} (a(x)\varphi_k, \varphi_k) = \overline{a}. \tag{7.3}$$

If φ_k consists of the standard complex exponentials,

$$(a(x)\varphi_k, \varphi_k) = \int\limits_{\mathbb{T}^n} a(x)|\varphi_k(x)|^2\, dx \equiv \overline{a}, \tag{7.4}$$

and Eq. 7.3 is trivial. However, in this setting eigenspaces of Δ have high multiplicities, and other forms of $\{\varphi_k\}$ can be produced. For them, Eq. 7.3 seems not to be trivial.

For the second example, we consider $M = S^n$, the unit sphere in $\mathbb{R}^{n+1}$, with its standard round metric. Again take $\Lambda = \sqrt{-\Delta}$. The geodesic flow is again integrable. In this case, G_t is periodic of period 2π, and $X = S^*S^n$ is foliated into circles, orbits of $\{G_t\}$. We have

$$Pa(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} a(G_t(x,\xi))\, dt. \tag{7.5}$$

In case $a \in C(S^n)$, i.e., $a = a(x)$, $Pa = \overline{a}$ provided

$$a(x) = \overline{a} + b(x), \quad b(-x) = -b(x). \tag{7.6}$$

(For general $a = a(x)$, Pa need not be a function of x alone.) Again in this setting, eigenspaces of Δ have high multiplicity, but here each eigenspace consists of functions that are either all even or all odd. Consequently, the results of Theorem 1.1 are trivial for $a(x, \xi) = a(x)$ in this case. However, there are many functions $a \in C^\infty(S^*S^n)$, lacking any symmetry, whose averages over the various closed orbits of $\{G_t\}$ all coincide, hence satisfying $Pa = \overline{a}$. Theorem 1.1 applied to these operators has a nontrivial conclusion.

The formulas (7.2) and (7.5) show that, for both families of examples mentioned above, with $X = S^*\mathbb{T}^n$ and $X = S^*S^n$, respectively, we have

$$P : C(X) \longrightarrow C(X), \tag{7.7}$$

and also

$$P : C^\infty(X) \longrightarrow C^\infty(X). \tag{7.8}$$

In general, there is a formula for Pa as a conditional expectation,

$$Pa = \mathbb{E}(a|\mathcal{F}), \tag{7.9}$$

where $\mathcal{F}$ is the σ-algebra of Borel sets in X that are G_t-invariant. In case (7.2), $X = \mathbb{T}^n \times S^{n-1}$ and

$$\mathcal{F} = \left\{\mathbb{T}^n \times B : B \subset S^{n-1} \text{ Borel}\right\}. \tag{7.10}$$

In case (7.5), $\mathcal{F}$ is the σ-algebra generated by

$$\left\{ \bigcup_{0 \le t \le 2\pi} G_t(B) : B \subset X \text{ compact}\right\}. \tag{7.11}$$

For the third example, we consider a non-flat torus, an “inner tube” $\mathcal{T}^2$, the image of $\mathbb{T}^2$ under the map

$$\Phi : \mathbb{T}^2 \longrightarrow \mathbb{R}^3, \quad \Phi(\theta, \omega) = \big((2 + \cos\theta)\cos\omega, (2 + \cos\theta)\sin\omega, \sin\theta\big). \tag{7.12}$$

We use (θ, ω) as coordinates on $\mathcal{T}^2$. Then $X = S^*\mathcal{T}^2 = \mathcal{T}^2 \times S^1$, and on the factor S^1 we use the coordinate φ, the angle a tangent vector makes with the vector $\partial_\theta\Phi(\theta, \omega)$ (and use the metric tensor to identify tangent vectors and cotangent vectors). Due to rotational symmetry about the x_3-axis, $\mathcal{T}^2$ also has an integrable geodesic flow. In this case, we have (7.9) with

$$\mathcal{F} = \left\{\Omega^{-1}(B) : B \subset \mathbb{R} \text{ Borel}\right\}, \tag{7.13}$$

where $\Omega : X \to \mathbb{R}$ is (the restriction to $X = S^*\mathcal{T}^2$ of) the angular momentum, i.e., the principal symbol of $i\partial/\partial\omega$. Writing

$$X = \left\{(\theta, \varphi, \omega) : \theta, \varphi, \omega \in \mathbb{T}^1\right\}, \tag{7.14}$$

we see that Ω is independent of ω, say $\Omega(\theta, \varphi, \omega) = \widetilde{\Omega}(\theta, \varphi)$, and

$$\mathcal{F} = \left\{\widetilde{\Omega}^{-1}(B) \times \mathbb{T}^1 : B \subset \mathbb{R} \text{ Borel}\right\}. \tag{7.15}$$

Thus, given $a \in C(X)$, say $a = a(\theta, \varphi, \omega)$, we have

$$Pa = P^b a^b, \quad a^b(\theta, \varphi) = \frac{1}{2\pi} \int_{\mathbb{T}^1} a(\theta, \varphi, \omega)\, d\omega, \tag{7.16}$$

and

$$P^b a^b(\theta, \omega) = \mathbb{E}\left(a^b|\mathcal{F}^b\right)(\theta, \omega), \tag{7.17}$$

where

$$\mathcal{F}^b = \left\{ \widetilde{\Omega}^{-1}(B) : B \subset \mathbb{R} \text{ Borel} \right\}, \quad \mathcal{F}^b \subset \mathbb{T}^2. \tag{7.18}$$

Regarding the level sets of $\widetilde{\Omega}$, we see that $\widetilde{\Omega}$ has four critical points:

$$\begin{gathered} \text{max at } \theta = 0,\ \varphi = \frac{\pi}{2}, \quad \text{min at } \theta = 0,\ \varphi = -\frac{\pi}{2}, \\ \text{saddles at } \theta = \pi,\ \varphi = \pm\frac{\pi}{2}. \end{gathered} \tag{7.19}$$

The max and min correspond to closed geodesics on the outer equator of $\mathcal{T}^2$, going counterclockwise or clockwise, and the saddles correspond to closed geodesics along the inner equator of $\mathcal{T}^2$, going counterclockwise or clockwise. There are four curves (on each of which $\widetilde{\Omega}$ is constant) that function as separatrices, two for each saddle, which separate $\mathbb{T}^2 = \{(\theta, \varphi)\}$ into four regions. Two regions correspond to geodesics on $\mathcal{T}^2$ that cross the inner equator infinitely often. The other two correspond to geodesics that never cross the inner equator.

Now, given $a \in C(X)$, we have $a^b \in C\left(\mathbb{T}^2\right)$. Formula (7.17) and the analysis above of $\mathcal{F}^b$ guarantee that $P^b a^b$ is continuous on the interior of each of the four regions described above, and extends to be continuous on the closure of each one of these. Furthermore, the limit on each separating curve from each side is the value of a^b at the corresponding saddle point. Hence $P^b a^b$ does not jump across these separating curves, so again (7.7) holds. On the other hand, Eq. 7.8 typically fails in this case. Instead, we have

$$a \in C^\infty(X) \Longrightarrow Pa \in C^\omega(X), \tag{7.20}$$

with

$$\omega(h) = \frac{1}{|\log h|}, \quad \text{for } h << 1. \tag{7.21}$$

See Appendix A for details.

Remark With a^b given by Eq. 7.16, we see that

$$P\left(a - a^b\right) = 0. \tag{7.22}$$

Thus Theorem 5.1 implies the following.

Proposition 7.1 *Given $A \in \Psi\left(\mathcal{T}^2\right)$ with symbol a, let $A^b \in \Psi\left(\mathcal{T}^2\right)$ have symbol a^b. If $\{\varphi_k\}$ is an orthonormal basis of $L^2\left(\mathcal{T}^2\right)$ consisting of eigenfunctions of Δ, or more generally satisfying (1.2), then there is a set $\mathcal{N} \subset \mathbb{N}$, of density 0, such that, for each $a \in C(X)$,*

$$\lim_{k \notin \mathcal{N}, K \to \infty} (A\varphi_k, \varphi_k) - \left(A^b \varphi_k, \varphi_k\right) = 0. \tag{7.23}$$

Remark Both $\mathcal{T}^2$ and S^2 are invariant under rotation about the x_3-axis. However, if one replaces $\mathcal{T}^2$ by S^2 in Proposition 7.1, the conclusion does not hold. This is because the σ-algebra of G_t-invariant Borel subsets of $S^* S^2$ in much richer than that for $S^*\mathcal{T}$, and Eq. 7.22 fails.

For the fourth example (or class of examples), we return to $M = S^2$, and consider other elliptic operators. Let Y_j be vector fields generating 2π-periodic rotation about the x_j-axis. For $c \in (0, \infty)$ take

$$\Lambda_c = \sqrt{-\left(Y_1^2 + Y_2^2 + cY_3^2\right)}. \tag{7.24}$$

Then $\Lambda_1 = \sqrt{-\Delta}$ is what we considered in Example 2. Now, Δ commutes with each Y_j. Hence Λ_c commutes with Y_3 for each $c \in \mathbb{R}^+$, but if $c \neq 1$, Λ_c does not commute with Y_1 or Y_2. Let $X_c \subset T^*S^2$ denote the set where $\sigma(\Lambda_c) = 1$, $G_{c,t}$ the flow on X_c generated by the Hamiltonian vector field associated to $\sigma(\Lambda_c)$. There is a natural probability measure on X_c, invariant under $\{G_{c,t} : t \in \mathbb{R}\}$. Let P_c denote the orthogonal projection of $L^2(X_c)$ onto the space of $G_{c,t}$-invariant functions. Let $\{\varphi_k : k \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\Lambda_c)$, consisting of eigenfunctions of Λ_c. (This time, we abuse notation, and do not call these functions $\varphi_{c,k}$.) By Theorem 6.1, there is a set $\mathcal{N}_c \subset \mathbb{N}$ of density 0 such that if $A \in \Psi\left(S^2\right)$ has principal symbol $a \in C(X_c)$, and if $P_c a \in C(X_c)$, forming then the principal symbol of $A_{p,c} \in \Psi(S^2)$, then

$$\lim_{k\notin\mathcal{N}_c, k\to\infty} (A\varphi_k, \varphi_k) - (A_{p,c}\varphi_k, \varphi_k) = 0. \tag{7.25}$$

When $c \neq 1$, the structure of P_c is quite a bit different from that of P_1, given by Eq. 7.5. By Eq. 7.9, we have

$$P_c a = \mathbb{E}(a|\mathcal{F}_c), \tag{7.26}$$

where $\mathcal{F}_c$ is the σ-algebra of Borel sets in X_c that are $G_{c,t}$-invariant. $\mathcal{F}_1$ is given by Eq. 7.11, but for other values of c, the situation is different.

Lemma 7.2 *If $c > 0$, then, modulo null sets,*

$$\mathcal{F}_c = \left\{\omega_c^{-1}(B) : B \subset \mathbb{R} \text{ Borel}\right\}, \quad \text{if } c \neq 1, \tag{7.27}$$

where ω_c is (the restriction to X_c of) the principal symbol of iY_3.

Proof Let us write $\sigma(\Lambda_c)$ as λ_c, where

$$\lambda_c^2 = \lambda^2 + (c-1)\omega^2, \tag{7.28}$$

with $\lambda = \lambda_1 = \sigma\left(\sqrt{-\Delta}\right)$ and $\omega = \sigma(iY_3)$. The set X_c is (up to a null set) foliated by tori $T_{c,\sigma} \subset X_c$, on which $\omega = \sigma$. The flow $G_{c,t}$, generated by the Hamiltonian vector field associated to λ_c, leaves each such torus invariant. To prove (7.27), it suffices to show that, if $c \neq 1$, one gets an irrational flow on almost all of these tori.

Note that the Hamiltonian vector fields H_λ and H_ω associated to λ and ω generate commuting flows $\mathcal{F}^t_{H_\lambda}$ and $\mathcal{F}^t_{H_\omega}$, each periodic of period 2π, and each leaving the tori $T_{c,\sigma}$ invariant. Since

$$\begin{aligned} H_{\lambda_c} &= \frac{1}{2} H_{\lambda_c^2} \quad \text{on } X_c \\ &= \lambda H_\lambda + (c-1)\omega H_\omega, \end{aligned} \tag{7.29}$$

we have

$$G_{c,t} = \mathcal{F}^t_{H_{\lambda_c}} = \mathcal{F}^{\lambda t}_{H_\lambda} \circ \mathcal{F}^{(c-1)\omega t}_{H_\omega}. \tag{7.30}$$

Now,

$$\omega = \sigma, \quad \lambda = \left(1 - (c-1)\sigma^2\right)^{1/2}, \quad \text{on } T_{c,\sigma}, \tag{7.31}$$

so, if $c \neq 1$,

$$\begin{aligned} \mathcal{F}^{\lambda t}_{H_\lambda}\Big|_{T_{c,\sigma}} &\quad \text{has period } \frac{2\pi}{\left(1-(c-1)\sigma^2\right)^{1/2}}, \\ \mathcal{F}^{(c-1)\omega t}_{H_\omega}\Big|_{T_{c,\sigma}} &\quad \text{has period } \frac{2\pi}{(c-1)\sigma} \end{aligned} \tag{7.32}$$

(if also $\sigma \neq 0$). Thus $G_{c,t}|_{T_{c,\sigma}}$ is ergodic unless

$$q(\sigma) = \frac{(c-1)\sigma}{(1-(c-1)\sigma^2)^{1/2}}$$

is rational. But (for $c \neq 1$) $q(\sigma)$ is irrational for almost all σ, so Lemma 7.2 holds. □

The flow on S^2 generated by Y_3 has a natural lift to a flow $\mathcal{H}_t$ on $T^*\left(S^2\right)$ (generated by the Hamiltonian vector field associated to $\sigma(iY_3)$). The flow $\mathcal{H}_t$ leaves X_c invariant. Furthermore, for $c \neq 1$, each Borel set in $\mathcal{F}_c$ is $\mathcal{H}_t$-invariant. It follows that if $a \in C(X_c)$,

$$P_c a = P_c a^b, \quad a^b = \frac{1}{2\pi}\int_0^{2\pi} a \circ \mathcal{H}_t \, dt. \tag{7.33}$$

Note the parallel to Eq. 7.16. The argument going from Eq. 7.22 to Proposition 7.1 applies here, to give (for $c \neq 1$)

$$\lim_{k \notin \mathcal{N}_c, k \to \infty} (A\varphi_k, \varphi_k) - \left(A^b\varphi_k, \varphi_k\right) = 0, \tag{7.34}$$

where $A^b \in \Psi(S^2)$ has principal symbol a^b, given by Eq. 7.33.

Since Λ_c and Y_3^2 commute, one can pick an orthonormal basis of $L^2\left(S^2\right)$ consisting of joint eigenfunctions for these two operators. In such a case, Eq. 7.34 seems relatively natural, at least when $Af(x) = a(x)f(x)$. One is tempted to wonder whether an eigenfunction of Λ_c must also be an eigenfunction of Y_3^2. As we will see, this holds for some values of c but fails in a big way for some other values of c.

Since

$$\Lambda_c^2 = -\Delta - (c-1)Y_3^2, \tag{7.35}$$

we can read off Spec Λ_c from the joint spectrum of Δ and Y_3. We recall that

$$L^2(S^2) = \bigoplus_{k=0}^{\infty} V_k, \quad -\Delta = k^2 + k \text{ on } V_k, \tag{7.36}$$

and

$$\operatorname{Spec} iY_3\Big|_{V_k} = \{j \in \mathbb{Z} : |j| \leq k\}, \quad \operatorname{Spec}\left(-Y_3^2\right)\Big|_{V_k} = \left\{j^2 : 0 \leq j \leq k\right\}. \tag{7.37}$$

Hence

$$\operatorname{Spec} \Lambda_c^2\big|_{V_k} = \{\lambda_{cjk} : 0 \leq j \leq k\} \quad \lambda_{cjk} = k^2 + k + (c-1)j^2. \tag{7.38}$$

If c is irrational, then the number λ_{cjk} uniquely determines j and k, given that they both are in $\mathbb{N} \cup \{0\}$. In view of the fact that each joint eigenspace of Δ and iY_3 is one dimensional, we have:

Proposition 7.3 *If $c \in \mathbb{R}^+$ is irrational, each eigenspace of Λ_c has dimension 1 or 2, and each eigenfunction of Λ_c is also an eigenfunction of Y_3^2.*

We now observe how very different the spectrum of Λ_c is when $c = 2$. By Eq. 7.38,

$$\operatorname{Spec} \Lambda_2^2\big|_{V_k} = \left\{k^2 + k + j^2 : 0 \leq j \leq k\right\}. \tag{7.39}$$

Now

$$n = k^2 + k + j^2 \Longrightarrow 4n + 1 = (2k+1)^2 + (2j)^2. \tag{7.40}$$

If we set

$$\begin{aligned} S_N &= \Big\{4n+1 : \exists\, j,k \text{ such that } 0 \le j \le k,\, 4n+1 = (2k+1)^2 + (2j)^2,\, n \le N\Big\}, \\ T_N &= \Big\{(j,k) : 0 \le j \le k,\, (2k+1)^2 + (2j)^2 \in S_N\Big\}, \end{aligned} \tag{7.41}$$

we see that the number of elements of T_N satisfies

$$\#(T_N) \sim \frac{\pi}{8} N, \quad \text{as } N \to \infty. \tag{7.42}$$

On the other hand, it is a classical result of lattice point counting that

$$\frac{\#(S_N)}{N} \longrightarrow 0 \text{ as } N \to \infty. \tag{7.43}$$

These two results give:

Proposition 7.4 *If the eigenvalues of* Λ_2 *are*

$$0 = \omega_1 < \omega_2 < \cdots < \omega_k \nearrow \infty, \tag{7.44}$$

with corresponding eigenspaces $\widetilde{V}_k$*, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \dim \widetilde{V}_k = \infty. \tag{7.45}$$

The argument proving Proposition 7.4 extends readily to treat

$$c - 1 = \frac{a^2}{b^2}, \quad a, b \in \mathbb{N}. \tag{7.46}$$

In such a case,

$$\lambda = k^2 + k + \frac{a^2}{b^2} j^2 \Longrightarrow (4\lambda + 1)b^2 = ((2k+1)b)^2 + (2ja)^2, \tag{7.47}$$

and considerations parallel to Eqs. 7.41–7.43 apply. We have:

Proposition 7.5 *The conclusion of Proposition 7.4 holds for* Λ_c *whenever*

$$\sqrt{c-1} \in \mathbb{Q}. \tag{7.48}$$

There is also an analogue of Proposition 7.5 for $c \in (0, 1)$.

Proposition 7.6 *If* $0 < c < 1$ *and*

$$\sqrt{1-c} \in \mathbb{Q}, \tag{7.49}$$

then Λ_c *has eigenspaces of arbitrarily high dimension.*

In such a case, in place of Eq. 7.46, we can write

$$1 - c = \frac{a^2}{b^2}, \quad a, b \in \mathbb{N},\ a < b, \tag{7.50}$$

and, with $\lambda = \lambda_{cjk}$ as in Eq. 7.38,

$$\lambda = k^2 + k - \frac{a^2}{b^2} j^2 \Longrightarrow (4\lambda + 1)b^2 = ((2k+1)b)^2 - (2ja)^2. \tag{7.51}$$

Thus Proposition 7.6 follows from the assertion that, given $a, b \in \mathbb{N}$, $a < b$, and

$$\begin{aligned} &\Phi_{a,b} : \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \le k\} \longrightarrow \mathbb{N}, \\ &\Phi_{a,b}(j,k) = ((2k+1)b)^2 - (2ja)^2, \end{aligned} \tag{7.52}$$

there is no upper bound on the size of $\Phi_{a,b}^{-1}(N)$, as N ranges over $\mathbb{N}$. We record a proof of this in Appendix C.

We briefly mention a further generalization of Eq. 7.24, namely

$$\Lambda_{\mathbf{c}} = \sqrt{-\left(c_1 Y_1^2 + c_2 Y_2^2 + c_3 Y_3^2\right)}, \quad \mathbf{c} = (c_1, c_2, c_3), \quad c_j > 0. \tag{7.53}$$

Since $\sqrt{-\Delta}$ commutes with each Y_j, we see that

$$\Lambda_{\mathbf{c}} \text{ and } \sqrt{-\Delta} \text{ commute}, \quad \forall\, \mathbf{c} \in \left(\mathbb{R}^+\right)^3. \tag{7.54}$$

Thus their symbols $\sigma(\Lambda_{\mathbf{c}})$ and $\sigma(\sqrt{-\Delta})$ Poisson commute, and $H_{\sigma(\Lambda_{\mathbf{c}})}$ is integrable. One could also look into how $\Lambda_{\mathbf{c}}$ operates on the spaces V_k introduced in Eq. 7.36, using basic representation theory of SO(3), but we will not take this up here.

8 Joint Eigenfunctions of Commuting Operators

Let $\Lambda \in OPS^1(M)$ be as in Section 1. Assume we have

$$Y = Y^* \in OPS^1(M), \quad \Lambda Y = Y \Lambda. \tag{8.1}$$

Then $L^2(M)$ has an orthonormal basis consisting of joint eigenfunctions of Λ and Y, but a random orthonormal basis consisting of eigenfunctions of Λ might not also be eigenfunctions of Y, as the example $M = S^2$, $\Lambda = \sqrt{-\Delta}$ considered in Section 7 shows. The following result is parallel to Proposition 7.3. To state it, note that there is an interval $I \subset \mathbb{R}$, containing 0, such that

$$\alpha \in I \Longrightarrow \Lambda + \alpha Y \in OPS^1(M) \text{ is elliptic.} \tag{8.2}$$

Proposition 8.1 *There is a countable set $\mathcal{C} \subset \mathbb{R}$ such that, for all $\alpha \in I \setminus \mathcal{C}$, if $u \in L^2(M)$ is an eigenfunction of $\Lambda + \alpha Y$, then u is an eigenfunction both of Λ and of Y.*

Proof We have

$$L^2(M) = \bigoplus_{k=0}^{\infty} V_k, \quad \Lambda = \lambda_k \text{ on } V_k, \quad 0 \le \lambda_0 < \lambda_1 < \lambda_2 < \cdots. \tag{8.3}$$

By Eq. 8.1, $Y : V_k \to V_k$. Say

$$\operatorname{Spec} Y\big|_{V_k} = \{\omega_{jk} : 1 \le j \le d_k\}, \tag{8.4}$$

where the numbers ω_{jk} are distinct, for each fixed k. Then

$$\operatorname{Spec}(\Lambda + \alpha Y)\big|_{V_k} = \{\lambda_k + \alpha\omega_{jk} : 1 \le j \le d_k\}. \tag{8.5}$$

To prove Proposition 8.1, we need to construct $\mathcal{C} \subset \mathbb{R}$ such that

$$\alpha \in I \setminus \mathcal{C}, \ \lambda_k + \alpha\omega_{jk} = \lambda_\ell + \alpha\omega_{m\ell} \tag{8.6}$$

implies $\lambda_k = \lambda_\ell$. Note that the hypotheses of Eq. 8.6 imply

$$\alpha(\omega_{jk} - \omega_{m\ell}) = \lambda_\ell - \lambda_k, \tag{8.7}$$

and if $\lambda_k \neq \lambda_\ell$, then also $\omega_{jk} \neq \omega_{m\ell}$, so

$$\alpha = \frac{\lambda_\ell - \lambda_k}{\omega_{jk} - \omega_{m\ell}}. \tag{8.8}$$

So to construct $\mathcal{C}$, first consider

$$\mathcal{S} = \left\{\lambda_k : k \in \mathbb{Z}^+\right\} \cup \left\{\omega_{jk} : k \in \mathbb{Z}^+, 1 \le j \le d_k\right\}, \tag{8.9}$$

which is countable. Then let $\mathcal{C} \subset \mathbb{R}$ be the field generated by $\mathcal{S}$, which is also countable. Alternatively, $\mathcal{C}$ could just be the set of quotients that appear on the right side of Eq. 8.8, with $\omega_{jk} \neq \omega_{m\ell}$. □

This spectral result suggests a dynamical counterpart. To state it, let

$$X_\alpha = \left\{(x,\xi) \in T^*M : \sigma(\Lambda + \alpha Y)(x,\xi) = 1\right\}, \tag{8.10}$$

which gets a natural probability measure, invariant under the flow generated by $H_{\sigma(\Lambda+\alpha Y)}$. Also, X_α is invariant under the flows generated by $H_{\sigma(\Lambda)}$ and $H_{\sigma(Y)}$.

Proposition 8.2 *There exists a countable set $\mathcal{C} \subset I$ such that, for $\alpha \in I \setminus \mathcal{C}$, the following holds. Given $b \in L^2(X_\alpha)$,*

$$H_{\sigma(\Lambda+\alpha Y)} b = 0 \Longrightarrow H_{\sigma(\Lambda)} b = 0 \ \text{ and } \ H_{\sigma(Y)} b = 0. \tag{8.11}$$

For such α, a Borel set $S \subset X_\alpha$ is invariant under the flow generated by $H_{\sigma(\Lambda+\alpha Y)}$ if and only if S is simultaneously invariant under the flow generated by $H_{\sigma(\Lambda)}$ and that generated by $H_{\sigma(Y)}$.

Note that, by homogeneity, we can simply work on $X = X_0$, rather than on X_α, in Proposition 8.2. In Appendix B, we establish a more general result, on commuting, measure-preserving flows, which implies Proposition 8.2.

9 Soft Chaos

Here, we take M, Λ, and X as in Section 1, but require

$$\dim M = 2, \tag{9.1}$$

so $\dim X = 3$. Assume the flow G_t has an elliptic periodic orbit, γ. Let T_0 denote its minimal period, so $p \in \gamma \Rightarrow G_{T_0} p = p$. We describe how "soft chaos" (a term used in [7]) can arise in this setting, as a consequence of KAM theory.

Pick $p \in \gamma$, and let $\Sigma \subset X$ be a 2-dimensional surface, transversal to γ, containing p. Let $R : \Sigma \to \Sigma$ be the Poincaré first return map. Thus $R(p) = p$. The symplectic form on T^*M pulls back to a nondegenerate, closed, 2-form on Σ, invariant under R. Thus Σ has an area element, and R is area-preserving. That γ is elliptic means $DR(p) : T_p\Sigma \to T_p\Sigma$ has eigenvalues of the form $\left\{e^{i\alpha}, e^{-i\alpha}\right\}$. In such a case, R is called an α-twist.

It is a classical result of G. Birkhoff that (under some extra hypotheses) R has a "normal form." The formulation below is from [1], p. 582.

Proposition 9.1 *If α is not 0 or an integral multiple of $\pi/2$ or $2\pi/3$, then (after perhaps shrinking Σ) there exist symplectic coordinates $u = u_1 + iu_2$ on Σ such that $u(p) = 0$ and*

$$R(u) = u\, e^{i\left(\alpha + \beta |u|^2\right)} + O\left(|u|^4\right). \tag{9.2}$$

One says the first return map R is an *elementary* twist map if α is as in Proposition 9.1 and Eq. 9.2 holds with $\beta \neq 0$.

We next recall a stability theorem, proved by J. Moser, in [11]. We say a *cycle* in Σ is a simple, closed, C^1 curve in Σ that encloses p.

Theorem 9.2 *Assume $R : \Sigma \to \Sigma$ is an elementary twist map. Then there is a collection $\{\sigma \in \mathcal{I}\}$ of invariant cycles, contained in Σ, such that*

(i) *For each $\sigma \in \mathcal{I}$, $R|_\sigma$ has irrational rotation number, and each orbit of $R|_\sigma$ is dense in σ.*
(ii) *$\bigcup_{\sigma \in \mathcal{I}} \sigma \subset \Sigma$ is closed.*
(iii) *For each $\varepsilon > 0$, there exists a neighborhood U of p in Σ such that the union $\cup\{\sigma \in \mathcal{I} : \sigma \subset U\}$ has 2D measure $\geq (1-\varepsilon)\,\mathrm{Area}(U)$.*

In addition to the proof in [11], there is a treatment, in the real analytic case, in [15], Sections 31–33. See also discussions in [12], Chapter 2, Section 4, and in [1]. The following result is perhaps implicit in these works, and is certainly implicit in illustrative figures. We make it explicit.

Proposition 9.3 *Given two invariant cycles $\sigma \neq \sigma'$, in $\mathcal{I}$, either σ encloses σ' or σ' encloses σ.*

Proof The invariant cycles in $\mathcal{I}$ are obtained as small perturbations of the cycles $|u| =$ small const., which are invariant under $R_0(u) = ue^{i(\alpha+\beta|u|^2)}$. Thus if $\sigma, \sigma' \in \mathcal{I}$, either one encloses the other or they have a nonempty intersection. Then, part (i) of Theorem 9.2 implies this intersection must be dense in σ, and in σ', which requires $\sigma = \sigma'$. ☐

At this point, we pick a "base" invariant cycle σ_0, and consider only those σ enclosed by (or equal to) σ_0.

Given an invariant cycle $\sigma \in \mathcal{I}$, let

$$T_\sigma = \{G_t(x) : x \in \sigma,\ t \in \mathbb{R}\} \subset X \tag{9.3}$$

be the tube swept out by σ under the flow G_t. Each such T_σ is homeomorphic to $\mathbb{T}^2$. Results in Theorem 9.2 and Proposition 9.3 imply the following.

(I) Each T_σ has zero 3D measure.
(II) The union of all such T_σ for $\sigma \in \mathcal{I}$ is closed.
(III) Given two invariant tubes $T_\sigma \neq T_{\sigma'}$, either T_σ encloses $T_{\sigma'}$ or $T_{\sigma'}$ encloses T_σ.

Now, let

$$T_\sigma^\# = \text{closed, solid tube enclosed by } T_\sigma. \tag{9.4}$$

Given $x \in X$, let

$$\mathcal{T}_x = \bigcup_{\sigma \in \mathcal{I}, x \notin T_\sigma^\#} T_\sigma, \tag{9.5}$$

and

$$\mathcal{T}_x^\# = \bigcup_{\sigma \in \mathcal{I}, x \notin T_\sigma^\#} T_\sigma^\#. \tag{9.6}$$

That is, $\mathcal{T}_x$ is the union of all 2D tubes T_σ that neither enclose x nor contain x, and $\mathcal{T}_x^\#$ is the union of all closed solid tubes $T_\sigma^\#$ that do not contain x.

Note that, given $x, y \in X$, possibly

$$\mathcal{T}_x = \mathcal{T}_y, \tag{9.7}$$

and if not, either

$$\mathcal{T}_x \subset \mathcal{T}_y, \quad \text{or } \mathcal{T}_y \subset \mathcal{T}_x. \tag{9.8}$$

If Eq. 9.7 holds, we write

$$x \sim y, \tag{9.9}$$

and if it fails, we write

$$x \prec y, \quad \text{or } y \prec x, \tag{9.10}$$

in the two cases in Eq. 9.8. We also have natural notions of $x \preceq y$ or $y \preceq x$.

Now, using the invariant volume element on X arising in Section 1, we define a function $\Phi : X \to \mathbb{R}$ by

$$\Phi(x) = \text{Vol } \mathcal{T}_x. \tag{9.11}$$

Note that $\mathcal{T}_x = \mathcal{T}_{G_t x}$, for all $t \in \mathbb{R},\ x \in X$, so $\Phi = \Phi \circ G_t$. We view the following as a key result.

Proposition 9.4 *The function Φ is continuous on X.*

Proof Let $x_k, x \in X,\ x_k \to x$. We need to show that $\Phi(x_k) \to \Phi(x)$. It suffices to show this in each of the following three cases:

$$x_k \prec x, \quad x \prec x_k, \quad x \sim x_k, \quad \forall k, \tag{9.12}$$

and $x_k \to x$.

Consider the first case, $x_k \prec x,\ \forall k$. Reordering the sequence (x_k) if necessary, we can assume

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots . \tag{9.13}$$

In such a case,

$$\mathcal{T}_{x_1} \subset \mathcal{T}_{x_2} \subset \mathcal{T}_{x_3} \subset \cdots , \tag{9.14}$$

hence

$$\Phi(x_1) \le \Phi(x_2) \le \Phi(x_3) \le \cdots \le \Phi(x). \tag{9.15}$$

Furthermore, the monotone convergence theorem implies

$$\Phi(x_k) \nearrow \text{Vol}\Big(\bigcup_{k\ge 1} \mathcal{T}_{x_k}\Big). \tag{9.16}$$

Of course,

$$\bigcup_{k\ge 1} \mathcal{T}_{x_k} \subset \mathcal{T}_x.$$

Recall that $\mathcal{T}_x$ is the union of all 2D tubes T_σ that neither enclose x nor contain it. If T_σ is such a tube, x has a positive distance from T_σ. Since $x_k \to x$, we deduce that for all sufficiently large k, T_σ will neither enclose nor contain x_k. Thus $T_\sigma \subset \mathcal{T}_{x_k}$ for all sufficiently large k, so in fact

$$\bigcup_{k\ge 1} \mathcal{T}_{x_k} = \mathcal{T}_x. \tag{9.17}$$

This proves Proposition 9.4 for the first case (and also the last case) of Eq. 9.12.

It remains to check the case $x \prec x_k,\ \forall k$. This time, we can reorder (x_k) to achieve

$$\cdots \preceq x_3 \preceq x_2 \preceq x_1. \tag{9.18}$$

In such a case,

$$\mathcal{T}_{x_1} \supset \mathcal{T}_{x_2} \supset \mathcal{T}_{x_3} \supset \cdots, \tag{9.19}$$

hence

$$\Phi(x_1) \geq \Phi(x_2) \geq \Phi(x_3) \geq \cdots \geq \Phi(x). \tag{9.20}$$

This time, the monotone convergence theorem implies

$$\Phi(x_k) \searrow \mathrm{Vol}\Big(\bigcap_{k\geq 1} \mathcal{T}_{x_k}\Big). \tag{9.21}$$

Clearly

$$\bigcap_{k\geq 1} \mathcal{T}_{x_k} \supset \mathcal{T}_x.$$

Recall that $\mathcal{T}_{x_k}$ is the union of all the 2D tubes T_σ that neither enclose nor contain x_k. Now suppose you take T_σ such that

$$T_\sigma \text{ is not a subset of } \mathcal{T}_x. \tag{9.22}$$

It follows that T_σ either contains x or encloses x. If T_σ encloses x, then x has a positive distance from T_σ, hence for all sufficiently large k, T_σ must enclose x_k, so T_σ is not a subset of $\mathcal{T}_{x_k}$ for large k. We deduce that, for each σ for which Eq. 9.22 holds,

$$T_\sigma \subset \bigcap_{k\geq 1} \mathcal{T}_{x_k} \Longrightarrow x \in T_\sigma. \tag{9.23}$$

Now condition (III) implies there can be at most one such σ. We conclude that either

$$\bigcap_{k\geq 1} \mathcal{T}_{x_k} = \mathcal{T}_x, \quad \text{or} \quad \bigcap_{k\geq 1} \mathcal{T}_{x_k} = \mathcal{T}_x \cup T_{\sigma_1}, \tag{9.24}$$

for an invariant cycle σ_1. Then condition (I) yields

$$\Phi(x_k) \searrow \Phi(x), \tag{9.25}$$

and the proof of Proposition 9.4 is complete. □

Proposition 9.4 gives a nontrivial function $\Phi \in C(X)$ that is invariant under G_t for all $t \in \mathbb{R}$. Of course, for each continuous $\beta : \mathbb{R} \to \mathbb{R}$, $\beta \circ \Phi$ belongs to $C(X)$ and is invariant under all G_t. Thus the range of P contains lots of elements of $C(X)$, where P is the orthogonal projection arising in Theorem 1.1. We are interested in the following complementary situation.

Problem Determine when one can take $a \in C(X)$ and guarantee that $Pa \in C(X)$.

Our quantum ergodic theorem points to the usefulness of obtaining results on this problem. We intend to address this in future work.

10 On the Existence of Soft Chaos

As stated in [7], p. 118, the term "soft chaos" is somewhat lacking in mathematical precision. A number of different varieties of soft chaos come to mind. To set definitions, let H be a Hamiltonian vector field on T^*M, X a constant energy surface, assumed to be compact, G_t the flow on X generated by H. Here is a weak notion of soft chaos.

$$\text{The flow } G_t \text{ is neither integrable nor ergodic.} \tag{10.1}$$

In this case, it has been proven in [10] that, in the generic case, Eq. 10.1 holds. Tools to show that, in certain cases, G_t is not ergodic include KAM theory. Tools to show that, in certain cases, G_t is not integrable include the detection of homoclinic tangles.

A stronger version of soft chaos is the following.

$$\begin{aligned}&\text{There is a partition } X = X_0 \cup X_1 \text{ with the following properties.}\\ &X_0 \subset X \text{ is closed and the union of invariant Lagrangian tori.}\\ &X_1 = \cup_\alpha X_\alpha, \text{ with } G_t \text{ acting ergodically on each } X_\alpha.\\ &\text{Finally, } X_0 \text{ and } X_1 \text{ both have positive measure.}\end{aligned} \tag{10.2}$$

It is widely believed (and this belief seems to be well supported by numerical evidence) that (10.2) holds generically, but proving this (or even proving some examples exist) has been a problem for some time, a problem mentioned in [12], p. 109, in 1973, and in [4] in 2008. Of course, KAM can establish that X_0 has positive measure, in certain cases. The problem is to show that $X \setminus X_0$ has positive measure. In this context, it should be mentioned that "Smale horseshoes" that arise from homoclinic tangles have measure zero.

The remarks above regarding (10.2) apply to cases where M has no boundary, which is the setting of this paper. There do exist bounded domains in Euclidean space, with piecewise smooth boundary, whose associated billiard ball maps have been proven to have property (10.2). More precisely, what is called the "mushroom domain" in $\mathbb{R}^2$ has been shown in [3] to have the property that $X = X_0 \cup X_1$ with the billiard ball map integrable on X_0 and ergodic on X_1, and both X_0 and X_1 have positive measure. Study of the mushroom domain and variants has produced a growing literature. As an example, we mention [2].

We now discuss some examples where one can expect to find soft chaos, and it is likely one can prove that (10.1) holds, though we do not propose proofs here. It also seems likely that (10.2) holds for these examples, but proofs of this will certainly have to wait!

Our first class of examples arise from geodesic flows on certain perturbations of the "inner tube" $\mathcal{T}^2 \subset \mathbb{R}^3$, given by Eq. 7.17. To begin, we consider $\mathcal{T}_a^2 \subset \mathbb{R}^3$, the image of $\mathbb{T}^2$ under the maps $\Phi_a : \mathbb{T}^2 \to \mathbb{R}^3$, given by

$$\Phi_a(\theta, \omega) = \big((2 + \cos\theta)\cos\omega, a(2 + \cos\theta)\sin\omega, \sin\theta\big). \tag{10.3}$$

In other words, $\mathcal{T}_a^2$ is the image of $\mathcal{T}^2$ under the map

$$M_a : \mathbb{R}^3 \to \mathbb{R}^3, \quad M_a(x, y, z) = (x, ay, z). \tag{10.4}$$

For each $a > 0$, $\mathcal{T}_a^2$ has four closed geodesics in the plane $\{z = 0\}$. Two (related by time reversal) are outer equatorial geodesics, and the other two (also related by time reversal) are inner equatorial geodesics. The outer equatorial geodesics are elliptic and the inner equatorial geodesics are hyperbolic. For (almost all?) $a > 0$, we expect the results of Section 9 to apply near the elliptic closed geodesics, yielding some nontrivial G_t-invariant functions in $C(X)$, $X = S^*\mathcal{T}_a^2$.

The results of Section 9 do not imply that there are stochastic regions in a small tubular neighborhood of the outer equatorial geodesics, though we expect this to be the case.

Another potential source of chaos is associated with the inner equatorial geodesics, which are hyperbolic. Identify an inner equatorial geodesic with the periodic G_t-orbit $\gamma \subset X$. Pick $p \in \gamma$, take a surface $\Sigma \subset X$, through p, transversal to γ, and let $R : \Sigma \to \Sigma$ be the Poincaré first return map. Then $R(p) = p$, and p is a hyperbolic fixed point. Thus there are invariant curves through p, one the stable manifold (near p) and one the unstable manifold. If $a = 1$, then, globally, these two coincide, and we have a homoclinic invariant curve. One would achieve chaos if, as a is moved from 1, these did not coincide, but intersected transversally, yielding a "homoclinic tangle." However, for M_a as in Eqs. 10.3–10.4, this does not happen.

In fact, for each $a > 0$, in this setup, the stable and unstable manifolds coincide. This can be established using the symmetries of $\mathcal{T}_a^2$. Each surface $\mathcal{T}_a^2$ is invariant under three reflections, namely reflection across $\{x = 0\}$, across $\{y = 0\}$, and across $\{z = 0\}$. Hence $\mathcal{T}_a^2$ is invariant under rotation by 180° about the x-axis (and also the y-axis, and for that matter, also the z-axis). Consider various geodesics in $\mathcal{T}_a^2$ starting at the point $e_1 = (1, 0, 0) \in \mathcal{T}_a^2$, where $\theta = \omega = 0$. If the velocity vector is close to $(0, 0, 1)$, the geodesic winds aronnd $\mathcal{T}_a^2$, crossing the inner equatorial geodesic infinitely often. If the velocity vector is close to $(0, 1, 0)$, the geodesic never crosses the inner equatorial geodesic. There are 4 critical velocity directions leading to geodesics through e_1 that tend toward the inner equatorial geodesic as $t \to +\infty$. Now the rotational symmetry mentioned above implies these critical directions occur in pairs, one velocity being the negative of the other. It follows that if a geodesic through e_1 tends toward the inner equatorial geodesic as $t \to +\infty$, it also tends toward same as $t \to -\infty$.

To address this phenomenon, we modify $\mathcal{T}_a^2$, obtaining surfaces $\mathcal{T}_{a,b}^2$, for which the reflection symmetries across $\{x = 0\}$ and $\{y = 0\}$ are destroyed, though we retain the reflection symmetry across $\{z = 0\}$. We take $\mathcal{T}_{a,b}^2 \subset \mathbb{R}^3$ to be the image of $\mathbb{T}^2$ under the map $\Phi_{a,b} : \mathbb{T}^2 \to \mathbb{R}^3$, with

$$\Phi_{a,b}(\theta, \omega) = \Phi_a(\theta, \omega) + b\Psi(\omega), \quad \Psi(\omega) = (\sin 3\omega, \cos 5\omega, 0). \tag{10.5}$$

We take

$$0 < b << a. \tag{10.6}$$

In such a case, $\mathcal{T}_{a,b}^2$ still has inner and outer equatorial geodesics, in $\{z = 0\}$.

Under such conditions, it is reasonable to expect that homoclinic tangles arise near the inner equatorial geodesic.

We turn to another class of examples. Let M_0 be a compact 2-dimensional Riemannian manifold for which the geodesic flow is known to be ergodic on S^*M_0. Take the standard unit sphere S^2. Cut a small open geodesic disk D from S^2, cut a small open geodesic disk D_0 from M_0, and join what remains by a neck N, obtaining

$$M = (S^2 \setminus D) \cup N \cup (M_0 \setminus D_0). \tag{10.7}$$

Endow M with a smooth metric tensor, agreeing with the original metric tensors on $S^2 \setminus D$ and on $M_0 \setminus D_0$. Let $D^* \subset S^2$ be the image of D under the antipodal map of S^2. The set of points in S^*S^2 whose image under the geodesic flow lies over $S^2 \setminus (D \cup D^*)$ for all t has positive measure. This gives rise to a subset Y of S^*M of positive measure (in fact, with nonempty interior) on which the geodesic flow is integrable. It is tempting to conjecture that $S^*M \setminus Y$ has a subset of positive measure on which the geodesic flow is ergodic. Needless to say, we do not have a proof of this.

We conclude with a description of a class of symbols to which Theorem 1.1 applies. With M as above, and $\beta \in \mathbb{R}$, take $a \in C(S^*M)$ such that

$$\begin{gathered} a \text{ averages to } \beta \text{ on each closed orbit in } Y, \\ a = \beta \text{ on } S^*M \setminus Y. \end{gathered} \tag{10.8}$$

Then $Pa \equiv \beta$, and Theorem 1.1 applies.

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Appendix A: Conditional Expectation Near a Hyperbolic Critical Point

In Eq. 7.17 we have the conditional expectation of a function on $\mathbb{T}^2$, with respect to the σ-algebra of sets of the form $\widetilde{\Omega}^{-1}(B)$, for Borel $B \subset \mathbb{R}$, where $\widetilde{\Omega} \in C^\infty\left(\mathbb{T}^2\right)$ has two hyperbolic critical points. We state there that this conditional expectation operator maps $C\left(\mathbb{T}^2\right)$ to $C\left(\mathbb{T}^2\right)$ and $C^\infty\left(\mathbb{T}^2\right)$ to $C^\omega\left(\mathbb{T}^2\right)$, with ω given by Eq. 7.21. Here we show how to establish these results.

To simplify the calculations, we look at a model case, with one hyperbolic critical point. Namely, we analyze $Pf = \mathbb{E}(f|\mathcal{F})$, on $Q = [-1,1] \times [-1,1] \subset \mathbb{R}^2$, with Lebesgue measure, where $\mathcal{F}$ is the σ-algebra of the form $g^{-1}(B)$, for Borel $B \subset \mathbb{R}$, with $g(x,y) = y^2 - x^2$, which has a critical point at $(0,0)$. We make a few preliminary observations. Clearly $P : L^\infty(Q) \to L^\infty(Q)$ has norm 1, so to show

$$P : C(Q) \longrightarrow C(Q), \tag{A.1}$$

it suffices to show

$$P : C^\infty(Q) \longrightarrow C(Q). \tag{A.2}$$

Also, clearly $f \in C(Q)$ (resp., $f \in C^\infty(Q)$) implies Pf is continuous (resp., Pf is smooth) on $Q \setminus X$, where

$$X = \{(x,y) \in Q : x = \pm y\}. \tag{A.3}$$

Consequently, the following result is useful.

Lemma A.1 *If $f \in C^1(Q)$, then*

$$z_k \in Q \setminus X, \;\; z_k \to z_0 \in X \Longrightarrow Pf(z_k) \to f(0,0). \tag{A.4}$$

Proof The set $Q \setminus X$ has four connected components. It will suffice to consider the case where each z_k lies in the upper quarter, where $y > |x|$, as similar arguments apply in the other cases. We may as well drop the subscripts, and take

$$y = \sqrt{\varepsilon + x^2}, \quad \varepsilon > 0. \tag{A.5}$$

Then $Pf(x,y)$ depends only on ε, and we have

$$Pf(x,y) = \frac{1}{A(\varepsilon)} \int_{-1}^{1} \frac{f\left(s, \sqrt{\varepsilon + s^2}\right)}{\sqrt{\varepsilon + s^2}}\, ds, \quad A(\varepsilon) = \int_{-1}^{1} \frac{ds}{\sqrt{\varepsilon + s^2}}. \tag{A.6}$$

□

Remark Actually, the domain of integration should be $s \in \left[-\sqrt{1-\varepsilon}, \sqrt{1-\varepsilon}\right]$. For notational convenience, we ignore this, here and below.

For the denominator $A(\varepsilon)$, we have

$$\begin{aligned} A(\varepsilon) &= \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \frac{du}{\sqrt{1+u^2}} \\ &= 2 \sinh^{-1} \frac{1}{\sqrt{\varepsilon}} \\ &= 2 \log\Big(\frac{1}{\sqrt{\varepsilon}} + \sqrt{\frac{1}{\varepsilon} - 1}\Big) \\ &= \Big(\log \frac{1}{\varepsilon}\Big)\big(1 + O(\varepsilon)\big). \end{aligned} \tag{A.7}$$

We can rewrite (A.6) as

$$Pf(x,y) = \frac{1}{A(\varepsilon)} \int_{-1}^{1} \frac{f_\varepsilon(s)}{\sqrt{\varepsilon + s^2}}\, ds, \tag{A.8}$$

where

$$f_\varepsilon(s) = f\left(s, \sqrt{\varepsilon + s^2}\right). \tag{A.9}$$

Note that, if $f \in C^1(Q)$, then the family f_ε is uniformly Lipschitz on $[-1, 1]$, for $\varepsilon \in (0, 1/2]$. We then have

$$\begin{aligned} Pf(x,y) &= f_\varepsilon(0) + \frac{1}{A(\varepsilon)} \int_{-1}^{1} \frac{f_\varepsilon(s) - f_\varepsilon(0)}{\sqrt{\varepsilon + s^2}}\, ds \\ &= f_\varepsilon(0) + \frac{1}{A(\varepsilon)} \int_{-1}^{1} \frac{s}{\sqrt{\varepsilon + s^2}} g_\varepsilon(s)\, ds, \end{aligned} \tag{A.10}$$

where

$$g_\varepsilon(s) = \frac{f_\varepsilon(s) - f_\varepsilon(0)}{s} = \frac{f\left(s, \sqrt{\varepsilon + s^2}\right) - f\left(0, \sqrt{\varepsilon}\right)}{s}. \tag{A.11}$$

Since

$$\sqrt{\varepsilon + s^2} - \sqrt{\varepsilon} = \frac{1}{2} \int_0^{s^2} \frac{dy}{\sqrt{\varepsilon + y}} \le \frac{1}{2} \int_0^{s^2} \frac{dy}{\sqrt{y}} = |s|, \tag{A.12}$$

and f is C^1, hence Lipschitz, we have

$$|g_\varepsilon(s)| \le L < \infty, \quad \forall\, \varepsilon. \tag{A.13}$$

Thus we have

$$\left| \int_{-1}^{1} \frac{s}{\sqrt{\varepsilon + s^2}} g_\varepsilon(s)\, ds \right| \le 2L, \quad \forall\, \varepsilon, \tag{A.14}$$

hence, for (x, y) as in Eq. A.5, $f \in C^1(Q)$,

$$\left| Pf(x,y) - f\left(0, \sqrt{\varepsilon}\right) \right| \le \frac{2L}{A(\varepsilon)}. \tag{A.15}$$

This establishes Lemma A.1. It hence yields (A.1), and also

$$P : C^1(Q) \longrightarrow C^\omega(Q), \tag{A.16}$$

with

$$\omega(h) = \frac{1}{|\log h|}, \quad 0 < h << 1. \tag{A.17}$$

Remark The estimate (A.13) and the Lebesgue dominated convergence theorem yield

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{-1}^{1} \frac{s}{\sqrt{\varepsilon + s^2}} g_\varepsilon(s)\, ds &= \int_{-1}^{1} (\operatorname{sgn} s) \frac{f(s, |s|) - f(0,0)}{s}\, ds \\ &= \int_{-1}^{1} \frac{f(s, |s|) - f(0,0)}{|s|}\, ds. \end{aligned} \tag{A.18}$$

Since there are functions $f \in C^1(Q)$ for which this last integral is $\neq 0$, this implies that (A.16–A.17) is sharp, as far as the range is concerned.

Appendix B: Invariance Properties of Commuting, Measure-Preserving Flows

Our goal here is to establish a general result that contains Proposition 8.2. To set things up, let (X, μ) be a probability space. Assume $L^2(X, \mu)$ is separable. For $j = 1, 2,\ t \in \mathbb{R}$, let $\mathcal{F}_j^t : X \to X$ be a 1-parameter group of measure-preserving transformations. Assume these transformations commute, i.e., $\mathcal{F}_1^t \mathcal{F}_2^s = \mathcal{F}_2^s \mathcal{F}_1^t$, for all $s, t \in \mathbb{R}$. Also assume the following continuity:

$$f \in L^2(X, \mu) \Longrightarrow t \mapsto f \circ \mathcal{F}_j^t \text{ is continuous from } \mathbb{R} \text{ to } L^2(X, \mu). \tag{B.1}$$

For $\alpha \in \mathbb{R}$, set

$$\mathcal{G}_\alpha^t = \mathcal{F}_1^t \circ \mathcal{F}_2^{\alpha t}. \tag{B.2}$$

The following result contains Proposition 8.2.

Proposition B.1 *Under the hypotheses stated above, there exists a countable set $\mathcal{C} \subset \mathbb{R}$ such that, for each $\alpha \in \mathbb{R} \setminus \mathcal{C}$,*

$$\begin{aligned} &f \in L^2(X, \mu),\ \ f \circ \mathcal{G}_\alpha^t = f,\ \ \forall t \in \mathbb{R} \\ &\qquad \Longrightarrow f \circ \mathcal{F}_j^t = f,\ \ \forall t \in \mathbb{R},\ \forall j. \end{aligned} \tag{B.3}$$

Proposition B.1 follows from the next proposition, an abstract result about commuting unitary groups. To state it, let $\left\{U_j^t : t \in \mathbb{R}\right\}$ be strongly continuous unitary groups on a separable Hilbert space H, and assume they commute, i.e., $U_1^t U_2^s = U_2^s U_1^t$, for all $s, t \in \mathbb{R}$. For $\alpha \in \mathbb{R}$, set

$$V_\alpha^t = U_1^t U_2^{\alpha t}. \tag{B.4}$$

Proposition B.2 *Given U_j^t as above, there exists a countable set $\mathcal{C} \subset \mathbb{R}$ such that, for each $\alpha \in \mathbb{R} \setminus \mathcal{C}$,*

$$\begin{aligned} &f \in H,\ V_\alpha^t f = f,\ \ \forall t \in \mathbb{R} \\ &\qquad \Longrightarrow U_j^t f = f,\ \ \forall t \in \mathbb{R},\ \forall j. \end{aligned} \tag{B.5}$$

To prove Proposition B.2, we use the following version of the Spectral Theorem. There exists a σ-finite measure space (Y, ν), a unitary map $W : H \to L^2(Y, \nu)$, and ν-measurable functions $a_j : Y \to \mathbb{R}$ such that

$$W\left(U_j^t f\right)(y) = e^{ita_j(y)} Wf(y), \quad \forall f \in H,\ t \in \mathbb{R}. \tag{B.6}$$

Note that

$$U_j^t f = f\ \ \forall t \Longleftrightarrow a_j Wf = 0\ (\nu\text{-a.e.}), \tag{B.7}$$

and

$$V_\alpha^t f = f\ \ \forall t \Longleftrightarrow (a_1 + \alpha a_2) Wf = 0\ (\nu\text{-a.e.}). \tag{B.8}$$

Hence Proposition B.2 is a consequence of the following.

Lemma B.3 *Given ν-measurable functions $a_j : Y \to \mathbb{R}$ as above, there exists a countable set $\mathcal{C} \subset \mathbb{R}$ such that for each $\alpha \in \mathbb{R} \setminus \mathcal{C}$,*

$$\begin{aligned} &g \in L^2(Y, \nu),\ (a_1 + \alpha a_2) g = 0\ (\nu\text{-a.e.}) \\ &\qquad \Longrightarrow a_1 g = a_2 g = 0\ (\nu\text{-a.e.}). \end{aligned} \tag{B.8}$$

Proof Consider the following subsets of Y:

$$\begin{aligned} S_\alpha &= \{y \in Y : a_1(y) = -\alpha a_2(y)\}, \\ S &= \{y \in Y : a_1(y) = a_2(y) = 0\}. \end{aligned} \tag{B.10}$$

Clearly $S \subset S_\alpha$, for each α. To prove Lemma B.3, it suffices to show that there exists a countable set $\mathcal{C} \subset \mathbb{R}$ such that

$$\alpha \in \mathbb{R} \setminus \mathcal{C} \Longrightarrow \nu(S_\alpha \setminus S) = 0. \tag{B.11}$$

Note that

$$S_\alpha \setminus S = T_\alpha = \{y \in Y : a_1(y) = -\alpha a_2(y),\ a_2(y) \neq 0\}. \tag{B.12}$$

On the other hand, clearly $\alpha \neq \alpha' \Rightarrow T_\alpha \cap T_{\alpha'} = \emptyset$. Hence $\nu(T_\alpha) \neq 0$ for at most countably many $\alpha \in \mathbb{R}$, and we are done. ☐

We note the following n-dimensional version of Lemma B.3.

Lemma B.4 *Given a ν-measurable $a : Y \to \mathbb{R}^n$, $a = (a_1, \dots, a_n)$, there exists $\mathcal{E} \subset \mathbb{R}^n$ of Lebesgue measure 0 such that, for each $\omega \in \mathbb{R}^n \setminus \mathcal{E}$,*

$$\begin{aligned} &g \in L^2(Y, \nu),\ (\omega \cdot a)g = 0\ (\nu\text{-a.e.}) \\ &\qquad \Longrightarrow a_1 g = \cdots = a_n g = 0\ (\nu\text{-a.e.}). \end{aligned} \tag{B.13}$$

This follows by induction on n, starting at $n = 2$, by Lemma B.3. This result leads to the following n-dimensional version of Proposition B.2.

Proposition B.5 *Let U be a strongly continuous unitary representation of $\mathbb{R}^n$ on a separable Hilbert space H. Then there exists a subset $\mathcal{E} \subset \mathbb{R}^n$, of Lebesgue measure 0, such that, for each $\omega \in \mathbb{R}^n \setminus \mathcal{E}$,*

$$\begin{aligned} &f \in H,\ U(t\omega)f = f, \quad \forall t \in \mathbb{R} \\ &\qquad \Longrightarrow U(\xi)f = f, \quad \forall \xi \in \mathbb{R}^n. \end{aligned} \tag{B.14}$$

This in turn leads to an n-dimensional version of Proposition B.1, which we leave to the reader.

Appendix C: Proof of Proposition 7.6

As discussed in Section 7, Proposition 7.6 is a consequence of the following.

Proposition C.1 *Given $a, b \in \mathbb{N}$, $a < b$, and*

$$\begin{aligned} &\Phi_{a,b} : \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \le k\} \longrightarrow \infty, \\ &\Phi_{a,b}(j,k) = ((2k+1)b)^2 - (2ja)^2, \end{aligned} \tag{C.1}$$

there is no upper bound on the number of elements of $\Phi_{a,b}^{-1}(N)$, as N ranges over $\mathbb{N}$.

We recall that Proposition 7.4 follows from an analogous result involving a sum of squares, rather than a difference. In that case, the desired conclusion followed from Eq. 7.43. That classical result is typically proven as a consequence of the fact that the ring $\mathbb{Z}[i]$

of Gaussian integers has the unique factorization property. The number theory behind Proposition C.1 is rather different, and arguably more elementary.

To start the proof of Proposition C.1, we write

$$\begin{aligned} N &= ((2k+1)b)^2 - (2ja)^2 \\ &= [(2k+1)b + 2ja] \cdot [(2k+1)b - 2ja] \\ &= (K+J)(K-J) \\ &= AB, \end{aligned} \tag{C.2}$$

with

$$\begin{aligned} \frac{A+B}{2} &= K = (2k+1)b, \\ \frac{A-B}{2} &= J = 2ja. \end{aligned} \tag{C.3}$$

It is convenient to write

$$a = 2^\mu \alpha, \quad \alpha \text{ odd}, \ \mu \in \{0, 1, 2, \dots\}. \tag{C.4}$$

We will look for numbers N that factorize (in many ways) as AB with

$$A = \alpha b q, \quad B = \alpha b r, \quad q > r. \tag{C.5}$$

Then (C.3) becomes

$$\alpha \frac{q+r}{2} = 2k+1, \quad b \frac{q-r}{2} = 2^{\mu+1} j, \tag{C.6}$$

so we need $(q+r)/2$ to be odd, and it will be convenient to have $(q-r)/2$ divisible by $2^{\mu+1}$. Say $(q-r)/2 = 2^{\mu+1}\ell, \ \ell \in \mathbb{N}$, so

$$q = r + 2^{\mu+2}\ell, \quad r \text{ odd}. \tag{C.7}$$

Then (C.6) holds with

$$\begin{aligned} 2k+1 &= \alpha\left(r + 2^{\mu+2}\ell\right), \\ j &= b\ell. \end{aligned} \tag{C.8}$$

In such a case, Eq. C.2 holds for

$$N = \alpha^2 b^2 r \left(r + 2^{\mu+2}\ell\right). \tag{C.9}$$

Recall that a (hence α) and b are given in $\mathbb{N}$ (with $a < b$), and we want (C.2) to hold for some (in fact, many) j, k satisfying $1 \le j \le k$. In light of the observations made above, Proposition C.1 is a consequence of the following.

Lemma C.2 *Take $\alpha, b \in \mathbb{N}, \ \mu \in \mathbb{Z}^+, \ 2^\mu \alpha < b$. Define*

$$\begin{aligned} &\Psi_{\mu\alpha b} : \left\{(r, \ell) \in \mathbb{N} \times \mathbb{N} : r \textit{ odd}, \ 2b\ell + 1 \le \alpha(r + 2^{\mu+1}\ell)\right\}, \\ &\Psi_{\mu\alpha b}(r, \ell) = r\left(r + 2^{\mu+2}\ell\right). \end{aligned} \tag{C.10}$$

Then there is no upper bound on the number of elements of $\Psi^{-1}_{\mu\alpha b}(\nu)$, as ν runs over $\mathbb{N}$.

Proof Given $K \in \mathbb{N}$, pick $\mu_1, \dots, \mu_K$ and $\tilde{\mu}_1, \dots, \tilde{\mu}_K \in \mathbb{N}$ such that

$$\mu_j = 1 \mod 2^{\mu+2}, \quad \tilde{\mu}_j = \mu_j + 2^{\mu+2}, \quad \mu_{j+1} > \tilde{\mu}_j. \tag{C.11}$$

Set

$$\nu = \mu_1^2 \tilde{\mu}_1^2 \cdots \mu_K^2 \tilde{\mu}_K^2. \tag{C.12}$$

Then, for each $k \in \{1, \dots, K\}$, we have

$$\nu = q_k r_k, \tag{C.13}$$

with

$$q_k = \tilde{\mu}_k^2 \left(\prod_{j \neq k} \mu_j \tilde{\mu}_j \right), \quad r_k = \mu_k^2 \left(\prod_{j \neq k} \mu_j \tilde{\mu}_j \right). \tag{C.14}$$

Note that (C.11) implies $q_k = 1 \mod 2^{\mu+2}$ and $r_k = 1 \mod 2^{\mu+2}$. Since $q_k > r_k$, we have

$$q_k = r_k + 2^{\mu+2} \ell_k, \tag{C.15}$$

for some $\ell_k \in \mathbb{N}$. It follows from Eqs. C.11 and C.14 that $j > k \Rightarrow r_k < r_j$, so the factorizations (C.13) are all distinct. Also, as long as we impose the requirement

$$\mu_1 >> 2^{\mu+2}, \tag{C.16}$$

we can satisfy the constraint

$$2b\ell_k + 1 \leq \alpha \left(r_k + 2^{\mu+1} \ell_k \right), \quad \forall k. \tag{C.17}$$

Thus we have Lemma C.2, and hence Proposition C.1. □

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