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Continuous Functions and Riesz Type Potentials in Homogeneous Spaces

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Abstract We develop a potential theory for a Riesz type kernel in a homogeneous space and characterize the compact sets K with capacity zero as the sets K for which every continous function f on K is the restriction to K of a continuous potential $U_k^{\sigma_f}$ of an absolutely continuous measure σ_f supported in an arbitrarily small neighbourhood of K. The measure σ_f can be choosen as a suitable restriction of a single measure σ that only depends on the set K and the kernel k.

Keywords Homogeneous space · Doubling measure · Kernel · Potential · Energy · Capacity · Capacitary potential · Approximate identity · Dyadic cubes

Mathematics Subject Classification (2010) Primary 31 B 15 · Secondary 46 E 35 · 42 B 25

1 Introduction

A classical result of G. C Evans [8] characterizes the compact sets K in R^n of $\alpha-$capacity zero as the sets for which there is a nonnegative measure σ such that the $\alpha-$potential $U_\alpha^\sigma(x)$ is infinite precisely on K. H. Wallin [21] proved a different characterization: K has $k-$capacity zero if and only if every continuous function on K is the restriction to K of a continuous potential in R^n, for suitable kernels k. Later, S. Ja Havinson [12] gave a unified treatment of both these results, removed some restrictions on the kernel and showed that it is sufficient to consider restrictions of a single measure that only depends on the set K and the kernel k. T. Sjödin generalized the problem described above to nonlinear L^p-potential

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theory and solved the extension problem for Bessel potentials [18] and Sobolev functions [19]. It is the purpose of this paper to extend the results described above from the Euclidean space R^n to an abstract space. A suitable setting for this is the homogeneous spaces of Coifman and Weiss [5] consisting of a quasimetric space (X, d) equipped with a nonnegative doubling measure μ. For more on such spaces, see [6] and the references contained there.

Let (X, d, μ) be a homogeneous space in the sense of [5], let $k(x, y)$ be a kernel and define the $k-$potential of a nonnegative Borel measure ν on X by

$$U_k^{\nu}(x) = \int k(x, y)\, d\nu(y). \tag{1.1}$$

B. Fuglede [9] developed a potential theory for U_k^{ν} and its related capacity C_k in a locally compact space and proved the basic existence theorem for capacitary measures and capacitary potentials. We show how [5] and [9] can be combined into a potential theory that extends the classical potential theory in R^n to the homogeneous spaces (X, d, μ). A natural analog of a Riesz type kernel in R^n is defined by

$$K \circ \mu(x, y) = K(\mu\overline{B}(x, d(x, y)) + \mu\overline{B}(y, d(x, y))),$$

where $K(r)$, $r > 0$, is a positive, continuous and nonincreasing function and $\mu\overline{B}(x, a) = \mu\{y \in X; d(y, x) \leq a\}$, c.f. [4] and [20]. See Section 3 for the exact definitions. The main tools in the proofs are our adoption of an approximation of the identity in Carleson [3] to the present setting and the construction of families of dyadic nets in M. Christ [4] and T. Hytönen, H. Kairema [13]. It allows us to mollify measures and potentials uniformly in X and build continuous potentials of measures that are absolutely continuous with respect to the measure μ. Theorems 2.1 and 2.2 are our principal results that give sufficient and necessary conditions, respectively, for every continuous function on K to be the restriction to K of a continous potential. Our main result (Theorems 2.3) extends the results of H. Wallin [21] and Ya. Havinson [12] described above to the homogeneous spaces (X, d, μ) and gives our characterization of compact sets with capacity zero. See Section 2 for details.

The plan of the paper is as follows. Section 2 states our results, that are proved in Sections 5 and 6. Our notation and definitions of the spaces (X, d, μ) are given in Section 3. We describe the potential theory in Section 4 and construct the potentials that will be our building blocks in the proofs of Theorems 2.1 and 2.2.

2 Main Results

We start with the extension theorem, which roughly states that if K is a compact subset of X with $C_k(K) = 0$ and $f \in C(K)$ then there is a nonnegative measure σ_f on X and a continuous potential $U_k^{\sigma_f}(x)$ in X such that $U_k^{\sigma_f}(x) = f(x), x \in K$, and σ_f is the restriction of a single measure σ on X. The theorem extends and unites results by H. Wallin [21], Theorem 1, and S. Ja Havinson [12], Theorem 1, to homogeneous spaces. More exactly, we have the following result.

Theorem 2.1 *Let (X, d, μ) be a complete homogeneous space of order γ, $0 < \sigma < 1$, satisfying (DC), and assume that μ satisfies (3.7). Let $K \circ \mu$ be a doubling kernel satisfying (4.6). Let K be a compact subset of X such that $C_K(K) = 0$, G an open set containing K and $\epsilon > 0$. Then there is $\sigma \in M_+(X)$ with the following properties:*

(a) *$supp(\sigma) \subset G$, $||\sigma||_1 < \epsilon$ and σ is absolutely continuous with respect to μ,*

(b) $U_K^\sigma(x) = \infty$, $x \in K$, $U_K^\sigma(x)$ *is finite and continuous outside* K *and* $\liminf_{x \to x_0} U_K^\sigma(x) = \infty$, $x_0 \in K$,

(c) *Let* $F(x)$ *be a positive and lower semicontinuous function in a neighbourhood of* K *and let* $f(x)$ *be the restriction of* $F(x)$ *to* K. *Then there exists a Borel subset* E_f *of* $supp(\sigma)$ *such that if* σ_f *is the restriction of* σ *to* E_f *then the potential* $U_K^{\sigma_f}(x)$ *is continuous outside* K, *lower semicontinuous in* X *and*

$$U_K^{\sigma_f}(x) = f(x), \ x \in K.$$

If $f \in C(K)$ *then* $U_K^{\sigma_f}(x)$ *is continuous in* X.

For the converse of Theorem 2.1 we assume that K is a compact subset of X with $C_K(K) > 0$. Then there exists a more singular kernel $K_0 \circ \mu$ such that $C_{K_0}(K) > 0$ and the potentials $U_K^\sigma(x)$ have a uniform continuity property outside a small subset of K (Lemma 6.1). However, this property fails in general for functions in $C(K)$ (Lemma 6.2), which proves Theorem 2.2. The Euclidean case was proved in H. Wallin [21], Theorem 2.

Theorem 2.2 *Let* (X, d, μ) *and* $K \circ \mu$ *be as in Theorem 2.1. Let* K *be a compact subset of* X *with* $C_K(K) > 0$. *Then there is* $f \in C(K)$ *that is not the restriction to* K *of a potential* $U_K^\sigma(x)$ *for any* $\sigma \in M_+(X)$ *with* $||\sigma||_1$ *finite.*

Combining Theorems 2.1 and 2.2 we get the following characterization of compact sets in X with C_K-capacity zero, which is our main result.

Theorem 2.3 *Let* (X, d, μ) *and* $K \circ \mu$ *be as in Theorem 2.1 and let* K *be a compact subset of* X. *Then the following statements (i)–(iii) are equivalent:*

(i) $C_K(K) = 0$,
(ii) *There exists* $\sigma \in M_+(X)$ *such that* $U_K^\sigma(x) = \infty$ *exactly for* $x \in K$,
(iii) *For every* $f \in C(K)$ *there is a continuous potential* $U_K^\sigma(x)$, $\sigma \in M_+(X)$, *such that* $f(x) = U_K^\sigma(x)$, $x \in K$.

In statements (ii) and (iii) we may require σ *to have arbitrarily small total mass* $||\sigma||_1$ *and support in an arbitrary neighbourhood of* K.

Remark Sets $\{x; U_K^\sigma(x) = \infty\}$, for some $\sigma \in M_+(X)$, are usually refered to as polar sets, see [15], Ch. III, §1. Theorem 2.3 (ii) says thet K is a polar set relative to the kernel $K \circ \mu$.

3 Notations and Definitions

A homogeneous space is a triple (X, d, μ), where (X, d) is a quasi–metric space satisfying

$$d(x, y) \leq K \cdot (d(x, z) + d(z, y)), \ x, y, z \in X, \tag{3.1}$$

for some constant $K \geq 1$, and μ is a nonnegative and nonatomic measure on the $\sigma-$algebra generated by all balls $B(x, r) = \{y \in X; d(y, x) < r\}$ and satisfies the doubling condition

$$0 < \mu B(x, 2r) \leq M \cdot \mu B(x, r), \ x \in X, \ r > 0. \tag{3.2}$$

The constant M is called the doubling constant of μ. It is an easy consequence of Eq. 3.2 that

$$\mu B(x,r) \geq C \cdot \left(\frac{r}{R}\right)^{\alpha} \cdot \mu B(x,R), \quad x \in X,\ 0 < r < R, \tag{3.3}$$

where the constants $C > 0$ and $\alpha > 0$ only depend on M. A relation which holds except in a set of $\mu-$measure zero is said to hold $\mu-$ a.e. See [5, 6, 16] for the basic properties of the homogeneous spaces (X, d, μ).

We give X the topology induced by the balls $B(x,r) = \{y \in X; d(y,x) < r\}$ and denote the closure and complement of a set $E \subset X$ by $\overline{E}$ and E^c respectively. Let $\overline{B}(x,r) = \{y \in X; d(y,x) \leq r\}$ and $S(x,r) = \{y \in X; d(y,x) = r\}$. It is a consequence of the doubling property of μ that there exists a positive integer N such that the following holds:

For every $m = 1, 2, \ldots$ and $x \in X$, $r > 0$ there are at most N^m points $x_i \in B(x,r)$ such that for all $i \neq j$ we have $d(x_i, x_j) \geq r \cdot 2^{-m}$,

c.f. [5], Ch. III. It follows that bounded sets are totally bounded. If (X, d) is complete then (X, d) is a locally compact and separable Hausdorff space, see [17], Ch. 4, Sec. 25, Theorem A. Every open set is a countable union of balls $B(x,r)$ and is therefore $\mu-$measurable. Hence μ is a regular Borel measure on X.

A homogeneous space (X, d, μ) is of order γ, $0 < \gamma < 1$, if there is a constant C such that

$$|d(x,z) - d(z,y)| \leq C \cdot R^{1-\gamma} \cdot d(x,y)^{\gamma}, \tag{3.4}$$

for all $x, y \in B(z,R)$, $z \in X$ and $R > 0$. Every homogeneous space has an equivalent quasi-norm satisfying (3.4), c.f. [16] Theorem 2. In that case all sets $B(x,r)$ and $\{y; d(y,x) > r\}$ are open sets in X. We say that a homogeneous space satisfies a density condition (DC) if there are constants $N \geq 2$ and $A > 1$ such that DC

$$\mu B(x,r) \geq A \cdot \mu B(x, r/N), \quad x \in X, r > 0. \qquad (DC)$$

The condition (DC) implies that

$$\mu B(x.r) \leq N^{\beta} \cdot (r/R)^{\beta} \cdot \mu B(x,R), \quad x \in X,\ 0 < r \leq R, \tag{3.5}$$

where $\beta > 0$ only depends on A and N, in analogy with Eq. 3.3.

A kernel $k(x,y)$ is a symmetric, $k(x,y) = k(y,x)$, and lower semicontinuous function $k : X \times X \to [0, \infty]$. We will mostly consider kernels of the type $K \circ \mu(x,y)$, where $K : (0, \infty) \to [0, \infty)$ is non–increasing and continuous and

$$K \circ \mu(x,y) = K(\mu\overline{B}(x, d(x,y)) + \mu\overline{B}(y, d(x,y))), \tag{3.6}$$

c.f. [20]. It is easy to see, assuming (3.4), that $K \circ \mu$ is symmetric and lower semicontinuous and hence is a kernel in our sense. We say that $K(r)$ is doubling if there is $B > 0$, the doubling constant, such that $K(r) \leq B \cdot K(2r)$, $r > 0$. Then also $K \circ \mu$ is doubling in the sense that $d(x,y) \leq 2 \cdot d(x,z)$ implies $K \circ \mu(x,z) \leq C \cdot K \circ \mu(x,y)$,where C depends on the doubling constants for μ and $K(r)$. See Section 4 for more on the various properties of our kernels.

In some of our results below we need the following continuity property for the measure μ,

$$\mu S(x,r) = \mu\{y \in X; d(y,x) = r\} = 0, \text{ for } x \in X,\ r > 0. \tag{3.7}$$

The following lemma is then an easy consequence of Eq. 3.7.

Lemma 3.1 *Let (X, d, μ) be a complete homogeneous space of order γ, $0 < \gamma < 1$, where* Eq. (3.7) *holds. Let S be a closed ball and $a > 0$, then*

$$|\mu B(x, r + h) - \mu B(x, r)| \leq g(h), \ \ \textit{for } x \in S \textit{ and } 0 \leq r \leq a, \tag{3.8}$$

where g is independent of x and r and $g(h) \to 0$, as $h \to 0$.

Proof Define $g(h)$ as the supremum of $|\mu B(x, r + h) - \mu B(x, r)|$ over $x \in S$, $0 \leq r \leq a$ and $r + h \geq 0$. We first show that $g(h) \to 0$ as $h \to 0$, $h > 0$. If this is not the case there are $h_i > 0$, $0 \leq r_i \leq a$ and $x_i \in S$, $i = 1, 2, \ldots$, such that $h_i \to 0$, as $i \to \infty$, and

$$c \leq \mu B(x_i, r_i + h_i) - \mu B(x_i, r_i), \ i = 1, 2, \ldots,$$

for some positive constant c. By compactness we may assume that $x_i \to x_0$ and $r_i \to r_0$, as $i \to \infty$. Let $r_0 > 0$, then the right hand side of the last expression is at most

$$\mu B\left(x_0, r_i + h_i + C \cdot d_i^{\gamma}\right) - \mu B\left(x_0, r_i - C \cdot d_i^{\gamma}\right), \ i = 1, 2, \ldots,$$

by Eq. 3.4, where $d_i = d(x_i, x_0)$. This expression tends to zero as $i \to \infty$ by Eq. 3.6, a contradiction. The cases $r_0 = 0$ and $h < 0$ are proved in the same way. Lemma 3.1 is proved. □

The assumptions of Lemma 3.1 also imply that $K \circ \mu(x, y)$ is continuous for $x \neq y$, which allows us to use continuous potentials as building blocks in the proof of Theorem 2.1.

Lemma 3.2 *Let (X, d, μ) be a homogeneous space of order γ, $0 < \gamma < 1$, where μ satisfies* (3.7) *and let $K \circ \mu$ be a kernel. Then $K \circ \mu(x, y)$ is continuous for $x \neq y$.*

Proof Since $K(r)$ is continuous and $K \circ \mu(x, y)$ is symmetric it suffices to prove that $g(x, y) = \mu \overline{B}(x, d(x, y))$ is continuous. Let $x_0, y_0 \in X$ be fixed, then $g(x, y) \leq \mu \overline{B}(x_0, d(x_0, y_0) + R(x, y))$, where $R(x, y) = C \cdot (d(x, x_0)^{\gamma} + d(y, y_0)^{\gamma})$, and hence $\limsup g(x, y) \leq g(x_0, y_0)$, as $(x, y) \to (x_0, y_0)$. Conversely, $\mu \overline{B}(x_0, d(x_0, y_0) - R(x, y)) \leq g(x, y)$ gives $g(x_0, y_0) \leq \liminf g(x, y)$, as $(x, y) \to (x_0, y_0)$, by Eq. 3.6. This proves Lemma 3.2. □

When (X, d, μ) is the Euclidean space R^n with Lebesgue measure and $K(r) = r^{-\alpha/n}$, $0 < \alpha < n$, we recover the usual Riesz kernel $K \circ \mu(x, y) = c_n \cdot |x - y|^{-\alpha}$, c.f. [3]. We use standard notation for measures and integrals. $M(X)$ denotes the class of Borel measures ν on a X, with finite mass on bounded sets, and $M_+(X)$ is the subclass of positive measures. If E is a Borel set $M(E)$ is the class of $\nu \in M(X)$ that are concentrated on E and analogously for $M_+(E)$. The closed support and total variation of a measure ν in $M(X)$ are denoted by $supp(\nu)$ and $||\nu||_1$, respectively. The class of all continuous functions on a set E is denoted by $C(E)$. Various constants, that may vary from one instance to another, are written $C, C_1, C_2, \ldots$.

4 Some Potential Theory

The potential theory for the Riesz kernel $|x|^{\rho - n}$, $0 < \rho < n$, in R^n has a long history that goes back at least to O. Frostman's thesis [10] from 1935. For its later development see for example N. S. Landkof's classical treatment [15], J. L. Doob [7], L. Carleson [3] and D. R. Adams, L-I Hedberg [1]. B. Fuglede [9] developed a potential theory in a locally compact

Hausdorff space in the 1960's. For the more recent harmonic analysis and potential theory in abstract spaces, see D. Deng, Y Han [6] and A. Björn, J. Björn [2] and the references contained there. Continuity properties of potentials on homogeneous spaces were studied in [11].

Let (X, d, μ) be a complete homogeneous space of order γ, $0 < \gamma < 1$, let $k(x, y)$ be any kernel and define the $k-$potential U_k^ν of a measure $\nu \in M_+(X)$ by Eq. 1.1. We define the $k-$energy $I_k(\nu)$ of ν by

$$I_k(\nu) = \int U_k^\nu(x)\, d\nu(x) = \int\int k(x, y)\, d\nu(x)\, d\nu(y)$$

and the $k-$capacity of a compact set K is defined by

$$C_k(K)^{-1} = \inf\{I_k(\nu);\ \nu \in M_+(K) \text{ and } ||\nu||_1 = 1\}.$$

For an arbitrary set A we define the inner capacity $\underline{C}_k(A)$ and outer capacity $\overline{C}_k(A)$ by $\underline{C}_k(A) = \sup\{C_k(K);\ K \subset A\}$ and $\overline{C}_k(A) = \inf\{\underline{C}_k(G);\ A \subset G\}$, where G and K denote open and compact sets, respectively. A set A is called capacitable if $\underline{C}_k(A) = \overline{C}_k(A)$. This common value is (with a slight abuse of notation) also denoted by $C_k(A)$. Open and compact sets are capacitable and we have $\underline{C}_k(A) = \sup\{C_k(K);\ K \subset A\}$ and $\overline{C}_k(A) = \inf\{C_k(G);\ A \subset G\}$, for arbitrary sets A. A relation which holds except in a set with C_k-capacity zero is said to hold C_k-quasi everywhere (C_k-q.e.) and analogously for $\underline{C}_k$ and $\overline{C}_k$, see [9] Ch. I. Clearly, $C_k(K) > 0$ if and only if there exists a nonzero measure $\nu \in M_+(K)$ with $||\nu||_1 < \infty$ and finite energy $I_k(\nu)$. The following basic existence theorem for the capacitary measure and the capacitary potential of a compact set K is proved in [9], Theorem 2.4.

Theorem 4.1 *Let (X, d, μ) be a complete homogeneous space of order γ, $0 < \gamma < 1$, and let $k(x, y)$ be a kernel. Then for every compact subset K of X with $C_k(K) > 0$ there exists $\nu = \nu_K \in M_+(K)$ such that*

$$I_k(\nu) = C_k(K)^{-1} \quad \text{and} \quad ||\nu||_1 = 1, \tag{4.1}$$

$$U_k^\nu(x) \geq C_k(K)^{-1}, \quad \underline{C}_k - q.e.\, x \in K, \tag{4.2}$$

$$U_k^\nu(x) \leq C_k(K)^{-1}, \quad x \in supp(\nu). \tag{4.3}$$

The measure $\nu = \nu_K$ and the potential U_k^ν in Theorem 4.1 are called the capacitary measure and the capacitary potential for K, respectively. In the case of the kernels $K \circ \mu$, potentials, energies and capacities are denoted by U_K^ν, $I_K(\nu)$ and C_K, respectively.

As in the Euclidean case, we need some basic compatibility between the capacity C_k and the doubling measure μ, see [21], p. 56. More exactly, we want the following to hold:

$$C_k(\{x\}) = 0, \qquad \text{for all } x \in X, \tag{4.4}$$

$$K \text{ a compact subset of } X \text{ and } \mu(K) > 0 \text{ implies that } C_k(K) > 0. \tag{4.5}$$

We show below that Eqs. 4.4 and 4.5 hold for the kernel $K \circ \mu$ if $K(r)$ satisfies

$$\lim_{r \to 0+} K(r) = \infty \quad \text{and} \int_0^1 K(r)\, dr < \infty, \tag{4.6}$$

using the following technical lemma.

Lemma 4.1 *Let (X, d, μ) be a homogeneous space satisfying* (DC) *and let $K \circ \mu$ be a doubling kernel. Then*

$$\int_{d(y,x)\le r} K \circ \mu(x, y)\, d\mu(y) \sim \int_0^{\mu\overline{B}(x,r)} K(t)\, dt,\ x \in X, r > 0,$$

with constants only depending on the doubling constants for μ and $K(r)$ and the constants A, N in the definition of (DC).

Proof Fix any $x \in X$, $r > 0$, let $N \ge 2$ be the constant in (DC) and define circular sets

$$E_\nu = \{y;\, N^{-\nu} \cdot r < d(y, x) \le N^{1-\nu} \cdot r\}, \quad \nu \text{ integer}.$$

Then $\mu E_\nu \sim \mu\overline{B}(x, N^{1-\nu} r)$ and for the integral in the left hand side of Eq. 4.2 we get

$$\sum_{\nu=1}^{\infty} \int_{E_\nu} K \circ \mu(x, y)\, d\mu(y) \sim \sum_{\nu=1}^{\infty} K(\mu\overline{B}(x, N^{-\nu} \cdot r)) \cdot \mu\overline{B}(x, N^{1-\nu} \cdot r).$$

Put $b_\nu = \mu\overline{B}(x, N^{-\nu} \cdot r)$, then for the integral on the right hand side of Eq. 4.2 we have

$$\sum_{\nu=1}^{\infty} \int_{b_\nu}^{b_{\nu-1}} K(t)\, dt \sim \sum_{\nu=1}^{\infty} K(\mu\overline{B}(x, N^{1-\nu} \cdot r)) \cdot (\mu\overline{B}(x, N^{1-\nu} \cdot r)) - \mu\overline{B}(x, N^{-\nu} \cdot r))$$

$$\sim \sum_{\nu=1}^{\infty} K(\mu\overline{B}(x, N^{-\nu} \cdot r)) \cdot \mu\overline{B}(x, N^{1-\nu} \cdot r),$$

with constants depending on the doubling constants of μ and $K(r)$ and the constants A, N in (DC). □

Lemma 4.2 *Let (X, d, μ) be a homogeneous space satisfying* (DC) *and let $K \circ \mu$ be a doubling kernel, where $K(r)$ satisfies* (4.6)*. Then* Eqs. 4.4 *and* 4.5 *hold.*

Proof The Dirac measure δ_x at x has infinite energy by Eq. 4.6, which implies that $C_K(\{x\}) = 0$, $x \in X$. To prove Eq. 4.5 we let K be a compact set with $\mu(K) > 0$ and let ν be the restriction of μ to K. If we show that $U_K\nu$ is bounded it follows that $I_K(\nu)$ is finite and hence $C_K(K) > 0$. It is sufficient to consider the case $K = \overline{B}(x_0, r_0)$. If $d(x, x_0) \le 4 \cdot K^2 \cdot r_0$ we have

$$U_K^\nu(x) \le \int_{\overline{B}(x,5K^3 r_0)} K \circ \mu(x, y)\, d\mu(y) \le C \cdot \int_0^{\mu(K)} K(t)\, dt,$$

by Lemma 4.1. When $d(x, x_0) > 4 \cdot K^2 \cdot r_0$ we get $U_K^\nu(x) \le K(\mu(K)) \cdot \mu(K)$, which completes the proof of Lemma 4.2. □

We will also need the following standard maximum principle, c.f. [3], § III, Theorem 1.

Lemma 4.3 *Let (X, d, μ) be a homogeneous space and let $K \circ \mu$ be a doubling kernel. If F is closed and $\nu \in M_+(F)$ then*

$$\sup_{x\in X} U_K^\nu(x) \le C \cdot \sup_{x\in F} U_K^\nu(x),$$

where C depends on the doubling constants of μ and $K(r)$.

Proof Let $x \in X \setminus F$ and let d be the distance between x and F. Take $x_0 \in F$ such that $d(x, x_0) \leq 2d$ then $2 \cdot d(y, x) \geq d(x, x_0)$, for all $y \in F$, and

$$U_K^\nu(x) = \int K \circ \mu(x, y)\, d\nu(y) \leq C \cdot \int K \circ \mu(x_0, y))\, d\nu(y) = C \cdot U_K^\nu(x_0),$$

where C depends on the doubling constants of μ and $K(r)$. □

The capacitary potential U_k^ν of a compact set K has a lower bound $\underline{C}_k$−q.e. in K (Theorem 4.1). In the classical case this lower bound holds everywhere in K, provided that K satisfies a cone condition, c.f. [3], Sec. III, Theorem 3. We next use the same idea to get a lower bound for $U_k^\nu(x)$ in the interiour points of K also in the present case, see also [1] Proposition 2.6.7.

Lemma 4.4 *Let (X, d, μ) be a homogeneous space of order γ, $0 < \gamma < 1$, satisfying (DC) and let $K \circ \mu$ be a doubling kernel. Let K be a compact set with nonempty interiour K^o and capacitary measure ν. Then*

$$U_K^\nu(x) \geq m \cdot C_K(K)^{-1}, \quad x \in K^o,$$

where $0 < m < 1$ depends on the doubling constants of μ and $K(r)$.

Proof The exceptional set in (4.2) is a F_σ set E such that $C_K(K) = 0$ and hence $\mu(K) = 0$, for all compact subsets of E, by Lemma 4.2. Thus (4.2) holds μ−a.e. by the regularity of μ. Define

$$q(z, x) = \frac{K \circ \mu(z, x)}{\int\limits_{0<d(w,x)<d(z,x)} K \circ \mu(w, x)\, d\mu(w)},$$

for $0 < a < d(z, x) < b$, and $q(z, x) = 0$ elsewhere, c.f. [3], Sec. III. For every $b > 0$ there is $0 < a < b$ such that $1 \leq \int q(z, x)\, d\mu(z) \leq C$, where C only depends on μ and $K(r)$. To see this we note that $\int q(z, x)\, d\mu(z) > 1$, if a is small enough, and $\int q(z, x)\, d\mu(z) \leq C$, if $a = b/N$ and N is the number in (DC).

Assume that $B(x, b) \subset K^o$, then we get

$$C_K(\overline{G})^{-1} \leq \int U_K^\nu(z) \cdot q(z, x)\, d\mu(z) = \int d\nu(y) \int K \circ \mu(z, y) \cdot q(z, x)\, d\mu(z).$$

Denote the inner integral by $I(x, y)$. It is enough to show that $I(x, y) \leq C \cdot K \circ \mu(x, y)$. Fix any $y \neq x$. If $d(z, x) \leq 2Kd(z, y)$ then $K \circ \mu(z, y) \leq C \cdot K \circ \mu(x, y)$ by the doubling property of k. Otherwise, $d(z, y) < d(z, x)/2K$ which implies that $d(x, y)/2K \leq d(z, x) \leq 2Kd(x, y)$ and $I(x, y) \leq C \cdot K \circ \mu(x, y)$, by Lemma 4.1. □

By Lemma 4.4 and 4.5 the capacitary potential of a compact set with positive capacity and interiour points is approximately constant on that set. We will sharpen this result and construct continuous potentials with suitable properties. The key step to do this is the following construction of an approximate identity $q(z, x)$ $x \in X$, acting uniformly in $x \in X$, inspired by the proof of [3], Theorem 3. We will need the following property on $K(r)$,

$$\frac{1}{r} \cdot \int_0^r K(t)\, dt \leq C \cdot K(r), \; r > 0, \tag{4.7}$$

for some positive constant C, c.f. [C], Section IV. The converse inequality holds trivially.

Lemma 4.5 *Let (X, d, μ) be a homogeneous space satisfying (DC) and let $K \circ \mu$ be a doubling kernel, where $K(r)$ satisfies (4.7). Define $q(z, x)$ as in Lemma 4.4 with $a = b/N$, where $b > 0$ and N is the constant in (DC). Then*

$$q(z, x) \sim \mu\overline{B}(x, b)^{-1} \quad \textit{and} \quad \int q(z, x)\, d\mu(z) \sim 1,$$

for $b/N < d(z, x) < b$, with constants that only depend on the doubling constants of μ and $K(r)$.

Proof Let $b > 0$, $b/N < d(z, x) < b$ and define $d = d(z, x)$. Then

$$q(z, x) \sim K \circ \mu(z, x) \cdot \left(\int_0^{\mu\overline{B}(x,d)} K(t)\, dt \right)^{-1} \sim$$

$$\sim K \circ \mu(z, x) \cdot \mu\overline{B}(x, d)^{-1} \cdot K(\mu\overline{B}(x, d))^{-1} \sim \mu\overline{B}(x, b)^{-1}$$

by Lemma 4.1, (4,7) and the doubling properties of μ and $K(r)$. This proves the lemma. □

We are now ready to construct the potentials that will be our building blocks in the proof of Theorem 2.1, c.f. [21], p. 58 and [12], Lemma 1. We do this in three steps. In the first step (Lemma 4.6) we construct a measure ν with arbitrary small total mass $||\nu||_1$ and support in an arbitrary neighbourhood of a compact set K with $C_K(K) = 0$, such that $U_K^\nu(x)$ is approximately constant near K. In the second step (Lemma 4.7) we use a smoothing method built on the function $q(z, x)$ in Lemma 4.5 to get a continuous potential with these properties and construct the approximating potentials in the third step (Lemma 4.8).

Lemma 4.6 *Let (X, d, μ) be a complete homogeneous space of order γ, $0 < \gamma < 1$, and let $K \circ \varphi$ be a doubling kernel. Let $K \subset X$ be a compact set with $C_K(K) = 0$, G an open set containing K, $a > 0$ and $\delta > 0$, Then there is $\nu \in M_+(X)$ such that $||\nu||_1 < \delta$, $supp(\nu) \subset G$, $U_K^\nu(x) \geq a$, $x \in K$ and $U_K^\nu(x) \leq D \cdot a$, $x \in X$, where D only depends on μ and $K(r)$.*

Proof By the outer regularity of C_K we can choose a finite union V of open balls such that $K \subset V$, $\overline{V} \subset G$ and $C_K(\overline{V}) = t$ arbitrarily small. Let ν_1 be the capacitary measure of $\overline{V}$. Then $U_K^{\nu_1}(x) \geq m/t$, $x \in V$, by Lemma 4.4 and $U_K^{\nu_1}(x) \leq C/t$, $x \in X$, by Lemma 4.3. Define $\nu = a \cdot t/m \cdot \nu_1$. Then $U_K^{\nu_1}(x) \geq a$, $x \in \overline{V}$, $U_K^{\nu_1}(x) \leq Ca/m = D \cdot a$, $x \in X$, $supp(\nu) \subset G$ and $||\nu||_1 = at/m < \delta$, provided $0 < t < \delta m/a$. □

Lemma 4.7 *Let (X, d, μ), $K \circ \mu$, K, G, $a > 0$ and $\delta > 0$ be as in Lemma 4.6. Further, assume that μ satisfies (3.7) and that $K(r)$ satisfies (4.7). Then there is $\sigma \in M_+(X)$, absolutely continuous with respect to μ, such that U_K^σ is continuous in X and σ has the same properties as ν in Lemma 4.6, possibly with another constant D.*

Proof Let K, G, a and δ be as in Lemma 4.7. For every $t > 0$ there are open sets W and V such that $K \subset W \subset V \subset G$, $\overline{W} \subset V$, $\overline{V} \subset G$, V is a finite union of open balls and the measure ν constructed for $\overline{V}$ in Lemma 4.6 satisfies $||\nu||_1 < t$. There is $b_0 > 0$ such that $B(w, b_0) \subset \overline{V}$, for all $w \in W$. Define $g(z) = \int q(z, x)\, d\nu(x)$ and $d\sigma(z) = g(z)\, d\mu(z)$, where $q(z, x)$ is the function in the proof of Lemma 4.5 with $0 < b < b_0$. Then σ is absolutely continuous with respect to μ and $||\sigma||_1 = ||g||_1 \leq C \cdot ||\nu||_1 < C \cdot t < \delta$ by Lemma 4.5, if t is small enough. Further, $U_K^\sigma(x) \geq a, x \in K$, and $U_K^\sigma(x) \leq C \cdot D \cdot a = D_1 \cdot a$ in X. The continuity of U_K^σ in X follows from the continuity of the kernel $K \circ \mu$. □

The following lemma constructs the approximating potentials, c.f. [21] and [12], Lemma 2. The standard net of dyadic cubes in R^n will here be replaced by the dyadic sets in [4], Theorem 2. See also [13] and [14] for similar constructions.

Lemma 4.8 *Let (X, d, μ) be a complete homogeneous space of order γ, $0 < \gamma < 1$, satisfying (DC) and let $K \circ \mu$ be a doubling kernel and assume that μ satisfies (3.7). Let K be a compact set contained in the interiour of a closed ball S and let f be a positiv continuous function on S. Then for every open set G containing K and every $\epsilon > 0$ there is a measure $\sigma \in M_+(X)$, absolutely continuous with respect to μ, such that U_K^σ is continuous in X,*

$$(a) \quad U_K^\sigma(x) < f(x), \; x \in S \quad b) \quad U_K^\sigma(x) > f(x) - \epsilon, \; x \in K, \tag{4.8}$$

and $supp(\sigma) \subset G$, $||\sigma||_1 < \epsilon$.

Proof We will use the standard collection of dyadic sets from [4], Theorem 11. For every $k \in Z$, I_k is an index set and $\mathcal{D}_k$ is a collection of open sets (called dyadic cubes in the following) B_α^k, $\alpha \in I_k$, in X with the following properties:

(a) $\mu(X \setminus \bigcup_\alpha B_\alpha^k) = 0$, for every $k \in Z$,
(b) If $l \geq k$ then either $B_\beta^l \subset B_\alpha^k$ or $B_\beta^l \bigcap B_\alpha^k = \phi$,
(c) For each (k, α) and $l < k$ there is a unique β such that $B_\alpha^k \subset B_\beta^l$,
(d) Diameter of B_α^k is at most $C \cdot \delta^k$,
(e) $\bigcup_\alpha \overline{B}_\alpha^k = X$, for every $k \in Z$.

Here $0 < \delta < 1$ and $C > 0$ are constants only depending on K and M in Section 3. Cubes in $\mathcal{D}_k$ are called cubes of generation k. If $B_\beta^{k+1} \subset B_\alpha^k$ we say that B_β^{k+1} is a child of B_α^k and B_α^k is a parent of B_β^{k+1}. Cubes in $\mathcal{D}_l$ are called ancestors or descendents of cubes in $\mathcal{D}_k$, depending on if $l < k$ or $l > k$. Each cube has at least one child and at most a fixed number of childs, depending on K, M and δ. Two cubes B_1 and B_2 in $\mathcal{D}_k$ are called neighbours if $\overline{B}_1 \bigcap \overline{B}_2 \neq \phi$. There exists a positive integer L such that any dyadic cube in $\mathcal{D}_k$ has at most L neighbours.

Let f be a positive and continuous function on S and let $\epsilon > 0$ be arbitrary. Put $\epsilon_1 = \epsilon/M$, where $M = L \cdot D + 4$ and D is the constant in Lemma 4.7. Take $k \in Z$ such that the oscillation of f is less than ϵ_1 for all cubes $\overline{B}_\alpha^k$ that intersect S and choose $\delta > 0$ such that $f(x) \geq \delta$ on all such cubes. Let $\mathcal{B}$ denote the collection of these cubes. We divide $\mathcal{B}$ into three subclasses. Let

$$\mathcal{B}_1 = \left\{ B \in \mathcal{B}; \sup_B f(x) > \epsilon \right\},$$

$$\mathcal{B}_2 = \{B \in \mathcal{B};\; B \notin \mathcal{B}_1 \text{ but } B \text{ has a neighbour in } \mathcal{B}_1\}$$

and $\mathcal{B}_3 = \mathcal{B} \setminus (\mathcal{B}_1 \bigcup \mathcal{B}_2)$. If $\mathcal{B}_1$ is empty then $\sigma = 0$ has the properties in the lemma and we are done. Otherwise, let $\mathcal{B}_1 = \{B_1, B_2, \ldots, B_s\}$ and take $B_i \in \mathcal{B}_1$. Since $C_K(\overline{B}_i \cap K) = 0$ there is $\nu_i \in M_+(X)$, absolutely continuous with respect to μ, such that $U_K^{\nu_i}(x) \geq \epsilon_1$ on

$\overline{B}_i \cap K$, $U_K^{\nu_i}(x) \leq D \cdot \epsilon_1$ in X, $U_K^{\nu_i}$ is continuous in X, $supp(\nu_i) \subset G$ and $||\nu_i||_1$ is arbitrary small. We can further choose ν_i such that

$$U_K^{\nu_i}(x) < \min(\delta/2s, \epsilon_i/s),$$

on every cube in $\mathcal{B}$ that is not a neighbour of B_i, since all such cubes have a positive distance to B_i. Finally we assume that $||\nu_i||_1 < \epsilon/2s$.

Now define $\tau_1 = \sum_{i=1}^{s} \nu_i$ and consider the potential $U_K^{\tau_1}$. Clearly, $U_K^{\tau_1}(x)$ is continuous in X, $supp(\tau_1) \subset G$ and $||\tau_1||_1 < \epsilon/2$. It remains to compare $U_K^{\tau_1}(x)$ and $f(x)$ and we begin with the upper bound for $U_K^{\tau_1}(x)$. If x belongs to a cube in $\mathcal{B}_1$ or $\mathcal{B}_2$ we have

$$f(x) > \epsilon - 2\epsilon_1 = (LD+2)\epsilon_1 \quad \text{and} \quad U_K^{\tau_1}(x) < LD\epsilon_1 + \epsilon_1 = (LD+1)\epsilon_1,$$

since any cube in $\mathcal{B}$ has at most L neighbours in $\mathcal{B}_1$. On the other hand, if x belongs to a cube in $\mathcal{B}_3$ then

$$f(x) > \delta \quad \text{and} \quad U_K^{\tau_1}(x) < s \cdot \delta/2s = \delta/2.$$

We get the upper bound $U_K^{\tau_1}(x) < f(x)$ in both cases.

Now we turn to the lower bound of $U_K^{\tau_i}(x)$ on K. If x belongs to $B_i \cap K$, $1 \leq i \leq s$, then $U_K^{\tau_i}(x) \geq \epsilon_1$ and if x belongs to a cube B in $\mathcal{B} \setminus \mathcal{B}_1$ then $f(x) \leq \epsilon = M \cdot \epsilon_1$, which gives $U_K^{\tau_i}(x) > f(x) - \epsilon \geq \min(f(x) - \epsilon, \epsilon_1)$. Summing up, we have proved that

$$U_K^{\tau_1}(x) < f(x),\ x \in S, \quad \text{and} \quad U_K^{\tau_1}(x) > \min(f(x) - \epsilon, \epsilon_1),\ x \in K.$$

If τ_1 satisfies (4.8) we stop and put $\sigma = \tau_1$. Otherwise we repeat the construction above with $f(x)$ replaced by $f(x) - U_K^{\tau_1}(x)$. This gives $\tau_2 \in M_+(X)$, absolutely continous with respect to μ, such that $supp(\tau_2) \subset G$, $||\tau_2||_1 < \epsilon/4$, $U_K^{\tau_2}(x)$ is continuous in X and

$$U_K^{\tau_2}(x) < f(x) - U_K^{\tau_1}(x), \quad x \in S,$$
$$U_K^{\tau_2}(x) > \min(f(x) - U_K^{\tau_1}(x) - \epsilon, \epsilon_1), \quad x \in K.$$

It follows that

$$U_K^{\tau_1+\tau_2}(x) < f(x),\ x \in S \quad \text{and} \quad U_K^{\tau_1+\tau_2}(x) > \min(f(x) - \epsilon, 2\epsilon_1),\ x \in K.$$

If $\tau_1 + \tau_2$ satisfies (4.8) we stop and put $\sigma = \tau_1 + \tau_2$. Otherwise we repeat the construction such that $supp(\tau_k) \subset G$ and $||\tau_k||_1 < \epsilon/2^k$ in step k. Put $N = \sup\{f(x); x \in S\}$ and let n be the smallest positive integer such that $n \cdot \epsilon_1 > N$. If the construction has not stopped before the $n-$th step we define $\sigma = \sum_{i=1}^{n} \tau_i$. Then $supp(\sigma) \subset G$, $||\sigma||_1 < \epsilon$,

$$U_K^{\sigma}(x) < f(x),\ x \in S \quad \text{and} \quad U_K^{\sigma}(x) > \min(f(x) - \epsilon, n\epsilon_1) = f(x) - \epsilon,\ x \in K,$$

and σ satisfies (4.8) by the construction. □

We conclude this section with a technical lemma, c.f. [12], Lemma 3. It states roughly that the potential of μ_D (the restriction of μ to D) is uniformly small in X, if only $\mu(D)$ is small enough, and makes it possible to construct measures with disjoint support in the proof of Theorem 2.1.

Lemma 4.9 *Let (X, d, μ) be a homogeneous space satisfying (DC) and let $K \circ \mu$ be a doubling kernel. Then for every $\epsilon > 0$ there is $\delta > 0$ such that if D is $\mu-$measurable and $\mu(D) < \delta$ then*

$$\int_D K \circ \mu(x, y)\, d\mu(y) < \epsilon, \text{ for all } x \in X.$$

Proof Let $\mu(D) = q$. For every $q > 0$ there is $Q > 0$ such that $K(r) < 1/\sqrt{q}$, for $r \geq Q$. Let $x \in X$ and consider

$$\int\limits_D K \circ \mu(x, y)\, d\mu(y) = \int\limits_{D \bigcap \overline{B}(x,R)} + \int\limits_{D \setminus \overline{B}(x,R)} = I + II.$$

Define $R = R(q, x) = \sup\{r; \mu\overline{B}(x, r)\} \leq Q\}$, then I is at most a constant times $\int\limits_0^Q K(t)\, dt$, by Lemma 4.1, and $II \leq \int\limits_D K(\overline{B}(x, R))\, d\mu(y) \leq K(Q) \cdot \mu(D) \leq \sqrt{q}$. The lemma now follows, since $q \to 0$ implies $Q \to 0$ and $K(t)$ satisfies (4.6). □

5 Proof of Theorem 2.1

Let K be a compact set with $C_K(K) = 0$. We are going to define the measure $\sigma \in M(X)$ in Theorem 2.1 as a sum $\sigma = \sum_{i=0}^{\infty} \sigma_i$ of measures with disjoint support, where each measure σ_i is constructed as in Lemma 4.8. Let S be a fixed closed ball containing K in its interior. The space $C(S)$ of continuous and real valued functions on S with supremum norm is separable [x.x]. Let $\{\phi_i(x)\}_{i=1}^{\infty}$ be a sequence of positive functions in $C(S)$ such that for every positive continuous function $f(x)$ in $C(S)$ there is a subsequence $\{\phi_{i_j}(x)\}_1^{\infty}$ that satisfies $\phi_{i_j}(x) < f(x)$, $x \in S$, and $\{\phi_{i_j}(x)\}_1^{\infty}$ converges uniformly to $f(x)$ on S, as $j \to \infty$. Define a new sequence $\{f_i\}_1^{\infty}$ by $f_1 = \phi_1$, $f_2 = \phi_2$, $f_3 = \phi_1$, $f_4 = \phi_2$, $f_5 = \phi_3$, $f_6 = \phi_1$ and so on, where every ϕ_i appears infinitely often. Now we use Lemma 4.8 to construct measures $\sigma_i \in M_+(X)$, $i \geq 1$, absolutely continuous with respect to μ, such that

$$supp(\sigma_i) \subset G \quad \text{and} \quad ||\sigma_i||_1 < \epsilon/2^i, \tag{5.1}$$

$$supp(\sigma_i) \cap K = \phi \quad \text{and} \quad supp(\sigma_i) \cap supp(\sigma_j) = \phi,\ i \neq j, \tag{5.2}$$

$$U_K^{\sigma_i}(x) < f_i(x),\ x \in S \quad \text{and} \quad U_K^{\sigma_i}(x) > f_i(x) - 1/2^i,\ x \in K. \tag{5.3}$$

The construction is done step by step. Since Eqs. 5.1 and 5.3 follow directly from Lemma 4.8 it is sufficient to describe how to get Eq. 5.2. Assume that $\sigma_1, \sigma_2, \ldots, \sigma_k$ have been defined such that Eq. 5.2 holds for $1 \leq i, j \leq k$. There is an open set V containing K that does not intersect $supp(\sigma_i)$, $1 \leq i \leq k$. Find σ_0 by Lemma 4.8 such that Eqs. 5.1 and 5.3 hold and $supp(\sigma_0) \subset V$. Put $d\sigma_0 = g_0\, d\mu$, where $g_0 \geq 0$, $supp(g_0) \subset V$ and $\int g_0\, d\mu < \epsilon/2^{k+1}$. Let W be another open set with $K \subset W \subset V$ and let g_1 be the restriction of g_0 to W. Then the potential $U_K^{g_1\, d\mu}(x)$ can be made arbitrarily small in X by taking μW small enough, by Lemma 4.9, since g_1 is bounded. This is possible, since $C_K(K) = 0$ implies $\mu(K) = 0$ by Lemma 4.2. Now define $d\sigma_{k+1} = (g_0 - g_1)d\mu$, then we can make σ_{k+1} satisfy (5.1)–(5.3). We can also choose $(\sigma_i)_1^{\infty}$ such that the distance between $supp(\sigma_i)$ and K tends to zero as $i \to \infty$.

Define $\sigma = \sum_{i=1}^{\infty} \sigma_i$, then σ is absolutely continuous with respect to μ, $supp(\sigma) \subset G$ and $||\sigma_1||_1 < \epsilon$, which proves (a). The potential $U_K^{\sigma}(x)$ is finite outside K by our construction. If $x \in K$ we choose $f(x) = N + 1$ and $f_i(x) > N$ on K. Then $U_K^{\sigma}(x) \geq U_K^{\sigma_i}(x) > N - 2^{-i}$, $x \in K$. Letting $N \to \infty$ gives that $U_K^{\sigma}(x) = \infty$, $x \in K$, which proves (b).

Let $F(x)$ be a positive and upper semicontinuous function in a neighbourhood of K. Then there are positive continuous functions $\{\psi_i\}_1^{\infty}$ such that $F(x) = \sum_{i=1}^{\infty} \psi_i(x)$. Choose

$f_{n_1}(x)$ such that $\psi_1(x) - 2^{-1} < f_{n_1}(x) < \psi_1(x), x \in S$, and consider the potential $U_K^{\sigma_{n_1}}(x)$. Put $g_1(x) = \psi_1(x)$ and $g_2(x) = \psi_1(x) + \psi_2(x) - U_K^{\sigma_{n_1}}(x)$. Now choose $f_{n_2}(x)$, $n_2 > n_1$, such that

$$g_2(x) - 2^{-2} < f_{n_2}(x) < g_2(x), \quad x \in S,$$

and consider the potential $U_K^{\sigma_{n_2}}(x)$. Put $g_3(x) = \psi_1(x) + \psi_2(x) + \psi_3(x) - U_K^{\sigma_{n_1}}(x) - U_K^{\sigma_{n_2}}(x)$ and continue this process. After k steps we have measures $\sigma_{n_1}, \sigma_{n_2}, \ldots, \sigma_{n_k}$ such that

$$U_K^{\sigma_{n_1}}(x) + \cdots + U_K^{\sigma_{n_k}}(x) < \psi_1(x) + \cdots + \psi_k(x), \; x \in S, \tag{5.4}$$

and

$$\psi_1(x) + \cdots + \psi_k(x) - 2^{-k} - 2^{-n_k} < U_K^{\sigma_{n_1}}(x) + \cdots + U_K^{\sigma_{n_k}}(x), \; x \in K. \tag{5.5}$$

Define $\sigma_f = \sum_{i=1}^{\infty} \sigma_{n_i}$, then σ_f is absolutely continuous with respect to μ and $U_K^{\sigma_f}(x) = \sum_{i=1}^{\infty} U_K^{\sigma_{n_i}}(x)$. It follows from Eqs. 5.4 and 5.5 and the definition of $F(x)$ that $U_K^{\sigma_f}(x) \leq F(x)$, $x \in S$, and $U_K^{\sigma_f}(x) = f(x)$, $x \in K$. Also, $U_K^{\sigma_f}(x)$ is continuous outside K by the construction of σ. It remains to prove that $U_K^{\sigma_f}(x)$ is continuous in X if $f \in C(K)$. Let $x_0 \in K$ then

$$\liminf_{x \to x_0} U_K^{\sigma_f}(x) \geq U_K^{\sigma_f}(x_0) = F(x_0) = f(x_0),$$

since $U_K^{\sigma_f}$ is lower semicontinuous, and

$$\limsup_{x \to x_0} U_K^{\sigma_f}(x) \leq \limsup_{x \to x_0} F(x) = F(x_0) = f(x_0).$$

Hence $U_K^{\sigma_f}$ is continuous also in K. This proves (c) and completes the proof of Theorem 2.1.

6 Proof of Theorem 2.2

We show that if a compact set K has positive capacity $C_K(K)$, then there is $f \in C(K)$ that is not the restriction to K of a continuous potential U_K^{ν} for any $\nu \in M_+(X)$. Our proof follows the idea in [21], Theorem 2. The main part of the proof is done in Lemma 6.1 and Lemma 6.2, that correspond to Lemma 2 and Lemma 3 in [21], respectively. The proof of Lemma 6.1 is fairly straight forward. To prove Lemma 6.2 we use recent results of T. Hyvönen, H. Martikainen [14] and T. Hyvönen, H. Kairema [13] for classes of dyadic nets in homogeneous spaces.

Lemma 6.1 *Let $K(r)$ and $K_0(r)$, $r > 0$, be positive, continuous and nonincreasing functions such that $\lim_{r \to 0} K(r) = \lim_{r \to 0} K_0(r) = \infty$ and*

$$\lim_{r \to 0} K_0(r)/K(r) = \infty$$

and let S be a closed ball with radius R. Then there exists a positive function $t(r)$, $r > 0$, with $\lim_{t \to 0} t(r) = 0$, depending on $K(r)$, $K_0(r)$ and μ, such that if $\nu \in M_+(X)$ has support in S, $||\nu||_1 \leq M_1$ and $U_{K_0}^{\nu}(x_i) \leq M_2$, $x_i \in S$, $i = 1, 2$, then

$$|U_K^{\nu}(x_1) - U_K^{\nu}(x_2)| \leq (M_1 + M_2) \cdot t(d(x_1, x_2)).$$

Proof Let $x_1, x_2 \in S$ be as in the lemma and put $r_0 = d(x_1, x_2)$. Define $E = \overline{B}(x_1, r_0 + r_1) \cup \overline{B}(x_2, r_0 + r_1)$, where $r_1 > 0$ will be defined below. Then

$$U_K^\nu(x_1) - U_K^\nu(x_2) = \int_S \big(K \circ \mu(x_1, y) - K \circ \mu(x_2, y)\big)\, d\nu(y) = \int_E + \int_{S \setminus E} = I + II.$$

Define $t_1(r) = K(r)/K_0(r)$, $r > 0$, then $t_1(r) \to 0$, as $r \to 0$, and

$$|I| \leq \int_E \big(K \circ \mu(x_1, y) + K \circ \mu(x_2, y)\big)\, d\nu(y) \leq$$

$$\leq M_2 \cdot \sup_{y \in E} \big(t_1(\varphi(x_1, y) + t_1(\varphi(x_2, y)\big) \leq 2 \cdot M_2 \cdot \sup_{0 \leq r \leq 2Kg(r_0 + r_1)} t_1(r),$$

where $\varphi(x, y) = \mu\overline{B}(x, d(x, y)) + \mu\overline{B}(y, d(x, y))$ and g is the function in Lemma 3.1.

Next we turn to the term II and note that, by Eq. 3.4 and set inclusion, $\varphi(x_1, y)$ is greater than or equal to

$$\mu\overline{B}(x_2, d(x_2, y) - C_1 \cdot d(x_1, x_2)^\gamma) + \mu\overline{B}(y, d(x_2, y) - C_1 \cdot d(x_1, x_2)^\gamma)$$
$$\geq \varphi(x_2, y) - 2 \cdot g(C_1 \cdot d(x_1, x_2)^\gamma),$$

which gives

$$II \leq \int_{S \setminus E} \big[K\big(\varphi(x_2, y) - 2 \cdot g(C_1 \cdot d(x_1, x_2))^\gamma\big) - K\big(\varphi(x_2, y)\big)\big]\, d\nu(y).$$

Now $\varphi(x_2, y) \geq C_2 \cdot \mu(S) \cdot R^{-\alpha} \cdot (r_0 + r_1)^\alpha \geq C_2 \cdot \mu(S) \cdot R^{-\alpha} \cdot r_1^\alpha$ by Eq. 3.3 and

$$\varphi(x_2, y) - 2 \cdot g(C_1 \cdot d(x_1, x_2))^\gamma \geq \frac{1}{2} \cdot C_2 \cdot \mu(S) \cdot R^{-\alpha} \cdot r_1^\alpha,$$

since we, without loss of generality, may assume that $r_1 >> r_0$. As in [21], p. 63, there is a function $t_2(r)$, $r > 0$, such that $t_2(r) \to 0$, as $r \to 0$, and

$$K(r) - K(r + \rho) \leq t_2(\eta), \text{ for } r \geq t_2(\eta) \text{ and } 0 \leq \rho \leq \eta.$$

Now put

$$\eta = 2 \cdot g(C_1 \cdot r_0^\gamma) \text{ and } \frac{1}{2} \cdot C_2 \cdot \mu(S) \cdot R^{-\alpha} \cdot r_1^\alpha = t_2(\eta),$$

which defines $r_1 = t_3(r_0)$ such that $t_3(r) \to 0$, as $r \to 0$, then $II \leq M_1 \cdot \frac{1}{2} \cdot C_2 \cdot \mu(S) \cdot R^{-\alpha} \cdot t_3(r_0)^\alpha$. Interchanging the roles of x_1 and x_2 combined with the estimate above for I gives

$$|U_K^\nu(x_1) - U_K^\nu(x_2)| \leq$$

$$\leq 4M_2 \cdot \sup_{0 \leq r \leq r_0 + t_3(r_0)} t_1(r) + M_1 \cdot \frac{1}{2} \cdot C_2 \cdot \mu(S) \cdot R^{-\alpha} \cdot t_3(r_0)^\alpha \leq (M_1 + M_2) \cdot t(r_0),$$

where $t(r) \to 0$, as $r \to 0$. Clearly, $t(r)$ only depends on $K(r)$, $K_0(r)$, μ and S. Lemma 6.1 is proved. □

In our last lemma we construct, given a nonatomic measure ν and a function $t^\star(r) \to 0$, as $r \to 0$, a continuous function with a bad continuity property outside every set with sufficiently small ν−measure. The novelty in the proof is the construction in [14] and [13] of dyadic nets $\mathcal{D}$ in a homogeneous space having the property that, for a given measure ν, $\nu(\partial B) = 0$ for every B in $\mathcal{D}$, [13], Corollary 6.4. The construction of such dyadic nets is done with a probabilistic method as well as with a deterministic method. The corresponding problem in [21] Lemma 3 was solved by induction over the dimension of the space R^n, which is clearly not applicable here.

Lemma 6.2 *Let (X, d, μ) be a homogeneous space of order γ, $0 < \gamma < 1$, and let $\nu \in M_+(X)$ be a nonatomic measure with support in a compact K. Let $t^\star(r)$, $r > 0$, be a positive and nondecreasing function such that $t^\star(r) \to 0$, as $r \to 0$. Then there is $f \in C(K)$ such that for every sufficiently small $\epsilon > 0$ and every Borel set E with $\nu(E) < \epsilon$ there are points x_1 and x_2 in $K \setminus E$ with $d(x_1, x_2)$ arbitrary small and such that*

$$|f(x_1) - f(x_2)| \geq t^\star(d(x_1, x_2)).$$

Proof Let $\nu \in M_+(X)$ have support in a compact set K contained in the interiour of a closed ball S. By [13], Corollary 6.4 there exists a dyadic net $\mathcal{D}$ in X, with the properties (a)–(e) in the proof of Lemma 4.8, such that $\nu(\partial B) = 0$ for every B in $\mathcal{D}$. Without loss of generality we may assume that every dyadic cube in $\mathcal{D}_k$ contains at least two cubes from $\mathcal{D}_{k+1}$. We follow the proof in [21], Lemma 3 and use two sequences $\{a_i\}_1^\infty$ and $\{b_i\}_1^\infty$ of positive numbers to construct the function $f(x) = \sum_{i=1}^{\infty} f_i(x)$ in the lemma. Let $i \geq 1$ be fixed and cover K with closed dyadic cubes from $\mathcal{D}$ with diameter less than a_i. Let $\overline{B}_{ij}$, $1 \leq j \leq n_i$, denote the cubes with $\nu(K \bigcup \overline{B}_{ij}) > 0$ and fix any such cube $\overline{B}_{ij}$. We now divide $\overline{B}_{ij}$ into a class $\mathcal{E} = \{E_1, E_2, \ldots, E_p\}$ of closed dyadic subcubes, of different sizes, that covers $K \bigcap \overline{B}_{ij}$ and satisfies $0 < \nu(K \bigcap E_l) < \nu(\overline{B}_{ij})/4$, $1 \leq l \leq p$. This is possible since B_{ij} has infinitely many childs and ν has no atoms. Divide $\mathcal{E}$ into two disjoint subclasses $\mathcal{E}_1$ and $\mathcal{E}_2$ such that the sum $\sum \nu(E_l)$ over each of the classes $\mathcal{E}_1$ and $\mathcal{E}_2$ lies between $\nu(\overline{B}_{ij})/4$ and $3\nu(\overline{B}_{ij})/4$. For each cube E_l there is an open set Δ_l containing ∂E_l such that $E_l \setminus \Delta_l$ has a positive distance to ∂E_l and

$$\nu(K \cap B_{ij})/4 < \sum_{\mathcal{E}_i} \nu(K \bigcap (E_l \setminus \Delta_l)) < 3 \cdot \nu(K \bigcap B_{ij})/4, \ i = 1, 2.$$

If we define $\Delta_{ij} = \bigcup_l \Delta_l$ and $\Delta = \bigcup_{i,j} \Delta_l$ we can choose Δ_l such that $\nu(\Delta) < \eta$, for any given $\eta > 0$.

Now define $f_i(x)$ on B_{ij}, $1 \leq j \leq n_i$, by $f_i(x) = 2 \cdot t^\star(a_i)$, $x \in E_l \setminus \Delta_l$, when $E_l \in \mathcal{E}_1$ and $f_i(x) = 0$, $x \in E_l \setminus \Delta_l$, when $E_l \in \mathcal{E}_2$, and extend $f_i(x)$ to a continuous function in S such that $0 \leq f_i(x) \leq 2 \cdot t^\star(a_i)$, $x \in S$. It remains to prove that $f(x) = \sum f_i(x)$ has the desired properties. Clearly, f is continuous of S, if the sequence $(a_i)_1^\infty$ is choosen small enough. Define b_{ij} as the minimal distance between $E_k \setminus \Delta_k$ and $\neq l$, and $b_i = \min\{b_{ij}, 1 \leq j \leq n_i\}$. Let $x_i \in E_l \setminus \Delta$, for $E_l \in \mathcal{E}_1$ and $x_i' \in E_l \setminus \Delta$, for $E_l \in \mathcal{E}_2$, then

$$|f_i(x_i) - f_i(y_i)| = 2 \cdot t^\star(a_i) \geq 2 \cdot t^\star(d(x_i, y_i)) \quad \text{and} \quad b_i \leq d(x_i, y_i) \leq a_i.$$

Now assume, without loss of generality, that $a_{i+1} < b_i$, $i = 1, 2, \ldots$ to get $f_j(x_i) - f_j(y_i) = 0$, $j < i$, and $|f_j(x_i) - f_j(y_i)| \leq 4 \cdot t^\star(a_i)$, $j > i$. This gives

$$|f(x_i) - f(y_i)| \geq t^\star(d(x_i, y_i)) \cdot \left(2 - 4 \cdot t^\star(b_i)^{-1} \cdot \sum_{j>i} t^\star(a_j)\right) \geq t^\star(d(x_i, y_i)),$$

if only $\sum_{j>i} t^\star(a_j) < t^\star(b_i)/4$, $i = 1, 2, \ldots$, which holds if $(a_i)_1^\infty$ is choosen small enough.

We finish the proof by showing that we can find such pairs x_i, y_i, with $d(x_i, y_i)$ arbitraly small, outside any Borel set with sufficiently small measure. More exactly we show that if $\epsilon > 0$ is small enough then for any Borel set $E \subset K$ with $\nu(E) < \epsilon$ and for every $i = 1, 2, \ldots$ there are x_i and y_i in $\left(\bigcup_{j=1}^{n_i} \overline{B}_{ij} \bigcap K\right) \setminus (\Delta \bigcup E)$ belonging to different cubes

$E_k \in \mathcal{E}_1$ and $E_l \in \mathcal{E}_2$. This implies $|f(x_i) - f(y_i)| \geq t^\star(d(x_i, y_i))$ and the lemma follows. Assume to the contrary that there is $i \geq 1$ such that for every $1 \leq j \leq n_i$ the set $(\overline{B}_{ij} \bigcap K) \setminus (\Delta \bigcup E)$ does not contain such a pair x_i, y_i. Then either all sets $K \bigcap (E_k \setminus \Delta_k)$, $E_k \in \mathcal{E}_1$ or all sets $K \bigcap (E_l \setminus \Delta_l)$, $E_l \in \mathcal{E}_2$, are contained in $E \bigcup \Delta$. In the first case we get for each such j

$$\frac{1}{4} \cdot \nu(\overline{B}_{ij} \bigcap K) \leq \sum_{E_k \in \overline{B}_{ij}} \nu(K \bigcap (E_k \setminus \Delta_k)) < \nu(\overline{B}_{ij} \bigcap (E \bigcup \Delta)).$$

Summing over $1 \leq j \leq n_i$ then gives $\nu(K)/4 < \nu(E \bigcup \Delta) < \epsilon + \eta$, which is a contradiction if $\epsilon < \nu(K)/4$ and η is small enough. The second case is handled in the same way. The proof of Lemma 6.2 is complete. □

Proof of Theorem 2.2 Let K be a compact set with $C_K(K) > 0$, then there is $\nu \in M_+(K)$ such that $||\nu||_1 = 1$ and

$$I_K(\nu) = \int \int K \circ \mu(x, y)\, d\nu(x)\, d\nu(y) < \infty.$$

A standard argument proves that there is $K_0(r)$, $r > 0$, with the same properties as $K(r)$, such that $I_{K_0}(\nu) < \infty$ and $K_0(r)/K(r) \to \infty$, as $r \to 0$. Then also $C_{K_0}(K) > 0$. Let $\sigma \in M_+(X)$ be a measure with $||\sigma||_1 = M_1$ finite, then $U^\sigma_{K_0}(x) < \infty$ except in a set of $\underline{C}_{K_0}-$capacity zero. It follows that $U^\sigma_{K_0}(x)$ is finite $\nu-$a.e., since $I_{K_0}(\nu) < \infty$, c.f. [15] Ch. III. Let $t(r)$, $r > 0$, be the function constructed in Lemma 6.1 and choose $t^\star(r)$, $r > 0$, such that $t^\star(r)/t(r) \to \infty$, as $r \to 0$. Define $E = \{x; U^\sigma_{K_0}(x) > M_2\}$, then by Lemma 6.1

$$|U^\sigma_K(x) - U^\sigma_K(y)| \leq (M_1 + M_2) \cdot t(d(x, y)),$$

for x, y in $K \setminus E$. Let $\epsilon > 0$ and choose M_2 such that $\nu(E) < \epsilon$. Then there are arbitrarily close points x and y in $K \setminus E$ such that

$$|f(x) - f(y)| \geq t^\star(d(x, y)),$$

provided only $\epsilon > 0$ is small enough, by Lemma 6.2. We conclude that there is no such measure σ for which $f(x)$ and $U^\sigma_K(x)$ coincide everywhere on K and Theorem 2.2 is proved. □

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