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On the L^1-Liouville Property of Stochastically Incomplete Manifolds

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Abstract A classical result by Alexander Grigor'yan states that on a stochastically complete manifold non-negative superharmonic L^1-functions are necessarily constant. In this paper we construct explicit examples showing that, in the presence of an anisotropy of the space, the reverse implication does not hold. We also consider natural geometric situations where stochastically incomplete manifolds do not posses the above mentioned L^1-Liouville property for superharmonic functions.

Keywords L^1-Liouville property · Stochastic completeness · Mean exit time

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1 Introduction

Let (M, g) be an m-dimensional Riemannian manifold. We use the symbol Δ to denote the negative-definite Laplace-Beltrami operator of M. Thus, if $M = \mathbb{R}$, $\Delta = +d^2/dx^2$. By a superharmonic function we mean a function $u \in C^0(M) \cap W^{1,2}_{loc}(M)$ satisfying $\Delta u \le 0$ in the sense of distributions, namely,

$$-\int_M \langle \nabla u, \nabla \varphi \rangle \, dv \le 0,$$

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for every $0 \leq \varphi \in W^{1,2}_{loc}$. By reversing the inequality we obtain the notion of subharmonic function and by replacing the inequality with an equality we get a harmonic function. By elliptic regularity, harmonic functions are necessarily smooth.

In general, there is no obstruction for a manifold to support many (super) harmonic functions. Indeed, according to a theorem due to Greene and Wu [4], any m-dimensional, non-compact Riemannian manifold can be embedded into $\mathbb{R}^{2m+1}$ by harmonic functions. On the other hand, the presence of superharmonic functions enjoying some special property is intimately related to the geometry of the underlying space. Thus, for instance, if the geodesically complete manifold (M, g) supports a non-constant, positive superharmonic function then M is non-parabolic and, in particular,

$$\int^{+\infty} \frac{r}{\mathrm{vol}\,(B_r\,(o))} dr < +\infty,$$

for some origin $o \in M$. Here, $B_r\,(o)$ denotes the geodesic ball of M centered at o and of radius $r > 0$. In this spirit one gives the following

Definition 1 A smooth Riemannian manifold (M, g) is said to satisfy the L^1-Liouville property, (shortly, M is L^1-Liouville), if every superharmonic function $0 \leq u \in L^1\,(M)$ must be constant.

According to a nice result by Alexander Grigor'yan [5], later extended to non-linear operators modeled on the p-Laplacian (see [7, 11]) in order to understand whether or not a manifold is L^1-Liouville one may simply consider the behavior of its Green kernel $G\,(x, y)$. We recall that this latter is the minimal, positive, fundamental solution of $-\Delta$.

Theorem 2 *The Riemannian manifold (M, g) is L^1-Liouville if and only if, for some (hence any) $x \in M$,*

$$\int_M G\,(x, y)\, dv\,(y) = +\infty.$$

Note that, in case that M is parabolic, we have $G \equiv +\infty$ and the integrability condition is trivially satisfied. However, in this case, we already know that positive superharmonic functions (without any further restriction) must be constant.

In [5], A. Grigor'yan makes a clever use of the equivalence established in Theorem 2 to obtain a neat geometric condition implying the L^1-Liouville property. This is achieved by relating the (non-)integrability of the Green function with a further stochastic property of the manifold, namely, its stochastic completeness. Recall that (M, g) is stochastically complete (for the Brownian motion with infinitesimal generator Δ) if for some (hence every) $x \in M$,

$$\int_M p_t\,(x, y)\, dv\,(y) = 1,$$

where $p_t\,(x, y)$ stands for the heat kernel of M, i.e., the minimal, positive fundamental solution of the heat operator $\Delta - \partial/\partial t$. From the probabilistic viewpoint, this means that the explosion time of the Brownian motion on M is almost surely infinite. Recall also that $G(x, y)$ and $p_t(x, y)$ are related by

$$G(x, y) = \int_0^{+\infty} p_t(x, y)\, dt.$$

Therefore, applying Tonelli's Theorem, from Theorem 2 we immediately deduce

Corollary 3 *A stochastically complete manifold is L^1-Liouville.*

In particular, since a geodesically complete manifold (M, g) is stochastically complete provided, for some origin $o \in M$,

$$\int^{+\infty} \frac{r}{\log(\mathrm{vol}(B_r(o)))} dr = +\infty \tag{1}$$

one may conclude the validity of the next

Corollary 4 *A geodesically complete Riemannian manifold (M, g) is L^1-Liouville provided the volume growth condition 1 is satisfied.*

So far we have essentially celebrated A. Grigor'yan work on the subject. From the above discussion, some natural questions arise.

Problems (a) *Does the converse of Corollary* 3 *hold?* (b) *If not, are there natural geometric situations where a given (necessarily stochastically incomplete) manifold is not L^1-Liouville?* (c) *More ambitiously, to what extent and under which conditions the validity of the L^1-Liouville property implies that the manifold is stochastically complete?*

In this paper we address questions (a) and (b). The more ambitious (c) will be the subject of future investigations.

In the case of a model manifold

$$M^m_\sigma = \left([0, +\infty) \times \mathbb{S}^{m-1}, dt^2 + \sigma(t)^2\, d\theta^2\right),$$

it is easy to see that stochastic completeness is in fact equivalent to the L^1-Liouville property. Indeed, the Green's kernel with pole at o of M^m_σ is given by

$$G(x, o) = c_m \int_r^{+\infty} \frac{1}{\sigma^{m-1}(t)} dt$$

so that, interchanging the order of integration,

$$\begin{aligned} \int_M G(x, o) dx &= c_m \int_0^{\infty} \sigma^{m-1}(r) dr \int_r^{\infty} \frac{1}{\sigma^{m-1}(t)} dt \\ &= c_m \int_0^{+\infty} dt \frac{\int_0^t \sigma^{m-1}(r) dr}{\sigma^{m-1}(t)}, \end{aligned}$$

which shows that the condition for stochastic completeness [6, Proposition 3.2] and that for the validity of the L^1-Liouville property of a model manifold coincide.

The investigation around these very natural questions would benefit of different viewpoints on the notion of stochastic completeness. We will make a constant use of the following equivalent description in the language of maximum principles at infinity (see [9, 10]).

Theorem 5 *A Riemannian manifold* (M, g) *is stochastically complete if and only if, for every* $u \in C^2(M)$ *satisfying* $\sup_M u = u^* < +\infty$, *there exists a sequence* $\{x_k\} \subset M$ *along which*

$$\text{(i) } u(x_k) > u^* - \frac{1}{k}, \qquad \text{(ii) } \Delta u(x_k) < \frac{1}{k}.$$

2 Two Examples

This section is devoted to show that, in general, an L^1-Liouville manifold may be stochastically incomplete. This answers in the negative Problem (a) stated in the previous section. To this purpose, we begin by constructing an explicit example with two ends and this will be accomplished in two steps.

First Step Recall that the connected sum $M_1 \# M_2$ of equidimensional Riemannian manifolds is stochastically incomplete provided either M_1 or M_2 are stochastically incomplete. See [1, Lemma 3.1]. This is a very special case of the following general fact which follows quite easily using the viewpoint of Theorem 5.

Proposition 6 *Let* (M, g) *be a complete manifold and let* $E_1, ..., E_k$ *be the ends of* M *with respect to any smooth, compact domain* $\Omega \subset M$. *Then* M *is stochastically complete if and only if, for every* $j = 1, ..., k$, *either of the following conditions is verified:*

(i) *There exists a compact domain* D_j *together with a diffeomorphism* $f_j : \partial D_j \to \partial E_j$ *such that the gluing* $M_j = D_j \cup_{f_j} E_j$ *is a stochastically complete manifold (without boundary).*
(ii) *The Riemannian double* $\mathcal{D}(E_j)$ *is a stochastically complete manifold (without boundary).*

In particular, consider the 2-dimensional model manifolds

$$M^2_{\sigma_j} = \left([0, +\infty) \times \mathbb{S}^1, dt^2 + \sigma_j(t)^2 \, d\theta^2\right),$$

$j = 1, 2$, where we require

$$\int^{+\infty} \sigma_1(t) \, dt = +\infty$$

and

$$\int^{+\infty} \frac{\int_0^r \sigma_2(t) \, dt}{\sigma_2(r)} dr < +\infty.$$

The first condition means that $M^2_{\sigma_1}$ has infinite volume. On the other hand, by a well known characterization, [6, Proposition 3.2], the second condition is equivalent to requiring that $M^2_{\sigma_2}$ be stochastically incomplete. Let

$$M = M^2_{\sigma_1} \# M^2_{\sigma_2}$$

where the connected sum is performed using embedded disks D centered at the poles of the manifolds. By the above considerations, (M, g) is stochastically incomplete. In particular, M is non-parabolic, therefore, it possesses a Green function $G < +\infty$.

Second Step We now perform a conformal change of the metric g. We define

$$\tilde{g} = \lambda^2 g,$$

where $\lambda > 0$ is any smooth function with the following properties:

(a) Outside a neighborhood of $M^2_{\sigma_1} \backslash D \subset M$, $\lambda \equiv 1$.

(b) Outside a neighborhood of $M^2_{\sigma_2} \backslash D \subset M$, λ satisfies

$$\lambda(t, \theta) \geq \frac{1}{\sqrt{\min_{[t_1,t]\times\mathbb{S}^1} G_{x_0}(x)}},$$

where x_0 is any point in $M^2_{\sigma_2} \backslash D \subset M$ and, without loss of generality, $[t_1, +\infty) \times \mathbb{S}^1 \subset M^2_{\sigma_1}$ has the original metric $dt^2 + \sigma_1^2(t)\, d\theta^2$.

Conclusion We claim that $\tilde{M} = (M, \tilde{g})$ is stochastically incomplete and possesses the L^1-Liouville property. Indeed, according to (a), and using Proposition 6, we see that $\tilde{M}$ is stochastically incomplete. In particular, $\tilde{M}$ is non-parabolic. Actually, since

$$\Delta_{\tilde{g}} = \frac{1}{\lambda^2} \Delta_g,$$

it follows that the Green function $\tilde{G}$ of $\tilde{M}$ satisfies

$$\tilde{G} = G.$$

Therefore,

$$\begin{aligned}
\int_{\tilde{M}} \tilde{G}(x_0, y)\, d\tilde{v}(y) &= \int_{\tilde{M}} G(x_0, y)\, \lambda^2(y)\, dv(y) \\
&\geq \lim_{t \to +\infty} \int_{[t_1,t]\times\mathbb{S}^1 \subset M^2_{\sigma_1}} G(x_0, y)\, \lambda^2(y)\, dv(y) \\
&\geq C \int_{t_1}^{+\infty} \sigma_1(t)\, dt \\
&= +\infty,
\end{aligned}$$

and by Theorem 2 we conclude that the Riemannian manifold $\tilde{M}$ is L^1-Liouville.

Actually, a variation of the above construction allows us to produce an example of a stochastically incomplete L^1-Liouville manifold with only one end. As above, we start with a 2-dimensional stochastically incomplete model

$$M^2_\sigma = \left([0, +\infty) \times \mathbb{S}^1, g = dr^2 + \sigma(r)^2 d\theta^2\right),$$

with $\sigma(r)$ increasing and diverging to infinity at infinity, and (radial) Green's function with pole at o, $G(o, x) = G(o, r(x))$. We perform the conformal change of metric

$$\tilde{g} = \lambda^2 g$$

with a conformality factor $\lambda(x) \geq 1$ such that $\lambda(x) = 1$ if $x = re^{i\theta}$ with $-\pi/2 \leq \theta \leq \pi/2$ and $\lambda(x) \geq G(o, r)^{-1/2}$ if $x = re^{i\theta}$ with $r > 1$ and $3\pi/4 \leq \theta \leq 5\pi/4$. Denoting as above with a tilde the quantities relative to the conformal metric $\tilde{g}$ and, using again the fact that $\tilde{G}(o, x) = G(o, x)$ and that $d\tilde{v} = \lambda^2 dv = \lambda^2 \sigma\, dr d\theta$ we see that

$$\begin{aligned}\int_{\tilde{M}} \tilde{G}(o, x) d\tilde{v} &\geq \int_{[1,\infty)\times[3\pi/4, 5\pi/4]} G\left(o, re^{i\theta}\right) \lambda\left(re^{i\theta}\right)^2 \sigma(r) dr d\theta \\ &\geq \pi/2 \int_1^\infty \sigma(r) dr = +\infty,\end{aligned}$$

and $\tilde{M}$ is L^1-Liouville. On the other hand, let

$$v_o(r) = \int_0^r \sigma(t)^{-1} \int_0^t \sigma(s) ds\, dt,$$

and let $v\left(re^{i\theta}\right) = v_o(r)\cos(\theta)$. Then v tends to its supremum along the ray re^{i0} and, using $\tilde{\Delta} = \lambda^{-2}\Delta$ and $\Delta v_o = 1$ we deduce that in the region where $-\pi/4 \leq \theta \leq \pi/4$ and $\sigma(r)^2 > 2 \sup v_o$ we have

$$\tilde{\Delta} v\left(re^{i\theta}\right) = \frac{1}{\lambda(re^{i\theta})^2}\left(\Delta v_o(r)\cos(\theta) - \frac{1}{\sigma(r)^2} v_o(r)\cos(\theta)\right) \geq \frac{\sqrt{2}}{4}.$$

Thus v does not satisfy the maximum principle at infinity and $\tilde{M}$ is not stochastically complete.

The examples above stress the fact that equivalence between stochastic completeness and the validity of the L^1-Liouville property depends very much on the rotational invariance of the models. In the presence of a strong anisotropy, it is possible that Brownian motion may explode in finite time in certain directions and yet the L^1-Liouville property holds, due to the fact that the Green's kernel is big enough in other regions (or ends of the manifold).

The first example constructed in Section 2 fits very well in this order of ideas. In fact, inspection of that example shows that the stochastically incomplete end remains essentially untouched whereas the background metric is conformally modified only on the end responsible for the validity of the L^1-Liouville property.

While, the second example shows that the L^1-Liouville property does not imply even a weak form of stochastic completeness where it is required that at least one of the ends of the manifold is stochastically complete.

3 Mean Exit Time and the L^1-Liouville Property

As remarked above, stochastic completeness and the L^1-Liouville property are equivalent on models, but, in general, a stochastically incomplete manifold may be L^1-Liouville. We are thus naturally led to investigating general geometric conditions that guarantee that a given (stochastically incomplete) manifold is not L^1-Liouville. In this section we will focus our attention on curvature conditions both of intrinsic and of extrinsic nature. In both cases we shall use the notion of "global mean exit time" that we are going to introduce.

Let (M, g) be a complete Riemannian manifold and let $o \in M$ be a fixed reference point. The mean exit time of the Brownian motion from the ball $B_R(o)$ is defined as the (positive) solution of the Dirichlet problem

$$\begin{cases} \Delta E_R = -1 & \text{on } B_R(o) \\ E_R = 0 & \text{on } \partial B_R(o) . \end{cases}$$

Note that E_R is a smooth function on $B_R(o)$. Moreover, if $G_R(x, y)$ denotes the Dirichlet Green function of $B_R(o)$, then, we have the representation formula

$$E_R(x) = \int_{B_R(o)} G_R(x, y)\, dv(y) .$$

Since $G_R(x, y) \nearrow G(x, y)$ as $R \nearrow +\infty$, by monotone convergence we deduce that

$$E_R(x) \nearrow E(x) = \int_M G(x, y)\, dv(y) .$$

We call $E(x)$ the *global mean exit time of* M. With this terminology and notation, M is not L^1-Liouville if and only if E is a genuine (say, finite) function. In particular, on a stochastically complete manifold, the global mean exit time must be infinite. On the other hand, we point out that, according to Section 2, there exist stochastically incomplete manifolds with infinite global mean exit time, thus showing that, in general, the global mean exit time does not carry enough information on the explosion of the Brownian motion because of the possible presence of direction along which explosion can occur in finite time.

3.1 Intrinsic Curvature Restrictions

We shall prove the following

Theorem 7 *Let (M, g) be a complete Riemannian manifold of dimension m with a pole o. Assume that the distance function $r(x) = d_M(x, o)$ satisfies*

$$\Delta r \geq (m-1)\frac{\sigma'}{\sigma}, \text{ on } M \tag{2}$$

where $\sigma : [0, +\infty) \to [0, +\infty)$ is the warping function of the m-dimensional model manifold M^m_σ. If M^m_σ is not L^1-Liouville (equivalently, stochastically incomplete) then M is not L^1-Liouville.

Remark 8 Since the model M^m_σ is stochastically incomplete, we already know by comparison arguments that M itself is stochastically incomplete. See [6] and, e.g., [3] where the more general case of weighted manifolds is considered.

Remark 9 By standard comparison arguments for the Laplacian of the distance function, condition 2 follows from the radial sectional curvature condition

$$Sec_{\text{rad}}(x) \le -\frac{\sigma''}{\sigma}(r(x)).$$

Note that the above result allows us to recover the already noted equivalence of stochastic completeness and L^1-Liouville property of model manifolds.

Proof Define

$$F_R(r) = \int_r^R \frac{\int_0^t \sigma^{m-1}(s)\,ds}{\sigma^{m-1}(t)}dt \tag{3}$$

and

$$F(r) = \int_r^{+\infty} \frac{\int_0^t \sigma^{m-1}(s)\,ds}{\sigma^{m-1}(t)}dt. \tag{4}$$

Since, by assumption, M_σ^m is stochastically incomplete, we have

$$F(r) < +\infty, \ \forall r.$$

Direct computations show that the transplanted function $F_R(r(x))$ satisfies

$$\begin{cases} \Delta F_R \le -1 \text{ on } B_R(o) \\ F_R \quad = 0 \quad \text{on } \partial B_R(o). \end{cases}$$

Therefore, by comparison on bounded domains,

$$E_R(x) \le F_R(r(x)) \text{ on } B_R,$$

and letting $R \to +\infty$ we conclude

$$E(x) \le F(r(x)).$$

This proves that M is not L^1-Liouville.

3.2 Minimal Submanifolds

This subsection aims to showing that the L^1-Liouville property of a proper minimal submanifold Σ of a manifold with a pole N depends on the curvature of the ambient space. In particular, in the case where N is a model with warping function σ, if the m-dimensional model manifold M_σ^m is not L^1-Liouville, and σ satisfies a technical isoperimetric condition, then Σ is not L^1-Liouville. As alluded to above, we shall use a global mean exit time comparison argument which extends a previous result by Markovsen, [8] (see also [2]).

Theorem 10 *Let $f : \Sigma \hookrightarrow N$ be an m-dimensional properly immersed minimal submanifold into a complete n-dimensional Riemannian manifold N. Assume that the sectional curvature of N satisfies*

$$K^N \le -G(\rho(y))$$

where G is a smooth even function on $\mathbb{R}$ and $\rho(y) = d_N(y, p)$ denotes the Riemannian distance function from a fixed point $p \in N$. Let σ be the solution of the initial value problem

$$\begin{cases} \sigma'' = G\sigma \\ \sigma(0) = 0,\ \sigma'(0) = 1 \end{cases}$$

and assume that

$$\sigma' \geq 0$$

and

$$t \to \frac{\int_0^t \sigma^{m-1}}{\sigma^m(t)} \text{ is non-increasing on } (0, +\infty). \tag{5}$$

Then, for every extrinsic ball B_R^N centered at p in N^n with radius $R < \mathrm{inj}_N(p)$, the mean exit time $E_{f^{-1}(B_R^N)}(x)$ satisfies

$$E_{f^{-1}(B_R^N)}(x) \leq F_R \circ \rho \circ f(x),$$

where F_R is defined in Eq. 3. In particular, if $\mathrm{inj}_N(p) = +\infty$, and

$$t \to \frac{\int_0^t \sigma^{m-1}}{\sigma^{m-1}} \in L^1(+\infty) \tag{6}$$

then Σ is not L^1-Liouville.

Proof Markvorsen [8], obtained the comparison result in the case of constant curvature reference spaces, which correspond to the choices

$$\sigma(t) = t, \quad \sigma(t) = k^{-1}\sin(kt), \quad \sigma(t) = k^{-1}\sinh(kt),$$

with $k > 0$. Actually, it is possible to extend Markvorsen arguments to the more general setting of Theorem 10 by using the function

$$\overline{F}_R(t) = F_R \circ \mathrm{i}^{-1}(t),$$

with

$$\mathrm{i}(t) = \int_0^t \sigma(s)\, ds.$$

We are going to exhibit a more straightforward argument which avoids the use of the auxiliary function $\overline{F}_R$.

Recall that if $f : M \to N$ is an isometric immersion and $\varphi : N \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ are smooth, then for every $X \in T_x M$ we have

$$\begin{aligned} \mathrm{Hess}(F \circ \varphi \circ f)(X, X) &= F''(\varphi(f(x)))\langle \nabla^N \varphi, df X\rangle^2 \\ &\quad + F'(\varphi(f(x)))\left[\mathrm{Hess}^N \varphi(df X, df X) + \langle \nabla^N \varphi, II(X, X)\rangle\right]. \end{aligned}$$

If $\varphi = \rho$ is the distance function, then the assumption on the sectional curvature of N implies

$$\mathrm{Hess}^N \rho(Y, Y) \geq \frac{\sigma'}{\sigma}\left[\langle Y, Y\rangle - \langle \nabla^N \rho, Y\rangle^2\right].$$

Thus, assuming that f is minimal and that $F' \leq 0$ and setting for ease of notation $\rho_x = \rho(f(x))$, we obtain

$$\Delta(F \circ \rho \circ f)(x) \leq m\left(F'\frac{\sigma'}{\sigma}\right)(\rho_x) + \left(F'' - F'\frac{\sigma'}{\sigma}\right)(\rho_x)\sum_{i=1}^{m}\langle\nabla^N\rho, dfX_i\rangle^2,$$

where $\{X_i\}$ is an orthonormal basis on T_xM.

Now, if $F = F_R$, then we have

$$F_R'(r) = -\frac{\int_0^r \sigma^{m-1}(s)\,ds}{\sigma^{m-1}(r)} < 0$$

and

$$F_R''(r) = -1 - (m-1)\frac{\sigma'}{\sigma}F_R',$$

so substituting,

$$\Delta(F \circ \rho \circ f)(x) \leq m\left(F'\frac{\sigma'}{\sigma}\right)(\rho_x) - \left(1 + m\frac{\sigma'}{\sigma}F_R'(\rho_x)\right)\sum_{i=1}^{m}\langle\nabla^N\rho, dfX_i\rangle^2. \qquad (7)$$

Complete $\{dfX_i\}_{i=1}^m$ to an orthonormal basis $\{dfX_i\}_{i=1}^m \cup \{X_j\}_{j=m+1}^n$ on $T_{f(x)}N$, and note that

$$\sum_i\langle\nabla^N\rho, dfX_i\rangle^2 + \sum_j\langle\nabla^N\rho, Y_j\rangle^2 = 1. \qquad (8)$$

On the other hand, using assumption (Eq. 5) in the form

$$\frac{\sigma^{m-1}}{\int_0^t \sigma^{m-1}} \leq m\frac{\sigma'}{\sigma},$$

we have

$$m\frac{\sigma'}{\sigma}F_R' = -m\frac{\sigma'}{\sigma}\frac{\int_0^t \sigma^{m-1}(s)\,ds}{\sigma^{m-1}(t)} \leq -1.$$

Inserting this latter and Eq. 8 into inequality 7 we finally obtain

$$\begin{aligned}\Delta(F \circ \rho \circ f)(x) &\leq -\sum_i\langle\nabla^N\rho, dfX_i\rangle^2 + m\frac{\sigma'}{\sigma}F_R'\sum_j\langle\nabla^N\rho, Y_j\rangle^2\\ &\leq -\sum_i\langle\nabla^N\rho, dfX_i\rangle^2 - \sum_j\langle\nabla^N\rho, Y_j\rangle^2 = -1.\end{aligned}$$

Thus,

$$\Delta\left(F_R \circ \rho \circ f\right) \leq -1 \text{ on } f^{-1}(B_R^{N_g}),$$

and since both $E_{f^{-1}(B_R^N)}$ and $F_R \circ \rho \circ f$ vanish on $\partial f^{-1}(B_R^N)$, the first assertion in the statement follows from the comparison principle.

The second assertion follows letting $R \to +\infty$, so that $f^{-1}(B_R^N) \nearrow M$ and $E_R(x) \nearrow E(x)$, while

$$F_R(r) \nearrow \int_r^{+\infty} \frac{\int_0^t \sigma^{m-1}(s)\, ds}{\sigma^{m-1}(t)} dt,$$

and, as seen above, the assumption that M_σ^m is not L^1-Liouville amounts to the fact that the integral on the right hand side is finite.

Note that Eq. 6 amounts to the fact that the m-dimensional model manifold M_σ^m is not L^1-Liouville.

We also remark that, since σ is non-decreasing, for every $n \geq m$

$$\int_r^R \frac{\int_0^t \sigma^{n-1}(s)\, ds}{\sigma^{n-1}(t)} dt \leq \int_r^R \frac{\int_0^t \sigma^{m-1}(s)\, ds}{\sigma^{m-1}(t)} dt,$$

so condition 6 also implies that the n-dimensional model N_σ^n is not L^1-Liouville, and therefore, by Theorem 7 the same holds for the manifold N.

Example 11 For $m = 2$, an admissible choice of the function $\sigma(r)$ is given by

$$\sigma(r) = r + 4r^3 e^{r^4}$$

which corresponds to

$$G(r) = \frac{8e^{r^4}}{4r^2 e^{r^4} + 1} \left(8r^8 + 18r^4 + 3\right) \asymp r^6.$$

Thus, from Theorem 10, if follows that if Σ is a 2-dimensional, properly immersed, minimal surface into an n-dimensional Cartan-Hadamard manifold N satisfying

$$K^N \leq -G(\rho(y))$$

then Σ is not L^1-Liouville.

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