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A Class of Second-Order Linear Elliptic Equations with Drift: Renormalized Solutions, Uniqueness and Homogenization

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Abstract In this paper a class of N-dimensional second-order linear elliptic equations with a drift is studied. When the drift belongs to L^2 the existence of a renormalized solution is proved. There is also uniqueness in the class of the renormalized solutions modulo L^∞, but the uniqueness is violated when the drift equation is regarded in the distributions sense. Then, considering a sequence of oscillating drifts which converges weakly in L^2 to a limit drift in L^q, with $q > N$, the homogenization process makes appear an extra zero-order term involving a non-negative Radon measure which does not load the zero capacity sets. This extends the homogenization result obtained in [3] by relaxing the equi-integrability of the drifts in L^2.

Keywords Second-order linear elliptic equation · Drift · Renormalized solution · Homogenization

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1 Introduction

This paper deals with the drift equation

$$\begin{cases} u \in H_0^1(\Omega) \\ -\Delta u + b\cdot\nabla u + \operatorname{div}(b\,u) = f \ \text{ in } \Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, the drift b is a function valued in $\mathbb{R}^N$ and the right-hand side f is in $H^{-1}(\Omega)$. As shown in [3], for $N = 3$, Eq. 1.1 can be regarded as the scalar version of the Stokes equation

$$\begin{cases} U \in H_0^1(\Omega)^3 \\ \operatorname{div} U = 0 & \text{in } \Omega \\ -\Delta U + (\operatorname{curl} B)\times U + \nabla P = F & \text{in } \Omega, \end{cases} \tag{1.2}$$

which was studied by Tartar [14, 15] in the context of homogenization in hydrodynamics. Indeed, noting that

$$(\operatorname{curl} B)\times U = (DU)^T B + \operatorname{Div}(B\otimes U) - \nabla(B\cdot U),$$

and inserting the last term in the pressure term of Eq. 1.2, the analogy between the two drift terms:

$$(DU)^T B + \operatorname{Div}(B\otimes U) \ \text{ in (1.2)} \quad \text{and} \quad b\cdot\nabla u + \operatorname{div}(b\,u) \ \text{ in (1.1)},$$

is clear.

It is well known that Eq. 1.1 admits a unique solution when the drift b belongs to $L^\infty(\Omega)^N$ (see, *e.g.*, [13]). The question is much more delicate when b is only in $L^2(\Omega)^N$. In this case Eq. 1.1 is comparable to a second-order elliptic equation with a right-hand side in $L^1(\Omega)$. This naturally suggests to study the existence and the uniqueness of solutions to Eq. 1.1 in the framework of *renormalized solutions* independently introduced by P.-L. Lions, F. Murat [9, 11], and P. Bénilan *et al.* [1]. In this perspective, we show (see Theorem 2.1) the existence of a renormalized solution u to the Eq. 1.1 for any $b \in L^2(\Omega)^N$. In particular, any renormalized solution u to Eq. 1.1 satisfies the inequalities

$$\lim_{k\to\infty}\left(k\int_{\{k<u\}} b\cdot\nabla u\,dx\right) \geq 0, \qquad \lim_{k\to\infty}\left(k\int_{\{u<-k\}} b\cdot\nabla u\,dx\right) \leq 0, \tag{1.3}$$

involving explicitly the drift term. The question of the uniqueness is much more intricate as for the nonlinear elliptic equations (see, *e.g.*, [7]). On the one hand, we show (see Theorem 2.5) that two renormalized solutions of Eq. 1.1 are equal if their difference is in $L^\infty(\Omega)$. On the other hand, we construct (see Theorem 2.6) a drift $b \in L^2(\Omega)^N$ and a right-hand side $f \in H^{-1}(\Omega)$, such that there are at least two solutions of Eq. 1.1 in the distributions sense. The first one is a non-explicit renormalized solution of Eq. 1.1, while the second one is an explicit solution of Eq. 1.1 which does not satisfy property Eq. 1.3 contrary to the renormalized solutions. This example is new and is quite different from Serrin's counter-example [12] for second-order linear elliptic equations with anisotropic discontinuous coefficients, but without first-order term. Here, the anisotropy and the discontinuity are given by the drift term.

The second part of the paper is devoted to the homogenization of the drift equation

$$\begin{cases} u_n \in H_0^1(\Omega) \\ -\Delta u_n + b_n\cdot\nabla u_n + \operatorname{div}(b_n u_n) = f_n \ \text{ in } \Omega, \end{cases} \tag{1.4}$$

where b_n is a sequence in $L^2(\Omega)^N$ which converges weakly to some b in $L^2(\Omega)^N$, and f_n converges strongly to some f in $H^{-1}(\Omega$. This problem has been studied in [3] assuming that b_n is equi-integrable in $L^2(\Omega)^N$, and in [2] assuming that b_n is a periodically oscillating sequence. In both cases the homogenized equation is a drift equation with an extra zero-order term of type

$$\begin{cases} u \in H_0^1(\Omega) \\ -\Delta u + b \cdot \nabla u + \operatorname{div}(b\,u) + \mu\,u = f \text{ in } \Omega, \end{cases} \tag{1.5}$$

where μ is a non-negative Radon measure independent of the right-hand side f. In the present case, assuming that the limit drift b is in $L^q(\Omega)^N$ for some $q > N$, with $N \in \{2, 3\}$, we obtain (see Theorem 3.1) the same homogenized Eq. 1.5 in which the measure μ is absolutely continuous with respect to the default measure induced by the limit of $|b_n - b|^2$. The general case with $b \in L^2(\Omega)^N$ and $N \geq 2$, remains open.

Following the famous Tartar method [14] of the oscillating test functions, and its adaptation by Dal Maso and Garroni [5] to perforated domains, the starting point of the proof is the use of the solution w_n to the dual equation of Eq. 1.4 with right-hand side 1. However, due to the drift term and contrary to [5], it is not immediate that the limit w of w_n satisfies the inequality

$$-\Delta w + b \cdot \nabla w + \operatorname{div}(b\,w) \leq 1 \quad \text{in } \Omega,$$

which is a crucial step to build the measure μ. To this end, we first obtain an inequality satisfied by w^2, and we need the maximum principle of Lemma 3.3. Because of Serrin's counter-example [12], this lemma turns out to be false if the Laplace operator is replaced by a second-order operator with discontinuous coefficients, except in dimension two with a Sobolev exponent close to 2 (see the final Remark 3.4). Note that the restriction to the Laplace operator is not present in the approach of [5], which shows that the drift problem is of different nature. Finally, to prove that μ is absolutely continuous with respect to the limit of $|b_n - b|^2$, we need to introduce a completely different battery of test functions, with a double index k, n, which are close to 1, considering the drift Eq. 1.4 penalized by a zero-order term weighted by large $k > 0$.

Therefore, taking into account the notion of renormalized solution combined with the question of uniqueness, and the restriction on the limit drift arising in the homogenization problem, the present contribution shows the delicate analysis in terms of well-posedness and homogenization, for a specific class of second-order linear elliptic equations with a first-order term.

Notations

- For any $u \in \mathbb{R}$, $u^{\pm} := \pm \max(\pm u, 0)$.
- For $k > 0$, T_k denotes the truncation at height k defined by $T_k(t) := \max(-k, \min(k, t))$, for $t \in \mathbb{R}$.
- For any $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, $\langle f, u \rangle$ simply denotes the duality $\langle f, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$.

2 Renormalized Solution of the Drift Equation

2.1 Existence of a Renormalized Solution

Along this section Ω is a bounded regular open set of $\mathbb{R}^N$, for $N \geq 2$, b_n is sequence in $L^\infty(\Omega)^N$ which converges strongly to b in $L^2(\Omega)^N$, and f_n is a sequence in $H^{-1}(\Omega)$ which strongly converges to f in $H^{-1}(\Omega)$.

Theorem 2.1 *The solution u_n to the drift Eq. 1.4 converges weakly in $H^1_0(\Omega)$, up to a subsequence, to a renormalized solution u of the problem*

$$\begin{cases} u \in H^1_0(\Omega) \\ -\Delta u + b \cdot \nabla u + div\,(b\,u) = f \ \ in\, \Omega, \end{cases} \tag{2.1}$$

in the following sense:

$$\begin{cases} u \in H^1_0(\Omega) \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega b \cdot (v\nabla u - u\nabla v)\, dx = \langle f, v\rangle, \\ \forall\, v \in H^1_0(\Omega) \cap L^\infty(\Omega), \ \ with \ \ u\, b \cdot \nabla v \in L^1(\Omega), \end{cases} \tag{2.2}$$

$$\int_\Omega |\nabla u^+|^2\, dx \leq \langle f, u^+\rangle \quad and \quad \int_\Omega |\nabla u^-|^2\, dx \leq -\,\langle f, u^-\,\rangle. \tag{2.3}$$

Moreover, any renormalized solution u of Eq. 2.2 satisfies

$$\lim_{k\to\infty} \left(k \int_{\{k<u\}} b \cdot \nabla u\, dx \right) \geq 0, \qquad \lim_{k\to\infty} \left(k \int_{\{u<-k\}} b \cdot \nabla u\, dx \right) \leq 0 \tag{2.4}$$

Remark 2.2 Note that, since a solution of Eq. 2.2 clearly exists if $b \in L^\infty(\Omega)^N$, the existence of a solution of Eq. 2.2 for any $b \in L^2(\Omega)^N$ follows from Theorem 2.1 taking a sequence $b_n \in L^\infty(\Omega)^N$ which converges strongly to b in $L^2(\Omega)^N$. Moreover, if $b \in L^q(\Omega)^2$ with $q > 2$ if $N = 2$, or $b \in L^N(\Omega)^N$ if $N > 2$, then there exists a unique renormalized solution of the problem Eq. 2.2 which is also the unique solution of Eq. 2.1 in the distributions sense (see [3], Theorem 2.4 ii)).

Proof of Theorem 2.1 The proof uses truncation techniques for renormalized solutions (see, *e.g.*, [7]), which are adapted to the drift Eq. 1.4.

First Step. Putting u_n in Eq. 1.4 we have

$$\int_\Omega |\nabla u_n|^2\, dx = \langle f, u_n\rangle, \tag{2.5}$$

which implies that u_n is bounded in $H^1_0(\Omega)$. Hence, we can assume that, up to a subsequence, u_n converges weakly to some function u in $H^1_0(\Omega)$.

Denote

$$R_m(s) := \begin{cases} 1 & \text{if } |s| \leq m, \\ 2 - \dfrac{|s|}{m} & \text{if } m < |s| < 2m, \\ 0 & \text{if } 2m \leq |s|. \end{cases}$$

Consider a function $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $u\, b \cdot \nabla v \in L^1(\Omega)$. Taking the function $R_m(u_n)\, v$ as test function in Eq. 1.4, we have

$$\begin{aligned}&\int_\Omega \nabla u_n \cdot \nabla v\, R_m(u_n)\, dx - \frac{1}{m} \int_{\{m<|u_n|<2m\}} \operatorname{sgn}(u_n)\, |\nabla u_n|^2\, v\, dx + \int_\Omega b_n \cdot \nabla u_n\, v\, R_m(u_n)\, dx \\ &+ \frac{1}{m} \int_{\{m<|u_n|<2m\}} u_n\, b_n \cdot \nabla u_n \operatorname{sgn}(u_n)\, v\, dx - \int_\Omega u_n\, b_n \cdot \nabla v\, R_m(u_n)\, dx = \langle f, R_m(u_n) \rangle.\end{aligned} \tag{2.6}$$

Using the weak convergence of u_n in $H^1(\Omega)$, the Rellich-Kondrachov compactness theorem, the boundedness of $u_n R_m(u_n)$ in $L^\infty(\Omega)$, and the strong convergences of b_n in $L^2(\Omega)^N$ and f_n in $H^{-1}(\Omega)$, it follows that

$$\int_\Omega \nabla u_n \cdot \nabla v\, R_m(u_n)\, dx \to \int_\Omega \nabla u \cdot \nabla v\, R_m(u)\, dx,$$

$$\int_\Omega b_n \cdot \nabla u_n\, v\, R_m(u_n)\, dx \to \int_\Omega b \cdot \nabla u\, v\, R_m(u)\, dx,$$

$$\int_\Omega u_n\, b_n \cdot \nabla v\, R_m(u_n)\, dx \to \int_\Omega u\, b \cdot \nabla v\, R_m(u)\, dx,$$

$$\langle f, R_m(u_n) \rangle \to \langle f, R_m(u) \rangle.$$

On the other hand, using that u_n is bounded in $H_0^1(\Omega)$, b_n converges strongly in $L^2(\Omega)^N$ and $v \in L^\infty(\Omega)$, we have

$$\left| \frac{1}{m} \int_{\{m<|u_n|<2m\}} \operatorname{sgn}(u_n)\, |\nabla u_n|^2\, v\, dx \right| \le \frac{C}{m}, \quad \forall n \in \mathbb{N},$$

$$\begin{aligned}\limsup_{n\to\infty} \left| \frac{1}{m} \int_{\{m<|u_n|<2m\}} u_n\, b_n \cdot \nabla u_n \operatorname{sgn}(u_n)\, v\, dx \right| &\le C \limsup_{n\to\infty} \int_{\{m<|u_n|<2m\}} |b_n||\nabla u_n|\, dx \\ &\le C \left(\int_{\{m\le|u|\le 2m\}} |b|^2 dx \right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, passing to the limit as $n \to \infty$ in Eq. 2.6 we get that

$$\begin{aligned}&\left| \int_\Omega \nabla u \cdot \nabla v\, R_m(u)\, dx + \int_\Omega v\, b \cdot \nabla u\, R_m(u)\, dx - \int_\Omega u\, b \cdot \nabla v\, R_m(u)\, dx - \langle f, R_m(u) \rangle \right| \\ &\le \frac{C}{m} + C \left(\int_{\{m\le|u|\le 2m\}} |b|^2 dx \right)^{\frac{1}{2}}.\end{aligned}$$

Since $v \in L^\infty(\Omega)$ and $u\, b \cdot \nabla v \in L^1(\Omega)$, we can pass to the limit in this inequality as $m \to \infty$ to deduce that u satisfies Eq. 2.2.

On the other hand, putting u_n^+ and u_n^- as test functions in Eq. 1.1 we get that

$$\int_\Omega |\nabla u_n^+|^2\, dx = \langle f_n, u_n^+ \rangle \quad \text{and} \quad \int_\Omega |\nabla u_n^-|^2\, dx = -\langle f_n, u_n^- \rangle.$$

Hence, passing to the liminf in these equalities and using the lower semi-continuity of the L^2-norm of the gradient in $H^1(\Omega)$, we deduce that u satisfies the inequalities Eq. 2.3.

Second step. Let us prove that any solution u of Eq. 2.2 satisfies inequality Eq. 2.4. Let $k > 0$. Putting the admissible test function $v = T_k(u)^+$ in Eq. 2.2, we have

$$\int_\Omega |\nabla T_k(u)^+|^2 + k \int_{\{k<u\}} b \cdot \nabla u\, dx = \langle f, T_k(u)^+ \rangle$$

Then, using the strong convergence of $T_k(u)^+$ to u^+ in $H_0^1(\Omega)$ and the first inequality in Eq. 2.3 we get that

$$\exists \lim_{k\to\infty}\left(k\int_{\{k<u\}} b\cdot\nabla u\,dx\right)=\langle f,u^+\rangle-\int_\Omega|\nabla u^+|^2dx\ge 0.$$

This proves the first inequality in Eq. 2.4. The second one can be proved analogously using $-T_k(u)^-$ as test function in Eq. 2.2. □

Corollary 2.3 *Let b_n be a sequence in $L^\infty(\Omega)^N$ which converges strongly to b in $L^2(\Omega)^N$, and let f_n be a sequence in $H^{-1}(\Omega)$ which converges strongly to f in $H^{-1}(\Omega)^N$. Consider the problem*

$$\begin{cases} u_n\in H_0^1(\Omega)\\ \displaystyle\int_\Omega \nabla u_n\cdot\nabla v\,dx+\int_\Omega b_n\cdot(v\nabla u_n-u_n\nabla v)\,dx=\langle f_n,v\rangle,\\ \forall\, v\in H_0^1(\Omega)\cap L^\infty(\Omega),\quad \textit{with}\ \ u_n\,b_n\cdot\nabla v\in L^1(\Omega).\end{cases}\tag{2.7}$$

Then, Eq. 2.7 admits a renormalized solution u_n which is bounded in $H_0^1(\Omega)$. Moreover, Eq. 2.7 is stable in the sense that any cluster point u of the sequence u_n for the weak topology of $H_0^1(\Omega)$ is a renormalized solution of Eq. 2.2 and satisfies Eq. 2.3.

Proof The existence of a renormalized solution u_n to Eq. 2.7 is a consequence of Theorem 2.1 approximating b_n, for a fixed n, by a sequence in $L^\infty(\Omega)^N$ which strongly converges to b_n in $L^2(\Omega)^N$. Moreover, the boundedness of u_n in $H_0^1(\Omega)$ follows from estimates Eq. 2.3 applied with u_n. Finally, if u_n converges weakly to some u in $H_0^1(\Omega)$, then the second step of the proof of Theorem 2.1 implies that u is a renormalized solution of Eq. 2.2 and satisfies the inequalities Eq. 2.3. □

2.2 Uniqueness of a Renormalized Solution Modulo L^∞

The main difficulty with the definition of a renormalized solution u of Eq. 2.2 is the nonlinearity of u with respect to the right-hand side f. In particular, the uniqueness of a solution cannot be deduced from the uniqueness with $f=0$. This non-linearity is strongly connected with inequalities Eq. 2.3 which turn out to be equalities when $b\in L^\infty(\Omega)^N$. However, we have the following result:

Lemma 2.4 *If u_1, u_2 are solutions of Eq. 2.2 with respective right-hand sides $f_1, f_2\in H^{-1}(\Omega)$, then, for any $\alpha_1,\alpha_2\in\mathbb{R}$, the function $\alpha_1u_1+\alpha_2u_2$ is a solution of Eq. 2.2 associated with the right-hand side $\alpha_1f_1+\alpha_2f_2$.*

Proof Since Eq. 2.2 is clearly homogeneous with respect its right-hand side, it is enough to prove the result for $\alpha_1=\alpha_2=1$.

Let $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$ be such that $(u_1+u_2)\,b\cdot\nabla v\in L^1(\Omega)$. We may consider, for a fixed $\varepsilon>0$, the function

$$v_\varepsilon:=\frac{v}{1+\varepsilon\,(|u_1|+|u_2|)}$$

as an admissible test function in the sum of the equations satisfied by u_1 and u_2, which gives

$$\begin{aligned}&\int_\Omega \frac{\nabla(u_1+u_2)\cdot\nabla v}{1+\varepsilon\,(|u_1|+|u_2|)}\,dx - \varepsilon\int_\Omega \frac{v\,\nabla(u_1+u_2)\cdot\big(\nabla u_1\,\mathrm{sgn}(u_1)+\nabla u_2\,\mathrm{sgn}(u_2)\big)}{\big(1+\varepsilon\,(|u_1|+|u_2|)\big)^2}\,dx\\&+\int_\Omega \frac{b\cdot\big(\nabla(u_1+u_2)\,v-\nabla v\,(u_1+u_2)\big)}{1+\varepsilon\,(|u_1|+|u_2|)}\,dx\\&-\varepsilon\int_\Omega \frac{v\,(u_1+u_2)\,b\cdot\big(\nabla u_1\,\mathrm{sgn}(u_1)+\nabla u_2\,\mathrm{sgn}(u_2)\big)}{\big(1+\varepsilon\,(|u_1|+|u_2|)\big)^2}\,dx\\&=\left\langle f_1+f_2,\frac{v}{1+\varepsilon\,(|u_1|+|u_2|)}\right\rangle.\end{aligned}\tag{2.8}$$

In this equality, thanks to the Lebesgue dominated convergence theorem we easily get as $\varepsilon\to 0$,

$$\begin{aligned}\int_\Omega \frac{\nabla(u_1+u_2)\cdot\nabla v}{1+\varepsilon(|u_1|+|u_2|)}\,dx &\longrightarrow \int_\Omega \nabla(u_1+u_2)\cdot\nabla v\,dx,\\ \varepsilon\int_\Omega \frac{v\,\nabla(u_1+u_2)\cdot\big(\nabla u_1\,\mathrm{sgn}(u_1)+\nabla u_2\,\mathrm{sgn}(u_2)\big)}{\big(1+\varepsilon\,(|u_1|+|u_2|)\big)^2}\,dx &\longrightarrow 0,\\ \left\langle f_1+f_2,\frac{v}{1+\varepsilon(|u_1|+|u_2|)}\right\rangle &\longrightarrow \langle f,v\rangle,\end{aligned}$$

and since $v\in L^\infty(\Omega)$, $(u_1+u_2)\,b\cdot\nabla v\in L^1(\Omega)$,

$$\int_\Omega \frac{b\cdot\big(\nabla(u_1+u_2)\,v-\nabla v\,(u_1+u_2)\big)}{1+\varepsilon\,(|u_1|+|u_2|)}\,dx \longrightarrow \int_\Omega b\cdot\big(\nabla(u_1+u_2)\,v-\nabla v\,(u_1+u_2)\big)\,dx.$$

In order to pass to the limit in the remaining term of Eq. 2.8, observe that

$$\left|\frac{\varepsilon\,(u_1+u_2)}{\big(1+\varepsilon\,(|u_1|+|u_2|)\big)^2}\right|\le 1 \quad\text{and}\quad \left|\frac{\varepsilon\,(u_1+u_2)}{\big(1+\varepsilon\,(|u_1|+|u_2|)\big)^2}\right|\longrightarrow 0 \quad \text{a.e. in } \Omega.$$

Then, we can again apply the Lebesgue dominated convergence theorem to get

$$\varepsilon\int_\Omega \frac{v\,(u_1+u_2)\,b\cdot\big(\nabla u_1\,\mathrm{sgn}(u_1)+\nabla u_2\,\mathrm{sgn}(u_2)\big)}{\big(1+\varepsilon\,(|u_1|+|u_2|)\big)^2}\,dx\longrightarrow 0.$$

Therefore, passing to the limit as $\varepsilon\to 0$ in Eq. 2.8 we obtain that u_1+u_2 is a solution of Eq. 2.2 with the right-hand side f_1+f_2. □

Lemma 2.4 allows us to prove the uniqueness of a renormalized solution to Eq. 2.2 modulo L^∞:

Theorem 2.5 *Assume that u_1, u_2 are solutions of Eq. 2.2 such that u_1-u_2 belongs to $L^\infty(\Omega)$, then $u_1=u_2$ a.e. in Ω.*

Proof It is enough to deduce from Lemma 2.4 that the difference u_1-u_2 is a solution of

$$\begin{cases} u_1-u_2\in H_0^1(\Omega)\\ \displaystyle\int_\Omega \nabla(u_1-u_2)\cdot\nabla v\,dx+\int_\Omega b\cdot\big(\nabla(u_1-u_2)\,v-(u_1-u_2)\nabla v\big)\,dx=0,\\ \forall\, v\in H_0^1(\Omega)\cap L^\infty(\Omega),\quad \text{with}\ \ (u_1-u_2)\,b\cdot\nabla v\in L^1(\Omega).\end{cases}$$

Since by assumption $u_1 - u_2$ is in $L^\infty(\Omega)$, we can take $v = u_1 - u_2$ as test function in the previous equation, and thus conclude that $u_1 - u_2 = 0$ a.e. in Ω. □

2.3 Non-uniqueness of a Solution of the Drift Equation

We have following result:

Theorem 2.6 *Let $\Omega = B(0,R)$ be the open ball of $\mathbb{R}^N$, for $N \in \{2,3\}$, centered at the origin and of radius $R \in (0,1)$. There exist $b \in L^2(\Omega)^N$ and $f \in H^{-1}(\Omega)$, such that the drift Eq. 2.1 has at least two solutions in the distributions sense, which belong to $H_0^1(\Omega)$. Moreover, one of the two solutions is not a renormalized solution of Eq. 2.2.*

Proof of Theorem 2.6. For $\alpha \in (\frac{1}{4}, \frac{1}{2})$, define the radial positive function $\hat u$ by ($r = |x|$)

$$\hat u(x) := \begin{cases} |\ln r|^\alpha & \text{if } N = 2 \\ \dfrac{1}{r^\alpha} & \text{if } N = 3 \end{cases} \quad \text{for } x \in \Omega, \tag{2.9}$$

and the vector-valued function b by

$$b(x) := \frac{1}{|S_{N-1}|\, r^{N-1}\, \hat u^2(x)}\, \frac{x}{r}, \quad \text{for } x \in \Omega, \tag{2.10}$$

where S_{N-1} denotes the $(N-1)$-dimensional sphere of $\mathbb{R}^N$.

The proof is divided in three steps:

- In the first step we prove that the function $\hat u$ satisfies

$$b \cdot \nabla \hat u + \operatorname{div}(b\, \hat u) = 0 \quad \text{in } \mathscr{D}'(\Omega), \tag{2.11}$$

$$k \int_{\{\hat u > k\}} b \cdot \nabla \hat u \, dx = -1, \quad \forall k \ge 1. \tag{2.12}$$

- In the second step we build from the function $\hat u$ suitable $u \in H_0^1(\Omega)$ and $f \in H^{-1}(\Omega)$ such that u is a solution of Eq. 2.1 in the distributions sense.
- The third step is devoted to the conclusion.

First step: It is easy to check that $u \in H^1(\Omega)$ and $b \in L^2(\Omega)^N$. Passing in polar coordinates and identifying abusively $\hat u(x)$ with $\hat u(r)$ (of derivative $\hat u'(r)$), we have for any $r \in (0,R)$,

$$\begin{aligned} |S_{N-1}| \big(b \cdot \nabla \hat u + \operatorname{div}(b\, \hat u)\big) &= \frac{\hat u'}{r^{N-1}\, \hat u^2} + \frac{1}{r^{N-1}} \left(r^{N-1} \frac{\hat u}{r^{N-1}\, \hat u^2} \right)' \\ &= \frac{\hat u'}{r^{N-1}\, \hat u^2} + \frac{1}{r^{N-1}} \left(\frac{1}{\hat u} \right)' = 0. \end{aligned} \tag{2.13}$$

Then, integrating by parts and using Eq. 2.13 we have for any $\varphi \in C_c^\infty(\Omega)$ and any $\varepsilon \in (0,R)$,

$$\int_\Omega \varphi\, b \cdot \nabla \hat u \, dx - \int_\Omega \hat u\, b \cdot \nabla \varphi \, dx = \int_{\{\varepsilon < r < R\}} \varphi\, b \cdot \nabla \hat u \, dx - \int_{\{\varepsilon < r < R\}} \hat u\, b \cdot \nabla \varphi \, dx + o(1)$$

$$= \int_{\{\varepsilon < r < R\}} \varphi\, b \cdot \nabla \hat u \, dx + \int_{\{\varepsilon < r < R\}} \varphi \operatorname{div}(\hat u\, b) dx - \int_{\{r = \varepsilon\}} \varphi\, \hat u\, b \cdot \frac{x}{r}\, d\sigma + o(1)$$

$$= -\varphi(0) \fint_{S_{N-1}} \frac{\hat u(\varepsilon)}{\varepsilon^{N-1}\, \hat u(\varepsilon)^2}\, \varepsilon^{N-1}\, d\sigma + o(1) = \frac{\varphi(0)}{\hat u(\varepsilon)} + o(1) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Therefore, $\hat{u}$ satisfies Eq. 2.11 in $\mathscr{D}'(\Omega)$.

On the other hand, $\hat{u}$ is a continuous one-to-one decreasing function from $(0, R)$ into $(0, \infty)$. Hence, by passing in polar coordinates we have for any $k \geq 1$,

$$\int_{\{\hat{u}>k\}} b \cdot \nabla \hat{u}\, dx = \fint_{S_{N-1}} \int_0^{\hat{u}^{-1}(k)} \frac{\hat{u}'(r)}{r^{N-1}\, \hat{u}^2(r)}\, r^{N-1}\, dr\, d\sigma = -\left[\frac{1}{\hat{u}(r)}\right]_{r=0}^{r=\hat{u}^{-1}(k)} = -\frac{1}{k},$$

that is Eq. 2.12.

Second step: Let ψ be a function in $W^{1,\infty}(\Omega)$ such that

$$\psi \equiv 0 \ \text{ in } \Omega \setminus B(0, 2R/3) \quad \text{and} \quad \psi \equiv 1 \ \text{ in } B(0, R/3), \tag{2.14}$$

and define the function

$$u := \psi\, \hat{u} \in H_0^1(\Omega). \tag{2.15}$$

Using Eq. 2.11 we have

$$\begin{aligned} -\Delta u + b \cdot \nabla u + \operatorname{div}(b\, u) &= -\Delta u + \psi\big(b \cdot \nabla \hat{u} + \operatorname{div}(b\, \hat{u})\big) + 2\, \hat{u}\, b \cdot \nabla \psi \\ &= -\Delta u + 2\, \hat{u}\, b \cdot \nabla \psi =: f. \end{aligned} \tag{2.16}$$

Since $u \in H_0^1(\Omega)$ and $\hat{u}\, b \cdot \nabla \psi \in L^{N'}(\Omega) \subset H^{-1}(\Omega)$ (recall that $N \leq 3$), the right-hand side f of Eq. 2.16 belongs to $H^{-1}(\Omega)$. Therefore, $u \in H_0^1(\Omega)$ is a solution of Eq. 2.1 in the distributions sense.

Third step: On the one hand, by virtue of Theorem 2.1 there exists a renormalized solution of Eq. 2.2 associated with $b \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)$ above defined, and thus in particular a solution in $H_0^1(\Omega)$ of Eq. 2.1 in the distributions sense. On the other hand, combining Eq. 2.12 with the equality $u = \hat{u}$ in $B(0, R/3)$, we get that

$$k \int_{\{u>k\}} b \cdot \nabla u\, dx = -1, \quad \text{for any large enough } k \geq 1,$$

which contradicts inequality Eq. 2.4. Therefore, again by Theorem 2.1 the function u defined by Eq. 2.15 is a solution of Eq. 2.1 in the distributions sense, but is not a renormalized solution of Eq. 2.2. The proof of Theorem 2.6 is now complete. □

3 Homogenization of the Drift Equation

Along this section Ω is a bounded regular open set of $\mathbb{R}^N$, for $N \in \{2, 3\}$, and b_n is a sequence in $L^\infty(\Omega)^N$. Assuming a better integrability of the limit drift we obtain the following result:

Theorem 3.1 *Assume that*

$$b_n \rightharpoonup b \ \text{ weakly in } L^2(\Omega)^N, \quad \text{with } b \in L^q(\Omega, \text{for some } q > N, \tag{3.1}$$

$$|b_n - b|^2 \rightharpoonup \vartheta \quad \text{weakly-}* \text{ in} \mathscr{M}(\Omega). \tag{3.2}$$

Then, there exist a subsequence of n, still denoted by n, and a non-negative Radon measure μ which does not load the zero capacity sets and is absolutely continuous with respect *to ϑ, such that for any sequence f_n converging strongly to f in $H^{-1}(\Omega)$, the solution u_n of the drift problem Eq. 1.4 converges weakly in $H^1(\Omega)$ to the solution u of the problem*

$$\begin{cases} u \in H_0^1(\Omega) \cap L_\mu^2(\Omega) \\ \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega \varphi \, b \cdot \nabla u \, dx - \int_\Omega u \, b \cdot \nabla \varphi \, dx + \int_\Omega \varphi \, u \, d\mu = \langle f, \varphi \rangle, \\ \forall \varphi \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases} \tag{3.3}$$

In the sequel we will often use the following comparison principle:

Lemma 3.2 *Let $c \in L^\infty(\Omega)^N$, let ν be a non-negative Borel measure on Ω which does not load the zero capacity sets, and let g be a non-negative distribution in $H^{-1}(\Omega)$. Consider the solution v of the equation*

$$\begin{cases} v \in H_0^1(\Omega) \cap L_\nu^2(\Omega) \\ -\Delta v + c \cdot \nabla v + div\,(c\, v) + v\, \nu = g \ \text{ in } \Omega \end{cases} \tag{3.4}$$

Then, $v \geq 0$ a.e. and ν-a.e. in Ω.

Proof Putting $v^- \in H_0^1(\Omega) \cap L_\nu^2(\Omega)$ as test function in Eq. 3.13 and integrating by parts we get that

$$\int_\Omega \nabla v \cdot \nabla v^- \, dx + \int_\Omega v^- \, c \cdot \nabla v \, dx - \int_\Omega v\, c \cdot \nabla v^- \, dx + \int_\Omega v^- \, v \, d\nu$$

$$= -\textstyle\int_\Omega |\nabla v^-|^2 \, dx - \int_\Omega (v^-)^2 \, d\nu = \langle g, v^- \rangle \geq 0,$$

which implies that $v^- = 0$ a.e. and ν-a.e. in Ω. Therefore, $v \geq 0$ a.e. in Ω. □

Proof of Theorem 3.1 First of all, writing

$$-\ \Delta u_n + (b_n - b) \cdot \nabla u_n + \text{div}\,\big((b_n - b)\, u_n\big) = f - b \cdot \nabla u_n - \text{div}\,(b\, u_n) =: F_n, \tag{3.5}$$

and due to the integrability Eq. 3.1 of b combined with the compact embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for any $p < \frac{2N}{N-2}$, the right-hand side F_n of Eq. 3.5 satisfies, up to a subsequence,

$$F_n \to F := f - b \cdot \nabla u - \text{div}\,(b\, u) \quad \text{strongly in } H^{-1}(\Omega). \tag{3.6}$$

We can thus replace b_n by $b_n - b$, and f_n by F_n. Moreover, using a density argument we can replace the right-hand side F_n by a fixed function f in $L^\infty(\Omega)$. On the other hand, let $u_{n,+}$ (resp. $u_{n,-}$) be the solution of the Eq. 1.4 associated with the non-negative (resp. non-positive) parts of f. By linearity we have $u_n = u_{n,+} - u_{n,-}$. Moreover, applying the comparison principle (Lemma 3.2) to $u_{n,\pm}$, we get that $u_{n,\pm} \geq 0$ a.e. in Ω. Therefore, from now on we assume that the limit b is zero, the right-side f_n is a fixed non-negative function f in $L^\infty(\Omega)$. As a consequence u_n and u are non-negative a.e. in Ω.

Consider the solution $w_n \in H_0^1(\Omega)$ of the dual equation

$$\begin{cases} w_n \in H_0^1(\Omega) \\ -\Delta w_n - b_n \cdot \nabla w_n - \text{div}(b_n w_n) = 1 \ \text{ in } \Omega, \end{cases} \tag{3.7}$$

which converges weakly (up to a subsequence) to some function w in $H^1(\Omega)$. By the comparison principle (Lemma 3.2) the function w_n is non-negative a.e. in Ω, so does its limit w. The proof is then divided in three steps:

- In the first step we prove that $w \in L^\infty(\Omega)$ and $-\Delta w \le 1$ in $\mathscr{D}'(\Omega)$.
- In the second step we follow the approach used by Dal Maso and Garroni [5] for an equation without drift term, to derive the limit problem Eq. 3.3 with a non-negative Borel measure μ.
- In the third step we prove, thanks to alternative test functions, that μ is actually a Radon measure which is absolutely continuous with respect to the measure ϑ of Eq. 3.2.

First step: Let $\psi \in C_c^\infty(\Omega)$, $\psi \ge 0$. Putting $\psi\, w_n$ in Eq. 3.7 and using the weak convergence of b_n to zero combined with the strong convergence of w_n in $L^p(\Omega)$ for $p < \frac{2N}{N-2}$, we have

$$\begin{aligned}&\int_\Omega |\nabla w_n|^2\, \psi\, dx + \int_\Omega w_n\, \nabla w_n \cdot \nabla \psi\, dx + \int_\Omega b_n \cdot \nabla \psi\, w_n^2\, dx\\ &= \int_\Omega |\nabla w_n|^2\, \psi\, dx + \int_\Omega w\, \nabla w \cdot \nabla \psi\, dx + o(1)\\ &= \int_\Omega \psi\, w\, dx + o(1).\end{aligned}$$

Hence, by the lower semi-continuity of the L^2-norm of the gradient in $H^1(\Omega)$, we deduce that

$$\begin{aligned}\int_\Omega w\, \nabla w \cdot \nabla \psi\, dx &\le \int_\Omega \nabla w \cdot \nabla(\psi\, w)\, dx\\ &\le \liminf_{n\to\infty} \left(\int_\Omega |\nabla w_n|^2\, \psi\, dx + \int_\Omega w\, \nabla w \cdot \nabla \psi\, dx \right)\\ &= \int_\Omega \psi\, w\, dx,\end{aligned} \tag{3.8}$$

which implies that

$$-\Delta(w^2) \le 2\, w \quad \text{in } \mathscr{D}'(\Omega). \tag{3.9}$$

Let y be the solution of the equation

$$\begin{cases} y \in H_0^1(\Omega) \\ -\Delta y - 2\, w = 0 \text{ in } \Omega \end{cases}$$

The function $w^2 - y$ is in $W_0^{1,3/2}(\Omega)$, and satisfies $-\Delta\big(w^2 - y\big) \le 0$ in $\mathscr{D}'(\Omega)$. Then, thanks to Lemma 3.3 below we get that $w^2 - y \le 0$ a.e. in Ω. Therefore, by the regularity results for the Laplace equation the function y belongs to $L^\infty(\Omega)$, so does w.

Lemma 3.3 *Let Ω be a regular open bounded set of $\mathbb{R}^N$. Let v be a function in $W_0^{1,p}(\Omega)$ with $p > 1$, such that $-\Delta v \le 0$ in $\mathscr{D}'(\Omega)$. Then, v is non-negative a.e. in Ω.* □

Let us now prove that $-\Delta w \le 1$ in $\mathscr{D}'(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$, $\varphi \ge 0$, and $\delta > 0$. Consider a non-negative and uniformly bounded sequence ψ_k in $C_c^\infty(\Omega)$ which converges strongly to the function $\varphi\,(w+\delta)^{-1}$ in $H_0^1(\Omega)$. Such a sequence can be obtained by using a smooth truncation of a sequence in $C_c^\infty(\Omega)$ which converges strongly in $H_0^1(\Omega)$ to $\varphi\,(w+\delta)^{-1} \in L^\infty(\Omega)$. Now, since $w \in L^\infty(\Omega)$, the sequence $\psi_k\, w$ converges weakly to $\varphi\, w\,(w+\delta)^{-1}$ in $H_0^1(\Omega)$. Then, passing to the limit as $k \to \infty$ in the second inequality of Eq. 3.8 with the test function $\psi = \psi_k$, we get that

$$\int_\Omega \nabla w \cdot \nabla \left(\varphi\, \frac{w}{w+\delta} \right) dx \le \int_\Omega \varphi\, \frac{w}{w+\delta}\, dx \le \int_\Omega \varphi\, dx. \tag{3.10}$$

Noting that

$$\int_\Omega \nabla w \cdot \nabla \left(\varphi \frac{w}{w+\delta}\right) dx = \int \Omega \frac{w}{w+\delta} \nabla w \cdot \nabla \varphi \, dx + \int_\Omega \varphi \left| \frac{\nabla w}{w+\delta} \right|^2 dx,$$

we deduce from Eq. 3.10 that

$$\int \Omega \frac{w}{w+\delta} \nabla w \cdot \nabla \varphi \, dx \leq \int_\Omega \varphi \, dx. \tag{3.11}$$

Finally, by the Lebesgue dominated convergence theorem the sequence $w\,(w+\delta)^{-1}\,\nabla w$ converges strongly to $\nabla w \, 1_{\{w>0\}}$ in $L^1(\Omega)^N$ as $\delta \to 0$, and $\nabla w \, 1_{\{w>0\}} = \nabla(w^+) = \nabla w$ a.e. in Ω. Therefore, passing to the limit as $\delta \to 0$ in Eq. 3.11 we obtain the inequality

$$\int_\Omega \nabla w \cdot \nabla \varphi \, dx \leq \int_\Omega \varphi \, dx, \quad \forall \varphi \in C_c^\infty(\Omega), \ \varphi \geq 0, \tag{3.12}$$

which yields the result.

Second step: First of all, the sequence v_n solution of the equation

$$\begin{cases} v_n \in H_0^1(\Omega) \\ -\Delta v_n + b_n \cdot \nabla v_n + \operatorname{div}(b_n v_n) = 1 \ \text{ in } \Omega, \end{cases} \tag{3.13}$$

satisfies the same properties as the sequence w_n defined by Eq. 3.7. In particular, by the first step the weak limit v of v_n in $H^1(\Omega)$ belongs to $L^\infty(\Omega)$. Then, by the comparison principle (Lemma 3.2) applied to the function $u_n - \|f\|_{L^\infty(\Omega)}\, v_n \in H_0^1(\Omega)$, we have $0 \leq u_n \leq \|f\|_{L^\infty(\Omega)}\, v_n$ a.e. in Ω, so that $u \in L^\infty(\Omega)$.

Now, apply the Tartar oscillating test functions method with the Eqs. 1.4 and 3.7. Let $\varphi \in C_c^\infty(\Omega)$, putting $\varphi\, w_n$ in Eq. 1.4, $\varphi\, u_n$ in Eq. 3.7, and equating the two formulas we get that

$$\int_\Omega \nabla u_n \cdot \nabla \varphi \, w_n \, dx - \int_\Omega \nabla w_n \cdot \nabla \varphi \, u_n - 2 \int_\Omega b_n \cdot \nabla \varphi \, u_n w_n \, dx = \int_\Omega f \, \varphi \, w_n \, dx - \int_\Omega \varphi \, u_n \, dx. \tag{3.14}$$

By virtue of the Sobolev embedding theorem, $u_n w_n$ converges strongly to $u\,w$ in $L^p(\Omega)$ for any $p < \frac{N}{N-2}$, thus in particular in $L^2(\Omega)$ since $N \leq 3$. Hence, $b_n u_n w_n$ converges to 0 in $\mathscr{D}'(\Omega)^2$ (recall that $b = 0$). Therefore, passing to the limit in Eq. 3.14 we obtain the limit variational problem

$$\begin{cases} u \in H^1(\Omega) \cap L^\infty(\Omega) \\ \displaystyle\int_\Omega w \, \nabla u \cdot \nabla \varphi \, dx - \int_\Omega \nabla w \cdot \nabla \varphi \, u = \int_\Omega f \, \varphi \, w \, dx - \int_\Omega \varphi \, u \, dx, \quad \forall \varphi \in C_c^\infty(\Omega). \end{cases} \tag{3.15}$$

From now on we may follow the Dal Maso and Garroni approach [5]. By the first step above and by Proposition 3.4 of [5], there exists a non-negative Borel measure μ which does not load the zero capacity sets, such that w is the unique solution in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$ of the problem

$$\int_\Omega \nabla w \cdot \nabla \varphi \, dx + \int_\Omega \varphi \, w \, d\mu = \int_\Omega \varphi \, dx, \quad \forall \varphi \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \tag{3.16}$$

On the other hand, by virtue of the Lax-Milgram theorem problem Eq. 3.3 (with $b = 0$) admits a unique solution $\tilde{u}$ in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$. Since f is non-negative a.e. in Ω, so is $\tilde{u}$ by the comparison principle (Lemma 3.2). Then, the function $\tilde{u}$ satisfies the inequality $-\Delta \tilde{u} \leq f$ in $\mathscr{D}'(\Omega)$, with $f \in L^\infty(\Omega)$. Hence, again using the comparison principle with the function $(-\Delta)^{-1}(f) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we get that $\tilde{u}$ belongs to $L^\infty(\Omega)$. Following Lemma 3.5 of [5] thanks to inequality Eq. 3.12, $\tilde{u}$ is also the unique solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of the variational problem Eq. 3.15. Since u is such a solution, the functions u and $\tilde{u}$ thus agree. We have just proved that u is the solution of the problem Eq. 3.3 with the non-negative Borel measure μ defined from the function w according to [5] (Lemma 3.5).

Third step: We will prove that μ is actually a Radon measure which is absolutely continuous with respect to the measure ϑ of Eq. 3.2. To this end consider the double index function z_n^k, for $k \geq 1$ and $n \in \mathbb{N}$, solution of

$$\begin{cases} -\Delta z_n^k + b_n \cdot \nabla z_n^k + \operatorname{div}(b_n\, z_n^k) + k\,(z_n^k - 1) = 0 \ \text{ in } \Omega \\ \hfill z_n^k = 1 \ \text{ on } \partial\Omega. \end{cases} \tag{3.17}$$

For $\varphi \in C^\infty(\bar{\Omega})$, putting $\varphi\,(z_n^k - 1)$ as test function in Eq. 3.17 we have

$$\begin{aligned} &\int_\Omega |\nabla z_n^k|^2\, \varphi\, dx + \int_\Omega (z_n^k - 1)\, \nabla z_n^k \cdot \nabla\varphi\, dx - \int_\Omega \varphi\, b_n \cdot \nabla z_n^k\, dx \\ &- \int_\Omega z_n^k\,(z_n^k - 1)\, b_n \cdot \nabla\varphi\, dx + k \int_\Omega (z_n^k - 1)^2\, \varphi\, dx = 0. \end{aligned} \tag{3.18}$$

In particular for $\varphi \equiv 1$, this yields

$$\int_\Omega |\nabla z_n^k|^2\, dx + k \int_\Omega (z_n^k - 1)^2\, dx = \int_\Omega b_n \cdot \nabla z_n^k\, dx,$$

hence

$$\limsup_{n\to\infty} \left(\frac{1}{2} \int_\Omega |\nabla z_n^k|^2 dx + k \int_\Omega (z_n^k - 1)^2\, dx \right) \leq \lim_{n\to\infty} \frac{1}{2} \int_\Omega |b_n|^2\, dx = \frac{1}{2}\, \vartheta(\bar{\Omega}) < \infty.$$

Using a diagonal argument we then deduce the existence of a subsequence of n, still denoted by n, such that for any $k \geq 1$, z_n^k converges weakly to some function z^k in $H^1(\Omega)$, and $b_n \cdot \nabla z_n^k$ converges weakly-$*$ to some Radon measure μ^k in $\mathscr{M}(\bar{\Omega})$. Moreover, as $k \to \infty$ the sequence z^k converges weakly to 1 in $H^1(\Omega)$, and up to a subsequence μ^k converges weakly-$*$ to some measure μ^∞ in $\mathscr{M}(\bar{\Omega})$. Passing to the limit as $n \to \infty$ in Eq. 3.18, we easily get that for any $\varphi \in C^\infty(\bar{\Omega})$,

$$\limsup_{n\to\infty} \int_\Omega |\nabla z_n^k|^2\, \varphi\, dx + \int_\Omega (z^k - 1)\, \nabla z^k \cdot \nabla\varphi\, dx - \int_{\bar{\Omega}} \varphi\, d\mu^k + k \int_\Omega (z^k - 1)^2\, \varphi\, dx = 0. \tag{3.19}$$

Hence, passing to the limit as $k \to \infty$ it follows that

$$\lim_{k\to\infty} \left[\limsup_{n\to\infty} \left(\int_\Omega |\nabla z_n^k|^2\, \varphi\, dx + k \int_\Omega (z^k - 1)^2\, \varphi\, dx \right) \right] = \int_{\bar{\Omega}} \varphi\, d\mu^\infty. \tag{3.20}$$

In particular, μ^∞ is a non-negative Radon measure on $\bar{\Omega}$. Moreover, by the Cauchy-Schwarz inequality and convergence Eq. 3.2 we have for any non-negative $\varphi \in C^\infty(\bar{\Omega})$,

$$\begin{aligned} &\int_{\bar{\Omega}} \varphi\, d\mu^\infty = \lim_{k\to\infty} \lim_{n\to\infty} \int_\Omega \varphi\, b_n \cdot \nabla z_n^k\, dx \\ &\leq \left(\lim_{n\to\infty} \int_\Omega \varphi\, |b_n|^2\, dx \right)^{\frac{1}{2}} \left(\lim_{k\to\infty} \lim_{n\to\infty} \int_\Omega \varphi\, |\nabla z_n^k|^2\, dx \right)^{\frac{1}{2}} \leq \left(\int_{\bar{\Omega}} \varphi\, d\vartheta \right)^{\frac{1}{2}} \left(\int_{\bar{\Omega}} \varphi\, d\mu^\infty \right)^{\frac{1}{2}}, \end{aligned}$$

which implies that

$$\int_{\bar{\Omega}} \varphi\, d\mu^\infty \leq \int_{\bar{\Omega}} \varphi\, d\vartheta, \quad \forall\, \varphi \in C^\infty(\bar{\Omega}),\ \varphi \geq 0. \tag{3.21}$$

Taking into account Eq. 3.21 it thus remains to compare the measures μ and μ^∞ in $\mathscr{M}(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$. The function φz_n^k is the solution in $H_0^1(\Omega)$ of the equation

$$\begin{aligned} &-\Delta(\varphi z_n^k) + b_n \cdot \nabla(\varphi z_n^k) + \operatorname{div}\big(b_n\, \varphi z_n^k\big) \\ &= k\,\varphi\,(1 - z_n^k) - 2\,\nabla z_n^k \cdot \nabla\varphi - z_n^k\,\Delta\varphi + 2\,z_n^k\, b_n \cdot \nabla\varphi =: \Phi_n^k \end{aligned} \quad \text{in } \mathscr{D}'(\Omega),$$

where Φ_n^k clearly converges strongly in $H^{-1}(\Omega)$ as $n \to \infty$. Then, by virtue of the second step the limit φz^k is the solution in $H_0^1(\Omega) \cap L^2_{\mu_k}(\Omega)$ of the following equation:

$$-\,\Delta(\varphi z^k) + \varphi z^k\,\mu = k\,\varphi\,(1 - z^k) - 2\,\nabla z^k \cdot \nabla\varphi - z^k\,\Delta\varphi \quad \text{in } \mathscr{D}'(\Omega). \tag{3.22}$$

On the other hand, passing to the limit as $n \to \infty$ in Eq. 3.17 we have

$$-\Delta z^k + \mu^k + k\,(z^k - 1) = 0 \quad \text{in } \mathscr{D}'(\Omega),$$

which implies that

$$-\,\Delta(\varphi z^k) + \varphi\,\mu^k = k\,\varphi\,(1 - z^k) - 2\,\nabla z^k \cdot \nabla\varphi - z^k\,\Delta\varphi \quad \text{in } \mathscr{D}'(\Omega). \tag{3.23}$$

Equating Eqs. 3.22 and 3.23 we get that $\varphi\,\mu^k = \varphi z^k\,\mu$ in $\mathscr{M}(\Omega)$. Moreover, since μ is a non-negative Borel measure which does not load the zero capacity sets, and since z^k converges weakly to 1 in $H^1(\Omega)$, by virtue of Propositions 3.3 and 3.5 of [4], z^k converges, up to a subsequence, to 1 μ-a.e. in Ω. Then, from the Fatou Lemma and Eq. 3.21 we deduce that for any non-negative function $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \varphi\,d\mu \le \liminf_{k\to\infty} \int_\Omega \varphi z^k\,d\mu = \liminf_{k\to\infty} \int_\Omega \varphi\,d\mu^k = \int_\Omega \varphi\,d\mu^\infty \le \int_\Omega \varphi\,d\vartheta.$$

Therefore, μ is a non-negative Radon measure which is absolutely continuous with respect to the measure ϑ of Eq. 3.2, which concludes the proof of Theorem 3.1.

Proof of Lemma 3.3. Using a partition of the unity $(\theta_j)_{0\le j\le m}$ for $\bar\Omega$, with $\operatorname{supp}(\theta_0) \subset \Omega$, and the regularity of $\partial\Omega$, we can construct a non-positive sequence g_ε in $C^\infty(\bar\Omega)$ which converges strongly to $g := -\,\Delta v$ in $W^{-1,p}(\Omega)$ $(p > 1)$. This result is known, but for the reader's convenience we will give now a short proof.

To this end, let ρ_ε, for $\varepsilon > 0$, be a non-negative mollifier the support of which is $B(0, \varepsilon)$. The function $\theta_0\,(\rho_\varepsilon * g)$ does the job for $\theta_0\,g$. On the other hand, if ω_j is a suitable open set containing $\operatorname{supp}\big(\theta_j\big)$, for $j = 1, \dots, m$, there exists a regular function ψ_j which maps ω_j onto the open ball $B(0, R)$, and $\omega_j \cap \Omega$ onto the half-ball $B^+(0, R) := B(0, R) \cap \{y_N > 0\}$, such that $(\theta_j\,g) \circ \psi_j^{-1} \equiv 0$ on a neighborhood of $\partial B(0, R) \cap \{y_N > 0\}$. Here, $g \circ \psi_j^{-1}$ is the distribution defined by the duality

$$\big\langle g \circ \psi_j^{-1}, \varphi \circ \psi_j^{-1} \big\rangle_{W^{-1,p}(B(0,R)^+),\, W_0^{1,p'}(B(0,R)^+)} := \langle g, \varphi \rangle_{W^{-1,p}(\omega_j\cap\Omega),\, W_0^{1,p'}(\omega_j\cap\Omega)},$$

for $\varphi \in W_0^{1,p'}(\omega_j \cap \Omega)$. Then, the "translated" distribution $\tilde g_{j,\varepsilon} : x \mapsto \big(g \circ \psi_j^{-1}\big)\big(y', y_N + \varepsilon\big)$, where $y := \psi_j(x)$, is defined in the enlarged domain $\omega_j \cap \big\{\psi_j(x)_N > -\,\varepsilon\big\}$ beyond $\omega_j \cap \Omega$. Therefore, the sequence

$$g_\varepsilon := \theta_0\,(\rho_\varepsilon * g) + \sum_{j=1}^m \theta_j\big(\rho_\varepsilon * \tilde g_{j,\varepsilon}\big)$$

has the desired properties.

Now, consider the solution $v_\varepsilon \in H_0^1(\Omega)$ of the equation $-\,\Delta v_\varepsilon = g_\varepsilon \le 0$ in $\mathscr{D}'(\Omega)$. By the maximum principle, $v_\varepsilon \le 0$ a.e. in Ω. Moreover, thanks to the Calderòn-Zygmund regularity for the Laplace operator (see, e.g., [8]) and the regularity of Ω, the operator $-\Delta$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$. Hence, the sequence v_ε converges strongly to v in $W_0^{1,p}(\Omega)$, which implies that $v \leq 0$ a.e. in Ω. □

Remark 3.4 Assume that the dimension is $N = 2$, and consider an equi-coercive matrix-valued function A in $L^\infty(\Omega)^{2\times 2}$. By virtue of Meyers' theorem [10] (Theorem 1), for any p close enough to 2, $\operatorname{div}(A\nabla\cdot)$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$. On the other hand, note that for any $v \in H^1(\Omega)$, the function v^2 belongs to $W^{1,p}(\Omega)$ for any $p < 2$. Then, Lemma 3.3, for p close to 2, and Theorem 3.1 – involving the function w satisfying the inequality (3.9), *i.e.* $-\Delta(w^2) \leq 2\,w$ – can be easily extended to the case where the Laplace operator is replaced by the operator $\operatorname{div}(A\nabla\cdot)$ in Eq. 1.4. So, for example the inequality (3.9) becomes $-\operatorname{div}\big(A\nabla(w^2)\big) \leq 2\,w$, which still implies that w belongs to $L^\infty(\Omega)$.

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References

1. Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vazquez, J.-L.: An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Scuola Norm. Sup. Pisa, Serie IV **22**(2), 241–273 (1995)
2. Briane M.: Homogenization with an oscillating drift: from L^2-bounded to unbounded drifts, 2d compactness results and 3d nonlocal effect. Ann. Mate. Pura Appl. **192**(5), 853–878 (2013)
3. Briane, M., Gérard, P.: A drift homogenization problem revisited. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **11**(5), 1–39 (2012)
4. Casado-Díaz, J., Dal Maso, G.: A weak notion of convergence in capacity with applications to thin obstacle problems, Calculus of Variations and Related Topics. In: Ioffe, A., Reich, S., Shafrir, I. (eds.) in Research Notes in Mathematics 410, Chapman and Hall, Boca Raton, 56–64 (2000)
5. Dal Maso, G., Garroni, A.: New results on the asymptotic behavior of Dirichlet problems in perforated domains. Math. Mod. Meth. Appl. Sci. **3**, 373–407 (1994)
6. Dal Maso, G., Murat, F.: Asymptotic behavior and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **24**(2), 239–290 (1997)
7. Dal Maso, G., Murat, F., Orsina, L., Prignet, A.: Renormalized solutions of elliptic equations with general measure data. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **28**(4), 741–808 (1999)
8. Geymonat, G.: G. Sul problema di Dirichlet per le equazioni lineari ellittiche. Ann. Scuola Norm. Sup. Pisa **16**(3), 225–284 (1962)
9. Lions, P.-L., Murat, F.: Sur les solutions renormalisées d'quations elliptiques non linéaires, unpublished.
10. Meyers, N.G.: An Lp-estimate for the gradient of solutions of second order elliptic divergence equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **3**(17), 189–206 (1963)
11. Murat, F.: Soluciones renormalizadas de EDP elipticas no lineales. Prépublication du laboratoire d'Analyse numérique de l'université Paris VI, France (1993)
12. Serrin, J.: Pathological solutions of elliptic differential equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **18**(3), 385–387 (1964)
13. Stampacchia, G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier **15**, 189–258 (1965)
14. Tartar, L.: Homogénéisation en hydrodynamique, in Singular Perturbation and Boundary Layer Theory, Lecture Notes Math., 597, Springer, Berlin-Heidelberg, 474–481 (1977)
15. Tartar, L.: Remarks on homogenization, in Homogenization and effective moduli of materials and media, IMA Vol. Math. Appl., 1, Springer, New-York, 228–246 (1986)