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Dirichlet Heat Kernel Estimates for Stable Processes with Singular Drift In Unbounded $C^{1,1}$ Open Sets

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Abstract Suppose $d \geq 2$ and $\alpha \in (1, 2)$. Let D be a (not necessarily bounded) $C^{1,1}$ open set in $\mathbb{R}^d$ and $\mu = (\mu^1, \dots, \mu^d)$ where each μ^j is a signed measure on $\mathbb{R}^d$ belonging to a certain Kato class of the rotationally symmetric α-stable process X. Let X^μ be an α-stable process with drift μ in $\mathbb{R}^d$ and let $X^{\mu,D}$ be the subprocess of X^μ in D. In this paper, we derive sharp two-sided estimates for the transition density of $X^{\mu,D}$.

Keywords Symmetric α-stable process · Gradient operator · Heat kernel · Transition density · Green function · Exit time · Lévy system · Boundary Harnack inequality · Kato class

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1 Introduction

Markov processes with discontinuous sample paths constitute an important family of stochastic processes in probability theory. Recently there has been intense interest in obtaining sharp two-sided estimates on the transition density $p_D(t, x, y)$ of Markov processes

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with discontinuous sample paths in open subset D of $\mathbb{R}^d$. If the Markov process is a (rotationally) symmetric α-stable process in $\mathbb{R}^d$, an explicit form of estimates in terms of t, $|x-y|$ and the distance to boundary of D was obtained in [7] when D is $C^{1,1}$ open set, and a Varopoulos type estimate in terms of surviving probabilities and the global transition density was obtained in [3] when D is a non-smooth κ-fat open set. Very recently in [8, 9, 11], sharp two-sided estimates on the transition density $p_D(t,x,y)$ were established for several symmetric Markov processes such as relativistic stable processes, mixed stable processes and censored stable processes in $C^{1,1}$ open subsets of $\mathbb{R}^d$, respectively.

When b is an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ belonging to a certain Kato class of the rotationally symmetric α-stable process, in [10], jointly with Zhen-Qing Chen, we showed that there is a non-symmetric Feller process with generator $\Delta^{\alpha/2} + b \cdot \nabla$ (called an α-stable process with drift b) and derived sharp two-sided estimates on the transition density of such process in a bounded $C^{1,1}$ open set D in $\mathbb{R}^d$. Independently in [5], sharp estimates on the Green functions of subprocesses, in bounded $C^{1,1}$ open sets in $\mathbb{R}^d$, of such process were investigated. The purpose of this paper is, through a somewhat different approach, to extend the main result of [10] to allow D being unbounded and the drift being a measure. This paper is a natural continuation of [21] where the existence and uniqueness of α-stable process with a singular measure-valued drift were established.

Throughout this paper we assume $d \geq 2, \alpha \in (1,2)$ and that X is a (rotationally) symmetric α-stable process in $\mathbb{R}^d$. The infinitesimal generator of X is $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. The transition density of X is denoted by $p(t,x,y)$. We will use $B(x,r)$ to denote the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$. The space of continuous functions on $\mathbb{R}^d$ will be denoted as $C(\mathbb{R}^d)$, while $C_\infty(\mathbb{R}^d)$ and $C_c^\infty(\mathbb{R}^d)$ denote the space of continuous functions on $\mathbb{R}^d$ that vanish at infinity and the space of smooth functions with compact supports respectively.

By a signed measure ν we mean in this paper the difference of two nonnegative σ-finite measures ν_1 and ν_2 in $\mathbb{R}^d$. Since there is an increasing sequence of subsets $\{F_k, k \geq 1\}$ of $\mathbb{R}^d$ such that $|\nu|_{F_k}$ is a finite measure, the positive and negative parts of ν are well defined on each F_k and hence on $\mathbb{R}^d$, which will be denoted as ν^+ and ν^-, respectively. We use $|\nu| = \nu^+ + \nu^-$ to denote the total variation measure of ν.

Definition 1.1 For any signed measure ν on $\mathbb{R}^d$, we define for any $r > 0$,

$$M_\nu^\alpha(r) = \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \frac{|\nu|(dy)}{|x-y|^{d+1-\alpha}}.$$

A signed measure ν on $\mathbb{R}^d$ is said to belong to the Kato class $\mathbb{K}_{d,\alpha-1}$ if $\lim_{r\downarrow 0} M_\nu^\alpha(r) = 0$. We say that an $\mathbb{R}^d$-valued signed measure $\mu = (\mu^1, \cdots, \mu^d)$ on $\mathbb{R}^d$ belongs to the Kato class $\mathbb{K}_{d,\alpha-1}$ if each μ^j belongs to the Kato class $\mathbb{K}_{d,\alpha-1}$.

Since $1 < \alpha < 2$, using Hölder's inequality, it is easy to see that, if $p > d/(\alpha-1)$, for every function $f \in L^\infty(\mathbb{R}^d; dx) + L^p(\mathbb{R}^d; dx)$, $f(x)dx$ is in the Kato class $\mathbb{K}_{d,\alpha-1}$. Note that any signed measure ν on $\mathbb{R}^d$ is Radon.

Throughout this paper we will assume that $\mu = (\mu^1, \cdots, \mu^d)$, where each μ^j is a signed measure on $\mathbb{R}^d$ belonging to $\mathbb{K}_{d,\alpha-1}$. Recently, in [21], we proved the existence and uniqueness of the α-stable process X^μ with drift μ in $\mathbb{R}^d$. Similar to [4], for small $t > 0$, the transition density $p^\mu(t,x,y)$ of X^μ can be expressed as an infinite series $\sum_{k=0}^\infty p_k^\mu(t,x,y)$, where $p_0^\mu(t,x,y)=p(t,x,y)$ and for $k\geq 1$,

$$p_k^\mu(t,x,y) := \int_0^t \int_{\mathbb{R}^d} p_{k-1}^\mu(s,x,z)\nabla_z p(t-s,z,y)\cdot\mu(dz). \tag{1.1}$$

We will use $\left\{P_t^\mu; t\geq 0\right\}$ to denote the transition semigroup of X^μ.

The following result is shown in [21]. Here and in the sequel, we use := as a way of definition. For $a,b\in\mathbb{R}$, $a\wedge b := \min\{a,b\}$ and $a\vee b := \max\{a,b\}$.

Theorem 1.2

(i) *There exist $T_0>0$ and $c_1>1$ depending on μ only via the rate at which $M_\mu^\alpha(r)$ goes to zero such that $\sum_{k=0}^\infty p_k^\mu(t,x,y)$ converges locally uniformly on $(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$ to a positive jointly continuous function $p^\mu(t,x,y)$ and that on $(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$c_1^{-1}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\leq p^\mu(t,x,y)\leq c_1\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right). \tag{1.2}$$

Moreover, $\int_{\mathbb{R}^d}p^\mu(t,x,y)dy=1$ for every $t\in(0,T_0]$ and $x\in\mathbb{R}^d$.

(ii) *The function $p^\mu(t,x,y)$ defined in (i) can be extended uniquely to a positive jointly continuous function on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ so that for all $s,t\in(0,\infty)$ and $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$, $\int_{\mathbb{R}^d}p^\mu(t,x,y)dy=1$ and*

$$p^\mu(s+t,x,y)=\int_{\mathbb{R}^d}p^\mu(s,x,z)p^\mu(t,z,y)dz. \tag{1.3}$$

(iii) *X^μ is a conservative Feller process with the strong Feller property admitting $p^\mu(t,x,y)$ as its transition density. It is also the unique weak solution to the stochastic differential equation $dX_t^\mu = dX_t+dA_t$, where, for $j=1,\cdots,d$, the j-th component of A_t is a continuous additive functional of finite variation with respect to X^μ and with Revuz measure μ^j.*

(iv) *For any $f\in C_c^\infty(\mathbb{R}^d)$ and $g\in C_\infty(\mathbb{R}^d)$,*

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^d}\left(P_t^\mu f(x)-f(x)\right)g(x)dx=\int_{\mathbb{R}^d}g(x)\Delta^{\alpha/2}f(x)dx+\int_{R^d}g(x)\nabla f(x)\cdot\mu(dx). \tag{1.4}$$

Here and in the rest of this paper, the meaning of the phrase "depending on μ only via the rate at which $M_\mu^\alpha(r)$ goes to zero" is that the statement is true for any $\mathbb{R}^d$-valued signed measure ν on $\mathbb{R}^d$ with

$$M_\nu^\alpha(r)\leq M_\mu^\alpha(r)\quad\text{for all } r>0.$$

For any open subset $D\subset\mathbb{R}^d$, we define $\tau_D^\mu=\inf\left\{t>0: X_t^\mu\notin D\right\}$. We will use $X^{\mu,D}$ to denote the subprocess of X^μ in D; that is, $X_t^{\mu,D}(\omega)=X_t^\mu(\omega)$ if $t<\tau_D^\mu(\omega)$ and $X_t^{\mu,D}(\omega)=\partial$ if $t\geq\tau_D^\mu(\omega)$, where ∂ is a cemetery state. The subprocess of X in D will be denoted by X^D. Throughout this paper, we use the convention that, for any function f, we extend its definition to ∂ by setting $f(\partial)=0$. The process $X^{\mu,D}$ has a transition density $p_D^\mu(t,x,y)$ with respect to the Lebesgue measure. (See Eq. 2.6 below.) The transition density of X^D is denoted by $p_D(t,x,y)$.

The purpose of this paper is to establish sharp two-sided estimates on $p_D^\mu(t,x,y)$ when D is a (possibly unbounded) $C^{1,1}$ open subset of $\mathbb{R}^d$. To state the main result of this paper, we first recall that an open set D in $\mathbb{R}^d$ is said to be a $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0, \dots, 0)$, $\|\nabla\phi\|_\infty \le \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(w)| \le \Lambda_0|x - w|$, and an orthonormal coordinate system CS_z: $y = (y_1, \cdots, y_{d-1}, y_d) := (\widetilde{y}, y_d)$ with its origin at z such that

$$B(z, R_0) \cap D = \{y \in B(0, R_0) \text{ in } CS_z : y_d > \phi(\widetilde{y})\}.$$

The pair (R_0, Λ_0) is called the characteristics of the $C^{1,1}$ open set D. We remark that in some literature, the $C^{1,1}$ open set defined above is called a *uniform* $C^{1,1}$ open set as (R_0, Λ_0) is universal for every $z \in \partial D$. For $x \in D$, let $\delta_D(x)$ denote the Euclidean distance between x and ∂D. Note that a $C^{1,1}$ open set may be disconnected.

Define

$$f_D(t, x, y) = \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).$$

Theorem 1.3 *Let D be a $C^{1,1}$ open subset of $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R_0, Λ_0). Suppose that X^μ is an α-stable process with drift μ in $\mathbb{R}^d$.*

(i) *For each $T > 0$, there exists a constant $c_1 = c_1(T, R_0, \Lambda_0, d, \alpha, \mu) \ge 1$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that on $(0, T] \times D \times D$,*

$$c_1^{-1} f_D(t, x, y) \le p^\mu_D(t, x, y) \le c_1 f_D(t, x, y).$$

(ii) *Suppose in addition that D is bounded. For each $T > 0$, there exists a constant $c_2 = c_2(diam(D), T, R_0, \Lambda_0, d, \alpha, \mu) \ge 1$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero so that for all $(t, x, y) \in [T, \infty) \times D \times D$,*

$$c_2^{-1} e^{-\lambda_1 t} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \le p^\mu_D(t, x, y) \le c_2 e^{-\lambda_1 t} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2},$$

where $-\lambda_1 := \sup \mathrm{Re}(\sigma(\mathcal{L}|_D)) < 0$ and $\mathcal{L}|_D$ is the generator of X^D.

Sharp two-sided estimates for $p_D(t, x, y)$, corresponding to the case $\mu = 0$ in Theorem 1.3, were first established in [7]. When D is a bounded $C^{1,1}$ open set and $\mu(dx) = b(x)dx$ for some $\mathbb{R}^d$-valued function $b(x)$ on $\mathbb{R}^d$ belonging to the Kato class $\mathbb{K}_{d,\alpha-1}$, Theorem 1.3 was established in [10]. However, the argument of [10] used the boundedness of D in an essential way and does not work when D is unbounded. Theorem 1.3 indicates that short time Dirichlet heat kernel estimates for the fractional Laplacian in $C^{1,1}$ open sets are stable under gradient perturbations. We also establish a boundary Harnack principle for X^μ (Theorem 5.8), which extends the corresponding result in [10]. We remark here that, unlike [10], the boundary Harnack principle will be used to prove Theorem 1.3

In the remainder of this paper, the constants $C_1, C_2, C_3, C_4, r_0, r_1, r_2, r_3, r_4$ will be fixed throughout this paper. The lower case constants $c_1, c_2, \dots$ can change from one appearance to another. The dependence of the constants on the dimension $d \ge 2$ and the stability index $\alpha \in (1, 2)$ will not be always mentioned explicitly. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure. For two non-negative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 so that $c_1 g(x) \le f(x) \le c_2 g(x)$ in the common domain of definition for f and g.

2 Stable Process with Drift μ

In this section we discuss some basic properties of the α-stable process X^μ with drift μ.

Recall that we always assume that $d \geq 2$ and $\alpha \in (1, 2)$. A (rotationally) symmetric α-stable process $X = \left\{X_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\right\}$ in $\mathbb{R}^d$ is a Lévy process such that

$$\mathbb{E}_x \left[e^{i\xi\cdot(X_t - X_0)}\right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

It is well-known that the symmetric stable process X has Lévy density

$$J(x, y) = \mathcal{A}(d, -\alpha)|x - y|^{-(d+\alpha)}$$

where $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma\left(\frac{d+\alpha}{2}\right)\Gamma\left(1 - \frac{\alpha}{2}\right)^{-1}$ with Γ being the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$ for every $\lambda > 0$.

The Lévy density gives rise to a Lévy system (N, H) for X, where $N(x, dy) = J(x, y)dy$ and $H_t = t$, which describes the jumps of the process X: for any $x \in \mathbb{R}^d$ and any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s, x, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ and stopping time T (with respect to the filtration of X),

$$\mathbb{E}_x \left[\sum_{s\leq T} f(s, X_{s-}, X_s)\right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X_s, y)J(X_s, y)dy\right) ds\right].$$

(See, for example, [13, Proof of Lemma 4.7] and [14, Appendix A].)

The infinitesimal generator of this process X is the fractional Laplacian $\Delta^{\alpha/2}$, which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2}u(x) = \lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^d:\, |y-x|>\varepsilon\}} (u(y) - u(x))J(x, y)\, dy. \tag{2.1}$$

Recall that $p(t, x, y)$ stands for the transition density of X (or equivalently the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$). It is well-known (see, e.g., [1, 13]) that

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \qquad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

Recall that X^D is the subprocess of X killed upon leaving an open set D. Denote the Green function of X^D by G_D. It is known that

$$|\nabla_z G_D(z, y)| \leq \frac{d}{|z - y| \wedge \delta_D(z)} G_D(z, y). \tag{2.2}$$

(See [6, Corollary 3.3].)

Recall that X^μ is the solution to the stochastic differential equation

$$dX_t^\mu = dX_t + dA_t, \tag{2.3}$$

where X_t is a symmetric α-stable process and, where, for $j = 1, \cdots, d$, the j-th component of A_t is a continuous additive functional of finite variation with respect to X^μ and with Revuz measure μ^j.

By the semigroup property of $p^\mu(t,x,y)$ and Eq. 1.2 (which are proved in [21]), there are constants $c \geq 1$ and $C_1 > 0$ depending only on d, α and μ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$,

$$c^{-1}e^{-C_1 t}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\leq p^\mu(t,x,y)\leq ce^{C_1 t}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right). \quad (2.4)$$

Recall that, for any open set D of $\mathbb{R}^d$, $X^{\mu,D}$ stands for the subprocess of X^μ killed upon exiting D. Let

$$k^\mu_D(t,x,y) := \mathbb{E}_x\left[p^\mu\left(t-\tau^\mu_D, X^\mu_{\tau^\mu_D}, y\right);\ \tau^\mu_D < t\right]$$

and

$$p^\mu_D(t,x,y) := p^\mu(t,x,y) - k^\mu_D(t,x,y). \quad (2.5)$$

Then $p^\mu_D(t,x,y)$ is the transition density of $X^{\mu,D}$. This is because by the strong Markov property of X^μ, for every $t>0$ and Borel set $A\subset\mathbb{R}^d$,

$$\mathbb{P}_x\left(X^{\mu,D}_t\in A\right) = \int_A p^\mu_D(t,x,y)dy. \quad (2.6)$$

Using the conservativeness of X^μ and Eq. 2.4, the proof of the next lemma is standard (for example, see [18, Lemma 6.1] and [10, Lemma 3.7]). So we omit the proof.

Lemma 2.1 *For any bounded open set D, there exist positive constants c_1 and c_2 depending only on d, α,* diam(D) *and μ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that*

$$p^\mu_D(t,x,y)\leq c_1e^{-c_2t},\quad (t,x,y)\in(1,\infty)\times D\times D.$$

Combining the result above with Eq. 1.2 we know that for every bounded open set D, there exists a positive constant $c_1 = c_1(\mathrm{diam}(D),\mu)$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that for any $(t,x,y)\in(0,\infty)\times D\times D$,

$$p^\mu_D(t,x,y)\ \leq\ c_1\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x-y|^{d+\alpha}}\right).$$

Therefore for every bounded open set D the Green function $G^\mu_D(x,y) := \int_0^\infty p^\mu_D(t,x,y)dt$ is finite and continuous off the diagonal of $D\times D$ and

$$G^\mu_D(x,y)\ \leq\ c_2\frac{1}{|x-y|^{d-\alpha}} \quad (2.7)$$

for some positive constant $c_2 = c_2(\mathrm{diam}(D),\mu)$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero.

Let $N(dt,dx)$ be the Poisson random measure describing the jumps of the stable process X, that is, for any $A\subset\mathbb{R}^d$ and $t>0$,

$$N(t,A) = \#\{s\leq t : X_s - X_{s-}\in A\}.$$

It is well-known that the intensity of the Poisson random measure N is $J(x)dxdt$. We will use $\widetilde{N}$ to denote the compensator of N:

$$\widetilde{N}(t, A) = N(t, A) - t \int_A J(x)dx.$$

Since X^μ is a solution of Eq. 2.3, by Ito's formula, we have that, for any $f \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} f\left(X_t^\mu\right) - f\left(X_0^\mu\right) = & \sum_{i=1}^d \int_0^t \partial_i f\left(X_{s-}^\mu\right) dA_s^i + \int_0^t \int_{|x|<1} \left[f\left(X_{s-}^\mu + x\right) - f\left(X_{s-}^\mu\right)\right] \widetilde{N}(ds, dx) \\ & + \int_0^t \int_{|x|\geq 1} \left[f\left(X_{s-}^\mu + x\right) - f\left(X_{s-}^\mu\right)\right] N(ds, dx) \\ & + \int_0^t \int_{|x|<1} \left[f\left(X_{s-}^\mu + x\right) - f\left(X_{s-}^\mu\right) - x \cdot \nabla f\left(X_{s-}^\mu\right)\right] J(x)dxds. \end{aligned} \tag{2.8}$$

The process

$$M_t^f := \int_0^t \int_{|x|<1} \left[f\left(X_{s-}^\mu + y\right) - f\left(X_{s-}^\mu\right)\right] \widetilde{N}(ds, dy) \tag{2.9}$$

is a $\mathbb{P}_x$-martingale for each $x \in \mathbb{R}^d$. Thus, using Eq. 2.8, we easily get the following Dynkin's formula.

Proposition 2.2 *For any $f \in C_c^\infty(\mathbb{R}^d)$, any open subset U of $\mathbb{R}^d$ and any $x \in U$,*

$$\mathbb{E}_x\left[f\left(X_{\tau_U^\mu}^\mu\right)\right] = f(x) + \sum_{i=1}^d \mathbb{E}_x \int_0^{\tau_U^\mu} \partial_i f\left(X_{s-}^\mu\right) dA_s^i + \mathbb{E}_x \int_0^{\tau_U^\mu} \Delta^{\alpha/2} f\left(X_s^\mu\right) ds. \tag{2.10}$$

3 Process Killed at an Independent Exponential Time

For each $q \geq 0$, we consider the subprocess $X_t^{\mu,q}$ of X_t^μ killed at an independent exponential time $\mathbf{e}$ of parameter q: $X_t^{\mu,q} = X_t^\mu$ when $t \leq \mathbf{e}$ and $X_t^{\mu,q} = \partial$ when $t \geq \mathbf{e}$, where ∂ is a cemetery point. By convention, an exponential random variable with parameter $q = 0$ is identically infinite, and so $X^{\mu,0}$ is simply X^μ. Let $\left\{P_t^{\mu,q}; t \geq 0\right\}$ be the transition semigroup of $X^{\mu,q}$. The transition density of $X^{\mu,q}$ is continuous and given by $p^{\mu,q}(t, x, y) = e^{-qt} p^\mu(t, x, y)$. Thus using the upper bound of $p^\mu(t, x, y)$, we have the following proposition. Since the proof is almost identical to that of [10, Proposition 2.3], we omit the proof.

Proposition 3.1 *For each $q \geq 0$, the semigroup $\left\{P_t^{\mu,q}; t \geq 0\right\}$ is a Feller semigroup. Moreover, it satisfies the strong Feller property; that is, for each $t > 0$, $P_t^{\mu,q} f$ maps bounded measurable functions to continuous functions.*

The following result gives the Lévy system of $X^{\mu,q}$. In the case when $q = 0$ and $\mu(dx) = b(x)dx$ for some $\mathbb{R}^d$-valued function b on $\mathbb{R}^d$ belonging to $\mathbb{K}_{d,\alpha-1}$, the following result was proved in [10] (see [10, Theorem 2.6]).

Theorem 3.2 *For each $q \geq 0$, the Lévy system of $X^{\mu,q}$ is given by (N^q, H^q), where $H_t^q = t$ and for any $x \in \mathbb{R}^d$,*

$$N^q(x, dy) = J(x, y)dy \text{ on } \mathbb{R}^d, \quad N^q(x, \{\partial\}) = q,$$

that is, for any $x \in \mathbb{R}^d$ and any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s, x, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ and stopping time T (with respect to the filtration of $X^{\mu,q}$),

$$\mathbb{E}_x\left[\sum_{s\leq T} f\left(s, X_{s-}^{\mu,q}, X_s^{\mu,q}\right)\right] = \mathbb{E}_x\left[\int_0^T \left(\int_{\mathbb{R}^d} f\left(s, X_s^{\mu,q}, y\right) N^q\left(X_s^{\mu,q}, y\right) dy\right) ds\right]. \tag{3.1}$$

Proof We first consider the case $q = 0$. The proof in this case is similar to that of [10, Theorem 2.6]. For $f \in C_c^\infty(\mathbb{R}^d)$, define M^f as in Eq. 2.9. Suppose that U and V are two compact subsets of $\mathbb{R}^d$ such that the distance between them is positive. Let $f \in C_c^\infty(\mathbb{R}^d)$ with $f = 0$ on U and $f = 1$ on V. Then we know that $N_t^f := \int_0^t \mathbf{1}_U(X_{s-}^\mu)dM_s^f$ is a martingale. Combining Eq. 2.8 with Eq. 2.1, we get that

$$\begin{aligned} N_t^f &= \sum_{s\leq t} \mathbf{1}_U\left(X_{s-}^\mu\right) f\left(X_s^\mu\right) - \int_0^t \mathbf{1}_U\left(X_s^\mu\right)\left(\Delta^{\alpha/2} f\left(X_s^\mu\right)\right) ds \\ &= \sum_{s\leq t} \mathbf{1}_U\left(X_{s-}^\mu\right) f(X_s^\mu) - \int_0^t \mathbf{1}_U\left(X_s^\mu\right) \int_{\mathbb{R}^d} f(y) J\left(X_s^\mu, y\right) dyds. \end{aligned}$$

By taking a sequence of functions $f_n \in C_c^\infty\left(\mathbb{R}^d\right)$ with $f_n = 0$ on U, $f_n = 1$ on V and $f_n \downarrow \mathbf{1}_V$, we get that, for any $x \in \mathbb{R}^d$,

$$\sum_{s\leq t} \mathbf{1}_U\left(X_{s-}^\mu\right) \mathbf{1}_V\left(X_s^\mu\right) - \int_0^t \mathbf{1}_U\left(X_s^\mu\right) \int_V J\left(X_s^\mu, y\right) dyds$$

is a martingale with respect to $\mathbb{P}_x$. Thus,

$$\mathbb{E}_x\left[\sum_{s\leq t} \mathbf{1}_U\left(X_{s-}^\mu\right) \mathbf{1}_V\left(X_s^\mu\right)\right] = \mathbb{E}_x\left[\int_0^t \int_{\mathbb{R}^d} \mathbf{1}_U\left(X_s^\mu\right) \mathbf{1}_V(y) J\left(X_s^\mu, y\right) dyds\right].$$

Using this and a routine measure theoretic arguments, we get

$$\mathbb{E}_x\left[\sum_{s\leq t} f\left(X_{s-}^\mu, X_s^\mu\right)\right] = \mathbb{E}_x\left[\int_0^t \int_{\mathbb{R}^d} f\left(X_s^\mu, y\right) J\left(X_s^\mu, y\right) dyds\right]$$

for any non-negative measurable function f on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$. Finally following the same arguments as in [13, Lemma 4.7] and [14, Appendix A], we get the theorem for $q = 0$.

Now we assume $q > 0$ and fix it. For any $x \in \mathbb{R}^d$, we will use $\widetilde{\mathbb{P}}_x$ to denote the product of the probability $\mathbb{P}_x$ with the probability measure for the independent random variable $\mathbf{e}$, and we will use $\widetilde{\mathbb{E}}_x$ to denote the expectation with respect to $\widetilde{\mathbb{P}}_x$. We will use $\mathbb{R}_\partial^d$ to denote

$\mathbb{R}^d \cup \{\partial\}$. Suppose that F is a nonnegative function on $\mathbb{R}^d \times \mathbb{R}^d_\partial$ which vanishes on the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$ such that $F(\cdot, \partial) = 0$. Then for any $x \in \mathbb{R}^d$ and $t > 0$,

$$\begin{aligned}
&\widetilde{\mathbb{E}}_x \sum_{s \le t} F\left(X^{\mu,q}_{s-}, X^{\mu,q}_s\right) \\
&= \widetilde{\mathbb{E}}_x \sum_{s \le t \wedge \mathbf{e}} F\left(X^{\mu,q}_{s-}, X^{\mu,q}_s\right) \\
&= \widetilde{\mathbb{E}}_x \left[\sum_{s \le t} F\left(X^{\mu,q}_{s-}, X^{\mu,q}_s\right); t < \mathbf{e}\right] + \widetilde{\mathbb{E}}_x \left[\sum_{s \le \mathbf{e}} F\left(X^{\mu,q}_{s-}, X^{\mu,q}_s\right); t \ge \mathbf{e}\right] \\
&= e^{-qt} \mathbb{E}_x \left[\sum_{s \le t} F\left(X^{\mu}_{s-}, X^{\mu}_s\right)\right] + \widetilde{\mathbb{E}}_x \left[\sum_{s < \mathbf{e}} F\left(X^{\mu,q}_{s-}, X^{\mu,q}_s\right); t \ge \mathbf{e}\right] + \widetilde{\mathbb{E}}_x \left[F\left(X^{\mu,q}_{\mathbf{e}-}, \partial\right); t \ge \mathbf{e}\right] \\
&= \mathbb{E}_x \left[e^{-qt} \int_0^t \int_{\mathbb{R}^d} F\left(X^{\mu}_s, y\right) J\left(X^{\mu}_s, y\right) dy ds\right] + \int_0^t q e^{-qr} \mathbb{E}_x \left[\sum_{s < r} F\left(X^{\mu}_{s-}, X^{\mu}_s\right)\right] dr \\
&\quad + \int_0^t q e^{-qr} \mathbb{E}_x \left[F\left(X^{\mu}_{r-}, \partial\right)\right] dr \\
&= \mathbb{E}_x \left[e^{-qt} \int_0^t \int_{\mathbb{R}^d} F\left(X^{\mu}_s, y\right) J\left(X^{\mu}_s, y\right) dy ds\right] + \mathbb{E}_x \left[\int_0^t q e^{-qr} \int_0^r \int_{\mathbb{R}^d} F\left(X^{\mu}_s, y\right) J\left(X^{\mu}_s, y\right) dy ds dr\right] \\
&\quad + \mathbb{E}_x \left[\int_0^t q e^{-qr} F\left(X^{\mu}_r, \partial\right) dr\right] \\
&= \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^d} e^{-qs} F\left(X^{\mu}_s, y\right) J\left(X^{\mu}_s, y\right) dy ds\right] + \mathbb{E}_x \left[\int_0^t q F\left(X^{\mu}_s, \partial\right) e^{-qs} ds\right] \\
&= \widetilde{\mathbb{E}}_x \left[\int_0^t \int_{\mathbb{R}^d} F\left(X^{\mu,q}_s, y\right) J\left(X^{\mu,q}_s, y\right) dy ds\right] + \widetilde{\mathbb{E}}_x \left[\int_0^t q F\left(X^{\mu,q}_s, \partial\right) ds\right] \\
&= \widetilde{\mathbb{E}}_x \left[\int_0^t \int_{\mathbb{R}^d} F\left(X^{\mu,q}_s, y\right) N\left(X^{\mu,q}_s, dy\right) ds\right].
\end{aligned}$$

Thus the assertion of the theorem is valid. □

For any open set D of $\mathbb{R}^d$, we will use $X^{\mu,q,D}$ to denote the subprocess of $X^{\mu,q}$ killed upon exiting D. It is easy to check from the definition that the process $X^{\mu,q,D}$ can also be obtained by killing the process $X^{\mu,D}$ at an independent exponential random variable $\mathbf{e}$. Thus the transition density $p^{\mu,q}_D$ is related to p^{μ}_D as follows:

$$p^{\mu,q}_D(t, x, y) = e^{-qt} p^{\mu}_D(t, x, y), \qquad (t, x, y) \in [0, \infty) \times D \times D. \tag{3.2}$$

For any Borel set $G \subset \mathbb{R}^d$, we define $\tau^{\mu,q}_G = \inf\left\{t > 0 : X^{\mu,q}_t \notin G\right\}$. A point z on the boundary ∂G is said to be a regular boundary point with respect to $X^{\mu,q}$ if $\mathbb{P}_z\left(\tau^{\mu,q}_G = 0\right) = 1$. A Borel set G is said to be regular with respect to $X^{\mu,q}$ if every point in ∂G is a regular boundary point with respect to $X^{\mu,q}$.

The next result follows from Eq. 1.2 and Blumenthal's zero-one law by a routine argument so we omit the proof. See [19, Proposition 2.2].

Proposition 3.3 *Suppose that $q \ge 0$ and that G is a Borel set of $\mathbb{R}^d$ and $z \in \partial G$. If there is a cone A with vertex z such that int$(A) \cap B(z, r) \subset G^c$ for some $r > 0$, then z is a regular boundary point of G with respect to $X^{\mu,q}$.*

This result implies that all Lipschitz open sets, and in particular, all $C^{1,1}$ open sets, are regular with respect to $X^{\mu,q}$.

We will use $\left\{P_t^{\mu,q,D}\right\}$ to denote the semigroup of $X^{\mu,q,D}$. Using some standard arguments (see [10, Theorem 3.4] and its proof), we can show the following. We omit the proof.

Theorem 3.4 *Let $q \geq 0$ and D be an open set in $\mathbb{R}^d$. The transition density $p_D^{\mu,q}(t,x,y)$ is jointly continuous on $(0,\infty) \times D \times D$. Thus for every x,y in D and $t, s > 0$,*

$$p_D^{\mu,q}(t+s,x,y) = \int_D p_D^{\mu,q}(t,x,z) p_D^{\mu,q}(s,z,y) dz. \tag{3.3}$$

The next result is a short time lower bound estimate for $p_D^{\mu,q}(t,x,y)$ near the diagonal. The technique used in its proof is well-known and the full detail is given in the proof of [10, Proposition 3.5].

Proposition 3.5 *For any $a_1 \in (0,1)$, $a_2 > 0$, $a_3 > 0$ and $R > 0$, there is a constant $c = c(a_1, a_2, a_3, R, \mu) > 0$ with the dependence on μ only via the rate at which $M_\mu^\alpha(r)$ goes to zero such that for all $x_0 \in \mathbb{R}^d$, $q \geq 0$ and $r \in (0, R]$,*

$$p_{B(x_0,r)}^{\mu,q}(t,x,y) \geq c e^{-qt} t^{-d/\alpha} \qquad \text{for all } x, y \in B(x_0, a_1 r) \text{ and } t \in \left[a_2 r^\alpha, a_3 r^\alpha\right]. \tag{3.4}$$

Corollary 3.6 *For every $q \geq 0$ and open subset $D \subset \mathbb{R}^d$, $p_D^{\mu,q}(t,x,y)$ is strictly positive.*

Proof See the proof of [10, Corollary 3.6]. □

4 Uniform Estimates on Green Functions

In this section we derive uniform sharp bounds on the Green function $G_U^{\mu,q}$ when U is some small $C^{1,1}$ open set. We first establish a Duhamel's principle for G_D^μ when $\mu|_D$ has compact support in D.

Proposition 4.1 *If D is a bounded open set and $\mu|_D$ has compact support in D, then G_D^μ satisfies*

$$G_D^\mu(x,y) = G_D(x,y) + \int_D G_D^\mu(x,z) \nabla_z G_D(z,y) \cdot \mu(dz). \tag{4.1}$$

Proof The proof of this proposition is similar to that of [10, Proposition 4.2]. Since X^μ is a solution of Eq. 2.3, by Ito's formula, we know that for any $f \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} M_t^f := f\left(X_t^\mu\right) - f\left(X_0^\mu\right) - \sum_{i=1}^d \int_0^t \partial_i f\left(X_s^\mu\right) dA_s^i - \int_0^t \int_{|y|\geq 1} \\ \times \left[f\left(X_{s-}^\mu + y\right) - f\left(X_{s-}^\mu\right)\right] N(ds, dy) \\ - \int_0^t \int_{|y|<1} \left[f\left(X_{s-}^\mu + y\right) - f\left(X_{s-}^\mu\right) - y \cdot \nabla f\left(X_{s-}^\mu\right)\right] J(y) dy ds \end{aligned}$$

is a $\mathbb{P}_x$-martingale for each $x \in \mathbb{R}^d$, where N is the Poisson random measure describing the jumps of the symmetric stable process X. Since $\mu|_D$ has compact support in D, in view of Eqs. 2.2, 2.7 and the fact that $\mu \in \mathbb{K}_{d,\alpha-1}$, $M_{t\wedge\tau_D^\mu}^f$ is a uniformly integrable martingale.

Define $D_j := \{x \in D : \text{dist}(x, D^c) > 1/j\}$. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\phi \geq 1$, supp$[\phi] \subset B(0, 1)$ and $\int_{\mathbb{R}^d} \phi(x)dx = 1$. Let $\phi_n(x) := n^d \phi(nx)$ and for any $\psi \in C_c(D)$, define $f_n := \phi_n * (G_D\psi)$. Clearly $f_n \in C_c^\infty(\mathbb{R}^d)$ and f_n converges uniformly to $G_D\psi$. Fix $j \geq 1$. Since $\mathbb{E}_x\left[M_0^{f_n}\right] = \mathbb{E}_x\left[M_{\tau_{D_j}^\mu}^{f_n}\right]$, and for every $y \in D_j$ and sufficiently large n, $\Delta^{\alpha/2} f_n(y) = \phi_n * \left(\Delta^{\alpha/2} G_D\psi\right)(y) = -\phi_n * \psi(y)$, we have, by Dynkin's formula (2.10), that for sufficiently large n,

$$\begin{aligned}\mathbb{E}_x\left[f_n\left(X^\mu_{\tau^\mu_{D_j}}\right)\right] - f_n(x) &= \int_{D_j} G^\mu_{D_j}(x, y)\Delta^{\alpha/2} f_n(y)dy + \int_{D_j} G^\mu_{D_j}(x, y)\nabla f_n(y) \cdot \mu(dy)\\ &= -\int_{D_j} G^\mu_{D_j}(x, y)\phi_n * \psi(y)dy\\ &\quad + \int_{D_j} G^\mu_{D_j}(x, y)\phi_n * \nabla(G_D\psi)(y) \cdot \mu(dy).\end{aligned}$$

Taking $n \to \infty$, we get, by Eqs. 2.2, 2.7 and the fact that $\mu \in \mathbb{K}_{d,\alpha-1}$,

$$\mathbb{E}_x\left[G_D\psi\left(X^\mu_{\tau^\mu_{D_j}}\right)\right] - G_D\psi(x) = -\int_D G^\mu_{D_j}(x, y)\psi(y)dy + \int_D G^\mu_{D_j}(x, y)\nabla(G_D\psi)(y)\cdot\mu(dy). \tag{4.2}$$

Now using the fact that $\mu|_D$ has compact support in D, taking $j \to \infty$, we have by Eqs.2.2, 2.7 and the fact that $\mu \in \mathbb{K}_{d,\alpha-1}$,

$$-G_D\psi(x) = -\int_D G^\mu_D(x, y)\psi(y)dy + \int_D G^\mu_D(x, y)\nabla(G_D\psi)(y) \cdot \mu(dy)$$

and the continuity of G^μ_D off the diagonal of $D \times D$ that, for each $x \in D$, Eq. 4.1 holds for all $x, y \in D$. □

We derive the following two-sided estimates on the Green functions of subprocesses of X^μ in certain nice open sets when the diameters of such open sets are less than or equal to some constant depending on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero. Using Proposition 4.1, the proofs of Theorems 4.2 and 4.3 below are almost identical to those of the corresponding results in [10]. Thus we omit the proof of Theorems 4.2 and only give a sketch of the proof of Theorem 4.3.

Theorem 4.2 *There exists a constant $r_1 = r_1(\mu) > 0$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that for any ball $B = B(x_0, r)$ of radius $r \leq r_1$,*

$$2^{-1}G_B(x, y) \leq G^\mu_B(x, y) \leq 2G_B(x, y), \quad x, y \in B.$$

For any bounded $C^{1,1}$ open set D with characteristic (R_0, Λ_0), it is well-known (see, for instance [25, Lemma 2.2]) that there exists $L = L(R_0, \Lambda_0, d) > 0$ such that for every $z \in \partial D$ and $r \leq R_0$, one can find a $C^{1,1}$ open set $U_{(z,r)}$ with characteristic $(rR_0/L, \Lambda_0 L/r)$ such that $D \cap B(z, r/2) \subset U_{(z,r)} \subset D \cap B(z, r)$. For the remainder of this paper, given a bounded $C^{1,1}$ open set D, $U_{(z,r)}$ always refers to the $C^{1,1}$ open set above.

Theorem 4.3 *For every $C^{1,1}$ open set D with the characteristic (R_0, Λ_0), there exists a constant $r_2 = r_2(R_0, \Lambda_0, \mu) \in (0, (R_0 \wedge 1)/8]$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that for any $z \in \partial D$ and $r \le r_2$, we have*

$$2^{-1} G_{U_{(z,r)}}(x, y) \le G^\mu_{U_{(z,r)}}(x, y) \le 2 G_{U_{(z,r)}}(x, y), \quad x, y \in U_{(z,r)}. \tag{4.3}$$

Proof Let $U := U(z, r)$ with $r \le R_0$. Using Proposition 4.1 and Eq. 2.7, one can follow the proof of [10, Proposition 4.4] and show that there exists $r_2 = r_2(R_0, \Lambda_0, \mu) \in (0, (R_0 \wedge 1)/8]$ such that Eq. 4.3 holds for $r \le r_2$ when μ is compactly supported in U.

Let

$$\mu_n(x) = \mu|_{U^c} + \mu|_{K_n} \tag{4.4}$$

with K_n being an increasing sequence of compact subsets of U such that $\cup_{n=1}^\infty K_n = U$. Define

$$\mathbb{N}_\mu(t) := \sum_{j=1}^d \sup_{w \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t \left(|w - z|^{-d-1} \wedge s^{-(d+1)/\alpha} \right) ds \left| \mu^j \right| (dz).$$

By following the proof of [4, Lemma 13] line by line, there exists a constant $C_2 > 0$ such that

$$\sum_{j=1}^d \int_{\mathbb{R}^d} \int_0^t p(t-s, x, z) |\nabla_z p(s, z, y)| ds \left| \mu^j \right| (dz) \le C_2 p(t, x, y) \mathbb{N}_\mu(t), \tag{4.5}$$

and so for every $n \ge 1$,

$$\sum_{j=1}^d \int_{\mathbb{R}^d} \int_0^t p(t-s, x, z) |\nabla_z p(s, z, y)| ds \left| \mu^j_n \right| (dz) \le C_2 p(t, x, y) \mathbb{N}_\mu(t). \tag{4.6}$$

Moreover, for every $n \ge 1$,

$$\begin{aligned} &\sum_{j=1}^d \int_{\mathbb{R}^d} \int_0^t p(t-s, x, z) |\nabla_z p(s, z, y)| ds \left| \mu^j - \mu^j_n \right| (dz) \\ &\le C_2 p(t, x, y) \mathbb{N}_{\mu - \mu_n}(t) \\ &= C_2 p(t, x, y) \sup_{w \in \mathbb{R}^d} \sum_{j=1}^d \int_{U \setminus K_n} \int_0^t \left(|w - z|^{-d-1} \wedge s^{-(d+1)/\alpha} \right) ds |\mu^j|(dz). \end{aligned} \tag{4.7}$$

Recall that $p^\mu_k(t, x, y)$, $k \ge 0$, was defined recursively by $p^\mu_0(t, x, y) := p(t, x, y)$ and Eq. 1.1. We define $p^{\mu_n}_k(t, x, y)$ similarly. By Eqs. 4.5–4.6 and induction we have

$$\left| p^\mu_k(t, x, y) \right| \vee \left(\sup_{n \ge 1} \left| p^{\mu_n}_k(t, x, y) \right| \right) \le (C_2 \mathbb{N}_\mu(t))^k p(t, x, y). \tag{4.8}$$

Choose $T_1 > 0$ small so that

$$C_2 \mathbb{N}_\mu(t) < \frac{1}{2}, \quad t \le T_1. \tag{4.9}$$

Then using Eqs. 4.7–4.9 and induction, one can show as in [10, Lemma 4.5] that for all $k \geq 1$ and $(t, x, y) \in (0, T_1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned}&\left|p_k^{\mu_n}(t, x, y) - p_k^{\mu}(t, x, y)\right|\\ \leq &kC_2 2^{-(k-1)} p(t, x, y) \sup_{w \in \mathbb{R}^d} \sum_{j=1}^{d} \int_{U \setminus K_n} \int_0^t \left(|w - z|^{-d-1} \wedge s^{-(d+1)/\alpha}\right) ds |\mu^j|(dz).\end{aligned}$$

Using this and Theorem 1.2(i), by following the first part of the proof of [10, Theorem 4.6], we see that $p^{\mu_n}(t, x, y)$ converges uniformly to $p^{\mu}(t, x, y)$ on any $[a, b] \times \mathbb{R}^d \times \mathbb{R}^d$, where $0 < a < b < \infty$.

Now using Eq. 1.2, Proposition 3.3 and the Lévy system for X^{μ}, one can follow the remainder part of the proof of [10, Theorem 4.6] and show that X^{μ_n} converges to X^{μ} weakly and the boundary of $\left\{t < \tau_U^{\mu}\right\}$ in Skorohod topology on $D\left([0, \infty), \mathbb{R}^d\right)$ is $\mathbb{P}_x$-null for every $x \in U$. Using these and Lemma 2.1 we finally can show that for any bounded continuous function f on $\overline{U}$,

$$\lim_{n \to \infty} \mathbb{E}_x \left[\int_0^{\infty} f\left(X_t^{\mu_n}\right) \mathbf{1}_{\{t < \tau_U^{\mu_n}\}} dt\right] = \lim_{n \to \infty} \int_0^{\infty} \mathbb{E}_x \left[f\left(X_t^{\mu_n}\right) \mathbf{1}_{\{t < \tau_U^{\mu_n}\}}\right] dt = \mathbb{E}_x \left[\int_0^{\infty} f\left(X_t^{\mu}\right) \mathbf{1}_{\{t < \tau_U^{\mu}\}} dt\right],$$

that is, $\lim_{n \to \infty} G_U^{\mu_n} f = G_U^{\mu} f$. Since Eq. 4.3 holds for μ_n, this implies the theorem. □

For the remainder of the paper we alway assume that $q_0 := 2C_1$, where C_1 is the constant in Eq. 2.4. It follows from Eq. 2.4 that there exists a positive constant $c_1 > 0$ such that for every $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_1^{-1} e^{-3C_1 t} \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right) \leq p^{\mu, q_0}(t, x, y) \leq c_1 e^{-C_1 t} \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right). \tag{4.10}$$

Consequently we have that for every $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_2^{-1} e^{-3C_1 t} p(t, x, y) \leq p^{\mu, q_0}(t, x, y) \leq c_2 e^{-C_1 t} p(t, x, y) \tag{4.11}$$

for some constant $c_2 > 1$.

It follows from Eq. 4.11 that, for any open subset D of $\mathbb{R}^d$, the Green function $G_D^{\mu, q_0}(x, y) = \int_0^{\infty} p_D^{\mu, q_0}(t, x, y) dt$ is finite and continuous off the diagonal of $D \times D$ and

$$G_D^{\mu, q_0}(x, y) \leq G^{\mu, q_0}(x, y) \leq c \frac{1}{|x - y|^{d-\alpha}} \tag{4.12}$$

for some positive constant $c = c(\mu)$ with the dependence on μ only via the rate at which $M_{\mu}^{\alpha}(r)$ goes to zero.

Theorem 4.4 *There exists a constant $r_3 = r_3(\mu) > 0$ with the dependence on μ only via the rate at which $M_{\mu}^{\alpha}(r)$ goes to zero such that for any ball $B = B(x_0, r)$ of radius $r \leq r_3$,*

$$4^{-1} G_B(x, y) \leq G_B^{\mu, q_0}(x, y) \leq 2 G_B(x, y), \quad x, y \in B.$$

Proof For any $z \in B(x_0, r)$, let $\left(\mathbb{P}_x^z, X_t^{\mu, B(x_0,r)}\right)$ be the $G^\mu_{B(x_0,r)}(\cdot, z)$-transform of $\left(\mathbb{P}_x, X_t^{\mu, B(x_0,r)}\right)$, that is, for any nonnegative Borel functions f in $B(x_0, r)$,

$$\mathbb{E}_x^z\left[f\left(X_t^{\mu, B(x_0,r)}\right)\right] = \mathbb{E}_x\left[\frac{G^\mu_{B(x_0,r)}\left(X_t^{\mu, B(x_0,r)}, z\right)}{G^\mu_{B(x_0,r)}(x, z)} f\left(X_t^{\mu, B(x_0,r)}\right)\right].$$

It is well-known that the following 3G-inequality holds: for all $r > 0, x_0 \in \mathbb{R}^d, x, y, z \in B(x_0, r)$,

$$\frac{G_{B(x_0,r)}(x, y) G_{B(x_0,r)}(y, z)}{G_{B(x_0,r)}(x, z)} \le c_1\left(|x-y|^{\alpha-d}+|y-z|^{\alpha-d}\right). \tag{4.13}$$

Thus, by applying Theorem 4.2, we have the following 3G-inequality for all $r \le r_1$, $x_0 \in \mathbb{R}^d, x, y, z \in B(x_0, r)$,

$$\frac{G^\mu_{B(x_0,r)}(x, y) G^\mu_{B(x_0,r)}(y, z)}{G^\mu_{B(x_0,r)}(x, z)} \le 8c_1\left(|x-y|^{\alpha-d}+|y-z|^{\alpha-d}\right). \tag{4.14}$$

Using Eq. 4.14 we choose a positive constant $r_3 \le r_1$ such that for any $r \in (0, r_3]$ and all $x, z \in B(x_0, r)$,

$$\mathbb{E}_x^z \tau^\mu_{B(x_0,r)} = \int_{B(x_0,r)} \frac{G^\mu_{B(x_0,r)}(x, y) G^\mu_{B(x_0,r)}(y, z)}{G^\mu_{B(x_0,r)}(x, z)} dy < q_0^{-1} \ln 2.$$

By Jensen's inequality this implies that for any $r \in (0, r_3]$,

$$\mathbb{E}_x^z\left[\exp\left(-q_0 \tau^\mu_{B(x_0,r)}\right)\right] \ge \exp\left(\mathbb{E}_x^z\left[-q_0 \tau^\mu_{B(x_0,r)}\right]\right) \ge 2. \tag{4.15}$$

Since

$$G^{\mu, q_0}_{B(x_0,r)}(x, z) = G^\mu_{B(x_0,r)}(x, z)\, \mathbb{E}_x^z\left[\exp\left(-q \tau^\mu_{B(x_0,r)}\right)\right], \quad x, z \in B(x_0, r),$$

by combining Theorem 4.2 with Eq. 4.15 we have proved theorem. □

Theorem 4.5 *For every $C^{1,1}$ open set D with the characteristic (R_0, Λ_0), there exists a constant $r_4 = r_4(R_0, \Lambda_0, \mu) \in (0, (R_0 \wedge 1)/8]$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that for any $z \in \partial D$ and $r \le r_4$, we have*

$$4^{-1} G_{U_{(z,r)}}(x, y) \le G^{\mu, q_0}_{U_{(z,r)}}(x, y) \le 2 G_{U_{(z,r)}}(x, y), \quad x, y \in U_{(z,r)}. \tag{4.16}$$

Proof Using [20, Theorem 3.3] and Theorem 4.3 instead of Eq. 4.13 and Theorem 4.2 respectively, the proof of the theorem is the same as that of Theorem 4.4. □

We will need the two results above later on.

5 Duality and Uniform Boundary Harnack Principle

Recall that $q_0 = 2C_1$, where C_1 is the constant in Eq. 2.4. We will discuss some basic properties of X^{μ, q_0} and its dual process with respect to some reference measure. The results of this section will be used later in this paper.

By Theorem 3.4 and Corollary 3.6, X^{μ,q_0} has a jointly continuous and strictly positive transition density $p^{\mu,q_0}(t,x,y)$. Thanks to Eq. 4.10, we can define a reference measure as follows.

$$h(x) := \int_{\mathbb{R}^d} G^{\mu,q_0}(y,x)dy \quad \text{and} \quad \xi(dx) := h(x)dx.$$

The following result says that ξ is a reference measure for X^{μ,q_0}.

Proposition 5.1 *ξ is an excessive measure with for X^{μ,q_0}, i.e., for every Borel function $f \geq 0$,*

$$\int_{\mathbb{R}^d} f(x)\xi(dx) \geq \int_{\mathbb{R}^d} \mathbb{E}_x\left[f\left(X_t^{\mu,q_0}\right)\right]\xi(dx).$$

Moreover, h is a strictly positive, bounded continuous function on $\mathbb{R}^d$, in fact, there exists a positive constant $c > 0$ such that $c^{-1} \leq h(x) \leq c$ for all $x \in \mathbb{R}^d$.

Proof The proof of the first claim is the same as the corresponding one in the proof of [10, Proposition 5.2]. So we only prove the second part.

By Eq. 4.11,

$$\begin{aligned} h(x) &= \int_0^\infty \int_{\mathbb{R}^d} p^{\mu,q_0}(t,x,y)dydt \leq c_1 \int_0^\infty e^{-C_1 t} \int_{\mathbb{R}^d} p(t,x,y)dydt \\ &= c_1 \int_0^\infty e^{-C_1 t}dt = c_1/C_1 < \infty \end{aligned}$$

and

$$h(x) \geq c_1^{-1} \int_0^\infty e^{-3C_1 t} \int_{\mathbb{R}^d} p(t,x,y)dydt = c_1^{-1} \int_0^\infty e^{-3C_1 t}dt = 1/(3C_1c_1) > 0.$$

The continuity of h now follows from the continuity of G^{μ,q_0}. □

We define a transition density with respect to the reference measure ξ by

$$\overline{p}^{\mu,q_0}(t,x,y) := \frac{p^{\mu,q_0}(t,x,y)}{h(y)}.$$

Since $p^{\mu,q_0}(t,x,y)$ is jointly continuous and strictly positive, $\overline{p}^{\mu,q_0}(t,x,y)$ is also jointly continuous and strictly positive by Proposition 5.1.

Let

$$\overline{G}^{\mu,q_0}(x,y) := \int_0^\infty \overline{p}^{\mu,q_0}(t,x,y)dt = \frac{G^{\mu,q_0}(x,y)}{h(y)}.$$

Then $\overline{G}^{\mu,q_0}(x,y)$ is the Green function of X^{μ,q_0} with respect to the reference measure ξ.

Before we discuss properties of $\overline{G}^{\mu,q_0}(x,y)$, let us first recall some definitions.

Definition 5.2 Suppose that $q \geq 0$ and that U is an open subset of $\mathbb{R}^d$. A Borel function u on $\mathbb{R}^d$ is said to be

(i) harmonic in U with respect to $X^{\mu,q}$ if

$$\mathbb{E}_x\left[\left|u\left(X^{\mu,q}_{\tau_B^{\mu,q}}\right)\right|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[u\left(X^{\mu,q}_{\tau_B^{\mu,q}}\right)\right], \qquad x \in B, \tag{5.1}$$

for every bounded open set B with $\overline{B} \subset U$;

(ii) excessive with respective to $X^{\mu,q}$ if u is non-negative and

$$u(x) \geq \mathbb{E}_x\left[u\left(X_t^{\mu,q}\right)\right] \quad \textit{and} \quad u(x) = \lim_{t\downarrow 0}\mathbb{E}_x\left[u\left(X_t^{\mu,q}\right)\right], \qquad t>0, x\in\mathbb{R}^d;$$

(iii) a potential with respect to $X^{\mu,q}$ if it is excessive with respect to $X^{\mu,q}$ and for every sequence $\{U_n\}_{n\geq 1}$ of open sets with $\overline{U_n}\subset U_{n+1}$ and $\cup_n U_n = \mathbb{R}^d$,

$$\lim_{n\to\infty}\mathbb{E}_x\left[u\left(X^{\mu,q}_{\tau^{\mu,q}_{U_n}}\right)\right] = 0; \qquad \xi\text{-a.e. } x\in\mathbb{R}^d;$$

(iv) a pure potential with respect to $X^{\mu,q}$ if it is a potential with respect to $X^{\mu,q}$ and

$$\lim_{t\to\infty}\mathbb{E}_x\left[u\left(X_t^{\mu,q}\right)\right] = 0, \qquad \xi\text{-a.e. } x\in\mathbb{R}^d;$$

(v) regular harmonic with respect to $X^{\mu,q}$ in U if u is harmonic with respect to $X^{\mu,q}$ in U and Eq. 5.1 is true for $B=U$.

The following properties of the Green function $\overline{G}^{\mu,q_0}(x,y)$ of X^{μ,q_0} hold.

(A1) $\overline{G}^{\mu,q_0}(x,y) > 0$ for all $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$; $\overline{G}^{\mu,q_0}(x,y)=\infty$ if and only if $x=y\in\mathbb{R}^d$;

(A2) For every $x\in\mathbb{R}^d$, $\overline{G}^{\mu,q_0}(x,\,\cdot\,)$ and $\overline{G}^{\mu,q_0}(\,\cdot\,,x)$ are extended continuous in $\mathbb{R}^d$;

(A3)

$$\sup_{x\in\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\overline{G}^{\mu,q_0}(x,y)\xi(dy) + \int_{\mathbb{R}^d}\overline{G}^{\mu,q_0}(y,x)\xi(dy)\right) < \infty.$$

Clearly (A3) follows from Eq. 4.11, and (A2) follows from the continuity of $\overline{p}^{\mu,q_0}(t,x,y)$ and Eq. 4.11. (A1) follows from Corollary 3.6, Proposition 5.1 and Eq. 4.11.

From (A1)–(A3), we know that the process X^{μ,q_0} satisfies the condition (R) on [16, p. 211] and the conditions (a)–(b) of [16, Theorem 5.4]. It follows from [16, Theorem 5.4] that X^{μ,q_0} satisfies Hunt's Hypothesis (B). Thus, by [16, Theorem 13.24], X^{μ,q_0} has a dual process $\widehat{X}^{\mu,q_0}$, which is a standard process.

Moreover, using Eqs. 4.12 and 3.1 and following the proof of (A4) in [10], we have the following.

(A4) For each y, $x\mapsto\overline{G}^{\mu,q_0}(x,y)$ is excessive with respect to X^{μ,q_0} and harmonic with respect to X^{μ,q_0} in $\mathbb{R}^d\setminus\{y\}$. Moreover, for every open subset U of $\mathbb{R}^d$, we have

$$\mathbb{E}_x\left[\overline{G}^{\mu,q_0}\left(X^{\mu,q_0}_{T_U^{\mu,q_0}},y\right)\right] = \overline{G}^{\mu,q_0}(x,y), \qquad (x,y)\in\mathbb{R}^d\times U, \tag{5.2}$$

where $T_U^{\mu,q_0} := \inf\{t>0 : X_t^{\mu,q_0}\in U\}$. In particular, for every $y\in E$ and $\varepsilon>0$, $\overline{G}^{\mu,q_0}(\,\cdot\,,y)$ is regular harmonic in $\mathbb{R}^d\setminus B(y,\varepsilon)$ with respect to X^{μ,q_0}.

Using our (A1)–(A2), (A4), the proof of the next result is the same as that of [10, Theorem 5.4]. We omit the proof.

Proposition 5.3 *For each $y\in\mathbb{R}^d$, $x\mapsto\overline{G}^{\mu,q_0}(x,y)$ is a pure potential with respect to X^{μ,q_0}. In fact, for every sequence $\{U_n\}_{n\geq 1}$ of open sets with $\overline{U_n}\subset U_{n+1}$ and $\cup_n U_n=\mathbb{R}^d$, and every $x\neq y$ in $\mathbb{R}^d$,*

$$\lim_{n\to\infty}\mathbb{E}_x\left[\overline{G}^{\mu,q_0}\left(X^{\mu,q_0}_{\tau^{\mu,q_0}_{U_n}},y\right)\right] = 0.$$

Moreover, for any $x,y\in\mathbb{R}^d$, we have $\lim_{t\to\infty}\mathbb{E}_x\left[\overline{G}^{\mu,q_0}\left(X_t^{\mu,q_0},y\right)\right]=0.$

Using (A1)–(A4), Eq. 2.7 and Proposition 5.3 we get from [22, 23] that X^{μ,q_0} has a transient Hunt process as a dual.

Theorem 5.4 *There exists a transient Hunt process $\widehat{X}^{\mu,q_0}$ in $\mathbb{R}^d$ such that $\widehat{X}^{\mu,q_0}$ is a strong dual of X^{μ,q_0} with respect to the measure ξ, that is, the density of the semigroup $\{\widehat{P}_t^{\mu,q_0}\}_{t\geq 0}$ of $\widehat{X}^{\mu,q_0}$ is given by $\overline{p}^{\mu,q_0}(t,y,x)$ and thus*

$$\int_{\mathbb{R}^d} f(x)P_t^{\mu,q_0}g(x)\xi(dx) = \int_{\mathbb{R}^d} g(x)\widehat{P}_t^{\mu,q_0}f(x)\xi(dx) \quad \text{for all } f,g\in L^2(\mathbb{R}^d,\xi).$$

In Theorem 3.2, we have determined a Lévy system (N^q, H^q) for $X^{\mu,q}$ with respect to the Lebesgue measure dx. To derive a Lévy system for $\widehat{X}^{\mu,q_0}$, we need to consider a Lévy system for X^{μ,q_0} with respect to the reference measure $\xi(dx)$. One can easily check that, if

$$\overline{N}_0^q(x,dy) := \frac{J(x,y)}{h(y)}\xi(dy) \quad \text{for } (x,y)\in\mathbb{R}^d\times\mathbb{R}^d, \quad \overline{N}_0^q(x,\{\partial\}) := q_0$$

and $\overline{H}_t^{q_0} := t$, then $\left(\overline{N}^{q_0}, \overline{H}^{q_0}\right)$ is a Lévy system for X^{μ,q_0} with respect to the reference measure $\xi(dx)$. It follows from [17] that a Lévy system $\left(\widehat{N}^{q_0}, \widehat{H}^{q_0}\right)$ for $\widehat{X}^{\mu,q_0}$ satisfies $\widehat{H}_t^{q_0} = t$ and

$$\widehat{N}^{q_0}(y,dx)\xi(dy) = \overline{N}^{q_0}(x,dy)\xi(dx).$$

Therefore, using $J(x,y)=J(y,x)$, we have for every stopping time T with respect to the filtration of $\widehat{X}^{\mu,q_0}$,

$$\begin{aligned}\mathbb{E}_x\left[\sum_{s\leq T} f\left(s,\widehat{X}_{s-}^{\mu,q_0},\widehat{X}_s^{\mu,q_0}\right)\right] &= \mathbb{E}_x\left[\int_0^T\left(\int_{\mathbb{R}^d} f(s,\widehat{X}_s^{\mu,q_0},y)\frac{J(\widehat{X}_s^{\mu,q_0},y)}{h(\widehat{X}_s^{\mu,q_0})}\xi(dy)\right)d\widehat{H}_s^{q_0}\right]\\ &= \mathbb{E}_x\left[\int_0^T\left(\int_{\mathbb{R}^d} f(s,\widehat{X}_s^{\mu,q_0},y)\frac{J(\widehat{X}_s^{\mu,q_0},y)h(y)}{h(\widehat{X}_s^{\mu,q_0})}dy\right)ds\right].\end{aligned} \tag{5.3}$$

That is,

$$\widehat{N}^{q_0}(x,dy) = \frac{J(x,y)h(y)}{h(x)}dy.$$

By the definition of ξ and $\overline{p}^{\mu,q_0}$, we have

$$P_t^{\mu,q_0}f(x) = \int_{\mathbb{R}^d}\overline{p}^{\mu,q_0}(t,x,y)f(y)\xi(dy).$$

Let

$$\widehat{P}_t^{\mu,q_0}f(x) := \int_{\mathbb{R}^d}\overline{p}^{\mu,q_0}(t,y,x)f(y)\xi(dy).$$

For any open subset U of $\mathbb{R}^d$, we use $\widehat{X}^{\mu,q_0,U}$ to denote the subprocess of $\widehat{X}^{\mu,q_0}$ in U, i.e., $\widehat{X}_t^{\mu,q_0,U}(\omega) = \widehat{X}_t^{\mu,q_0}(\omega)$ if $t<\widehat{\tau}_U^{\mu,q_0}(\omega)$ and $\widehat{X}_t^{\mu,q_0,U}(\omega)=\partial$ if $t\geq\widehat{\tau}_U^{\mu,q_0}(\omega)$, where

$\widehat{\tau}_U^{\mu,q_0} := \inf\{t > 0 : \widehat{X}_t^{\mu,q_0} \notin U\}$ and ∂ is the cemetery state. Then by [24, Theorem 2 and Remark 2], $X^{\mu,q_0,U}$ and $\widehat{X}^{\mu,q_0,U}$ are dual processes with respect to ξ. Now we let

$$\widehat{p}_U^{\mu,q_0}(t,x,y) := \frac{p_U^{\mu,q_0}(t,y,x)h(y)}{h(x)}. \tag{5.4}$$

By the joint continuity of $p_U^{\mu,q_0}(t,x,y)$ (Theorem 3.4) and the continuity and positivity of h (Proposition 5.1), we know that $\widehat{p}_U^{\mu,q_0}(t,\cdot,\cdot)$ is jointly continuous on $U \times U$. Thus we have the following.

Theorem 5.5 *For every open subset U, $\widehat{p}_U^{\mu,q_0}(t,x,y)$ is strictly positive and jointly continuous on $U \times U$ and is the transition density of $\widehat{X}^{\mu,q_0,U}$ with respect to the Lebesgue measure. Moreover,*

$$\widehat{G}_U^{\mu,q_0}(x,y) := \frac{G_U^{\mu,q_0}(y,x)h(y)}{h(x)} \tag{5.5}$$

is the Green function of $\widehat{X}^{\mu,q_0,U}$ with respect to the Lebesgue measure so that for every nonnegative Borel function f,

$$\mathbb{E}_x\left[\int_0^{\widehat{\tau}_U^{\mu,q_0}} f\left(\widehat{X}_t^{\mu,q_0}\right)dt\right] = \int_U \widehat{G}_U^{\mu,q_0}(x,y)f(y)dy.$$

In the remainder of this section, we will establish a uniform boundary Harnack principle on D for certain harmonic functions of X^{μ}. Since the arguments are mostly similar to those in [10]. We only give a sketch.

A real-valued function u on $\mathbb{R}^d$ is said to be harmonic in an open set $U \subset \mathbb{R}^d$ with respect to $\widehat{X}^{\mu,q_0}$ if for every relatively compact open subset B with $\overline{B} \subset U$,

$$\mathbb{E}_x\left[\left|u\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}_B^{\mu,q_0}}\right)\right|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}_B^{\mu,q_0}}\right)\right] \qquad \text{for every } x \in B. \tag{5.6}$$

A real-valued function u on $\mathbb{R}^d$ is said to be regular harmonic in an open set $U \subset \mathbb{R}^d$ with respect to $\widehat{X}^{\mu,q_0}$ if Eq. 5.6 is true with $B = U$. Clearly, a regular harmonic function in U is harmonic in U.

For any open set U, define the Poisson kernel for X of U as

$$K_U(x,z) := \int_U G_U(x,y)J(y,z)dy, \quad (x,z) \in U \times \left(\mathbb{R}^d \setminus \overline{U}\right), \tag{5.7}$$

the Poisson kernel for $X^{\mu,q}$ of U as

$$K_U^{\mu,q}(x,z) := \int_U G_U^{\mu,q}(x,y)J(y,z)dy, \quad (x,z) \in U \times \left(\mathbb{R}^d \setminus \overline{U}\right) \tag{5.8}$$

and the Poisson kernel for $\widehat{X}^{\mu,q_0}$ of U as

$$\widehat{K}_U^{\mu,q_0}(x,z) := \frac{h(z)}{h(x)}\int_U G_U^{\mu,q_0}(y,x)J(z,y)dy, \quad (x,z) \in U \times \left(\mathbb{R}^d \setminus \overline{U}\right). \tag{5.9}$$

By Eqs. 3.1 and 5.3, we have

$$\mathbb{E}_x\left[f\left(X^{\mu,q_0}_{\tau_U^{\mu,q_0}}\right);\ X^{\mu,q_0}_{\tau_U^{\mu,q_0}-} \neq X^{\mu,q_0}_{\tau_U^{\mu,q_0}}\right] = \int_{\overline{U}^c} K_U^{\mu,q_0}(x,z)f(z)dz$$

and

$$\mathbb{E}_x\left[f\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_U}\right);\ \widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_U-}\neq \widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_U}\right]=\int_{\overline{U}^c}\widehat{K}^{\mu,q_0}_U(x,z)f(z)dz. \tag{5.10}$$

Define

$$M:=M(\mu,q_0):=\sup_{x,y\in\mathbb{R}^d}\frac{h(x)}{h(y)}. \tag{5.11}$$

Note that, by Proposition 5.1, we have

$$1\le M(\mu,q_0)<\infty. \tag{5.12}$$

By Eq. 5.12, Theorems 4.2 and 4.4, we have that for every $r\in(0,r_3]$ and every $x\in\mathbb{R}^d$,

$$\mathbb{E}_z\left[\tau^{\mu}_{B(x,r)}\right]\asymp\mathbb{E}_z\left[\tau^{\mu,q_0}_{B(x,r)}\right]\asymp\mathbb{E}_z\left[\widehat{\tau}^{\mu,q_0}_{B(x,r)}\right]\asymp\mathbb{E}_z\left[\tau_{B(x,r)}\right],\quad z\in B(x,r). \tag{5.13}$$

Since, using the results we have obtained so far, the proofs in the remainder of this section are almost identical to those in [10], we give details only for parts that require extra explanation.

Lemma 5.6 *Suppose that U is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with* $\mathrm{diam}(U)\le 3r_3$ *where r_3 is the constant in Theorem 4.4. Then*

$$\mathbb{P}_x\left(X^{\mu}_{\tau^{\mu}_U}\in\partial U\right)=\mathbb{P}_x\left(X^{\mu,q_0}_{\tau^{\mu,q_0}_U}\in\partial U\right)=\mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_U}\in\partial U\right)=0\qquad \textit{for every } x\in U.$$

Proof Using Eqs. 5.12, 5.13 and 5.10, the proof of the lemma is the same as that of [10, Lemma 6.1] and [2, Lemma 6]. We omit the proof. □

By Eqs. 5.12, 5.7–5.9, Lemmas 4.4 and 5.6, we have for every $r\in(0,r_3]$ and every $x\in\mathbb{R}^d$ and $(y,z)\in B(x,r)\times\left(\mathbb{R}^d\setminus\overline{B(x,r)}\right)$,

$$K_{B(x,r)}(y,z)\asymp K^{\mu}_{B(x,r)}(y,z)\asymp K^{\mu,q_0}_{B(x,r)}(y,z)\asymp\widehat{K}^{\mu,q_0}_{B(x,r)}(y,z). \tag{5.14}$$

Using Eq. 5.14, Lemma 5.6 and a standard chain argument, we get the following form of Harnack inequality.

Theorem 5.7 *For every $R>0$ and $a\in(0,1)$, there exists $c=c(a,R)>0$ such that for every $r\in(0,R]$, $x_0\in\mathbb{R}^d$, and any function u which is nonnegative on $\mathbb{R}^d$ and harmonic with respect to X^{μ} (or X^{μ,q_0}, or $\widehat{X}^{\mu,q_0}$) in $B(x_0,r)$, we have*

$$u(x)\ \le\ c\,u(y),\quad \textit{for all } x,y\in B(x_0,ar)\,.$$

Let $z\in\partial D$. We will say that a function $u:\mathbb{R}^d\to\mathbb{R}$ vanishes continuously on $D^c\cap B(z,r)$ if $u=0$ on $D^c\cap B(z,r)$ and u is continuous at every point of $\partial D\cap B(z,r)$.

Note that, by the same proof as that of [12, Lemma 4.2], every nonnegative function u in $\mathbb{R}^d$ that is harmonic with respect to X^{μ} (or X^{μ,q_0}, or $\widehat{X}^{\mu,q_0}$, respectively) in $D\cap B(0,r)$ and vanishes continuously on D^c is regular harmonic in $D\cap B(0,r)$ with respect to X^{μ} (or X^{μ,q_0}, or $\widehat{X}^{\mu,q_0}$, respectively).

Theorem 5.8 (Boundary Harnack principle) *Suppose $d\ge 2$ and $\alpha\in(1,2)$. Let D be a (not necessarily bounded) $C^{1,1}$ open set in $\mathbb{R}^d$ and $\mu=\left(\mu^1,\ldots,\mu^d\right)$ where each μ^j is a signed measure on $\mathbb{R}^d$ belonging to the Kato class $\mathbb{K}_{d,\alpha-1}$. There exists a positive constant $c=c(R_0,\Lambda_0,\mu)$ with the dependence on μ only via the rate at which $M^{\alpha}_{\mu}(r)$ goes to* *zero such that for all $z \in \partial D$, $r \in (0, R_0]$ and all function $u \geq 0$ on $\mathbb{R}^d$ that is positive harmonic with respect to X^μ (or X^{μ,q_0}, or $\widehat{X}^{\mu,q_0}$) in $D \cap B(z,r)$ and vanishes continuously on $D^c \cap B(z,r)$ we have*

$$\frac{u(x)}{u(y)} \leq c\frac{\delta_D(x)^{\alpha/2}}{\delta_D(y)^{\alpha/2}}, \qquad x, y \in D \cap B(z, r/4).$$

Proof Using Eqs. 5.12, 5.3, Lemma 5.6, then using Theorems 4.3 and 4.5 and the boundary Harnack principle for X in $C^{1,1}$ open sets (see [15, 26]), we obtain the conclusion of the theorem for $r \leq r_3 \wedge r_4$ by the same argument of [10, Theorem 6.2]. Using the fact that D is a $C^{1,1}$ open set, now the theorem for all $r \leq R_0$ follows from the result for $r \leq r_3 \wedge r_4$, Theorem 5.7 and a standard chain argument. □

6 Proof of Theorem 1.3

The strategy used in [7] to establish short time sharp two-sided heat kernel estimates is to first establish sharp two-sided estimates on $p^\mu_D(t,x,y)$ at time $t=1$ and then use a scaling argument to establish estimates for $t \leq T$.

Unfortunately due to the appearance of q_0, one cannot use the scaling property of $X^{\mu,q}$ to deduce the sharp two-sided estimates on $p^{\mu,q_0}_D(t,x,y)$ for $t \leq T$ from these at time $t=1$. Our strategy is to establish sharp two-sided estimates on $p^{\mu,q_0}_D(t,x,y)$ for $t \leq T$ directly without using a scaling argument.

Recall that $M \geq 1$ is the constant defined in Eq. 5.11. The next result follows from Proposition 3.5, Eqs. 5.4 and 5.11.

Proposition 6.1 *For all $a_1 \in (0,1)$, $a_2, a_3, R > 0$, there exists $c_1 = c_1(a_1, a_2, a_3, R, M, \mu) > 0$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that for all open ball $B(x_0, r) \subset \mathbb{R}^D$ with $r \leq R$,*

$$\widehat{p}^{\,\mu,q_0}_{B(x_0,r)}(t,x,y) \geq c_1\, t^{-d/\alpha} \qquad \text{for all } x, y \in B(x_0, a_1 r) \text{ and } t \in \left[a_2 r^\alpha,\ a_3 r^\alpha\right].$$

Lemma 6.2 *Suppose that U_1, U_3, U are open subsets of $\mathbb{R}^d$ with $U_1, U_3 \subset U$ and* $\mathrm{dist}(U_1, U_3) > 0$. *Let $U_2 := U \setminus (U_1 \cup U_3)$. If $x \in U_1$ and $y \in U_3$, then for all $t > 0$,*

$$p^{\mu,q_0}_U(t,x,y) \leq \mathbb{P}_x\left(X^{\mu,q_0}_{\tau^{\mu,q_0}_{U_1}} \in U_2\right) \cdot \sup_{s<t,\, z\in U_2} p^{\mu,q_0}_U(s,z,y) + \left(t \wedge \mathbb{E}_x\left[\tau^{\mu,q_0}_{U_1}\right]\right) \cdot \sup_{u\in U_1,\, z\in U_3} J(u,z), \tag{6.1}$$

$$\begin{aligned} p^{\mu,q_0}_U(t,y,x) \leq\, & M\mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_{U_1}} \in U_2\right) \cdot \sup_{s<t,\, z\in U_2} \widehat{p}^{\mu,q_0}_{E,U}(s,z,y) \\ & + M\left(t \wedge \mathbb{E}_x\left[\widehat{\tau}^{\mu,q_0}_{U_1}\right]\right) \cdot \sup_{u\in U_1,\, z\in U_3} J(u,z) \end{aligned} \tag{6.2}$$

and

$$p^{\mu,q_0}_U(t,x,y) \geq \frac{t}{M}\mathbb{P}_x\left(\tau^{\mu,q_0}_{U_1} > t\right)\mathbb{P}_y\left(\widehat{\tau}^{\mu,q_0}_{U_3} > t\right) \cdot \inf_{u\in U_1,\, z\in U_3} J(u,z). \tag{6.3}$$

Proof For Eq. 6.1, see the proofs of [3, Lemma 2] and [7, Lemma 2.2]. For Eqs. 6.2–6.3, see the proof of [10, Lemma 7.3]. □

Lemma 6.3 *If* $T > 0$*, then there is a constant* $c = c(T, M, \mu) > 0$ *with the dependence on* μ *only via the rate at which* $M_\mu^\alpha(r)$ *goes to zero such that for every* $t \le T$ *and* $u, v \subset \mathbb{R}^d$*,*

$$p^{\mu,q_0}_{B(u,t^{1/\alpha})\cup B(v,t^{1/\alpha})}(t/3, u, v) \ \ge\ c\left(t^{-d/\alpha} \wedge \frac{t}{|u-v|^{d+\alpha}}\right).$$

Proof If $|u - v| \le t^{1/\alpha}/2$, by Proposition 3.5,

$$p^{\mu,q_0}_{B(u,t^{1/\alpha})\cup B(v,t^{1/\alpha})}(t/3, u, v) \ge \inf_{|u-v|\le t^{1/\alpha}/2} p^{\mu,q_0}_{B(u,t^{1/\alpha})}(t/3, u, v) \ge c_1 t^{-d/\alpha}.$$

On the other hand, by Propositions 3.5 and 6.1,

$$\inf_{(t,x)\in(0,T]\times\mathbb{R}^d}\left(\int_{B(x,t^{1/\alpha}/16)} p^{\mu,q_0}_{B(x,t^{1/\alpha}/8)}(t/3, x, z)dz \wedge \right.$$
$$\left.\int_{B(v,t^{1/\alpha}/16)} \widehat{p}^{\ \mu,q_0}_{\ B(u,t^{1/\alpha}/8)}(t/3, v, z)dz\right) \ge c_2 > 0.$$

Thus, if $|u - v| \ge t^{1/\alpha}/2$, by Eq. 6.3,

$$p^{\mu,q_0}_{B(u,t^{1/\alpha})\cup B(v,t^{1/\alpha})}(t/3, u, v) \ge \frac{t}{3M}\mathbb{P}_u\left(\tau^{\mu,q_0}_{U_1} > t/3\right)\mathbb{P}_v\left(\widehat{\tau}^{\ \mu,q_0}_{\ U_3} > t/3\right)\inf_{w\in U_1,\, z\in U_3} J(w, z)$$
$$\ge c_3 t\int_{B(u,t^{1/\alpha}/16)} p^{\mu,q_0}_{B(u,t^{1/\alpha}/8)}(t/3, u, z)dz\int_{B(v,t^{1/\alpha}/16)} \widehat{p}^{\mu,q_0}_{B(u,t^{1/\alpha}/8)}(t/3, v, z)dz\frac{1}{|u-v|^{d+\alpha}}$$
$$\ge c_4\frac{t}{|u-v|^{d+\alpha}} \ge c_4\left(t^{-d/\alpha} \wedge \frac{t}{|u-v|^{d+\alpha}}\right).$$

□

We now fix a $C^{1,1}$ open set $D \subset \mathbb{R}^d$ with characteristics (R_0, Λ_0). It is well-known that any $C^{1,1}$ open set D satisfies the *uniform interior ball condition*: there exists $r_0 < R_0$ such that for every $x \in D$ with $\delta_D(x) < r_0$ there is $z_x \in \partial D$ so that $|x - z_x| = \delta_D(x)$ and that $B(x_0, r_0) \subset D$ for $x_0 = z_x + r_0(x - z_x)/|x - z_x|$. For the remainder of the paper, we fix such r_0 and use z_x as above. For $x \in D$ with $\delta_D(x) < r_0$, let

$$U_x(t) := B(z_x, t) \cap D. \tag{6.4}$$

Lemma 6.4 *For every* $T > 0$*, there is* $c = c(R_0, T, M, \Lambda_0, \mu) > 0$ *with the dependence on* μ *only via the rate at which* $M_\mu^\alpha(r)$ *goes to zero such that for all* $x \in D$ *and* $t \in (0, T]$*,*

$$\mathbb{P}_x\left(\tau^{\mu,q_0}_D > t/4\right) \le \frac{4}{t}\mathbb{E}_x\left[\tau^{\mu,q_0}_{U_x((t/T)^{1/\alpha}r_0/8)}\right] + \mathbb{P}_x\left(X_{\tau^{\mu,q_0}_{U_x((t/T)^{1/\alpha}r_0/8)}}^{\mu,q_0} \in D\right) \le c\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \tag{6.5}$$

and

$$\mathbb{P}_x\left(\widehat{\tau}^{\mu,q_0}_D > t/4\right) \le \frac{4}{t}\mathbb{E}_x\left[\widehat{\tau}^{\mu,q_0}_{U_x((t/T)^{1/\alpha}r_0/8)}\right] + \mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_{U_x((t/T)^{1/\alpha}r_0/8)}} \in D\right) \le c\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right). \tag{6.6}$$

Proof Since the arguments of the proofs of Eqs. 6.5 and 6.6 are same, we only give the proof of Eq. 6.6.

Fix $T > 0$ and $t \in (0, T]$. Clearly we only need to show the theorem for $x \in D$ with $\delta_D(x) < (t/T)^{1/\alpha} r_0/(16) \le r_0/(16)$, which we will assume throughout the proof. Let $U_1 := U_x((t/T)^{1/\alpha} r_0/8)$. Take $x_0 \in \mathbb{R}^d$ so that

$$B(x_0, (t/T)^{1/\alpha} r_0/(16)) \subset B(z_x, (t/T)^{1/\alpha} r_0/4) \setminus B(z_x, (t/T)^{1/\alpha} r_0/8).$$

Then, by the Lévy system in Eq. 5.10, we have

$$\mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_{U_1}} \in D\right) \ge \mathbb{E}_x\left[\int_0^{\widehat{\tau}^{\mu,q_0}_{U_1}} \int_{B(x_0,(t/T)^{1/\alpha} r_0/(16))} \frac{J(\widehat{X}^{\mu,q_0}_s, y) h(y)}{h(\widehat{X}^{\mu,q_0}_s)} dy ds\right]$$
$$\ge c_1 t^{1/\alpha} |B(x_0, (t/T)^{1/\alpha} r_0/(16))| |(t/T)^{1/\alpha} r_0/(16)|^{-d-\alpha} \mathbb{E}_x\left[\widehat{\tau}^{\mu,q_0}_{U_1}\right] \ge c_2 t^{-1} \mathbb{E}_x\left[\widehat{\tau}^{\mu,q_0}_{U_1}\right].$$

Thus

$$\mathbb{P}_x\left(\widehat{\tau}^{\mu,q_0}_D > t/4\right) \le \mathbb{P}_x\left(\widehat{\tau}^{\mu,q_0}_{U_1} > t/4\right) + \mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_{U_1}} \in D\right)$$
$$\le \frac{4}{t} \mathbb{E}_x\left[\widehat{\tau}^{\mu,q_0}_{U_1}\right] + \mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_{U_1}} \in D\right) \le c_3 \mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_{U_1}} \in D\right).$$

Now with $x_1 = z_x + t^{1/\alpha} r_0 16^{-1} T^{-1/\alpha} \mathbf{n}(z_x)$, where $\mathbf{n}(z_x)$ is the unit inward normal to ∂D at the point z_x, by applying our boundary Harnack principle (Theorem 5.8) for $\widehat{X}^{\mu,q_0}$, we get

$$\mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_{U_1}} \in D\right) \le c_4 \mathbb{P}_{x_1}\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}^{\mu,q_0}_{U_1}} \in D\right) \frac{\delta_D(x)^{\alpha/2}}{\delta_D(x_1)^{\alpha/2}} \le c_5 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.$$

□

Lemma 6.5 *For every $T > 0$, there is a positive constant $c = c(T, R_0, \Lambda_0, M, \mu)$ with the dependence on μ only via the rate at which $M^\alpha_\mu(r)$ goes to zero such that for all $x, y \in D$,*

$$p^{\mu,q_0}_D(t/2, x, y) \le c \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \tag{6.7}$$

and

$$p^{\mu,q_0}_D(t/2, x, y) \le c \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{6.8}$$

Proof We only give the proof of Eq. 6.8.

When $|x - y| \le (t/T)^{1/\alpha} r_0$, by the semigroup property (3.3), (4.10), (5.4) and (6.6),

$$p^{\mu,q_0}_D(t/2, x, y) \le \int_D p^{\mu,q_0}(t/4, x, z) \widehat{p}^{\mu,q_0}_D(t/4, y, z) \frac{h(y)}{h(z)} dz$$
$$\le c_2 M t^{-d/\alpha} \mathbb{P}_y(\widehat{\tau}^{\mu,q_0}_D > t/4) \le c_3 \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) t^{-d/\alpha}.$$

By this and Eq. 4.10, we only need to consider the case that $|x - y| \ge (t/T)^{1/\alpha} r_0$ and $y \in D$ with $\delta_D(y) < (t/T)^{1/\alpha} r_0/(16) \le r_0/(16)$ which we assume throughout the proof. Let $U_1 = U_x((t/T)^{1/\alpha} r_0/8)$, $U_3 := \{z \in D : |z - x| > |x - y|/2\}$ and $U_2 := D \setminus (U_1 \cup U_3)$.

If $u \in U_1$ and $z \in U_3$, then

$$|u - z| \ge |z - x| - |z_x - x| - |z_x - u| \ge |z - x| - (t/T)^{1/\alpha} r_0/4 \ge \frac{1}{2}|z - x| \ge \frac{1}{4}|x - y|.$$

Thus,

$$\sup_{u\in U_1,\, z\in U_3} J(u,z) \le c_5 \sup_{(u,z):|u-z|\ge \frac{1}{4}|x-y|} |u-z|^{-d-\alpha} \le c_6|x-y|^{-d-\alpha}. \tag{6.9}$$

Since $|z-y| \ge |x-y| - |x-z| \ge |x-y|/2$ for $z \in U_2$, by Eq. 4.10,

$$\sup_{s\le t,\, z\in U_2} p^{\mu,q_0}(s,z,y) \le c_7 \sup_{s\le t,\, |z-y|\ge|x-y|/2} \big(sJ(z,y)\big) \le c_5 t|x-y|^{-d-\alpha}. \tag{6.10}$$

Using Eq. 6.2 and then applying Eqs. 6.9, 6.10 and 6.6, we conclude that

$$p_D^{\mu,q_0}(t,x,y) \le c_9\left(\mathbb{E}_x\left[\widehat{\tau}_{U_1}^{\mu,q_0}\right] + t\mathbb{P}_x\left(\widehat{X}^{\mu,q_0}_{\widehat{\tau}_{U_1}^{\mu,q_0}} \in U_2\right)\right)|x-y|^{-d-\alpha} \le \left(1\wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)|x-y|^{-d-\alpha}.$$

□

Lemma 6.6 *For every $T>0$, there is a positive constant $c = c(T, R_0, \Lambda_0, M, \mu)$ with the dependence on μ only via the rate at which $M_\mu^\alpha(r)$ goes to zero such that for all $x, y \in D$,*

$$p_D^{\mu,q_0}(t,x,y) \le c\left(1\wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{6.11}$$

Proof Using Eqs. 6.7–6.8, the semigroup property (3.3) and the two-sided estimates of $p(t,x,y)$,

$$\begin{aligned} p_D^{\mu,q_0}(t,x,y) &= \int_{\mathbb{R}^d} p_D^{\mu,q_0}(t/2,x,z)p_D^{\mu,q_0}(t/2,z,y)dz \\ &\le c\left(1\wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\int_{\mathbb{R}^d} p(t/2,x,z)p(t/2,z,y)dz \\ &= c\left(1\wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) p(t,x,y) \\ &\le c\left(1\wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right). \end{aligned}$$

□

Lemma 6.7 *For every $T>0$, there is a positive constant $c_1 = c_1(T, R_0, \Lambda_0, M, \mu)$ with the dependence on μ only via the rate at which $M_\mu^\alpha(r)$ goes to zero such that*

$$p_D^{\mu,q_0}(t,x,y) \ge c_1\left(1\wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

Proof Assume first that $t \le T_0 := \left(\frac{r_0}{16}\right)^\alpha$. Since D satisfies the uniform interior ball condition with radius r_0 and $0 < t \le T_0$, we can choose ξ_x^t as follows: if $\delta_D(x) \le 3t^{1/\alpha}$, let $\xi_x^t = z_x + (9/2)t^{1/\alpha}\mathbf{n}(z_x)$ (so that $B\left(\xi_x^t, (3/2)t^{1/\alpha}\right) \subset B\left(z_x + 3t^{1/\alpha}\mathbf{n}(z_x), 3t^{1/\alpha}\right)\setminus\{x\}$ and $\delta_D(z) \ge 3t^{1/\alpha}$ for every $z \in B(\xi_x^t, (3/2)t^{1/\alpha})$). If $\delta_D(x) > 3t^{1/\alpha}$, choose $\xi_x^t \in B(x, \delta_D(x))$ so that $\left|x - \xi_x^t\right| = (3/2)t^{1/\alpha}$. Note that in this case, $B(\xi_x^t, (3/2)t^{1/\alpha}) \subset B(x, \delta_D(x)) \setminus \{x\}$ and $\delta_D(z) \ge t^{1/\alpha}$ for every $z \in B(\xi_x^t, 2^{-1}t^{1/\alpha})$. We also define ξ_y^t the same way.

For every $(u, v) \in B\left(\xi_x^t, 2^{-1}t^{1/\alpha}\right) \times B\left(\xi_y^t, 2^{-1}t^{1/\alpha}\right)$, we have $|u-v| \le 21(|x-y| \wedge t^{1/\alpha})$. Thus using Lemma 6.3, for such u and v we have

$$p^{\mu,q_0}_{B(u,2^{-1}t^{1/\alpha})\cup B(v,2^{-1}t^{1/\alpha})}(t/3,u,v) \ge c_1\left(\frac{t}{|u-v|^{d+\alpha}} \wedge t^{-d/\alpha}\right) \ge c_2\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{6.12}$$

Now by the semigroup property (3.3),

$$\begin{aligned}
&p_D^{\mu,q_0}(t,x,y)\\
\ge&\int_{B(\xi_y^t,2^{-1}t^{1/\alpha})}\int_{B(\xi_x^t,2^{-1}t^{1/\alpha})} p_D^{\mu,q_0}(t/3,x,u)\,p^{\mu,q_0}_{B(u,2^{-1}t^{1/\alpha})\cup B(v,2^{-1}t^{1/\alpha})}(t/3,u,v)\,p_D^{\mu,q_0}(t/3,v,y)dudv\\
\ge& c_2\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)\left(\int_{B(\xi_x^t,2^{-1}t^{1/\alpha})} p_D^{\mu,q_0}(t/3,x,u)du\right)\left(\int_{B(\xi_y^t,2^{-1}t^{1/\alpha})} p_D^{\mu,q_0}(t/3,v,y)dv\right).
\end{aligned} \tag{6.13}$$

We claim that

$$\begin{aligned}
\int_{B(\xi_y^t,2^{-1}t^{1/\alpha})} p_D^{\mu,q_0}(t/3,x,u)du &\ge c_3\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right),\\
\int_{B(\xi_y^t,2^{-1}t^{1/\alpha})} p_D^{\mu,q_0}(t/3,v,y)dv &\ge c_3\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right),
\end{aligned} \tag{6.14}$$

which, combined with Eqs. 6.12–6.13, proves the theorem for $t \le T_0$.

We only give the proof of the second inequality in Eq. 6.14. Recall that $z_y \in \partial D$ be such that $|y - z_y| = \delta_D(y)$ and $U_y(t)$ is defined in (6.4). Let

$$V_1 = U_y\left(13t^{1/\alpha}/4\right) \quad \text{and} \quad V_2 = \begin{cases} U_y\left(15t^{1/\alpha}/4\right) & \text{when } \delta_D(y) \le 3t^{1/\alpha}\\ B\left(y, t^{1/\alpha}\right) & \text{when } \delta_D(y) > 3t^{1/\alpha}.\end{cases}$$

By Eq. 6.3 and Proposition 3.5,

$$\begin{aligned}
&\int_{B(\xi_y,t^{1/\alpha}/2)} p_D^{\mu,q_0}\left(t/3,v,y\right)dv\\
\ge&\frac{t}{3M}\left(\int_{B(\xi_y^t,4^{-1}t^{1/\alpha})}\mathbb{P}_v\left(\tau^{\mu,q_0}_{B(\xi_y^t,2^{-1}t^{1/\alpha})} > t/3\right)dv\right)\mathbb{P}_y\left(\widehat{\tau}^{\mu,q_0}_{V_2} > t/3\right)\inf_{w\in B(\xi_y^t,2^{-1}t^{1/\alpha}),\, z\in V_2} J(w,y)\\
\ge& c_3\,\mathbb{P}_y\left(\widehat{\tau}^{\mu,q_0}_{V_2} > t/3\right),
\end{aligned} \tag{6.15}$$

which is bounded above by some positive constant if $\delta_D(y) > 3t^{1/\alpha}$ by Proposition 6.1.

We now assume $\delta_D(y) \leq 3t^{1/\alpha}$ and let $B\left(y_0, c_4 t^{1/\alpha}\right)$ be a ball in $D \cap \left(B\left(z_y, 15t^{1/\alpha}/4\right) \setminus B\left(z_y, 7t^{1/\alpha}/2\right)\right)$ where $c_4 = c_4(\Lambda_0, R_0, d) > 0$. By the strong Markov property,

$$\begin{aligned}
&\left(\inf_{w\in B\left(y_0, c_4 t^{1/\alpha}/2\right)} \mathbb{P}_w\left(\widehat{\tau}^{\mu,q_0}_{B(w,c_4 t^{1/\alpha}/2)} > t/3\right)\right)\mathbb{P}_y\left(\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right)\\
&\leq \mathbb{E}_y\left[\mathbb{P}_{\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)}\left(\widehat{\tau}^{\mu,q_0}_{B\left(\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right), c_4 t^{1/\alpha}/2\right)} > t/3\right); \widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right]\\
&\leq \mathbb{E}_y\left[\mathbb{P}_{\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)}\left(\widehat{\tau}^{\mu,q_0}_{V_2} > t/3\right); \widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right]\\
&= \mathbb{P}_y\left(\widehat{\tau}^{\mu,q_0}_{V_2} > t/3, \widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right) \leq \mathbb{P}_y\left(\widehat{\tau}^{\mu,q_0}_{V_2} > t/3\right).
\end{aligned}$$

Using Proposition 6.1, we get

$$\mathbb{P}_y\left(\widehat{\tau}^{\mu,q_0}_{V_2} > t/3\right) \geq c_5 \mathbb{P}_y\left(\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right). \tag{6.16}$$

Let $B\left(y_1, c_6 t^{1/\alpha}\right)$ be a ball in V_1 where $c_6 = c_6\left(\Lambda_0, r_1, d, r_3\right) \in \left(0, r_3/T_0^{1/\alpha}\right)$. Recall that r_3 is the constant from Theorem 4.4. Applying Theorem 5.8, we have

$$\begin{aligned}
&\mathbb{P}_y\left(\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right) \geq c_7 \mathbb{P}_{y_1}\left(\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right)\frac{\delta_D(y)^{\alpha/2}}{\delta_D(y_1)^{\alpha/2}}\\
&\geq c_8 \mathbb{P}_{y_1}\left(\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right)\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.
\end{aligned}$$

By the Lévy system in Eq. 5.10, we have

$$\begin{aligned}
\mathbb{P}_{y_1}\left(\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right) &= \mathbb{E}_{y_1}\left[\int_0^{\widehat{\tau}^{\mu,q_0}_{V_1}}\int_{B(y_0,c_4 t^{1/\alpha}/2)}\frac{J\left(\left|\widehat{X}^{\mu,q_0}_s - y\right|\right)h(y)}{h(\widehat{X}^{\mu,q_0}_s)}dyds\right]\\
&\geq c_9 t^{1/\alpha}\left|B\left(y_0, c_4 t^{1/\alpha}/2\right)\right| t^{-d-\alpha}\mathbb{E}_{y_1}\left[\widehat{\tau}^{\mu,q_0}_{V_1}\right]\\
&\geq c_{10} t^{-1}\mathbb{E}_{y_1}\left[\widehat{\tau}^{\mu,q_0}_{V_1}\right] \geq c_{10} t^{-1}\mathbb{E}_{y_1}\left[\widehat{\tau}^{\mu,q_0}_{B(y_1,c_6 t^{1/\alpha})}\right] \geq c_{11}.
\end{aligned}$$

In the last inequality we have used Eq. 5.13. Therefore

$$\mathbb{P}_y\left(\widehat{X}^{\mu,q_0}\left(\widehat{\tau}^{\mu,q_0}_{V_1}\right)\in B\left(y_0, c_4 t^{1/\alpha}/2\right)\right) \geq c_{12}\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}. \tag{6.17}$$

Combining Eqs. 6.15–6.17, we have proved Eq. 6.14.

To get the theorem for $T > T_0$, it is enough to handle the case $T = 2T_0$ and the proof of this case is the same as the one in [7, page 1323–1324]. □

Proof of Theorem 1.3

(i) follows immediately from the two lemmas above and Eq. 3.2.

(ii) We need to redefine dual process as [10] without introducing q_0. Since the argument is same as that in [10]. here we provide the sketch of the proof.

Choose a ball E large enough so that $D \subset \frac{1}{4}E$. Define

$$h_E(x) := \int_E G_E^\mu(y,x)dy \quad \text{and} \quad \xi_E(dx) := h_E(x)dx.$$

Then $h_E(y)$ is strictly positive and continuous on E and ξ_E is an excessive measure for $X^{\mu,E}$. We define a transition density with respect to the reference measure ξ_E by

$$\overline{p}_E^\mu(t,x,y) := \frac{p_E^\mu(t,x,y)}{h_E(y)}.$$

Then one can show that there exists a transient Hunt process $\widehat{X}^{\mu,E}$ in E such that $\widehat{X}^{\mu,E}$ is a strong dual of $X^{\mu,E}$ with respect to the measure ξ_E. Let

$$\overline{p}_D^{\mu,E}(t,x,y) := \frac{p_D^\mu(t,x,y)}{h_E(y)},$$

which is strictly positive, bounded and continuous on $(t,x,y) \in (0,\infty) \times D \times D$ because $p_D^\mu(t,x,y)$ is strictly positive, bounded and continuous on $(t,x,y) \in (0,\infty) \times D \times D$ and $h_E(y)$ is strictly positive and continuous on E. For each $x \in D$, $(t,y) \mapsto \overline{p}_D^{\mu,E}(t,x,y)$ is the transition density of $\left(X^{\mu,D}, \mathbb{P}_x\right)$ with respect to the reference measure ξ_E and, for each $y \in D$, $(t,x) \mapsto \overline{p}_D^{\mu,E}(t,x,y)$ is the transition density of $\left(\widehat{X}^{\mu,E,D}, \mathbb{P}_y\right)$, the dual process of $X^{\mu,D}$ with respect to the reference measure ξ_E.

By using the same argument as that in [10, Section 8], one can show that the semigroups $\left\{P_t^{\mu,E,U}\right\}$ and $\left\{\widehat{P}_t^{\mu,E,U}\right\}$ of $X^{\mu,D}$ and $X^{\mu,E,D}$ with respect to the reference measure ξ_E are intrinsically ultracontractive. Using this, now (ii) follows from (i) and the argument in the proof of [10, Theorem 1.3 (ii)]. □

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