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On Riesz Decomposition for Super-Polyharmonic Functions in $\mathbb{R}^n$

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Abstract The classical Riesz Decomposition Theorem is a powerful tool describing superharmonic functions on compact subsets of $\mathbb{R}^n$. There is also the global version of this result dealing with functions superharmonic in $\mathbb{R}^n$ and satisfying an additional condition. Recently, a generalization of this result for superbiharmonic functions in $\mathbb{R}^n$ was obtained by (J. Anal. Math. **60**, 113–133 2006). We consider its further generalization for m-superharmonic functions.

Keywords Riesz potential · Riesz decomposition · Super-polyharmonic function · Polyharmonic function · Integral means

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1 Introduction and Main Results

Subharmonic and superharmonic functions are very important classes of functions since they enjoy many properties of harmonic functions, but unlike the latter, they are more flexible. There are several equivalent definitions of a superharmonic function on an open subset $\Omega \subset \mathbb{R}^n$ (see, e.g., [5, Ch. 2], [1, Ch. 3], [7, Ch. III]). The class of subharmonic functions in Ω is denoted by $\mathcal{S}(\Omega)$. For superharmonic functions, we use $\mathcal{SH}(\Omega)$.

Let us note that if $s \in C^2(\Omega)$, then it is subharmonic if and only if its Laplacian Δs is non-negative in Ω. Moreover, for an arbitrary $s \in \mathcal{S}(\Omega)$, and an open subset ω such that $\overline{\omega} \subset \Omega$, there exists a decreasing sequence of functions $s_n \in \mathcal{S}(\omega) \cap C^\infty(\omega)$ convergent to

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s pointwise on ω (see, e.g., [1, Th. 3.3.3]). This result and Green's formula suggest to consider Laplacian in the distributional sense to give an equivalent definition of a subharmonic function (see, e.g., [1, § 4.3]).

Definition 1.1 For an open set $\Omega \subset \mathbb{R}^n$, we use $C_0(\Omega)$ to denote the vector space (over $\mathbb{R}$) of all real-valued functions continuous and compactly supported in Ω. Furthermore, $C_0^\infty(\Omega) := C_0(\Omega) \cap C^\infty(\Omega)$. If $u : \Omega \to [-\infty, \infty]$ is locally integrable on Ω, then the linear functional

$$L_u(\varphi) := \int_\Omega u(x)\Delta\varphi(x)\, dx, \quad \varphi \in C_0^\infty(\Omega), \tag{1}$$

is called the distributional Laplacian of u.

Using Green's formula, it is easy to conclude (see, e.g., [1, § 4.3]) that if $u \in C^2(\Omega)$, then $L_u(\varphi) = \int_\Omega \varphi(x)\Delta u(x)\, dx$. In general, if $s \in \mathcal{S}(\Omega)$, then L_s is a positive linear functional on $C_0^\infty(\Omega)$, and there is a unique measure μ_s on Ω, such that

$$a_n^{-1} L_s(\varphi) = \int_\Omega \varphi(x)\, d\mu_s(x), \quad \varphi \in C_0^\infty(\Omega),$$

where $a_n = \sigma_n \max\{1, n-2\}$, and σ_n is the surface measure of the unit sphere in $\mathbb{R}^n$, i.e.,

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

The measure μ_s is called *the Riesz measure associated with s*. For a superharmonic function u, the Riesz measure is defined to be the one associated with the subharmonic function $-u$. In both cases, Riesz measure is a non-negative measure, which characterizes the function. Namely, if $u, v \in \mathcal{S}(\Omega)$, (or $\mathcal{SH}(\Omega)$) are such that $L_u = L_v$ on $C_0^\infty(\Omega)$, then $u - v$ is harmonic in Ω (see, e.g., [5, Ch. 3, Lemma 3.7]).

The Riesz Decomposition Theorem gives even more. This theorem in various forms and for various Ω could be found in any book on Potential Theory (see, e.g., [1, Th. 4.4.1], or [11, Th. 3.7.9]). The most classical ones (see, e.g., [5, Ch. 3, Th. 3.9]) describes superharmonic functions on compact subsets of Ω.

There are several versions of the Riesz Decomposition Theorem for functions superharmonic in a ball, half-space, etc. (see, e.g., [1, Ch. 4, § 4.4]). However, we are interested in generalizations of the following global type of result (see, e.g., [9, Ch. I, § 5, Ths. 1.24 and 1.25]).

Theorem 1.1 (Riesz Decomposition Theorem, "Global Version") *Suppose u is superharmonic in $\mathbb{R}^n$, $n \geq 3$. Then, there is a harmonic function h in $\mathbb{R}^n$ such that*

$$u(x) = (\sigma_n(n-2))^{-1} \int_{\mathbb{R}^n} K_2(x-y)\, d\mu_u(y) + h(x),$$

if and only if

$$\lim_{r\to\infty} M(r, u) > -\infty.$$

Here and in what follows we use the following notations.

For a measurable function g, the *spherical mean* over the sphere $S(0, r)$ of radius $r > 0$ centered at the origin is defined by

$$M(r, g) = \frac{1}{\sigma_n r^{n-1}} \int_{S(0,r)} g(x)\, d\sigma(x),$$

where $d\sigma$ is the surface measure in $\mathbb{R}^n$.

The *Riesz Kernels* are given by:

$$K_\alpha(x) := |x|^{\alpha-n}, \quad \alpha > 0.$$

As a corollary of Theorem 1.1, one can obtain (see [1, Cor. 4.4.2]) that if u is superharmonic in $\mathbb{R}^n$, $n \geq 3$, $u \geq 0$, and $u \not\equiv \infty$, then

$$u(x) = (\sigma_n\,(n-2))^{-1} \int_{\mathbb{R}^n} K_2\,(x-y)\; d\mu_u(y) + c, \quad x \in \mathbb{R}^n,$$

where c is a non-negative constant.

We are interested in a generalization of the Riesz Decomposition Theorem for m-superharmonic functions (see Definition 1.3 below). Recently, for $m = 2$ (superbiharmonic functions) the generalization we are looking for was obtained by K. Kitaura and Y. Mizuta [8]. Let us introduce precise definitions first.

Definition 1.2 Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. A function $u : \Omega \to \mathbb{R}$ is called $m-harmonic$ ($m \in \mathbb{N}$), or *polyharmonic of order* m, in Ω if $u \in C^{2m}(\Omega)$, and $\Delta^m u \equiv 0$ in Ω. The set of all functions m-harmonic in Ω is denoted by $\mathcal{H}^m(\Omega)$.

Polyharmonic functions have many interesting properties. The monograph [2] is an excellent source of information about them.

Definition 1.3 Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. A function $u : \Omega \to (-\infty, \infty]$ is called m*-superharmonic* if
(i) u is locally integrable on Ω;
(ii) u is lower semicontinuous in Ω;
(iii) $\mu_u := (-\Delta)^m u$ is a positive Radon measure in Ω in the sense of distributions, i.e.,

$$\int_\Omega \varphi(x)\, d\mu_u(x) = \int_\Omega u(x)\,(-\Delta)^m\,\varphi(x)\, dx \geq 0, \quad \forall \varphi \in C_0^\infty(\Omega)\,,\ \varphi \geq 0;$$

(iv) Every point of Ω is the Lebesgue point of u.
The class of all m-superharmonic functions in Ω is denoted by $\mathcal{SH}^m(\Omega)$. If $m = 2$, we have the class of superbiharmonic functions;

The generalization of the Riesz Decomposition Theorem for superbiharmonic functions is given by the following theorem

Theorem 1.2 (K. Kitaura, Y. Mizuta [8, Th. 1.2]) *Let* $n \geq 5$, $u \in \mathcal{SH}^2(\mathbb{R}^n)$*, and* $\mu_u = \Delta^2 u$*. Then* $M(2r, u) - 4M(r, u)$ *is bounded when* $r > 1$ *if and only if* u *is of the form*

$$u(x) = (2\sigma_n\,(n-4)\,(n-2))^{-1} \int_{\mathbb{R}^n} K_4(x-y)\, d\mu_u(y) + h(x),$$

where $h \in \mathcal{H}^2(\mathbb{R}^n)$*, and*

$$\int_{\mathbb{R}^n} (1+|y|)^{4-n}\; d\mu_u(y) < \infty.$$

Moreover, in [8], the authors consider the case of lower dimensions too. However, they use some modification of the Riesz kernels in that case.

The main point is that the possibility for a superbiharmonic function to possess a Riesz decomposition is given in terms of boundedness of a linear combination of spherical means.

For the m-superharmonic case, the appropriate linear combination of spherical means is more complicated. It is defined in the following proposition.

Proposition 1.1 *Let $m \in \mathbb{N}$, $m \geq 2$, and let $\alpha_{m,1} = 1$. Then there are unique real constants $\alpha_{m,2}, \ldots, \alpha_{m,m}$ such that for every polynomial of the form*

$$F_m(r) := \sum_{k=0}^{m-1} a_k r^{2k},$$

we have

$$\sum_{j=1}^{m} \alpha_{m,j} F_m\left(2^{m-j} r\right) = a_0 \sum_{j=1}^{m} \alpha_{m,j}, \quad r \in \mathbb{R}. \tag{2}$$

The constants are given by

$$\alpha_{m,k+1} = (-1)^{k+\frac{m-1}{2}(m-2)} 4^{\frac{m}{2}(m-1)-(m-k-1)} \frac{\prod\limits_{1\leq l<j\leq m-1} \left(\theta_{m,j,k} - \theta_{m,l,k}\right)}{\prod\limits_{1\leq l<j\leq m-1} \left(4^j - 4^l\right)}, \tag{3}$$

where

$$\theta_{m,j,k} = \begin{cases} 4^{m-j}, & 1 \leq j \leq k, \\ 4^{m-1-j}, & k+1 \leq j \leq m-1, \end{cases} \qquad 1 \leq k \leq m-1.$$

To formulate the main result, we need to introduce $\mathcal{R}$ – the class of functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying:

(i) $\varphi(x) \equiv 1$ in $\overline{B(0,1)}$ (as usual, $B(0,r)$ denotes the ball in $\mathbb{R}^n$ of radius r centered at the origin);
(ii) $\operatorname{supp} \varphi \subset \overline{B(0,2)}$;
(iii) $0 \leq \varphi(x) \leq 1$, $x \in \mathbb{R}^n$.

Such functions are often used for regularization purposes.

Our main result is given by the following theorem.

Theorem 1.3 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, $\mu_u = (-\Delta)^m u$, and $\varphi \in \mathcal{R}$ is chosen arbitrarily. Furthermore, let $\alpha_{m,j}$ be the absolute constants from Proposition 1.1. Then*

$$\sup_{r>1} \left| \sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) \right| < \infty \quad \text{and} \quad \sup_{r>1} \int_{1\leq|t|\leq 2} u(rt)\,(-\Delta)^m \varphi(t)\,dt < \infty \tag{4}$$

if and only if

$$\int_{\mathbb{R}^n} (1+|y|)^{2m-n}\, d\mu_u(y) < \infty, \tag{5}$$

and u is of the form

$$u(x) = c_{m,n} \int_{\mathbb{R}^n} K_{2m}(x-y)\, d\mu_u(y) + h(x), \tag{6}$$

where $h \in \mathcal{H}^m(\mathbb{R}^n)$, and

$$c_{m,n} = \left(2^{m-1}(m-1)!\sigma_n \prod_{0\leq j\leq m-1,\, j\neq m-n/2} (n-2m+2j) \right)^{-1}. \tag{7}$$

Note that (5) is the condition for existence of the potential in (6). Furthermore, the normalizing coefficients $c_{m,n}$ are chosen so that $c_{m,n}\,(-\Delta)^m\,K_{2m}$ is the delta-function δ_0 (see [6] and [4, § 3]).

Comparing Theorems 1.2 and 1.3, one can observe that the first condition in (4) is exactly the condition on the boundedness of $M\,(2r,u)-4M\,(r,u)$ used in Theorem 1.2. The second one is an extra condition. However, for $m=2$, the second condition in (4) follows from the first one. This seems to be false for $m\geq 3$.

Moreover, for the case $2m\geq n$, one needs to consider different kernels. For example, K. Kitaura and Y. Mizuta [8] considered special kernels wich are products of the Riesz kernels and $\ln\frac{1}{|x|}$. It was shown that if $u\in\mathcal{SH}^2\,(\mathbb{R}^n)$ and $n\leq 4$, then the linear combination of spherical means $M\,(2r,u)-4M\,(r,u)$ is bounded on $r>1$ if and only if $u\in\mathcal{H}^2\,(\mathbb{R}^n)$. The authors investigate the case for each n between 2 and 4 separately. The Riesz decomposition for superharmonic functions in $\mathbb{R}^n$ ($m=1$) is also proven in [8].

The following corollary gives an easy to use sufficient condition for an m-superharmonic function to have the representation (6).

Corollary 1.1 *Let $m,n\in\mathbb{N}$, $2m<n$, $u\in\mathcal{SH}^m\,(\mathbb{R}^n)$, $\mu_u=(-\Delta)^m\,u$. If*

$$\sup_{r>1}\left|\sum_{j=1}^{m}\alpha_{m,j}\,M\left(2^{m-j}r,u\right)\right|<\infty,$$

and one of the conditions

$$(a)\ \ \sup_{r>1}\frac{1}{r^n}\int\limits_{r\leq|x|\leq 2r}|u(x)|^p\,dt<\infty,\ \textit{for some}\ p\in[1,\infty);$$

$$(b)\ \ \frac{u(x)}{|x|^{n/p}}\in L^p\left(\mathbb{R}^n\setminus B(0,1)\right),\qquad \textit{for some}\ p\in[1,\infty],$$

is satisfied, then (5) and (6) hold.

Let us also note that there is a generalization of the Riesz Decomposition Theorem for α-superharmonic functions (fractional $\alpha\in(1,2)$) obtained by N. S. Landkof [9, Ch. I, § 6, Th. 1.30]. It would be very interesting to get an analogous result for fractional $\alpha>2$. However, the method developed by N. S. Landkof seems to be non-applicable in the latter case.

2 Lemmas on Riesz Kernels

We will assume that x,y are vectors in $\mathbb{R}^n$, $m,n,L\in\mathbb{N}$, $n\geq 2$, and that $2m<n$ or $2m-n$ is a positive odd integer.

Following [8], we consider the *generalized Riesz kernels*

$$K_{2m,L}(x,y):=\begin{cases}K_{2m}(x-y), & |y|<1,\\ K_{2m}(x-y)-\sum\limits_{|\nu|\leq L}\frac{x^\nu}{\nu!}\,(D^\nu K_{2m})\,(-y), & |y|\geq 1,\end{cases}\qquad L\in\mathbb{Z}_+.$$

Here and in the sequel, for a multi-index $\nu=(\nu_1,\ldots,\nu_n)$, $\nu_j\in\mathbb{Z}_+$,

$$x^\nu=x_1^{\nu_1}\cdot\dots\cdot x_n^{\nu_n},\quad \nu!=\nu_1!\cdot\dots\cdot\nu_n!,\quad |\nu|=\nu_1+\dots+\nu_n,\quad D^\nu f(x)=\frac{\partial^{|\nu|}f}{\partial x_1^{\nu_1}\ldots\partial x_n^{\nu_n}}.$$

We will also use Δ_x to denote the n-dimentional Laplace operator applied with respect to the variable $x \in \mathbb{R}^n$.

Lemma 2.1 *If $2m < n$ or $2m - n$ is a positive odd integer, then for any $k \in \mathbb{Z}_+$,*

$$\Delta^k K_{2m}(x) = c_{m,n,k} K_{2(m-k)}(x), \tag{8}$$

$$\Delta_x^k K_{2m,2(m-1)}(x, y) = c_{m,n,k} K_{2(m-k),2(m-k-1)}(x, y), \tag{9}$$

where

$$c_{m,n,k} := \begin{cases} 1, & k = 0, \\ 2^k \prod_{j=0}^{k-1} \left((2(m-j)-n)(m-j-1)\right), & 1 \le k \le m-1, \\ 0, & k \ge m. \end{cases}$$

In particular, $K_{2m}(x)$ and $K_{2m,2(m-1)}(x, y)$ (with y as a parameter) are m-harmonic functions in $\mathbb{R}^n \setminus \{0\}$.

Proof First, assume $k = 1$. Since $2m < n$ or $2m - n$ is a positive odd integer, we obtain

$$\frac{\partial^2}{\partial x_j^2}\left(|x|^{2m-n}\right) = (2m-n)\left(|x|^{2m-n-2} + x_j^2(2m-n-2)\,|x|^{2m-n-4}\right). \tag{10}$$

Hence,

$$\Delta_x\left(|x|^{2m-n}\right) = (2m-n)(2m-2)|x|^{2m-n-2}.$$

This gives (8) for $k = 1$. Now, for $|y| \ge 1$, we get

$$\frac{\partial^2}{\partial x_j^2}\left(\sum_{|\nu|\le 2m-2} \frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)$$

$$= \sum_{|\nu|\le 2m-2}\left(\frac{1}{\nu!}\left(D^\nu K_{2m}\right)(-y)\nu_j(\nu_j-1)x^{\nu_j-2}\prod_{k=\overline{1,n},\, k\ne j} x_k^{\nu_k}\right)$$

$$= \sum_{\nu_1+\cdots+\nu_n\le 2(m-1),\ \nu_j\ge 2} \frac{x_1^{\nu_1}}{\nu_1!}\cdots\frac{x_{j-1}^{\nu_{j-1}}}{\nu_{j-1}!}\frac{x_j^{\nu_j-2}}{(\nu_j-2)!}\frac{x_{j+1}^{\nu_{j+1}}}{\nu_{j+1}!}\cdots\frac{x_n^{\nu_n}}{\nu_n!}\left(D^{\nu_1\ldots\nu_n}K_{2m}\right)(-y).$$

Replacing the multi-index ν by $\tilde\nu = (\tilde\nu_1, \ldots, \tilde\nu_n)$ with

$$\tilde\nu_k = \begin{cases} \nu_k, & k \ne j, \\ \nu_j - 2, & k = j, \end{cases}$$

we obtain

$$\frac{\partial^2}{\partial x_j^2}\left(\sum_{|\nu|\le 2m-2} \frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)$$

$$= \sum_{|\tilde\nu|\le 2(m-2)} \frac{x^{\tilde\nu}}{\tilde\nu!}\left(D^{\tilde\nu_1\ldots\tilde\nu_{j-1}(\tilde\nu_j+2)\tilde\nu_{j+1}\ldots\tilde\nu_n}K_{2m}\right)(-y). \tag{11}$$

From (10), it is clear that

$$\left(D^{\tilde\nu_1\ldots\tilde\nu_{j-1}(\tilde\nu_j+2)\tilde\nu_{j+1}\ldots\tilde\nu_n}K_{2m}\right)(-y)=D^{\tilde\nu}\left(\frac{\partial^2}{\partial y_j^2}K_{2m}\right)(-y)$$
$$=(2m-n)D^{\tilde\nu}\left(K_{2(m-1)}+y_j^2(2m-n-2)K_{2(m-2)}\right)(-y).$$

Setting $\nu:=\tilde\nu$ in the right-hand side of (11), and taking the sum over $j=1,\ldots,n$, we deduce

$$\Delta_x\left(\sum_{|\nu|\le 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)$$
$$=(2m-n)(2m-2)\sum_{|\nu|\le 2(m-2)}\frac{x^\nu}{\nu!}\left(D^\nu K_{2(m-1)}\right)(-y). \tag{12}$$

Thus, considering (8), we obtain (9) for $k=1$.

For $k>1$, the statement follows by applying (8) and (9) with $k=1$ repeatedly. □

Lemma 2.2 *If $2m<n$ or $2m-n$ is a positive odd integer, then for any $r>0$,*

$$M\left(r,K_{2m}(\cdot-y)\right)=\begin{cases}\Gamma\left(\frac{n}{2}\right)\sum\limits_{k=0}^{m-1}\left(\frac{|y|}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(\frac{n}{2}+k\right)}r^{2(m-k)-n}, & |y|\le r,\\ \Gamma\left(\frac{n}{2}\right)\sum\limits_{k=0}^{m-1}\left(\frac{r}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(\frac{n}{2}+k\right)}|y|^{2(m-k)-n}, & |y|>r,\end{cases}$$

where $c_{m,n,k}$ are defined in Lemma 2.1.

Moreover, for any $y\neq 0$ and $r>0$,

$$\frac{1}{\sigma_n r^{n-1}}\int_{S(0,r)}\sum_{|\nu|\le 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\,d\sigma(x)$$
$$=\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\left(\frac{r}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(k+\frac{n}{2}\right)}|y|^{2(m-k)-n}. \tag{13}$$

Proof We will use formula (7.11) from [12, Ch. 1.7]:

$$\int_{B(0,r)}f(x+y)\,dx=\sum_{k=0}^{m-1}\frac{\pi^{n/2}r^{2k+n}\left(\Delta^k f\right)(y)}{2^{2k}k!\Gamma\left(k+\frac{n}{2}+1\right)}, \tag{14}$$

which is valid for any function $f\in\mathcal{H}^m(\mathcal{U})$ for some domain $\mathcal{U}$, $y\in\mathcal{U}$, and any $r\in(0,\operatorname{dist}(y,\partial\mathcal{U}))$.

Assume $|y|>r$. Applying (14) with $f=K_{2m},\mathcal{U}=\mathbb{R}^n\setminus\{0\}$, and using Lemma 2.1, we get

$$\int_{B(0,r)}K_{2m}(x-y)\,dx=\sum_{k=0}^{m-1}\frac{\pi^{n/2}r^{2k+n}c_{m,n,k}K_{2(m-k)}(-y)}{2^{2k}k!\Gamma\left(k+\frac{n}{2}+1\right)}$$
$$=\sum_{k=0}^{m-1}\frac{\pi^{n/2}r^{2k+n}c_{m,n,k}|y|^{2(m-k)-n}}{2^{2k}k!\Gamma\left(k+\frac{n}{2}+1\right)}. \tag{15}$$

If we fix y and let $r < |y|$ to be arbitrary, then differentiating the last equality with respect to r, we obtain

$$M\left(r, K_{2m}(\cdot - y)\right) = \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \frac{r^{2k} c_{m,n,k} |y|^{2(m-k)-n}}{2^{2k} k! \Gamma\left(k + \frac{n}{2}\right)}. \tag{16}$$

Now, let $0 < |y| < r$. We cannot apply the above approach since we have a singularity in $B(0, r)$. To get rid of it, we will use the reflection technique as in Kelvin transform, described in [1, Ch. 1, § 1.6]. For $w \neq 0$, we will consider its inverse with respect to the unit sphere $S(0, 1)$:

$$w^* := \frac{1}{|w|^2} w.$$

If $x \in S(0, 1)$, and $y \neq 0$, then

$$|y| \left|x - y^*\right| = |x - y|. \tag{17}$$

Using (17) and making a simple change of variable, we obtain

$$M\left(r, K_{2m}(\cdot - y)\right) = |y|^{2m-n} M\left(1, K_{2m}\left(\cdot - \left(\frac{y}{r}\right)^*\right)\right).$$

Since $\left|\left(\frac{y}{r}\right)^*\right| = \frac{r}{|y|} > 1$, we can apply (16) with $r = 1$ to get

$$M\left(r, K_{2m}(\cdot - y)\right) = \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \left(\frac{|y|}{2}\right)^{2k} \frac{c_{m,n,k}}{k! \Gamma\left(k + \frac{n}{2}\right)} r^{2(m-k)-n}. \tag{18}$$

For $|y| = r$, let $y_l := \left(1 + \frac{1}{l}\right) y$, $l \in \mathbb{N}$. Note that $|x - y| < |x - y_l|$ provided $|x| = r$. Thus, if $2m < n$, we obtain $|x - y_l|^{2m-n} \leq |x - y|^{2m-n}$. Since the function $|x - y|^{2m-n}$ (as a function of x) is in $L^1\left(S(0, r)\right)$, we can apply the Lebesgue Dominated Convergence Theorem to get

$$M\left(r, K_{2m}(\cdot - y)\right) = \lim_{l \to \infty} M\left(r, K_{2m}(\cdot - y_l)\right).$$

If $2m - n \geq 0$, then $|x - y_l|^{2m-n}$ converges to $|x - y|^{2m-n}$ uniformly on $S(0, r)$, and the last equality is obviously justified. Therefore, in either case, applying (16) with $y = y_l$, we deduce

$$\begin{aligned} M\left(r, K_{2m}(\cdot - y)\right) &= \lim_{l \to \infty} \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \frac{r^{2k} c_{m,n,k} |y_l|^{2(m-k)-n}}{2^{2k} k! \Gamma\left(k + \frac{n}{2}\right)} \\ &= \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \left(\frac{|y|}{2}\right)^{2k} \frac{c_{m,n,k}}{k! \Gamma\left(\frac{n}{2} + k\right)} r^{2(m-k)-n}, \quad y \neq 0. \end{aligned}$$

If $y = 0$, then (18) is obvious.

To obtain (13), we shall use (12) to conclude that

$$\Delta_x^k \left(\sum_{|\nu| \leq 2m-2} \frac{x^\nu}{\nu!} \left(D^\nu K_{2m}\right)(-y) \right) = c_{m,n,k} \sum_{|\nu| \leq 2(m-k-1)} \frac{x^\nu}{\nu!} \left(D^\nu K_{2(m-k)}\right)(-y),$$

and then apply (14) with $\mathcal{U} = \mathbb{R}^n$, $y = 0$, and $f(x) = \sum_{|\nu| \leq 2m-2} \frac{x^\nu}{\nu!} \left(D^\nu K_{2m}\right)(-y)$, where y is considered as a constant. Thus, we get

$$\int_{B(0,r)} \sum_{|\nu| \leq 2m-2} \frac{x^\nu}{\nu!} \left(D^\nu K_{2m}\right)(-y)\, dx = \sum_{k=0}^{m-1} \frac{\pi^{n/2} r^{2k+n} c_{m,n,k}}{2^{2k} k! \Gamma\left(k + \frac{n}{2} + 1\right)} |y|^{2(m-k)-n}.$$

This equality is valid for any $y \neq 0$ and $r > 0$. Differentiation on r gives (13). □

Note. There is even more general result on spherical means of the Riesz kernels due to J. S. Brauchart, P. D. Dragnev, E. B. Saff [3, Th. 2]. Their statement covers fractional powers of $|x - y|$, but the answer is given in terms of a hypergeometric function, which makes it more complicated to apply in our proofs.

Integrating the formula for spherical means in Lemma 2.2 on r, we arrive at the following statement.

Lemma 2.3 *If $2m < n$ or $2m - n$ is a positive odd integer, then for any $R > 0$,*

$$\int_{B(0,R)} K_{2m}(x-y)\, dx$$

$$= \begin{cases} 2\pi^{n/2} \sum_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k! \Gamma(\frac{n}{2}+k)} \left(|y|^{2m} \left(\frac{1}{2k+n} - \frac{1}{2(m-k)} \right) + \frac{|y|^{2k} R^{2(m-k)}}{2(m-k)} \right), & |y| \le R, \\ \pi^{n/2} \sum_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k! \Gamma(\frac{n}{2}+k+1)} |y|^{2(m-k)-n} R^{2k+n}, & |y| > R, \end{cases}$$

where $c_{m,n,k}$ are as in Lemma 2.1.

3 Proof of Proposition 1.1

Proof of Proposition 1.1. Note that for $\alpha_{m,1} = 1$ and any $\alpha_{m,j}$, $j \ge 2$, we have

$$\sum_{j=1}^{m} \alpha_{m,j} F_m\left(2^{m-j} r\right) = \sum_{k=0}^{m-1} a_k r^{2k} \left(4^{(m-1)k} + \sum_{j=2}^{m} 4^{(m-j)k} \alpha_{m,j} \right). \tag{19}$$

Let us show that there is a unique set of $\alpha_{m,2}, \ldots, \alpha_{m,m}$, such that

$$4^{(m-1)k} + \sum_{j=2}^{m} 4^{(m-j)k} \alpha_{m,j} = 0, \quad k = 1, \ldots, m-1,$$

which is equivalent to (2) holding for every r and $a_0, \ldots, a_m$. As we will also see, these $\alpha_{m,j}$'s satisfy (3).

We can rewrite the last system as

$$\sum_{j=2}^{m} 4^{(m-j)k} \alpha_{m,j} = -4^{(m-1)k}, \quad k = 1, \ldots, m-1. \tag{20}$$

This is a linear system of $(m-1)$ equations for $(m-1)$ unknowns, whose matrix is

$$\left(\begin{array}{ccccccc|c} 4^{m-2} & 4^{m-3} & \ldots & 4^{m-1-j} & \ldots & 4 & 1 & -4^{m-1} \\ 4^{2(m-2)} & 4^{2(m-3)} & \ldots & 4^{2(m-1-j)} & \ldots & 4^2 & 1 & -4^{2(m-1)} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 4^{l(m-2)} & 4^{l(m-3)} & \ldots & 4^{l(m-1-j)} & \ldots & 4^l & 1 & -4^{l(m-1)} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 4^{(m-1)(m-2)} & 4^{(m-1)(m-3)} & \ldots & 4^{(m-1)(m-1-j)} & \ldots & 4^{m-1} & 1 & -4^{(m-1)(m-1)} \end{array} \right). \tag{21}$$

To evaluate the main determinant of this matrix, let us make a reflection in horizontal direction, so that the last column becomes first, next to the last becomes second, etc.:

$$A := \begin{pmatrix} 1 & 4 & \dots & 4^{j-1} & \dots & 4^{m-3} & 4^{m-2} \\ 1 & 4^2 & \dots & 4^{2(j-1)} & \dots & 4^{2(m-3)} & 4^{2(m-2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots \\ 1 & 4^l & \dots & 4^{l(j-1)} & \dots & 4^{l(m-3)} & 4^{l(m-2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots \\ 1 & 4^{m-1} & \dots & 4^{(m-1)(j-1)} & \dots & 4^{(m-1)(m-3)} & 4^{(m-1)(m-2)} \end{pmatrix}.$$

The main determinant D of system (21) and the determinant of A are related by

$$D = (-1)^{\frac{m-1}{2}(m-2)} \det(A),$$

and the matrix A is a Vandermonde matrix, whose determinant is well known. Thus, we obtain

$$D = (-1)^{\frac{m-1}{2}(m-2)} \prod_{1\le l<j\le m-1} \left(4^j - 4^l\right). \tag{22}$$

Since $D \neq 0$, system (20) has a solution $\alpha_{m,2}, \dots, \alpha_{m,m}$, and this solution is unique.

Now, for $k = 1, \dots, m-1$, let us evaluate the determinant of the left-hand side of the matrix in (21) with k-th column replaced by the right-hand side:

$$D_k = \begin{vmatrix} 4^{m-2} & \dots & 4^{m-k} & -4^{m-1} & 4^{m-k-2} & \dots & 4 & 1 \\ 4^{2(m-2)} & \dots & 4^{2(m-k)} & -4^{2(m-1)} & 4^{2(m-k-2)} & \dots & 4^2 & 1 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 4^{l(m-2)} & \dots & 4^{l(m-k)} & -4^{l(m-1)} & 4^{l(m-k-2)} & \dots & 4^l & 1 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 4^{(m-1)(m-2)} & \dots & 4^{(m-1)(m-k)} & -4^{(m-1)(m-1)} & 4^{(m-1)(m-k-2)} & \dots & 4^{m-1} & 1 \end{vmatrix}.$$

Multiplying the k-th column by -1 and then each column by the reciprocal of its first entry (i.e., multiplying j-th column by the reciprocal of $(1, j)$-entry), we get

$$D_k = -4^{m-2} \dots 4^{m-k} 4^{m-1} 4^{m-k-2} \dots 4 \times$$

$$\begin{vmatrix} 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 4^{m-2} & \dots & 4^{m-k} & 4^{m-1} & 4^{m-k-2} & \dots & 4 & 1 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 4^{(l-1)(m-2)} & \dots & 4^{(l-1)(m-k)} & 4^{(l-1)(m-1)} & 4^{(l-1)(m-k-2)} & \dots & 4^{l-1} & 1 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 4^{(m-2)(m-2)} & \dots & 4^{(m-2)(m-k)} & 4^{(m-2)(m-1)} & 4^{(m-2)(m-k-2)} & \dots & 4^{m-2} & 1 \end{vmatrix}.$$

Taking the transposition and moving the k-th row to the first place, we again arrive at a Vandermonde matrix, whence

$$D_k = (-1)^k 4^{\frac{m}{2}(m-1)-(m-k-1)} \prod_{1\le l<j\le m-1} \left(\theta_{m,j,k} - \theta_{m,l,k}\right). \tag{23}$$

Finally, using Kramer's rule, we conclude that

$$\alpha_{m,k+1} = \frac{D_k}{D}, \quad k = 1, \dots, m-1,$$

whence (3) follows immediately from (22) and (23).

Conversely, if $\alpha_{m,2}, \ldots, \alpha_{m,m}$ satisfy (20), representation (19) yields

$$\sum_{j=1}^{m} \alpha_{m,j} F_m\left(2^{m-j} r\right) = a_0 \left(1 + \sum_{j=2}^{m} \alpha_{m,j}\right) = a_0 \sum_{j=1}^{m} \alpha_{m,j}.$$

□

Note. We can give an explicit representation in (2) for some values of m:

$$\begin{aligned} m = 2: &\quad F_2(2r) - 4F_2(r) = -3a_0; \\ m = 3: &\quad F_3(4r) - 20F_3(2r) + 64F_3(r) = 45a_0; \\ m = 4: &\quad F_4(8r) - 84F_4(4r) + 1344F_4(2r) - 4096F_4(r) = -2835a_0. \end{aligned}$$

4 Spherical Means of *m*-Superharmonic Functions

The key ingredient to the proof of Theorem 1.3 is the following formula for spherical means.

Lemma 4.1 *Let $u \in \mathcal{SH}^m(\mathbb{R}^n)$, and let $\mu_u = (-\Delta)^m u$. Then for $r > 1$,*

$$M(r, u) = \int_{B(0,r)} f(r, y)\, d\mu_u(y) + \sum_{k=0}^{m-1} a_k r^{2k},$$

where a_k's are constants independent of r,

$$\begin{aligned} f(r, y) &= c_{m,n} \Gamma\left(\frac{n}{2}\right) \\ &\times \begin{cases} \sum_{k=0}^{m-1} \left(\frac{|y|}{2}\right)^{2k} \frac{c_{m,n,k}}{k!\Gamma(\frac{n}{2}+k)} r^{2(m-k)-n}, & |y| < 1, \\ \sum_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k!\Gamma(\frac{n}{2}+k)} \left(|y|^{2k} r^{2(m-k)-n} - r^{2k}|y|^{2(m-n)-k}\right) & 1 \le |y| < r, \\ 0, & |y| \ge r, \end{cases} \end{aligned}$$

$c_{m,n,k}$ are as in Lemma 2.1, and $c_{m,n}$ are given by (7), so that

$$c_{m,n}(-\Delta)^m K_{2m,L}(\cdot, y) = \delta_y. \tag{24}$$

Proof It follows from the Riesz decomposition that (see [4, Representation (3.1)]) if $v \in \mathcal{SH}^m(\mathbb{R}^n)$, then

$$v(x) = c_{m,n} \int_{B(0,R)} K_{2m,2(m-1)}(x, y)\, d\mu_v(y) + h_R(x), \quad x \in B(0, R),$$

where $h_R \in \mathcal{H}^m(B(0, R))$. (For (24), see [4, § 3].) Indeed, let us consider the following positive linear functional on $C_0^\infty(B(0, R))$:

$$L_p(\varphi) := \int_{B(0,R)} p(x)(-\Delta)^m \varphi(x)\, dx, \quad \varphi \in C_0^\infty(B(0, R)),$$

where

$$p(x) := c_{m,n} \int_{B(0,R)} K_{2m,2(m-1)}(x, y)\, d\mu_v(y).$$

Using Fubini's theorem and (24), we have

$$\begin{aligned} L_p(\varphi) &= \int_{B(0,R)} \left(c_{m,n} \int_{B(0,R)} K_{2m,2(m-1)}(x,y)\, d\mu_v(y) \right) (-\Delta)^m \varphi(x)\, dx \\ &= \int_{B(0,R)} \varphi(y)\, d\mu_v(y) = L_v(\varphi). \end{aligned}$$

This implies that for a.e. $x \in B(0,R)$, $v(x) - p(x)$ coincides with a function from $\mathcal{H}^m(B(0,R))$. Let us call it $h_R(x)$. Thus, $v(x) = p(x) + h_R(x)$, a.e. Since two m-superharmonic functions, which are equal a.e., are equal identically, we conclude that $v(x) = p(x) + h_R(x)$ in $B(0,R)$.

Therefore, since $u \in \mathcal{SH}^m(\mathbb{R}^n)$, then for any $r_2 > r_1 > 0$

$$u(x) = c_{m,n} \int_{B(0,r_j)} K_{2m,2(m-1)}(x,y)\, d\mu_u(y) + h_{r_j}(x), \quad x \in B\left(0, r_j\right), \quad j = 1, 2, \tag{25}$$

where $h_{r_j} \in \mathcal{H}^m\left(B\left(0, r_j\right)\right)$.

Let us fix two arbitrary r_1 and r_2 (assume $r_1 < r_2$), and take an arbitrary r with $1 < r < r_1 < r_2$. Integrating the last equality over the sphere of radius r, we obtain

$$\begin{aligned} M(r,u) &= \frac{c_{m,n}}{\sigma_n r^{n-1}} \int_{S(0,r)} \int_{B(0,r_j)} K_{2m,2(m-1)}(x,y)\, d\mu_u(y)\, d\sigma(x) \\ &\quad + \frac{1}{\sigma_n r^{n-1}} \int_{S(0,r)} h_{r_j}(x)\, d\sigma(x). \end{aligned} \tag{26}$$

Since $h_{r_j} \in \mathcal{H}^m\left(B\left(0, r_j\right)\right)$, the Almansi expansion (see, e.g., [2, Ch. I, Prop. 1.3]) implies that there exist functions $g_{0,j}, \ldots, g_{m-1,j}$ harmonic in $B\left(0, r_j\right)$, such that

$$h_{r_j}(x) = \sum_{k=0}^{m-1} |x|^{2k} g_{k,j}(x), \quad x \in B\left(0, r_j\right). \tag{27}$$

The mean-value property for harmonic functions yields

$$\frac{1}{\sigma_n r^{n-1}} \int_{S(0,r)} h_{r_j}(x)\, d\sigma(x) = \sum_{k=0}^{m-1} r^{2k} g_{k,j}(0). \tag{28}$$

Changing the order of integration in the first summand of (26) using Fubini's theorem, and applying (28) we get

$$M(r,u) = c_{m,n} \int_{B(0,r_j)} M\left(r, K_{2m,2(m-1)}(\cdot, y)\right) d\mu_u(y) + \sum_{k=0}^{m-1} r^{2k} g_{k,j}(0).$$

Now, Lemma 2.2 implies

$$M(r,u) = \int_{B(0,r_j)} f(r,y)\, d\mu_u(y) + \sum_{k=0}^{m-1} r^{2k} g_{k,j}(0).$$

Since $f(r,y) = 0$ when $|y| > r$, the last equality can be rewritten as

$$M(r,u) - \int_{B(0,r)} f(r,y)\, d\mu_u(y) = \sum_{k=0}^{m-1} r^{2k} g_{k,j}(0). \tag{29}$$

Since the left-hand side is independent of $j \in \{1, 2\}$, so is the right-hand side. But for each $j \in \{1, 2\}$, the expression in the right-hand side is a polynomial in r. Thus, we can rewrite (29) as

$$M(r, u) = \int_{B(0,r)} f(r, y)\, d\mu_u(y) + \sum_{k=0}^{m-1} a_k r^{2k},$$

where

$$a_k := g_{k,1}(0), \quad k = 0, \ldots, m-1, \tag{30}$$

and $r_1 > 1$ is taken arbitrarily. □

Corollary 4.1 *Let $u \in \mathcal{SH}^m(\mathbb{R}^n)$, $2m < n$, and let $\mu_u = (-\Delta)^m u$. Then for any $r > 1$,*

$$\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) = \int_{B(0,r)} \sum_{j=1}^{m} \alpha_{m,j} f\left(2^{m-j} r, y\right) d\mu_u(y)$$
$$+ \sum_{l=1}^{m-1} \int_{B(0,2^l r)\setminus B(0,2^{l-1} r)} \sum_{j=1}^{m-l} \alpha_{m,j} f\left(2^{m-j} r, y\right) d\mu_u(y) + a_0 \sum_{j=1}^{m} \alpha_{m,j}, \tag{31}$$

where $f(r, y)$ is defined in Lemma 4.1, $\alpha_{m,1} = 1$, $\alpha_{m,2}, \ldots, \alpha_{m,m}$ are given by (3) in Proposition 1.1, and a_0 is from the proof of Lemma 4.1.

Furthermore, if $u(0) \neq \infty$, then

$$a_0 = u(0) - c_{m,n} \int_{B(0,1)} |y|^{2m-n}\, d\mu_u(y), \tag{32}$$

where $c_{m,n}$ are given by (7).

Proof Since $f(R, y) = 0$ when $|y| \geq R$, then (31) follows immediately from Lemma 4.1 and Proposition 1.1.

To get a_0, we need to refer the proof of Lemma 4.1. Assume that $u(0) \neq \infty$. Using (25) with some $r_1 > 1$, we conclude that

$$u(0) = c_{m,n} \int_{B(0,r_1)} K_{2m,2(m-1)}(0, y)\, d\mu_u(y) + h_{r_1}(0).$$

Since

$$K_{2m,2(m-1)}(0, y) = \begin{cases} |y|^{2m-n}, & |y| < 1, \\ 0, & |y| \geq 1, \end{cases}$$

we obtain

$$u(0) = c_{m,n} \int_{B(0,1)} |y|^{2m-n}\, d\mu_u(y) + h_{r_1}(0).$$

Now, (32) follows from (27) and (30). □

Note. It is clear that if $h \in \mathcal{H}^m(\mathbb{R}^n)$, then μ_h is a zero measure. Moreover, using the same reasoning as in the proof of Lemma 4.1, we obtain that for any $r > 0$, $M(r, h) = \sum_{k=0}^{m-1} a_k r^{2k}$. Therefore, Proposition 1.1 and (32) imply

$$\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, h\right) = h(0) \sum_{j=1}^{m} \alpha_{m,j}, \quad r > 0. \tag{33}$$

5 Proof of the Riesz Decomposition

The following statement is straightforward.

Lemma 5.1 *Let* $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, $\mu_u = (-\Delta)^m u$, $k = 0, \dots, m-1$, *and*

$$\sup_{r>1} r^{2m-n} \mu_u(B(0,r)) < \infty. \tag{34}$$

Let also $1 \le a \le b$ *and*

$$c_1(b,r,m,n,k) := \int_{B(0,br)\setminus B(0,1)} |y|^{2k} r^{2(m-k)-n}\, d\mu_u(y),$$
$$c_2(a,b,r,m,n,k) := \int_{B(0,br)\setminus B(0,ar)} |y|^{2k} r^{2(m-k)-n}\, d\mu_u(y),$$
$$c_3(a,b,r,m,n,k) := \int_{B(0,br)\setminus B(0,ar)} |y|^{2(m-k)-n} r^{2k}\, d\mu_u(y).$$

Then

$$\sup_{r>1} |c_1(b,r,m,n,k)| < \infty, \quad \sup_{r>1} |c_2(a,b,r,m,n,k)| < \infty, \quad \sup_{r>1} |c_3(a,b,r,m,n,k)| < \infty.$$

Lemma 5.2 *Let* $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, *and* $\mu_u = (-\Delta)^m u$. *If*

$$\sup_{r>1} \left| \sum_{j=1}^m \alpha_{m,j} M\left(2^{m-j} r, u\right) \right| < \infty \quad \text{and} \quad \sup_{r>1} r^{2m-n} \mu_u(B(0,r)) < \infty, \tag{35}$$

then

$$\sup_{r>1} \int_{B(0,r)\setminus B(0,1)} |y|^{2m-n}\, d\mu_u(y) < \infty.$$

Proof Corollary 4.1 implies that

$$\sum_{j=1}^m \alpha_{m,j} M\left(2^{m-j} r, u\right) = \int_{B(0,r)} \sum_{j=1}^m \alpha_{m,j} f\left(2^{m-j} r, y\right) d\mu_u(y) +$$
$$\sum_{l=1}^{m-1} \int_{B(0,2^l r)\setminus B(0,2^{l-1} r)} \sum_{j=1}^{m-l} \alpha_{m,j} f\left(2^{m-j} r, y\right) d\mu_u(y) + a_0 \sum_{j=1}^m \alpha_{m,j}.$$

Let us denote

$$\beta_{m,n,k} := \Gamma\left(\frac{n}{2}\right) \frac{c_{m,n,k}}{4^k k! \Gamma\left(\frac{n}{2} + k\right)}, \tag{36}$$

where $c_{m,n,k}$ are defined in Lemma 2.1.

According to (20), $\sum_{j=1}^m \alpha_{m,j} 4^{(m-j)k} = 0$, $k = 1, \dots, m-1$. Hence,

$$\sum_{j=1}^m \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} \left(2^{m-j} r\right)^{2k} |y|^{2(m-k)-n} = |y|^{2m-n} \sum_{j=1}^m \alpha_{m,j}. \tag{37}$$

Using the representation of $f(r, y)$ given by Lemma 4.1, and (37), we get

$$\sum_{j=1}^m \alpha_{m,j} M\left(2^{m-j} r, u\right) = \int_{B(0,1)} \sum_{j=1}^m \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} |y|^{2k} \left(2^{m-j} r\right)^{2(m-k)-n} d\mu_u(y)$$

$$+\int\limits_{B(0,r)\setminus B(0,1)}\sum_{j=1}^{m}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\,d\mu_u(y)$$

$$-\int\limits_{B(0,r)\setminus B(0,1)}|y|^{2m-n}\,d\mu_u(y)\sum_{j=1}^{m}\alpha_{m,j}$$

$$+\sum_{l=1}^{m-1}\int\limits_{B(0,2^l r)\setminus B(0,2^{l-1}r)}\sum_{j=1}^{m-l}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\,d\mu_u(y)$$

$$-\sum_{l=1}^{m-1}\int\limits_{B(0,2^l r)\setminus B(0,2^{l-1}r)}\sum_{j=1}^{m-l}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}\left(2^{m-j}r\right)^{2k}|y|^{2(m-k)-n}\,d\mu_u(y)$$

$$+a_0\sum_{j=1}^{m}\alpha_{m,j}. \tag{38}$$

It is easy to see that

$$\left|\int\limits_{B(0,1)}\sum_{j=1}^{m}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\,d\mu_u(y)\right|$$

$$\leq r^{2m-n}\mu_u\left(B(0,1)\right)\sum_{j=1}^{m}\left|\alpha_{m,j}\right|\sum_{k=0}^{m-1}\left|\beta_{m,n,k}\right|\left(2^{m-j}\right)^{2(m-k)-n}\to 0,\quad r\to\infty.$$

Hence

$$c_0\left(r,m,n\right):=\int\limits_{B(0,1)}\sum_{j=1}^{m}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\,d\mu_u(y)$$

is bounded as a function of r for $r>1$.

In terms of Lemma 5.1, we can rewrite (38) as

$$\sum_{j=1}^{m}\alpha_{m,j}M\left(2^{m-j}r,u\right)=c_0\left(r,m,n\right)$$

$$+\sum_{j=1}^{m}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}2^{(m-j)(2(m-k)-n)}c_1\left(1,r,m,n,k\right)$$

$$-\int\limits_{B(0,r)\setminus B(0,1)}|y|^{2m-n}\,d\mu_u(y)\sum_{j=1}^{m}\alpha_{m,j}$$

$$+\sum_{l=1}^{m-1}\sum_{j=1}^{m-l}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}2^{(m-j)(2(m-k)-n)}c_2\left(2^{l-1},2^l,r,m,n,k\right)$$

$$-\sum_{l=1}^{m-1}\sum_{j=1}^{m-l}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}4^{k(m-j)}c_3\left(2^{l-1},2^l,r,m,n,k\right)+a_0\sum_{j=1}^{m}\alpha_{m,j}.$$

Thus, Lemma 5.1 and boundedness of $c_0\,(r,m,n)$ imply that

$$\sum_{j=1}^{m}\alpha_{m,j}M\left(2^{m-j}r,u\right)=c\,(r,m,n)-\int\limits_{B(0,r)\setminus B(0,1)}|y|^{2m-n}\,d\mu_u(y)\sum_{j=1}^{m}\alpha_{m,j},$$

where $\sup_{r>1}|c\,(r,m,n)|<\infty$. It is clear from (3) that for any fixed m, $\alpha_{m,j}$'s alternate in sign and grow in absolute value when j increases. Hence $\sum_{j=1}^{m}\alpha_{m,j}\neq 0$. Therefore, if $\sup_{r>1}\left|\sum_{j=1}^{m}\alpha_{m,j}M\left(2^{m-j}r,u\right)\right|<\infty$, we obtain that

$$\sup_{r>1}\int_{B(0,r)\setminus B(0,1)}|y|^{2m-n}\,d\mu_u(y)<\infty.$$

□

As a corollary, we immediately get the following lemma.

Lemma 5.3 *Let $m,n\in\mathbb{N}$, $2m<n$, $u\in\mathcal{SH}^m(\mathbb{R}^n)$, and $\mu_u=(-\Delta)^m u$. If (35) holds, then*

$$\int_{\mathbb{R}^n}(1+|y|)^{2m-n}\,d\mu_u(y)<\infty.$$

Theorem 5.1 *Let $m,n\in\mathbb{N}$, $2m<n$, $u\in\mathcal{SH}^m(\mathbb{R}^n)$, and $\mu_u=(-\Delta)^m u$. Then (35) holds if and only if*

$$\int_{\mathbb{R}^n}(1+|y|)^{2m-n}\,d\mu_u(y)<\infty,\tag{39}$$

and u is of the form

$$u(x)=c_{m,n}\int_{\mathbb{R}^n}K_{2m}(x-y)\,d\mu_u(y)+h(x),\quad x\in\mathbb{R}^n,\tag{40}$$

where $h\in\mathcal{H}^m(\mathbb{R}^n)$, and $c_{m,n}$ are given by (7).

Proof Suppose that (35) holds. Consider the following function

$$U_{2m}^{\mu_u}(x):=\int_{\mathbb{R}^n}|x-y|^{2m-n}\,d\mu_u(y).$$

Let us show that $U_{2m}^{\mu_u}$ is locally integrable in $\mathbb{R}^n$. Indeed, choose an arbitrary $R>0$. It follows from Lemma 2.3 that $\int_{B(0,R)}|x-y|^{2m-n}\,dx$ is continuous on $\mathbb{R}^n$, and

$$\int_{B(0,R)}|x-y|^{2m-n}\,dx\leq\begin{cases}R^{2m}2\pi^{n/2}\sum\limits_{k=0}^{m-1}\frac{|c_{m,n,k}|}{4^k k!\Gamma\left(\frac{n}{2}+k\right)}\left(\frac{1}{m-k}-\frac{1}{2k+n}\right), & |y|\leq R,\\ |y|^{2m-n}R^n\pi^{n/2}\sum\limits_{k=0}^{m-1}\frac{|c_{m,n,k}|}{4^k k!\Gamma\left(\frac{n}{2}+k+1\right)}, & |y|>R.\end{cases}$$

Lemma 5.3 also implies that

$$\int_{\mathbb{R}^n}(1+|y|)^{2m-n}\,d\mu_u(y)<\infty.$$

Hence, for any $R>0$,

$$\int_{\mathbb{R}^n}\left(\int_{B(0,R)}|x-y|^{2m-n}\,dx\right)d\mu_u(y)<\infty.\tag{41}$$

Now, Tonelli-Fubini's Theorem yields that $U_{2m}^{\mu_u} \in L_{loc}^1(\mathbb{R}^n)$. In particular, we have that $U_{2m}^{\mu_u}(x) \neq \infty$ a.e. (in the Lebesgue measure sense) in $\mathbb{R}^n$.

Theorem 1.2 of [10, Ch. 2, S 2.1] implies that $U_{2m}^{\mu_u}$ is lower semicontinuous in $\mathbb{R}^n$.

Furthermore, since $|\cdot - y|^{2m-n}\, in \mathcal{SH}^m(\mathbb{R}^n)$, we conclude that $(-\Delta)^m U_{2m}^{\mu_u}$ is a positive Radon measure in $\mathbb{R}^n$. It is also clear that every point of $\mathbb{R}^n$ is the Lebesgue point of $U_{2m}^{\mu_u}$. Thus, $U_{2m}^{\mu_u} in \mathcal{SH}^m(\mathbb{R}^n)$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Since $c_{m,n}(-\Delta)^m K_{2m} = \delta_0$ (see, e.g., [4]), we may apply Tonelli-Fubini's Theorem (justified by (41)) to get

$$c_{m,n}\int_{\mathbb{R}^n} U_{2m}^{\mu_u}(x)\,(-\Delta)^m\varphi(x)\,dx = \int_{\mathbb{R}^n}\varphi(y)\,d\mu_u(y) = \int_{\mathbb{R}^n} u(x)\,(-\Delta)^m\varphi(x)\,dx.$$

Thus, we have two functions, $U_{2m}^{\mu_u}$ and u, from the class $\mathcal{SH}^m(\mathbb{R}^n)$, such that $(-\Delta)^m\left[c_{m,n}U_{2m}^{\mu_u}\right] = (-\Delta)^m u$ in distributional sense. Using the same reasoning as in the proof of Lemma 4.1, we conclude that $h := u - c_{m,n}U_{2m}^{\mu_u} \in \mathcal{H}^m(\mathbb{R}^n)$. Thus, (40) follows.

Conversely, let $u \in \mathcal{SH}^m(\mathbb{R}^n)$ be of the form (40), where μ_u satisfies (39). Applying Tonelli-Fubini's Theorem, and Lemma 2.2, we obtain that

$$\begin{aligned}
M\left(r, U_{2m}^{\mu_u}\right) &= \int_{\mathbb{R}^n} M\left(r, K_{2m}(\cdot - y)\right)\,d\mu_u(y)\\
&= \int_{B(0,r)} \Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\left(\frac{|y|}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(\frac{n}{2}+k\right)} r^{2(m-k)-n}\,d\mu_u(y)\\
&\quad + \int_{\mathbb{R}^n\setminus B(0,r)} \Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\left(\frac{r}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(\frac{n}{2}+k\right)}|y|^{2(m-k)-n}\,d\mu_u(y)
\end{aligned}$$

$$\leq r^{2m-n}\sum_{k=0}^{m-1}\left|\beta_{m,n,k}\right|\int_{B(0,r)} d\mu_u(y) + \sum_{k=0}^{m-1}\left|\beta_{m,n,k}\right|\int_{\mathbb{R}^n\setminus B(0,r)}|y|^{2m-n}\,d\mu_u(y),$$

where $\beta_{m,n,k}$ are defined by (36). Now, if $r > 1$, we get

$$\int_{B(0,r)\setminus B(0,1)} r^{2m-n}\,d\mu_u(y) \leq \int_{B(0,r)\setminus B(0,1)}|y|^{2m-n}\,d\mu_u(y).$$

Therefore,

$$M\left(r, U_{2m}^{\mu_u}\right) \leq \left(r^{2m-n}\mu_u\left(B(0,1)\right) + \int_{\mathbb{R}^n\setminus B(0,1)}|y|^{2m-n}\,d\mu_u(y)\right)\sum_{k=0}^{m-1}\left|\beta_{m,n,k}\right|.$$

Since (39) holds, we conclude that $\sup_{r>1} M\left(r, U_{2m}^{\mu_u}\right) < \infty$, whence

$$\sup_{r>1}\left|\sum_{j=1}^{m}\alpha_{m,j}M\left(2^{m-j}r, U_{2m}^{\mu_u}\right)\right| < \infty. \tag{42}$$

Now, from (40), (42) and (33), we deduce that

$$\sup_{r>1}\left|\sum_{j=1}^{m}\alpha_{m,j}M\left(2^{m-j}r, u\right)\right| < \infty.$$

Finally, for any $r > 1$ we have

$$r^{2m-n}\int_{B(0,r)} d\mu_u(y) \leq \mu_u\left(B(0,1)\right) + 2^{n-2m}\int_{B(0,r)\setminus B(0,1)}(1+|y|)^{2m-n}\,d\mu_u(y).$$

Hence,

$$\sup_{r>1} r^{2m-n}\mu_u(B(0,r)) \le \mu_u(B(0,1)) + 2^{n-2m}\int_{\mathbb{R}^n}(1+|y|)^{2m-n}\,d\mu_u(y) < \infty.$$

□

To prove Theorem 1.3, it remains to replace the condition

$$\sup_{r>1} r^{2m-n}\mu_u(B(0,r)) < \infty$$

by another one which would be easier to check having a particular function $u \in \mathcal{SH}^m(\mathbb{R}^n)$. The replacement is given by the following lemma.

Lemma 5.4 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, and $\mu_u = (-\Delta)^m u$. The following are equivalent:*

$$\begin{aligned}
&(a)\ \sup_{r>1} r^{2m-n}\mu_u(B(0,r)) < \infty;\\
&(b)\ \sup_{r>1}\int_{1\le|t|\le 2} u(rt)(-\Delta)^m\varphi(t)\,dt < \infty,\ \textit{for some } \varphi\in\mathcal{R};\\
&(c)\ \sup_{r>1}\int_{1\le|t|\le 2} u(rt)(-\Delta)^m\varphi(t)\,dt < \infty,\ \textit{for any } \varphi\in\mathcal{R}.
\end{aligned}$$

Proof Since $u \in \mathcal{SH}^m(\mathbb{R}^n)$, it is locally integrable, and $d\mu_u(x)$ is a positive Borel measure on $\mathbb{R}^n$. Take any $\varphi \in \mathcal{R}$, $r > 0$, and let $\Phi(x) := \varphi(x/r)$. Since $\Phi \in C_0^\infty(\mathbb{R}^n)$ and $\Phi(x) \equiv 1$ in $\overline{B(0,r)}$, we obtain that

$$\begin{aligned}
\mu_u(B(0,r)) &= \int_{B(0,r)}\Phi(x)\,d\mu_u(x) \le \int_{B(0,2r)}\Phi(x)\,d\mu_u(x)\\
&= \int_{\mathbb{R}^n} u(x)(-\Delta)^m\Phi(x)\,dx = r^{-2m}\int_{r\le|x|\le 2r} u(x)\left[(-\Delta)^m\varphi\right]\left(\frac{x}{r}\right)dx.
\end{aligned}$$

Making the substitution $t := x/r$ in the last integral, we get

$$r^{2m-n}\mu_u(B(0,r)) \le \int_{1\le|t|\le 2} u(rt)(-\Delta)^m\varphi(t)\,dt, \quad r>0. \tag{43}$$

Analogously, since $0 \le \Phi(x) \le 1$,

$$\mu_u(B(0,2r)) \ge \int_{B(0,2r)}\Phi(x)\,d\mu_u(x) = r^{-2m}\int_{r\le|x|\le 2r} u(x)\left[(-\Delta)^m\varphi\right]\left(\frac{x}{r}\right)dx.$$

Using the substitution $t := x/r$ in the last integral, we arrive at

$$(2r)^{2m-n}\mu_u(B(0,2r)) \ge 2^{2m-n}\int_{1\le|t|\le 2} u(rt)(-\Delta)^m\varphi(t)\,dt, \quad r>0. \tag{44}$$

Now, assume (a) holds. Taking an arbitrary $\varphi \in \mathcal{R}$, we conclude from (44) that

$$\sup_{r>1/2}\int_{1\le|t|\le 2} u(rt)(-\Delta)^m\varphi(t)\,dt \le 2^{n-2m}\sup_{r>1} r^{2m-n}\mu_u(B(0,r)) < \infty,$$

which implies (c), and then, trivially, (b).

If (b) holds with some $\varphi \in \mathcal{R}$, then (43) yields (a) immediately. □

Thus, Theorem 1.3 follows from Theorem 5.1 and Lemma 5.4.

Furthermore, we may use (43) to get easy-to-check sufficient conditions on u to have Riesz representation (6).

Proof of Corollary 1.1. Applying Hölder's inequality to the right-hand side of (43), and using the substitution $x = rt$, we conclude that for any $p \in [1, \infty)$ and q, such that $1/p + 1/q = 1$,

$$r^{2m-n} \mu\left(B(0,r)\right) \leq \left\| (-\Delta)^m \varphi \right\|_{L^q\left(\overline{B(0,2)} \setminus B(0,1)\right)} \left(\frac{1}{r^n} \int_{r \leq |x| \leq 2r} |u(x)|^p \, dt \right)^{1/p}$$
$$\leq 2^{n/p} \left\| (-\Delta)^m \varphi \right\|_{L^q\left(\overline{B(0,2)} \setminus B(0,1)\right)} \left(\int_{|x| \geq 1} \frac{|u(x)|^p}{|x|^n} \, dt \right)^{1/p}.$$

If $p = \infty$, then clearly,

$$r^{2m-n} \mu_u\left(B(0,r)\right) \leq \left\| (-\Delta)^m \varphi \right\|_{L^1\left(\overline{B(0,2)} \setminus B(0,1)\right)} \operatorname*{ess\,sup}_{r \leq |x| \leq 2r} |u(x)|.$$

Thus, if either condition, (a) or (b) is satisfied, then $\sup_{r>1} r^{2m-n} \mu_u\left(B(0,r)\right) < \infty$. Applying Theorem 5.1, we get relation (5), and representation (6). □

Open Problem. It would be interesting to generalize Theorem 1.3 to the case of α-superharmonic functions in $\mathbb{R}^n$. We have already mentioned a formula for spherical means of Riesz kernels obtained in [3], which could be a good starting point. Although it is unclear what should be a condition replacing the boundedness of the linear combination of spherical means $\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right)$ in the case of a fractional power of Laplacian $\alpha/2$ instead of m.

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