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Orlicz–Morrey Spaces and Fractional Operators

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Abstract By taking an interest in a natural extension to the small parameters of the trace inequality for Morrey spaces, Orlicz–Morrey spaces are introduced and some inequalities for generalized fractional integral operators on Orlicz–Morrey spaces are established. The local boundedness property of the Orlicz maximal operators is investigated and some Morrey-norm equivalences are also verified. The result obtained here sharpens the one in our earlier papers.

Keywords Orlicz–Morrey space · Generalized fractional integral operator · Generalized fractional maximal operator · Orlicz maximal operator · Trace inequality · Olsen's inequality · Wiener–Stein equivalence

Mathematics Subject Classifications (2010) Primary 42B35; Secondary 42B25

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1 Introduction

In this paper we investigate some boundedness properties of the generalized fractional integral operators on Orlicz–Morrey spaces. The fractional integral operator I_α, $0 < \alpha < 1$, is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n(1-\alpha)}}\, dy,$$

and the fractional maximal operator M_α, $0 \le \alpha < 1$, is defined by

$$M_\alpha f(x) := \sup_{x \in Q \in \mathcal{Q}} \frac{1}{|Q|^{1-\alpha}} \int_Q |f(y)|\, dy.$$

Here, we use the notation $\mathcal{Q}$ to denote the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ to denote the volume of Q. Let $0 < p \le p_0 < \infty$. For an L^p locally integrable function f on $\mathbb{R}^n$ we set

$$\|f\|_{\mathcal{M}^{p,p_0}} := \sup_{Q \in \mathcal{Q}} |Q|^{1/p_0} \left(\frac{1}{|Q|} \int_Q |f(x)|^p\, dx \right)^{1/p}.$$

We will call the (classical) Morrey space $\mathcal{M}^{p,p_0}(\mathbb{R}^n) = \mathcal{M}^{p,p_0}$ the subset of all L^p locally integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{\mathcal{M}^{p,p_0}}$ is finite. Applying Hölder's inequality, we see that

$$\|f\|_{\mathcal{M}^{p_1,p_0}} \ge \|f\|_{\mathcal{M}^{p_2,p_0}} \text{ for all } p_0 \ge p_1 \ge p_2 > 0.$$

This tells us that

$$L^{p_0} = \mathcal{M}^{p_0,p_0} \subset \mathcal{M}^{p_1,p_0} \subset \mathcal{M}^{p_2,p_0} \text{ for all } p_0 \ge p_1 \ge p_2 > 0.$$

Morrey spaces, which were introduced by C. Morrey in order to study regularity questions which appear in the Calculus of Variations, describe local regularity more precisely than Lebesgue spaces and are widely used not only in harmonic analysis but also in partial differential equations (c.f. [4]).

The positivity of the Schrödinger operator $L = -\Delta - |v|^2$ holds if, for $u \in C_0^\infty(\mathbb{R}^n)$ and $0 < K < 1$,

$$\|uv\|_{L^2} \le K \|\, |\nabla u|\, \|_{L^2}. \tag{1.1}$$

Indeed, integration by parts says that

$$\langle Lu, u \rangle = \|\, |\nabla u|\, \|_{L^2}^2 - \|uv\|_{L^2}^2 \ge (1-K^2) \|\, |\nabla u|\, \|_{L^2}^2 > 0.$$

Thus, it is important to determine the smallest constant $K = K_v$ such that Eq. 1.1 holds. One way to prove Eq. 1.1 is by means of the inequality $|f(x)| \le C I_{1/n}(|\nabla f|)(x)$, which follows from the classical Sobolev integral representation (c.f. [18, p. 125]). Therefore one is now led to consider weighted inequality of Lebesgue spaces (so-called the trace inequality)

$$\|g \cdot I_\alpha f\|_{L^p} \le K_g \|f\|_{L^p}, \quad 1 < p < 1/\alpha. \tag{1.2}$$

This inequality was studied by many authors (see [5, Introduction]) and Kerman and Sawyer established the following.

The trace inequality 1.2 holds if and only if for all cubes $Q \in \mathcal{Q}$ there exists a constant $K > 0$ such that

$$\int_Q M_\alpha[|g|^p \chi_Q](x)^{p'}\, dx \le K^{p'} \int_Q |g(x)|^p\, dx < \infty, \tag{1.3}$$

where χ_Q denotes the characteristic function of cube Q and $p' = p/(p-1)$ is the conjugate exponent number of p.

We shall now define the modified (classical) Morrey space $\tilde{\mathcal{M}}^{p,\,1/\alpha}(\mathbb{R}^n) = \tilde{\mathcal{M}}^{p,\,1/\alpha}$ by the set of all L^p locally integrable functions g on $\mathbb{R}^n$ for which Eq. 1.3 holds, and define the norm $\|g\|_{\tilde{\mathcal{M}}^{p,1/\alpha}}$ by the smallest constant that satisfies Eq. 1.3. We would like to know the relation between the class $\tilde{\mathcal{M}}^{p,\,1/\alpha}$ and the corresponding ordinary Morrey spaces.

It follows that

$$\|f\|_{\mathcal{M}^{p,1/\alpha}} \le \|f\|_{\tilde{\mathcal{M}}^{p,1/\alpha}} \le C\|f\|_{\mathcal{M}^{q,1/\alpha}}, \quad 1 < p < q \le 1/\alpha. \tag{1.4}$$

Indeed,

$$\begin{aligned}\left(\frac{|Q|^{\alpha p}}{|Q|}\int_Q |f(x)|^p\, dx\right)^{p'} &= |Q|^{\alpha p}\left(\frac{|Q|^{\alpha}}{|Q|}\int_Q |f(x)|^p\, dx\right)^{p'}\\ &\le \frac{|Q|^{\alpha p}}{|Q|}\int_Q M_\alpha[|f|^p\chi_Q](x)^{p'}\, dx\\ &\le \|f\|^{p'}_{\tilde{\mathcal{M}}^{p,1/\alpha}}\frac{|Q|^{\alpha p}}{|Q|}\int_Q |f(x)|^p\, dx < \infty.\end{aligned}$$

This yields that $|Q|^\alpha\left(\frac{1}{|Q|}\int_Q |f(x)|^p\, dx\right)^{1/p} \le \|f\|_{\tilde{\mathcal{M}}^{p,1/\alpha}}$ and that the left inequality of Eq. 1.4 by taking the supremum over Q. Using Hölder's inequality, we have

$$\frac{|Q|^\alpha}{|Q|}\int_Q |f(x)|^p\, dx \le |Q|^\alpha\left(\frac{1}{|Q|}\int_Q |f(x)|^q\, dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q |f(x)|^{(p-1)q'}\, dx\right)^{1/q'}.$$

This implies

$$M_\alpha[|f|^p\chi_Q](x) \le \|f\|_{\mathcal{M}^{q,1/\alpha}}\left(M[|f|^{(p-1)q'}\chi_Q](x)\right)^{1/q'},$$

where $M = M_0$ denotes the Hardy–Littlewood maximal operator. The right inequality of Eq. 1.4 then follows by this and the $L^{p'/q'}$-boundedness of M (see also Claim 4.5 to follow). We emphasize that in the right inequality of Eq. 1.4 the parameter q of the integration satisfies proper inequality $p < q$.

The extension of the trace inequality 1.2 from Lebesgue spaces to Morrey spaces was due to Olsen in [11]. He showed that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{p,p_0}} \le C\|g\|_{\mathcal{M}^{q,1/\alpha}}\|f\|_{\mathcal{M}^{p,p_0}}, \quad 1 < p \le p_0 < 1/\alpha,\ 1 < p < q \le 1/\alpha. \tag{1.5}$$

We emphasize again that the parameter q of the integration satisfies the proper inequality $p < q$.

The third author showed further in [19] that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{p,p_0}} \le C\|g\|_{\mathcal{M}^{p,1/\alpha}}\|f\|_{\mathcal{M}^{q,p_0}}, \quad 0 < p \le 1 < q \le p_0 < 1/\alpha. \tag{1.6}$$

We emphasize now that $1 < q$.

The purpose of this paper is, motivated by these three proper inequalities contained in Eqs. 1.4–1.6, to study the boundedness properties of the generalized fractional integral operators on Orlicz–Morrey spaces. In general, Orlicz–Morrey spaces can describe more accurately the local regularity with the parameter q close to p or 1 (c.f. [15]), and, were introduced and studied by Nakai in [9] and [10]. However, our definition of the spaces is different from that due to Nakai.

The remainder of this paper is organized as follows: Main results can be found in the beginning of Sections 2 and 3. In Section 2 we describe Orlicz–Morrey spaces and establish some norm inequalities (the trace inequality and the Olsen inequality) for the generalized fractional integral operators. We give further a necessary and sufficient condition for which the Orlicz maximal operator is "locally bounded". Section 3 is devoted to investigating Morrey spaces with small parameters. We introduce some Morrey-norm equivalences and verify the boundedness properties of the generalized fractional integral operators for the small parameters. The accurate description of Orlicz–Morrey spaces works well in this problem. Section 4 has several examples of our main results. Finally, in Section 5 we state and prove some additional results. Throughout this paper all the notations are standard or will be defined as needed.

2 Orlicz–Morrey Spaces

In this section we define Orlicz–Morrey spaces and establish some norm inequalities for the generalized fractional integral operators. Especially, we give a necessary and sufficient condition for which the Orlicz maximal operator is locally bounded.

2.1 Definitions and Results

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. By $A \approx B$ we mean that $c^{-1}B \le A \le cB$ with some positive constant c independent of appropriate quantities. For any $1 < p < \infty$ we will write p' for the conjugate exponent number given by $1/p + 1/p' = 1$. All "cubes" in $\mathbb{R}^n$ are assumed to have their sides parallel to the coordinate axes, $\mathcal{Q}$ to denote the family of all such cubes and $\ell(Q)$ to denote the side-length of Q. For $Q \in \mathcal{Q}$ we use cQ to denote the cube with the same center as Q, but with side-length $c\ell(Q)$. We denote $|E|$ by the Lebesgue measure of $E \subset \mathbb{R}^n$. We will denote by $\mathcal{D}$ the family of all dyadic cubes in $\mathbb{R}^n$. We will also write, for the sake of simplicity, for any cube $Q \in \mathcal{Q}$ and any locally integrable function f

$$m_Q(f) := \frac{1}{|Q|}\int_Q f(x)\,dx.$$

Let $\rho : [0, \infty) \to [0, \infty]$ be a suitable function. We define the generalized fractional integral operator T_ρ and the generalized fractional maximal operator M_ρ by

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy,$$
$$M_\rho f(x) := \sup_{x \in Q \in \mathcal{Q}} \rho(\ell(Q)) m_Q(|f|).$$

If $\rho(t) \equiv t^{n\alpha}$, $0 < \alpha < 1$, then $T_\rho = I_\alpha$ and $M_\rho = M_\alpha$. We now define the condition that we need to postulate of a function ρ for the generalized fractional integral operator T_ρ.

Definition 2.1 By the "Dini condition" we mean that

$$\int_0^1 \frac{\rho(s)}{s} \, ds < \infty, \tag{2.1}$$

while the "weaker growth condition" is that there are constants $\delta, c > 0$, $0 \le \varepsilon < 1$ with the property that

$$\sup_{s \in (t/2,\, t]} \rho(s) \le c \int_{\frac{\delta(1-\varepsilon)}{2} t}^{\delta(1+\varepsilon)t} \frac{\rho(s)}{s} \, ds \text{ for all } t > 0. \tag{2.2}$$

In the sequel, for the generalized fractional integral operator T_ρ, we always assume that ρ satisfies Eqs. 2.1 and 2.2, and, then denote the set of all such functions by $\mathcal{G}_0$. We will write, when $\rho \in \mathcal{G}_0$,

$$\tilde{\rho}(t) := \int_0^t \frac{\rho(s)}{s} \, ds.$$

Remark 2.2 Typical examples of $\rho(t)$ that we envisage are, for $0 < \alpha < 1$,

$$\rho(t) \equiv \begin{cases} \dfrac{t^{n\alpha}}{\log(e/t)}, & 0 < t \le 1, \\ t^{n\alpha} \log(et), & 1 \le t < \infty, \end{cases}$$

and, for $c > 0$,

$$\rho(t) \equiv \begin{cases} t^{n\alpha}, & 0 < t \le 1, \\ e^c t^{-n} e^{-ct^2}, & 1 \le t < \infty. \end{cases}$$

The second one is used to control the Bessel potential (see [18]). In our previous papers [16] and [17] we have assumed that ρ satisfies

$$\frac{1}{C} \le \frac{\rho(s)}{\rho(t)} \le C, \text{ if } \frac{1}{2} \le \frac{s}{t} \le 2. \tag{2.3}$$

This implies

$$\sup_{s \in (t/2,\, t]} \rho(s) \le C \int_{t/2}^t \frac{\rho(s)}{s} \, ds.$$

However, this condition cannot reflect the rapid decay of integral kernel at infinity such as the Bessel potential. For this reason, following mainly [12], we postulate the "weaker growth condition" on ρ. As we will see below, ρ enjoys the same sufficient conditions for the boundedness properties.

To describe Orlicz–Morrey spaces, we recall some definitions and notation.

A function $\Phi : [0, \infty) \to [0, \infty]$ is said to be a Young function if it is left-continuous, convex and increasing, and if $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. We say that Φ is a normalized Young function when Φ is a Young function and $\Phi(1) = 1$. It is easy to see that t^p, $1 \leq p < \infty$, is a normalized Young function.

A Young function Φ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if for some $K > 1$

$$\Phi(2t) \leq K\Phi(t) \text{ for all } t > 0.$$

Meanwhile, a Young function Φ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if for some $K > 1$

$$\Phi(t) \leq \frac{1}{2K}\Phi(Kt) \text{ for all } t > 0.$$

The function $\Phi(t) \equiv t$ satisfies the Δ_2-condition but fails the ∇_2-condition. If $1 < p < \infty$, then $\Phi(t) \equiv t^p$ satisfies both conditions. The complementary function $\bar{\Phi}$ of a Young function Φ is defined by

$$\bar{\Phi}(t) := \sup\{ts - \Phi(s) : s \in [0, \infty)\}.$$

Then $\bar{\Phi}$ is also a Young function and $\bar{\bar{\Phi}} = \Phi$. Notice that $\Phi \in \nabla_2$ if and only if $\bar{\Phi} \in \Delta_2$. For the other properties of Young functions and the examples, see [9, p. 196] or the book [14]. In Section 5 we collect some examples as well.

Given a Young function Φ, define the Orlicz space $\mathcal{L}^\Phi(\mathbb{R}^n) = \mathcal{L}^\Phi$ by the Luxemberg norm

$$\|f\|_{\mathcal{L}^\Phi} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

When $\Phi(t) \equiv t^p$, $1 \leq p < \infty$, $\|f\|_{\mathcal{L}^\Phi} = \|f\|_{L^p}$. We need the following basic two facts.

Generalized Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq C\|f\|_{\mathcal{L}^\Phi}\|g\|_{\mathcal{L}^{\bar{\Phi}}};$$

The dual equation:

$$\|f\|_{\mathcal{L}^\Phi} \approx \sup\left\{\|fg\|_{L^1} : \|g\|_{\mathcal{L}^{\bar{\Phi}}} \leq 1\right\}.$$

Given a Young function Φ, define the mean Luxemburg norm of f on a cube $Q \in \mathcal{Q}$ by

$$\|f\|_{\Phi, Q} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

When $\Phi(t) \equiv t^p$, $1 \leq p < \infty$,

$$\|f\|_{\Phi, Q} = \left(\frac{1}{|Q|}\int_Q |f(x)|^p\, dx\right)^{1/p},$$

that is, the mean Luxemburg norm coincides with the (normalized) L^p norm. It should be noticed that

$$\|f\|_{\Phi, Q} = \|\tau_{\ell(Q)}[f\chi_Q]\|_{\mathcal{L}^\Phi}, \tag{2.4}$$

where τ_δ, $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$. It follows from this relation and generalized Hölder's inequality that for any cube $Q \in \mathcal{Q}$

$$m_Q(|fg|) \leq C\|f\|_{\Phi, Q}\|g\|_{\bar{\Phi}, Q}. \tag{2.5}$$

The Orlicz maximal operator, for any Young function Ψ, is defined by

$$M^\Psi f(x) := \sup_{x\in Q\in\mathcal{Q}} \|f\|_{\Psi, Q}.$$

Now let us introduce Orlicz–Morrey spaces.

Definition 2.3 Let $\mathcal{G}_1$ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t)$ is nondecreasing but that $\phi(t)t^{-n}$ is nonincreasing. Let $\phi \in \mathcal{G}_1$ and let Φ be a Young function. The Orlicz–Morrey space $\mathcal{L}^{\Phi,\phi}(\mathbb{R}^n) = \mathcal{L}^{\Phi,\phi}$ consists of all locally integrable functions f on $\mathbb{R}^n$ for which the norm

$$\|f\|_{\mathcal{L}^{\Phi,\phi}} := \sup_{Q\in\mathcal{Q}} \phi(\ell(Q))\|f\|_{\Phi, Q}$$

is finite. In particular, in order that the characteristic function of the unit cubes belongs to $\mathcal{L}^{\Phi,\phi}$, it should be always assumed that

$$\sup_{t>1} \frac{\phi(t)}{\Phi^{-1}(t^n)} < \infty.$$

If $\Phi(t) \equiv t^p$ and $\phi(t) \equiv t^{n/p_0}$, $1 \leq p \leq p_0 < \infty$, then $\mathcal{L}^{\Phi,\phi} = \mathcal{M}^{p,p_0}$. That is, then Orlicz–Morrey spaces coincide with (classical) Morrey spaces. When $\Phi(t) \equiv t^p$, $1 \leq p < \infty$, we will denote $\mathcal{L}^{\Phi,\phi}$ by $\mathcal{M}^{p,\phi}$. In this case we will call it the (generalized) Morrey space. In Section 3 we consider $\mathcal{M}^{p,\phi}$ even for $0 < p < 1$.

Remark 2.4

(1) The class $\mathcal{G}_1$ is a natural one for defining $\mathcal{L}^{\Phi,\phi}$. We shall verify that, for any suitable function $\phi : [0, \infty) \to [0, \infty)$, the norm defined by

$$\|f\|_{\mathcal{L}^{\Phi,\phi}} := \sup_{Q\in\mathcal{Q}} \phi(\ell(Q))\|f\|_{\Phi, Q}$$

is equivalent to a norm $\|f\|_{\mathcal{L}^{\Phi,\phi_2}}$ for some $\phi_2 \in \mathcal{G}_1$. Indeed, if we let

$$\phi_1(t) = \sup_{t'\in[0, t]} \phi(t'),$$

then $\phi_1(t)$ is nondecreasing and $\|f\|_{\Phi,\phi} \approx \|f\|_{\Phi,\phi_1}$. Similarly, if we let

$$\phi_2(t) = t^n \sup_{t'\geq t} \phi_1(t')t'^{-n},$$

then $\phi_2(t)$ is nondecreasing but $\phi_2(t)t^{-n}$ is nonincreasing and $\| f\|_{\Phi,\,\phi_1} \approx \| f\|_{\Phi,\,\phi_2}$. These hold from the fact that, for any cube $Q \in \mathcal{Q}$ and any positive number $t' \le \ell(Q)$,

$$\| f\|_{\Phi,\,Q} \le 2^{n+1} \sup_{Q' \in \mathcal{Q}:\, Q' \subset Q,\, \ell(Q')=t'} \| f\|_{\Phi,\,Q'}$$

and that, for all cubes $Q_1 \subset Q_2$,

$$|Q_1|\,\| f\|_{\Phi,\,Q_1} \le C|Q_2|\,\| f\|_{\Phi,\,Q_2},$$

which can be proved by using the simple geometric fact and another characterization of the Luxemburg norm (c.f. [14, p. 69])

$$\| f\|_{\Phi,\,Q} \le \inf_{s>0} \left\{ s + \frac{s}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{s}\right) dx \right\} \le 2\| f\|_{\Phi,\,Q}.$$

(2) We also remark that if $\phi \in \mathcal{G}_1$ then it automatically satisfies the doubling condition

$$\phi(2t) \le 2^n \phi(t) \text{ for all } t > 0.$$

(3) As a special case when $\psi(t) \equiv 1$, we have $\mathcal{L}^{\Phi,\,\phi} = L^\infty$ with norm equivalence. Indeed, it is not so hard to see that $\mathcal{L}^{\Phi,\,\phi} \hookrightarrow L^\infty$. To see the converse, we take $f \in \mathcal{L}^{\Phi,\,\phi}$ arbitrarily. Then we have

$$\| f\|_{\Phi,\,Q} \le \| f\|_{\mathcal{L}^{\Phi,\,\phi}}$$

for all cubes $Q \in \mathcal{Q}$, which implies

$$\frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\| f\|_{\mathcal{L}^{\Phi,\,\phi}}}\right) dx \le 1.$$

If a sequence $\{Q_j\}_{j=1}^\infty$ of cubes shrinks to a Lebesgue point x of $\Phi \circ f$, then we have

$$\Phi\left(\frac{|f(x)|}{\| f\|_{\mathcal{L}^{\Phi,\,\phi}}}\right) = \lim_{j\to\infty} \frac{1}{|Q_j|} \int_{Q_j} \Phi\left(\frac{|f(y)|}{\| f\|_{\mathcal{L}^{\Phi,\,\phi}}}\right) dy \le 1.$$

The set of all Lebesgue points of $\Phi \circ f$ being almost equal to $\mathbb{R}^n$, we see that $\mathcal{L}^{\Phi,\,\phi} \hookleftarrow L^\infty$.

(4) The class $\mathcal{G}_1$ is a good class of describing intersection spaces. Indeed, we have that $\max(\phi_1,\,\phi_2) \in \mathcal{G}_1$ whenever $\phi_1, \phi_2 \in \mathcal{G}_1$ and that $\mathcal{L}^{\Phi,\,\phi_1} \cap \mathcal{L}^{\Phi,\,\phi_2} = \mathcal{L}^{\Phi,\,\max(\phi_1,\,\phi_2)}$ with norm equivalence.

We define an auxiliary space too.

Definition 2.5 Let $\phi \in \mathcal{G}_1$ and let Φ be a Young function. The space $\tilde{\mathcal{L}}^{\Phi,\,\phi}(\mathbb{R}^n) = \tilde{\mathcal{L}}^{\Phi,\,\phi}$ consists of all locally integrable functions g on $\mathbb{R}^n$ for which the norm

$$\|g\|_{\tilde{\mathcal{L}}^{\Phi,\,\phi}} := \sup\left\{\| M_\phi[gw\chi_Q]\|_{\bar{\Phi},\,Q} \;:\; Q \in \mathcal{Q},\; \|w\|_{\bar{\Phi},\,Q} \le 1\right\}$$

is finite. Here, M_ϕ is the generalized fractional maximal operator.

Related to the space $\tilde{\mathcal{L}}^{\Phi,\,\phi}$, we need the following notion too.

Definition 2.6 Let Φ and Ψ be Young functions. We say that "the Orlicz maximal operator M^{Ψ} is locally bounded in the norm determined by Φ", when it satisfies

$$\|M^{\Psi}[g\chi_Q]\|_{\Phi,\,Q} \le C\|g\|_{\Phi,\,Q} \text{ for all cubes } Q \in \mathcal{Q}.$$

Remark 2.7 When $\phi(t) \equiv t^{n\alpha}$, $0 < \alpha < 1$, and $\Phi(t) \equiv t^p$, $1 < p < 1/\alpha$, the space $\tilde{\mathcal{L}}^{\Phi,\,\phi}$ can be characterized by the condition 1.3 (see [5, Theorem 2.3]). That is, then we have $\tilde{\mathcal{L}}^{\Phi,\,\phi} = \tilde{\mathcal{M}}^{p,\,1/\alpha}$. We do not know whether the Orlicz counterpart of Eq. 1.3 is available or not. Following [12], using the local boundedness property of the Orlicz maximal operator, we can find Orlicz–Morrey spaces which are embeded into $\tilde{\mathcal{L}}^{\Phi,\,\psi}$ (see Claim 2.13 to follow).

We now state our first results, which are the extension of those in [16, 17] to Orlicz–Morrey spaces.

Theorem 2.8 *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$ and $\Phi \in \nabla_2$. Suppose that*

$$\int_t^{\infty} \frac{\rho(s)}{s\phi(s)}\, ds \le C\frac{\tilde{\rho}(t)}{\phi(t)} \text{ for all } t > 0. \tag{2.6}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\,\phi}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Phi,\,\tilde{\rho}}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorem 2.9 *Let Ψ be a Young function. With the same condition posed in Theorem* 2.8, *if, in addition, $M^{\bar{\Psi}}$ is locally bounded in the norm determined by $\bar{\Phi}$, then we have*

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\,\phi}} \le C\|g\|_{\mathcal{L}^{\Psi,\,\tilde{\rho}}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorems 2.8 and 2.9 are the trace inequalities of the generalized fractional integral operators for Orlicz–Morrey spaces.

Theorem 2.10 *Let $\rho \in \mathcal{G}_0$, $\phi, \psi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \le 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^{\infty} \frac{\rho(s)}{s\phi(s)}\, ds \le C\frac{\psi(t)}{\eta(t)} \text{ for all } t > 0. \tag{2.7}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Psi,\,\eta}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Psi,\,\psi}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorem 2.10 is a general form of Theorem 2.8 (letting $a \equiv 1$) and is the Olsen inequality of the generalized fractional integral operators for Orlicz–Morrey spaces. In Section 4 we will encounter some examples.

Letting $g(x) \equiv 1$ and $\psi(t) \equiv 1$ in Theorem 2.10, we can recover the boundedness property of T_ρ by noticing that (see Corollary 2.19 to follow) if $\bar{\Psi} \in \nabla_2$ then

$$\sup\left\{\|M[w\chi_Q]\|_{\bar{\Psi},\,Q} : Q \in \mathcal{Q},\ \|w\|_{\bar{\Psi},\,Q} \le 1\right\} = C,$$

where M is the Hardy–Littlewood maximal operator.

Corollary 2.11 *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \le 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that $\bar{\Psi} \in \nabla_2$ and that

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le \frac{C}{\eta(t)} \text{ for all } t > 0.$$

Then

$$\|T_\rho f\|_{\mathcal{L}^{\Psi,\eta}} \le C\|f\|_{\mathcal{L}^{\Phi,\phi}}.$$

In Theorem 5.4, using the method developed in the last part of the proof of Lemma 2.22, we reprove this corollary directly without the assumption $\bar{\Psi} \in \nabla_2$. Corollary 2.11 generalizes [16, Corollary 1.7]. In [8] Nakai studied the boundness of the generalized fractional integral operator T_ρ on Orlicz spaces. Since, we cannot recover Orlicz spaces as a special case of our Orlicz–Morrey spaces, we dare not compare Corollary 2.11 with [8, Theorem 3.1].

2.2 Principal Lemma

The proof of the previous results relies upon the following principal lemma (Lemma 2.12). We shall make some remarks since it would be somehow complicated.

Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$ and $\Phi \in \nabla_2$. Then, by letting $\psi(t) \equiv \tilde{\rho}(t)$, (2) of Lemma 2.12 yields

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\phi}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Phi,\tilde{\rho}}}\|Mf\|_{\mathcal{L}^{\Phi,\phi}}, \tag{2.8}$$

if Eq. 2.6 holds. Once we verify Eq. 2.8, Theorem 2.8 will have been an immediate consequence of the boundedness of the Hardy–Littlewood maximal operator M on the Orlicz–Morrey space $\mathcal{L}^{\Phi,\phi}$ (see Corollary 2.21 to follow). While, by letting $g(x) \equiv 1$ and $\psi(t) \equiv 1$ and noticing that $\|1\|_{\tilde{\mathcal{L}}^{\Phi,1}} = C$ when $\bar{\Phi} \in \nabla_2$, (2) of Lemma 2.12 yields

$$\|T_\rho f\|_{\mathcal{L}^{\Phi,\phi}} \le C\|M_{\tilde{\rho}} f\|_{\mathcal{L}^{\Phi,\phi}}, \tag{2.9}$$

if $\bar{\Phi} \in \nabla_2$ and

$$\int_t^\infty \frac{\rho(s)}{s\tilde{\rho}(s)\phi(s)}\,ds \le \frac{C}{\phi(t)} \text{ for all } t > 0.$$

The inequality 2.9 means that the Orlicz–Morrey norm of the generalized fractional integral operator $T_\rho f$ can be controled by that of the generalized fractional maximal operator $M_{\tilde{\rho}} f$. Sometimes, assuming Eq. 2.3 for example, one can verify the converse. In general, in spite of losing the linearity, one could expect less singularity to $M_{\tilde{\rho}}$ than T_ρ. This Morrey norm equivalence is first proved by Adams and Xiao in [2]. To prove Theorem 2.10, we need more infomation between the generalized fractional integral operators and the generalized fractional maximal operators in the framework of Orlicz–Morrey spaces. In some sense, Lemma 2.12 is the link between Eqs. 2.8 and 2.9.

Lemma 2.12 *Let $\rho \in \mathcal{G}_0$, $\phi, \psi, \eta \in \mathcal{G}_1$ and let Φ be a Young function.*

(1) *Assume that*

$$\int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le C\frac{\psi(t)}{\eta(t)} \text{ for all } t > 0. \tag{2.10}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\eta}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}}\left(\|M_{\tilde{\rho}/\psi} f\|_{\mathcal{L}^{\Phi,\eta}} + \|f\|_{\mathcal{M}^{1,\phi}}\right).$$

(2) *Assume that*

$$\int_t^\infty \frac{\rho(s)\psi(s)}{s\tilde{\rho}(s)\phi(s)}\,ds \le C\frac{\psi(t)}{\phi(t)} \text{ for all } t > 0. \tag{2.11}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\phi}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}}\|M_{\tilde{\rho}/\psi} f\|_{\mathcal{L}^{\Phi,\phi}}.$$

Proof We denote by $\mathcal{D}$ the family of all dyadic cubes in $\mathbb{R}^n$. First of all we notice that, in general, if $\phi \in \mathcal{G}_1$ then it automatically satisfies the doubling condition

$$\phi(2t) \le 2^n\phi(t) \text{ for all } t > 0.$$

A geometric observation shows that

$$\|f\|_{\mathcal{L}^{\Phi,\phi}} \approx \sup_{Q\in\mathcal{D}} \phi(\ell(Q))\|f\|_{\Phi,\,Q}. \tag{2.12}$$

We assume that f and g are nonnegative. Noticing Eq. 2.12, for any $Q_0 \in \mathcal{D}$ we wish to estimate

$$\|g \cdot T_\rho f\|_{\Phi,\,Q_0}.$$

By a duality argument, noticing Eq. 2.4, it suffices to estimate

$$\frac{1}{|Q_0|}\int_{Q_0} w(x)g(x)T_\rho f(x)\,dx$$

for all nonnegative measurable functions w such that

$$\|w\|_{\bar{\Phi},\,Q_0} \le 1.$$

We now set, for all $t > 0$,

$$\hat{\rho}(t) := \int_{\frac{\delta(1-\varepsilon)}{2}t}^{\delta(1+\varepsilon)t} \frac{\rho(s)}{s}\,ds.$$

Then, by Eq. 2.2 we have

$$\sup_{s\in(t/2,\,t]} \rho(s) \le c\hat{\rho}(t). \tag{2.13}$$

For simplicity, we will write

$$c_1 \equiv \frac{\delta(1-\varepsilon)}{2} \text{ and } c_2 \equiv \delta(1+\varepsilon).$$

It follows that

$$\begin{aligned}
T_\rho f(x) &= \sum_{\nu\in\mathbb{Z}} \int_{2^{\nu-1}<|x-y|\le 2^\nu} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\,dy \\
&\le C\sum_{\nu\in\mathbb{Z}} \frac{\hat\rho(2^\nu)}{2^{n\nu}} \int_{|x-y|\le 2^\nu} f(y)\,dy \\
&\le C\sum_{\nu\in\mathbb{Z}} \sum_{Q\in\mathcal{D}:\,Q\ni x,\,\ell(Q)=2^\nu} \frac{\hat\rho(\ell(Q))}{|Q|}\int_{3Q} f(y)\,dy \\
&= C\sum_{Q\in\mathcal{D}} \frac{\hat\rho(\ell(Q))}{|Q|}\int_{3Q} f(y)\,dy\cdot\chi_Q(x) \\
&= C\sum_{Q\in\mathcal{D}} \hat\rho(\ell(Q))m_{3Q}(f)\cdot\chi_Q(x),
\end{aligned}$$

where we use the notation $m_Q(f) := \dfrac{1}{|Q|}\displaystyle\int_Q f(x)\,dx$.

We now let

$$\begin{cases} \mathcal{D}_1(Q_0) := \{Q\in\mathcal{D}:\ Q\subset Q_0\} \\ \mathcal{D}_2(Q_0) := \{Q\in\mathcal{D}:\ Q\supsetneq Q_0\} \end{cases}$$

and evaluate the quantities, for $i=1,2$,

$$J_i := \frac{1}{|Q_0|}\int_{Q_0} w(x)g(x)\left(\sum_{Q\in\mathcal{D}_i(Q_0)} \hat\rho(\ell(Q))m_{3Q}(f)\chi_Q(x)\right)dx.$$

Denote by N_0 an integer such that

$$N_0 \approx 1+\log_2\frac{1+\varepsilon}{1-\varepsilon}.$$

Then a geometric observation shows that

$$\sum_{\nu\in\mathbb{Z}} \chi_{(c_1,\,c_2)}(2^\nu)\le N_0.$$

Consequently, for any dyadic cube $R\in\mathcal{D}$, we have

$$\begin{aligned}
\sum_{q\in\mathcal{D}:\,Q\subset R} \hat\rho(\ell(Q))\int_{3Q} f(y)\,dy &= \sum_{\nu=-\infty}^{\log_2\ell(R)} \hat\rho(2^\nu)\left(\sum_{Q\in\mathcal{D}:\,Q\subset R,\,\ell(Q)=2^\nu}\int_{3Q} f(y)\,dy\right) \\
&\le C\int_{3R} f(y)\,dy\left(\sum_{\nu=-\infty}^{\log_2\ell(R)}\hat\rho(2^\nu)\right) \\
&\le CN_0\tilde\rho(\ell(c_2R))\int_{3R} f(y)\,dy.
\end{aligned}$$

Once we verify this fact, it has been essentially shown in [17] (see also Lemma 3.8 below) that there exist a collection of dyadic cubes $\{Q_{k,j}\} \subset \mathcal{D}_1(Q_0)$ and a collection of disjoint measurable sets $\{E_0\} \cup \{E_{k,j}\}$ such that

$$E_0 \subset Q_0,\ E_{k,j} \subset Q_{k,j}, \quad |Q_0| \le 2|E_0|,\ |Q_{k,j}| \le 2|E_{k,j}|, \quad Q_0 = E_0 \cup \bigcup_{k,j} E_{k,j}$$

and that

$$\begin{aligned} C|Q_0|J_1 \le\ & \tilde{\rho}(\ell(c_2 Q_0)) m_{Q_0}(wg) m_{3Q_0}(f)|E_0| \\ & + \sum_{k,j} \tilde{\rho}(\ell(c_2 Q_{k,j})) m_{Q_{k,j}}(wg) m_{3Q_{k,j}}(f)|E_{k,j}|. \end{aligned}$$

Noticing, by use of the doubling condition of ψ,

$$\begin{aligned} & \tilde{\rho}(\ell(c_2 Q_{k,j})) m_{Q_{k,j}}(wg) m_{3Q_{k,j}}(f) \\ & \le C\psi(\ell(Q_{k,j})) m_{Q_{k,j}}(wg) \cdot \frac{\tilde{\rho}(\ell(c_0 Q_{k,j}))}{\psi(\ell(c_0 Q_{k,j}))} m_{c_0 Q_{k,j}}(f), \end{aligned}$$

where $c_0 \equiv \max(c_2, 3)$, and then

$$\tilde{\rho}(\ell(c_2 Q_{k,j})) m_{Q_{k,j}}(wg) m_{3Q_{k,j}}(f)|E_{k,j}| \le C \int_{E_{k,j}} M_\psi[wg](x) M_{\tilde{\rho}/\psi} f(x)\, dx,$$

we have

$$J_1 \le \frac{C}{|Q_0|} \int_{Q_0} M_\psi[wg](x) M_{\tilde{\rho}/\psi} f(x)\, dx.$$

It follows from generalized Hölder's inequality 2.5 that

$$J_1 \le C\| M_\psi[wg]\|_{\bar{\Phi}, Q_0} \| M_{\tilde{\rho}/\psi} f\|_{\Phi, Q_0}.$$

Recalling that $\|w\|_{\bar{\Phi}, Q_0} \le 1$, we have (see Definition 2.5)

$$\| M_\psi[wg]\|_{\bar{\Phi}, Q_0} \le \|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}}$$

and, hence, we obtain

$$J_1 \le C\|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}} \| M_{\tilde{\rho}/\psi} f\|_{\Phi, Q_0}. \tag{2.14}$$

Thus, the estimate for J_1 is now valid.

Let us turn to the estimate of J_2. It follows for any $Q \in \mathcal{Q}$ that

$$\hat{\rho}(\ell(Q)) m_{3Q}(f) \le \|f\|_{\mathcal{M}^{1,\phi}} \frac{\hat{\rho}(\ell(Q))}{\phi(\ell(3Q))} \le C \int_{\ell(c_1 Q)}^{\ell(c_2 Q)} \frac{\rho(s)}{s\phi(s)}\, ds,$$

where we have used the fact that $\phi(t)$ is nondecreasing and the doubling condition of ϕ when $c_2 > 3$. As a consequence

$$\begin{aligned} J_2 & \le C\|f\|_{\mathcal{M}^{1,\phi}} m_{Q_0}(wg) \sum_{Q \in \mathcal{D}_2(Q_0)} \int_{\ell(c_1 Q)}^{\ell(c_2 Q)} \frac{\rho(s)}{s\phi(s)}\, ds \\ & \le C\|f\|_{\mathcal{M}^{1,\phi}} m_{Q_0}(wg) \int_{\ell(c_1 Q_0)}^{\infty} \frac{\rho(s)}{s\phi(s)}\, ds \le C\|f\|_{\mathcal{M}^{1,\phi}} m_{Q_0}(wg) \frac{\psi(\ell(Q_0))}{\eta(\ell(Q_0))}. \end{aligned}$$

Here, we have invoked the condition 2.10 and the doubling condition of ψ and η. It follows immediately from the definition of the mean Luxemburg norm that

$$\|\chi_{Q_0}\|_{\Phi,\,Q_0} = \Phi^{-1}(1).$$

These imply that

$$J_2 \leq C\eta(\ell(Q_0))^{-1}\|f\|_{\mathcal{M}^{1,\phi}}\psi(\ell(Q_0))m_{Q_0}(wg)$$

and that

$$\begin{aligned}\psi(\ell(Q_0))m_{Q_0}(wg) &\leq m_{Q_0}\left(M_\psi[wg\chi_{Q_0}]\right)\\ &\leq C\Phi^{-1}(1)\|M_\psi[wg]\|_{\tilde{\Phi},\,Q_0}\\ &\leq C\Phi^{-1}(1)\|g\|_{\tilde{\mathcal{L}}(\Phi,\,\psi)},\end{aligned}$$

where in the second inequality we have used generalized Hölder's inequality 2.5. Thus, we obtain

$$J_2 \leq C\eta(\ell(Q_0))^{-1}\|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}}\|f\|_{\mathcal{M}^{1,\phi}}. \tag{2.15}$$

It follows from Eqs. 2.14 and 2.15 that

$$\begin{aligned}C\eta(\ell(Q_0))\|g\cdot T_\rho f\|_{\Phi,\,Q_0} &\leq \|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}}\left(\eta(\ell(Q_0))\|M_{\tilde{\rho}/\psi}f\|_{\Phi,\,Q_0} + \|f\|_{\mathcal{M}^{1,\phi}}\right)\\ &\leq \|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}}\left(\|M_{\tilde{\rho}/\psi}f\|_{\mathcal{L}^{\Phi,\eta}} + \|f\|_{\mathcal{M}^{1,\phi}}\right).\end{aligned}$$

By taking the supremum over all dyadic cubes $Q_0 \in \mathcal{D}$ in the left side, (1) of the theorem is now verified.

Finally, (2) of the theorem holds from the following two facts. First, for any $Q \in \mathcal{Q}$ we have, recalling that $c_0 \equiv \max(c_2, 3)$,

$$\begin{aligned}\hat{\rho}(\ell(Q))m_{3Q}(f) &\leq C\frac{\hat{\rho}(\ell(Q))\psi(\ell(c_0Q))}{\tilde{\rho}(\ell(c_0Q))}\cdot\frac{\tilde{\rho}(\ell(c_0Q))}{\psi(\ell(c_0Q))}m_{c_0Q}(f)\\ &\leq C\frac{\hat{\rho}(\ell(Q))\psi(\ell(c_0Q))}{\tilde{\rho}(\ell(c_0Q))}\cdot\frac{1}{|Q|}\int_Q M_{\tilde{\rho}/\psi}f(x)\,dx\\ &\leq C\|M_{\tilde{\rho}/\psi}f\|_{\mathcal{L}^{\Phi,\phi}}\frac{\hat{\rho}(\ell(Q))\psi(\ell(c_0Q))}{\tilde{\rho}(\ell(c_0Q))\phi(\ell(Q))}.\end{aligned}$$

Second, by use of the facts that $\psi(t)$, $\phi(t)$ and $\tilde{\rho}(t)$ are nondecreasing and ψ and ϕ satisfy the doubling condition and use of the condition 2.11 we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}_2(Q_0)}\frac{\hat{\rho}(\ell(Q))\psi(\ell(c_0Q))}{\tilde{\rho}(\ell(c_0Q))\phi(\ell(Q))} &\leq C\sum_{\nu=1+\log_2\ell(Q_0)}^{\infty}\int_{c_12^{n\nu}}^{c_22^{n\nu}}\frac{\rho(s)\psi(s)}{s\tilde{\rho}(s)\phi(s)}\,ds\\ &\leq C\frac{\psi(\ell(Q_0))}{\phi(\ell(Q_0))}.\end{aligned}$$

These imply

$$J_2 \leq C\phi(\ell(Q_0))^{-1}\|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}}\|M_{\tilde{\rho}/\psi}f\|_{\mathcal{L}^{\Phi,\phi}}. \tag{2.16}$$

The desired inequality then follows from Eqs. 2.14 and 2.16. $\square$

Following [12], we wish to find Orlicz–Morrey spaces which are embeded into $\tilde{\mathcal{L}}^{\Phi,\psi}$.

Claim 2.13 Let $\psi \in \mathcal{G}_1$ and let Φ and Ψ be Young functions. Assume that $M^{\bar{\Psi}}$ is locally bounded in the norm determined by $\bar{\Phi}$. Then

$$\|g\|_{\tilde{\mathcal{L}}^{\Phi,\psi}} \le C\|g\|_{\mathcal{L}^{\Psi,\psi}}.$$

Proof Fix a cube R. By generalized Hölder's inequality 2.5 and the definition of the Orlicz–Morrey norm, we have, for any $Q \in \mathcal{Q}$,

$$\psi(\ell(Q))m_Q(wg\chi_R) \le C\|g\|_{\mathcal{L}^{\Psi,\psi}}\|w\chi_R\|_{\bar{\Psi},Q}.$$

This yields

$$M_\psi[wg\chi_R](x) \le C\|g\|_{\mathcal{L}^{\Psi,\psi}} M^{\bar{\Psi}}[w\chi_R](x).$$

Then, using the assumption $M^{\bar{\Psi}}$ is locally bounded in the norm determined by $\bar{\Phi}$, we have

$$\|M_\psi[wg\chi_R]\|_{\bar{\Phi},R} \le C\|g\|_{\mathcal{L}^{\Psi,\psi}}\|M^{\bar{\Psi}}[w\chi_R]\|_{\bar{\Phi},R} \le C\|g\|_{\mathcal{L}^{\Psi,\psi}}\|w\|_{\bar{\Phi},R}.$$

This means the statement of the lemma (see Definition 2.5). □

2.3 The Local Boundedness Property

In this part, motivated by Claim 2.13, we investigate the local boundedness property of the Orlicz maximal operators. Recall that Φ is a normalized Young function when Φ is a Young function and $\Phi(1) = 1$. In general, for a Young function Φ it should be remarked that

$$\begin{cases} \theta\Phi(t) \ge \Phi(\theta t) & \text{if } 0 < \theta < 1, \\ \theta\Phi(t) \le \Phi(\theta t) & \text{if } 1 < \theta < \infty. \end{cases}$$

Indeed, since Φ is convex and $\Phi(0) = 0$, for $0 < \theta < 1$,

$$\Phi(\theta t) = \Phi((1-\theta)0 + \theta t) \le (1-\theta)\Phi(0) + \theta\Phi(t) = \theta\Phi(t)$$

and, for $1 < \theta < \infty$,

$$\theta\Phi(t) = \theta\Phi(\theta^{-1}\theta t) \le \theta\theta^{-1}\Phi(\theta t) = \Phi(\theta t).$$

The following lemma is the localized version of so-called the "Wiener–Stein equivalence" and is essentially found in [12, Lemma 4.1].

Lemma 2.14 *Suppose that Ψ is a normalized Young function and that f is a nonnegative locally integrable function. Then, for any cube Q and any number $t > \|f\|_{\Psi,Q}$,*

$$\begin{aligned} \int_{\{x\in Q:\, f(x)>t\}} \Psi\left(\frac{f(y)}{2^n t}\right) dy &\le \left|\{x \in Q : M^\Psi[f\chi_Q](x) > t\}\right| \\ &\le C\int_{\{x\in Q:\, f(x)>t/2\}} \Psi\left(\frac{2f(y)}{t}\right) dy. \end{aligned} \tag{2.17}$$

In particular, if $\Psi \in \Delta_2$ *then*

$$C^{-1}\int_{\{x\in Q:\ f(x)>t\}} \Psi\left(\frac{f(y)}{t}\right) dy \le \left|\left\{x \in Q:\ M^{\Psi}[f\chi_Q](x) > t\right\}\right|$$
$$\le C\int_{\{x\in Q:\ f(x)>t/2\}} \Psi\left(\frac{f(y)}{t}\right) dy.$$

Proof First, we notice that the maximal operator M^{Ψ} is subadditive and that for all cubes R $\|\chi_R\|_{\Psi,\,R} = \Psi^{-1}(1) = 1$. Using these observations and the standard idea of writing f as $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $f(x) > t/2$ and $f_1(x) = 0$ otherwise, we have

$$M^{\Psi}[f\chi_Q](x) \le M^{\Psi}[f_1\chi_Q](x) + t/2.$$

This implies

$$\left\{x \in Q:\ M^{\Psi}[f\chi_Q](x) > t\right\} \subset \left\{x \in Q:\ M^{\Psi}[f_1\chi_Q](x) > t/2\right\}.$$

It follows from the well-known Wiener covering lemma that there exists the set of disjoint cubes $\{Q_j\}$ such that

$$Q_j \subset Q, \quad t/2 < \|f_1\|_{\Psi,\,Q_j}, \quad \left\{x \in Q:\ M^{\Psi}[f_1\chi_Q](x) > t/2\right\} \subset \bigcup_j 3Q_j.$$

We now see that $\dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} \Psi\left(\frac{2f_1(x)}{t}\right) dx > 1$; if it were not, then by the definition of the mean Luxemberg norm we would have $t/2 \ge \|f_1\|_{\Psi,\,Q_j}$, which is a contradiction. This implies that $|Q_j| < \displaystyle\int_{Q_j} \Psi\left(\frac{2f_1(x)}{t}\right) dx$ and that the right inequality of Eq. 2.17.

Let $\mathcal{D}(Q)$ be the collection of all dyadic subcubes of Q, that is, all those cubes obtained by dividing Q into 2^n congruent cubes of half its length, dividing each of those into 2^n congruent cubes, and so on. By convention, Q itself belongs to $\mathcal{D}(Q)$. Since, $t > \|f\|_{\Psi,\,Q}$, by use of standard argument of the Calderón-Zygmund decomposition, there exists a disjoint collection of maximal (with inclusion) dyadic cubes $\{Q_j\} \subset \mathcal{D}(Q)$ such that

$$t < \|f\|_{\Psi,\,Q_j}, \quad \left\{x \in Q:\ M^{\Psi}[f\chi_Q](x) > t\right\} \supset \bigcup_j Q_j.$$

Let $\tilde{Q}_j \in \mathcal{D}(Q)$ be the unique dyadic cube containing Q_j with side-length twice that of Q_j. In general, one knows that $\|f\|_{\Phi,\,Q} \le \delta^n \|f\|_{\Phi,\,\delta Q}$ for any cube Q and any $\delta > 1$. Thus, we have

$$\|f\|_{\Psi,\,Q_j} \le 2^n \|f\|_{\Psi,\,\tilde{Q}_j}$$

and by the maximality of Q_j

$$\|f\|_{\Psi,\,Q_j} \le 2^n \|f\|_{\Psi,\,\tilde{Q}_j} \le 2^n t.$$

It follows from the definition of the mean Luxemberg norm and the fact that $\Psi(t)$ is a nondecreasing function that

$$1 \geq \frac{1}{|Q_j|}\int_{Q_j}\Psi\left(\frac{f(x)}{\|f\|_{\Psi,\,Q_j}}\right)dx \geq \frac{1}{|Q_j|}\int_{Q_j}\Psi\left(\frac{f(x)}{2^n t}\right)dx.$$

This yields that $|Q_j| \geq \displaystyle\int_{Q_j}\Psi\left(\frac{f(x)}{2^n t}\right)dx$ and that the left inequality of Eq. 2.17, by observing further that $\displaystyle\bigcup_j Q_j \supset \{x \in Q : f(x) > t\}$, which holds by the Lebesgue differential theorem. □

Remark 2.15 Following the proof of Lemma 2.14, we see that

$$\left|\left\{x \in Q : M^{\Psi}[f\chi_Q](x) > t\right\}\right| \leq C\int_{Q\cap\{|f|>t/2\}}\Psi\left(\frac{2|f(x)|}{t}\right)dx \text{ for all } t > 0.$$

Lemma 2.16 *Let Ψ be a normalized Young function. Suppose that normalized Young functions Φ, Φ_1 and Φ_2 fulfill, for some positive constants C_1 and C_2,*

$$\Phi_1(C_1 t) - 1 \leq \int_1^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\,ds \leq \Phi_2(C_2 t) \textit{ for all } t > 1.$$

Then, for any $Q \in \mathcal{Q}$,

$$C^{-1}\|f\|_{\Phi_1,\,Q} \leq \|M^{\Psi}[f\chi_Q]\|_{\Phi,\,Q} \leq C\|f\|_{\Phi_2,\,Q}.$$

In particular, the boundedness of M^{Ψ} does not depend on the values of $\Phi(t)$ and $\Psi(t)$, t less than one (see also Claim 5.1).

Proof First, we verify

$$\|M^{\Psi}[f\chi_Q]\|_{\Phi,\,Q} \leq C\|f\|_{\Phi_2,\,Q}.$$

By Remark 2.15 for all $t > 0$ we have

$$\left|\left\{x \in Q : M^{\Psi}[f\chi_Q](x) > t\right\}\right| \leq C\int_{Q\cap\{|f|>t/2\}}\Psi\left(\frac{2|f(x)|}{t}\right)dx.$$

It follows from this inequality that

$$\begin{aligned}
\int_Q \Phi\left(M^{\Psi}[f\chi_Q](x)\right)dx &= \int_0^{\infty}|\{x \in Q : M^{\Psi}[f\chi_Q](x) > t\}|\Phi'(t)\,dt \\
&\leq |Q| + C\int_1^{\infty}\left(\int_{Q\cap\{|f|>t/2\}}\Psi\left(\frac{2|f(x)|}{t}\right)dx\right)\Phi'(t)\,dt \\
&= |Q| + C\int_{Q\cap\{|f|>1/2\}}\left(\int_1^{2|f(x)|}\Psi\left(\frac{2|f(x)|}{t}\right)\Phi'(t)\,dt\right)dx \\
&\leq |Q| + C\int_Q \Phi_2(2C_2|f(x)|)\,dx.
\end{aligned}$$

For any $\lambda > 0$, replacing $|f(x)|$ by $\dfrac{|f(x)|}{\lambda}$, we have

$$\frac{1}{|Q|}\int_Q \Phi\left(\frac{M^\Psi[f\chi_Q](x)}{\lambda}\right)dx \le 1 + C\frac{1}{|Q|}\int_Q \Phi_2\left(\frac{2C_2|f(x)|}{\lambda}\right)dx.$$

This yields, by the definition of the mean Luxemburg norm, for some $C > 1$

$$\frac{1}{|Q|}\int_Q \Phi\left(\frac{M^\Psi[f\chi_Q](x)}{C\|f\|_{\Phi_2,\,Q}}\right)dx \le 1,$$

which proves the desired inequality.

Next, we verify the converse. Without loss of generality, we may assume that $\|M^\Psi[f\chi_Q]\|_{\Phi,\,Q} = 1$. This means that

$$\frac{1}{|Q|}\int_Q \Phi\left(M^\Psi[f\chi_Q](x)\right)dx \le 1.$$

We now claim that then $\|f\|_{\Psi,\,Q} \le 1$. If it were not, then we must have the above integral mean is bigger than one, which contradicts to our normalization above, by virtue of the fact that for almost every $x \in Q$ $\|f\|_{\Psi,\,Q} \le M^\Psi[f\chi_Q](x)$.

We wish to prove $\|f\|_{\Phi_1,\,Q} \le C$. Lemma 2.14 yields that, by noticing $\|f\|_{\Psi,\,Q} \le 1$,

$$\begin{aligned}
|Q| &\ge \int_Q \Phi\left(M^\Psi[f\chi_Q](x)\right)dx \\
&= \int_0^\infty |\{x \in Q:\ M^\Psi[f\chi_Q](x) > t\}|\Phi'(t)\,dt \\
&\ge \int_1^\infty \left(\int_{Q\cap\{|f|>t\}} \Psi\left(\frac{|f(x)|}{2^n t}\right)dx\right)\Phi'(t)\,dt \\
&= \int_{Q\cap\{|f|>1\}} \left(\int_1^{|f(x)|} \Psi\left(\frac{|f(x)|}{2^n t}\right)\Phi'(t)\,dt\right)dx
\end{aligned}$$

and further that

$$\begin{aligned}
|Q| &\ge \int_{Q\cap\{|f|>2^n\}} \left(\int_1^{|f(x)|/2^n} \Psi\left(\frac{|f(x)|}{2^n t}\right)\Phi'(t)\,dt\right)dx \\
&\ge \int_{Q\cap\{|f|>2^n\}} \Phi_1(2^{-n}C_1|f(x)|)\,dx - |Q\cap\{|f| > 2^n\}| \\
&\ge \int_Q \Phi_1(2^{-n}C_1|f(x)|)\,dx - (\Phi_1(C_1)+1)|Q|.
\end{aligned}$$

This yields, for some $C > 0$,

$$\frac{1}{|Q|}\int_Q \Phi_1\left(\frac{|f(x)|}{C}\right)dx \le 1$$

and, hence, $\|f\|_{\Phi_1,\,Q} \le C$. The proof is now complete. □

Proposition 2.17 *Let Φ and Ψ be normalized Young functions. Then the following are equivalent.*

(1) *The maximal operator M^{Ψ} is locally bounded in the norm determined by Φ;*
(2) *The functions Φ and Ψ satisfy*

$$\int_1^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\,ds \le \Phi(Ct) \text{ for some } C>0 \text{ and for all } t>1;$$

(3) *The functions Φ and Ψ satisfy*

$$\int_1^t \Phi\left(\frac{t}{s}\right)\Psi'(s)\,ds \le \Phi(Ct) \text{ for some } C>0 \text{ and for all } t>1.$$

In [6, 7] Kita established similar results for $\Psi(t)\equiv t$ on $\mathcal{L}^{\Phi}$.

Proof We have already proved that (2) implies (1). First, we verify (1) implies (2). Suppose that (1), namely, assume that there exists a constant $C_0 \ge 1$ such that

$$\frac{1}{|Q|}\int_Q \Phi\left(\frac{M^{\Psi}[f\chi_Q](x)}{C_0\|f\|_{\Phi,Q}}\right)dx \le 1.$$

We now claim that then $\|f\|_{\Psi,Q} \le C_0\|f\|_{\Phi,Q}$. If it were not, then we would have the above integral mean is bigger than one, which contradicts to our normalization above, by virtue of the fact that for almost every $x \in Q$ $\|f\|_{\Psi,Q} \le M^{\Psi}[f\chi_Q](x)$. Thus, we have by Lemma 2.16

$$\begin{aligned}
|Q| &\ge \int_Q \Phi\left(\frac{M^{\Psi}[f\chi_Q](x)}{C_0\|f\|_{\Phi,Q}}\right)dx\\
&= \int_0^{\infty}\left|\left\{x\in Q:\ M^{\Psi}[f\chi_Q](x) > C_0\|f\|_{\Phi,Q}\cdot s\right\}\right|\Phi'(s)\,ds\\
&\ge \int_1^{\infty}\int_{Q\cap\{|f|>C_0\|f\|_{\Phi,Q}\cdot s\}}\Psi\left(\frac{|f(x)|}{2^nC_0\|f\|_{\Phi,Q}\cdot s}\right)dx\,\Phi'(s)\,ds\\
&\ge \int_{Q\cap\{|f|>2^nC_0\|f\|_{\Phi,Q}\}}\int_1^{|f(x)|/(2^nC_0\|f\|_{\Phi,Q})}\Psi\left(\frac{|f(x)|}{2^nC_0\|f\|_{\Phi,Q}\cdot s}\right)\Phi'(s)\,ds\,dx.
\end{aligned}$$

If we set $f(x)\equiv\chi_R(x)$ with $R\in\mathcal{Q}$ contained in Q and let $t=(2^nC_0\|f\|_{\Phi,Q})^{-1}$, then we have

$$\int_1^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\,ds \le \frac{|Q|}{|R|}.$$

Observing that

$$\Phi(2^nC_0t) = \Phi\left(\frac{1}{\|f\|_{\Phi,Q}}\right) = \frac{|Q|}{|R|},$$

we obtain the desired inequality.

Next, we verify (3) implies (2). Carrying out integration by parts, we have

$$\begin{aligned}\int_1^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\,ds &= \left[\Psi\left(\frac{t}{s}\right)\Phi(s)\right]_1^t + t\int_1^t \Psi'\left(\frac{t}{s}\right)\Phi(s)\,\frac{ds}{s^2}\\ &= \Phi(t) - \Psi(t) + \int_1^t \Phi\left(\frac{t}{s}\right)\Psi'(s)\,ds\\ &\le \Phi(t) + \Phi(Ct) \le \Phi(Ct).\end{aligned}$$

Here, we have used changed variables $\frac{t}{s} \mapsto s$. The converse, (2) implies (3), is similar, once we notice that

$$\begin{aligned}\Psi(t) &\le \frac{1}{\Phi(2)-\Phi(1)}\int_1^2 \Psi\left(\frac{2t}{s}\right)\Phi'(s)\,ds\\ &\le \frac{1}{\Phi(2)-\Phi(1)}\int_1^{2t} \Psi\left(\frac{2t}{s}\right)\Phi'(s)\,ds\\ &\le \frac{1}{\Phi(2)-\Phi(1)}\Phi(2Ct)\\ &\le \Phi(Ct),\ t>1.\end{aligned}$$

□

Claim 2.18

(1) Let Φ be a normalized Young function and $\Psi(t) \equiv t^p$, $1 \le p < \infty$. Suppose that $s^{1-p}\Phi'(s)$, $s > 1$, is nondecreasing. Then the following are equivalent.

(a) The maximal operator M^Ψ is locally bounded in the norm determined by Φ;

(b) There exists some constant $C > 0$ such that

$$t^p \int_1^t \frac{\Phi'(s)}{s^p}\,ds \le \Phi(Ct) \text{ for all } t > 1;$$

(c) There exists some constant $K > 1$ such that

$$\Phi(t) \le \frac{1}{2K^p}\Phi(Kt) \text{ for all } t > 1.$$

(3) The normalized Young functions Φ and Ψ satisfy

$$\int_1^t \Phi\left(\frac{t}{s}\right)\Psi'(s)\,ds \le \Phi(Ct) \text{ for some } C > 0 \text{ and for all } t > 1$$

provided that there exist some constant $K > 1$ and some positive summable sequence $\{a_j\}$ such that, for all $j \in \mathbb{N}$ and for all $t > 1$,

$$\Psi(K^{j+1})\Phi(K^{-j}t) \le a_j\Phi(t).$$

Proof We first prove (1). We have already proved the equivalence of (a) and (b). We verify (b) implies (c). By the assumption, $s^{1-p}\Phi'(s)$ is nondecreasing, we have for $\mu > 1$

$$\Phi(C\mu t) \geq (\mu t)^p \int_t^{\mu t} \frac{\Phi'(s)}{s^p}\, ds \geq \mu^p t \Phi'(t) \int_t^{\mu t} \frac{1}{s}\, ds \geq \mu^p \Phi(t) \log \mu,$$

where we have used the fact that $\dfrac{\Phi(t)}{t} \leq \Phi'(t)$ which holds from the convexity. It follows by letting μ big enough so that $C^{-p}\log\mu > 2$ and then by setting $K = C\mu$ that

$$2K^p\Phi(t) \leq \Phi(Kt).$$

This is the desired inequality. We verify the converse. It follows for an appropriate $N > 1$ that

$$\begin{aligned}\int_1^t \frac{\Phi'(s)}{s^p}\, ds &\leq \sum_{j=0}^N \int_{K^{-j-1}t}^{K^{-j}t} \frac{\Phi'(s)}{s^p}\, ds \leq \sum_{j=0}^N \frac{\Phi(K^{-j}t)}{(K^{-j-1}t)^p} \\ &= \frac{K^p}{t^p} \sum_{j=0}^N K^{pj}\Phi(K^{-j}t) \leq \frac{K^p}{t^p}\Phi(t)\sum_{j=0}^N 2^{-j} \leq \frac{1}{t^p}\Phi(2K^p t).\end{aligned}$$

This proves the desired inequality.

Next we prove (2). It follows for an appropriate $N > 1$ that

$$\begin{aligned}\int_1^t \Phi\left(\frac{t}{s}\right)\Psi'(s)\, ds &\leq \sum_{j=0}^N \int_{K^j}^{K^{j+1}} \Phi\left(\frac{t}{s}\right)\Psi'(s)\, ds \leq \sum_{j=0}^N \Phi\left(\frac{t}{K^j}\right)\int_{K^j}^{K^{j+1}} \Psi'(s)\, ds \\ &\leq \sum_{j=0}^N \Phi\left(\frac{t}{K^j}\right)\Psi(K^{j+1}) \leq \Phi(t)\left(\Psi(K) + \sum_{j=1}^N a_j\right) \leq \Phi(Ct).\end{aligned}$$

This is the desired inequality. □

As a special case we can recover the classical result on the ∇_2-condition.

Corollary 2.19 *Let $\Phi \in \nabla_2$. Then*

$$\|M[f\chi_Q]\|_{\Phi,Q} \approx \|f\|_{\Phi,Q},$$

where M is the Hardy–Littlewood maximal operator. Moreover, the converse is also true.

Proof We need only notice that $\Phi'(s)$ is a nondecreasing function. □

2.4 Some Additional Lemmas

Using Lemma 2.16, we show that the condition which gives the boundedness of the Orlicz maximal operators on Orlicz–Morrey spaces.

Proposition 2.20 *Let $\phi \in \mathcal{G}_1$. Suppose that normalized Young functions Ψ, Φ, Φ_1 and Φ_2 satisfy the same condition posed in Lemma* 2.16. *Then*

$$\|f\|_{\mathcal{L}^{\Phi_1,\phi}} \leq \|M^\Psi f\|_{\mathcal{L}^{\Phi,\phi}} \leq C\left(\|f\|_{\mathcal{L}^{\Phi_2,\phi}} + \|f\|_{\mathcal{L}^{\Psi,\phi}}\right). \tag{2.18}$$

Proof Since, for any cube $Q \in \mathcal{Q}$ and any point $x \in Q$, $M^\Psi f(x) \geq M^\Psi[f\chi_Q](x)$, we see that the left inequality of Eq. 2.18 by Lemma 2.16. Thus, we verify the right inequality.

It follows from the subadditivity of M^Ψ that

$$\|M^\Psi f\|_{\mathcal{L}^{\Phi,\phi}} \leq \sup_{Q\in\mathcal{Q}} \phi(\ell(Q))\|M^\Psi[f\chi_{3Q}]\|_{\Phi,\,Q} + \sup_{Q\in\mathcal{Q}} \phi(\ell(Q))\|M^\Psi[f\chi_{(3Q)^c}]\|_{\Phi,\,Q}.$$

Using Lemma 2.16, the fact that $\phi(t)$ is nondecreasing and that, for any $\delta > 1$, $\|f\|_{\Phi,\,Q} \leq \delta^n\|f\|_{\Phi,\,\delta Q}$, we have

$$\sup_{Q\in\mathcal{Q}} \phi(\ell(Q))\|M^\Psi[f\chi_{3Q}]\|_{\Phi,\,Q} \leq 3^n \sup_{Q\in\mathcal{Q}} \phi(\ell(Q))\|M^\Psi[f\chi_Q]\|_{\Phi,\,Q} \leq C\|f\|_{\mathcal{L}^{\Phi_2,\phi}}.$$

We notice that

$$\sup_{x\in Q} M^\Psi[f\chi_{(3Q)^c}](x) \leq 3^n \inf_{x\in Q} M^\Psi[f\chi_{(3Q)^c}](x). \tag{2.19}$$

This holds from the fact that, for fixed $x \in Q$, if $R \in \mathcal{Q}$ satisfies $R \ni x$ and $R \cap (3Q)^c \neq \emptyset$, then $3R$ must contain Q. It follows from Eq. 2.19 that

$$\|M^\Psi[f\chi_{(3Q)^c}]\|_{\Phi,\,Q} \leq C \inf_{x\in Q} M^\Psi[f\chi_{(3Q)^c}](x).$$

Since $\phi(t)$ is nondecreasing,

$$\sup_{Q\in\mathcal{Q}} \phi(\ell(Q)) \inf_{x\in Q} M^\Psi[f\chi_{(3Q)^c}](x) \leq \|f\|_{\mathcal{L}^{\Psi,\phi}}$$

and we obtain the right inequality of Eq. 2.18. □

Corollary 2.21 *If $\Phi \in \nabla_2$ then*

$$\|Mf\|_{\mathcal{L}^{\Phi,\phi}} \approx \|f\|_{\mathcal{L}^{\Phi,\phi}},$$

where M is the Hardy–Littlewood maximal operator.

Proof Notice that when $\Psi(t) \equiv t$

$$\|f\|_{\mathcal{L}^{\Psi,\phi}} = \|f\|_{\mathcal{M}^{1,\phi}} \leq C\|f\|_{\mathcal{L}^{\Phi,\phi}}. \tag{2.20}$$

This inequality, Claim 2.18 and Proposition 2.20 yield the corollary. □

Lemma 2.22 *Let $\phi \in \mathcal{G}_1$, $0 < a \leq 1$ and let Φ be a Young function. Set*

$$\psi(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Then

$$\|M_{\phi/\psi} f\|_{\mathcal{L}^{\Psi,\psi}} \leq C\|Mf\|_{\mathcal{L}^{\Phi,\phi}}.$$

Proof Fix $x \in \mathbb{R}^n$. Then for any cube $Q \ni x$ we see that

$$\phi(\ell(Q))^{1-a} m_Q(|f|) \leq \phi(\ell(Q))^{1-a} Mf(x)$$

and that

$$\phi(\ell(Q))^{1-a} m_Q(|f|) = \phi(\ell(Q))^{-a} \phi(\ell(Q)) m_Q(|f|) \leq C\phi(\ell(Q))^{-a} \|Mf\|_{\mathcal{L}^{\Phi,\phi}}.$$

These imply

$$\begin{aligned}\phi(\ell(Q))^{1-a} m_Q(|f|) &\leq C \min\left(\phi(\ell(Q))^{1-a} Mf(x)\,,\ \phi(\ell(Q))^{-a} \|Mf\|_{\mathcal{L}^{\Phi,\phi}}\right)\\ &\leq C \sup_{t>0} \min\left(t^{1-a} Mf(x)\,,\ t^{-a} \|Mf\|_{\mathcal{L}^{\Phi,\phi}}\right)\\ &= C \|Mf\|_{\mathcal{L}^{\Phi,\phi}}^{1-a} Mf(x)^a,\end{aligned}$$

where in the last inequality we have used $0 < a \leq 1$. This yields, for every cube $Q \in \mathcal{Q}$,

$$\|M_{\phi^{1-a}} f\|_{\Psi,\,Q} \leq C \|Mf\|_{\mathcal{L}^{\Phi,\phi}}^{1-a} \|(Mf)^a\|_{\Psi,\,Q}.$$

It follows from the definition of the mean Luxemburg norm that

$$\begin{aligned}\|(Mf)^a\|_{\Psi,\,Q} &= \inf\left\{\lambda > 0 : \frac{1}{|Q|} \int_Q \Psi\left(\frac{Mf(x)^a}{\lambda}\right) dx \leq 1\right\}\\ &= \inf\left\{\lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{Mf(x)}{\lambda^{1/a}}\right) dx \leq 1\right\} = \|Mf\|_{\Phi,\,Q}^a.\end{aligned}$$

Thus, we have

$$\phi(\ell(Q))^a \|M_{\phi^{1-a}} f\|_{\Psi,\,Q} \leq C \|Mf\|_{\mathcal{L}^{\Phi,\phi}}^{1-a} \left(\phi(\ell(Q)) \|Mf\|_{\Phi,\,Q}\right)^a \leq C \|Mf\|_{\mathcal{L}^{\Phi,\phi}}.$$

Taking the supremum over all cubes $Q \in \mathcal{Q}$ in the left side, we have the desired inequality. □

2.5 Proof of Theorems 2.8–2.10

Combining the previous results, we can prove the theorems.

First, noticing that $M_1 = M$ (the Hardy–Littlewood maximal operator), Theorem 2.8 follows from (2) of Lemma 2.12 and Corollary 2.21. Similarly, Theorem 2.9 follows from Claim 2.13. Next, Theorem 2.10 holds from the following.

We notice that the condition 2.7 yields

$$\int_t^\infty \frac{\rho(s)}{s\phi(s)}\, ds \leq C \frac{\psi(t)}{\eta(t)} \text{ for all } t > 0.$$

Thus, it follows from (1) of Lemma 2.12 that

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Psi,\eta}} \leq C \|g\|_{\tilde{\mathcal{L}}^{\Psi,\psi}} \left(\|M_{\tilde{\rho}/\psi} f\|_{\mathcal{L}^{\Psi,\eta}} + \|f\|_{\mathcal{M}^{1,\phi}}\right).$$

Also, the condition 2.7 implies

$$\frac{\tilde{\rho}(t)}{\psi(t)} \leq C \frac{\phi(t)}{\eta(t)}.$$

Thus, it follows from Lemma 2.22 that

$$\|M_{\tilde{\rho}/\psi} f\|_{\mathcal{L}^{\Psi,\eta}} \le C\|M_{\phi/\eta} f\|_{\mathcal{L}^{\Psi,\eta}} \le C\|Mf\|_{\mathcal{L}^{\Phi,\phi}}.$$

Because $\Phi \in \nabla_2$, by Corollary 2.21 we have

$$\|Mf\|_{\mathcal{L}^{\Phi,\phi}} \le C\|f\|_{\mathcal{L}^{\Phi,\phi}}.$$

Noticing also Eq. 2.20, we obtain the desired inequality

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Psi,\eta}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Psi,\psi}} \|f\|_{\mathcal{L}^{\Phi,\phi}}$$

and complete the proof of Theorem 2.10.

3 Morrey Spaces with Small Parameters

In this section, focusing on Morrey spaces with small parameters we establish some norm inequalities for the generalized fractional integral operators. The accurate description of Orlicz–Morrey spaces works well in this problem.

Let $\phi \in \mathcal{G}_1$ and $0 < p < \infty$. Set

$$\|f\|_{\mathcal{M}^{p,\phi}} := \sup_{Q \in \mathcal{Q}} \phi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p}$$

and define the (generalized) Morrey spaces $\mathcal{M}^{p,\phi}(\mathbb{R}^n) = \mathcal{M}^{p,\phi}$ by this quasi norm $\| \cdot \|_{\mathcal{M}^{p,\phi}}$. We write the Orlicz–Morrey space $\mathcal{L}^{\Phi,\phi}$ as $\mathcal{M}^{L(\log L)^j,\phi}$, $j \in \mathbb{N}$, in the case when $\Phi(t) \equiv t(\log(2+t))^j$. When $j = 1$ we simply write $\mathcal{M}^{L(\log L),\phi}$ for $\mathcal{M}^{L(\log L)^1,\phi}$.

We now state our second results.

Theorem 3.1 *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$ and $0 < p < 1$. Assume that the condition 2.6;*

$$\int_t^\infty \frac{\rho(s)}{s\phi(s)} \, ds \le C \frac{\tilde{\rho}(t)}{\phi(t)} \text{ for all } t > 0.$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{M}^{p,\phi}} \le C\|g\|_{\mathcal{M}^{p,\tilde{\rho}}} \|f\|_{\mathcal{M}^{1,\phi}}.$$

Theorem 3.2 *With the same condition posed in Theorem 3.1, we have*

$$\|g \cdot T_\rho f\|_{\mathcal{M}^{1,\phi}} \le C\|g\|_{\mathcal{M}^{1,\tilde{\rho}}} \|f\|_{\mathcal{M}^{L(\log L),\phi}}.$$

Theorem 3.3 *Let $\rho \in \mathcal{G}_0$, $\phi, \psi \in \mathcal{G}_1$ and $0 < p < r \le 1$. Assume that*

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)} \, ds \le C \frac{\psi(t)}{\phi(t)^{p/r}} \text{ for all } t > 0. \quad (3.1)$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{M}^{r,\phi^{p/r}}} \le C\|g\|_{\mathcal{M}^{r,\psi}} \|f\|_{\mathcal{M}^{1,\phi}}.$$

In Section 4 we will encounter some examples.

3.1 Some Lemmas

To prove the theorems we need the following basic lemmas.

Lemma 3.4 *Let* $\phi \in \mathcal{G}_1$ *and* $0 < p < 1$. *Then*

$$\|Mf\|_{\mathcal{M}^{p,\phi}} \approx \|f\|_{\mathcal{M}^{1,\phi}}.$$

Proof Since, $0 < p < 1$, $(Mf)^p$ is an A_1-weight, that is, for any $Q \in \mathcal{Q}$, we have

$$m_Q((Mf)^p) \le C \inf_{x\in Q} Mf(x)^p$$

(c.f. [3, Chapter II]). This implies

$$\|Mf\|_{\mathcal{M}^{p,\phi}} \le C \sup_{Q\in\mathcal{Q}} \phi(\ell(Q)) \inf_{x\in Q} Mf(x).$$

We can easily obtain the reverse inequality of this one;

$$\begin{aligned}
\phi(\ell(Q)) \inf_{x\in Q} Mf(x) &= \phi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q \inf_{x\in Q} Mf(x)^p \, dy \right)^{1/p} \\
&\le \phi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q Mf(y)^p \, dy \right)^{1/p} \\
&\le \|Mf\|_{\mathcal{M}^{p,\phi}}.
\end{aligned}$$

Consequently we have

$$\|Mf\|_{\mathcal{M}^{p,\phi}} \approx \sup_{Q\in\mathcal{Q}} \phi(\ell(Q)) \inf_{x\in Q} Mf(x).$$

Because the right-hand side dominates $\|f\|_{\mathcal{M}^{1,\phi}}$, we consider the converse.

First, we notice that by the geometric observation

$$\sup_{x\in Q} M[f\chi_{(3Q)^c}](x) \le 3^n \inf_{x\in Q} M[f\chi_{(3Q)^c}](x). \tag{3.2}$$

This follows from the fact that, for a fixed point $x \in Q$, if $R \in \mathcal{Q}$ satisfies $R \ni x$ and $R \cap (3Q)^c \ne \emptyset$, then $3R$ engulfs Q. It follows from Eq. 3.2 and the subadditivity of M that

$$\phi(\ell(Q)) \inf_{x\in Q} Mf(x) \le \phi(\ell(Q)) \inf_{x\in Q} M[f\chi_{3Q}](x) + 3^n \phi(\ell(Q)) \inf_{x\in Q} M[f\chi_{(3Q)^c}](x).$$

Since $\phi(t)$ is nondecreasing we have

$$\phi(\ell(Q)) \inf_{x\in Q} M[f\chi_{(3Q)^c}](x) \le \|f\|_{\mathcal{M}^{1,\phi}}.$$

For any $t > 0$ one knows that (c.f. Remark 2.15)

$$t\,|\{x \in Q : \ M[f\chi_{3Q}](x) > t\}| \le 3^n \int_{3Q} |f(y)| \, dy.$$

Taking $t = 12^n m_{3Q}(|f|)$, we see that there exists a point $x_0 \in Q$ so that $Mf(x_0) \le t$. This yields

$$\phi(\ell(Q)) \inf_{x \in Q} M[f\chi_{3Q}](x) \le 12^n \phi(\ell(3Q)) m_{3Q}(|f|) \le 12^n \|f\|_{\mathcal{M}^{1,\phi}}.$$

These complete the proof. □

Lemma 3.5 *Let $\phi \in \mathcal{G}_1$ and $j \in \mathbb{N}$. Then*

$$\|M^j f\|_{\mathcal{M}^{1,\phi}} \approx \|f\|_{\mathcal{M}^{L(\log L)^j,\phi}},$$

where M^j denotes the j-fold composition of the Hardy–Littlewood maximal operator M. In particular,

$$\|Mf\|_{\mathcal{M}^{1,\phi}} \approx \|f\|_{\mathcal{M}^{L(\log L),\phi}}.$$

Proof We have for $t > 1$

$$1 + \int_1^t \frac{t}{s}\{s(\log(2+s))^{j-1}\}'\,ds \approx t(\log(2+t))^j.$$

Thus, applying Proposition 2.20 with $\Psi(t) = t$, $\Phi(t) = t(\log(2+t))^{j-1}$ and $\Phi_1(t) = \Phi_2(t) = t(\log(2+t))^j$, we obtain (see also Eq. 2.20)

$$\|Mf\|_{\mathcal{M}^{L(\log L)^{j-1},\phi}} \approx \|f\|_{\mathcal{M}^{L(\log L)^j,\phi}}.$$

This yields the lemma by an inductive argument. □

Lemma 3.6 *Let $\phi \in \mathcal{G}_1$ and $0 < p \le r < \infty$. Then*

$$\|M_{\phi^{1-p/r}} f\|_{\mathcal{M}^{r,\phi^{p/r}}} \le C\|Mf\|_{\mathcal{M}^{p,\phi}}.$$

Proof Let $x \in \mathbb{R}^n$ be a fixed point. For every cube $Q \ni x$ we see that

$$\phi(\ell(Q))^{1-p/r} m_Q(|f|) \le \phi(\ell(Q))^{1-p/r} Mf(x)$$

and that

$$\begin{aligned}\phi(\ell(Q))^{1-p/r} m_Q(|f|) &\le \phi(\ell(Q))^{1-p/r} \inf_{y \in Q} Mf(y)\\ &\le \phi(\ell(Q))^{-p/r}\phi(\ell(Q)) \left(\frac{1}{|Q|}\int_Q Mf(y)^p\,dy\right)^{1/p}\\ &\le \phi(\ell(Q))^{-p/r}\|Mf\|_{\mathcal{M}^{p,\phi}}.\end{aligned}$$

These imply

$$\begin{aligned}\phi(\ell(Q))^{1-p/r} m_Q(|f|) &\le \min\left(\phi(\ell(Q))^{1-p/r} Mf(x),\ \phi(\ell(Q))^{-p/r}\|Mf\|_{\mathcal{M}^{p,\phi}}\right)\\ &\le \sup_{t \ge 0} \min\left(t^{1-p/r} Mf(x),\ t^{-p/r}\|Mf\|_{\mathcal{M}^{p,\phi}}\right)\\ &= \|Mf\|_{\mathcal{M}^{p,\phi}}^{1-p/r} Mf(x)^{p/r}.\end{aligned}$$

This yields

$$M_{\phi^{1-p/r}} f(x)^r \le \|Mf\|_{\mathcal{M}^{p,\phi}}^{r-p} Mf(x)^p$$

and

$$\phi(\ell(Q))^{p/r} \left(\frac{1}{|Q|} \int_Q M_{\phi^{1-p/r}} f(x)^r \, dx \right)^{1/r}$$
$$\le \|Mf\|_{\mathcal{M}^{p,\phi}}^{1-p/r} \left(\phi(\ell(Q)) \left(\frac{1}{|Q|} \int_Q Mf(x)^p \, dx \right)^{1/p} \right)^{p/r} \le \|Mf\|_{\mathcal{M}^{p,\phi}}.$$

Taking the supremum over all cubes $Q \in \mathcal{Q}$ in the left side, we have the desired inequality. □

3.2 Principal Lemma Revisited

The same as before the proof of the previous results relies upon the following principal lemma.

Lemma 3.7 *Let $\rho \in \mathcal{G}_0$, $\phi, \psi, \eta \in \mathcal{G}_1$ and $0 < p \le 1$.*

(1) *Assume that the condition 2.10;*

$$\int_t^\infty \frac{\rho(s)}{s\phi(s)} \, ds \le C \frac{\psi(t)}{\eta(t)} \text{ for all } t > 0.$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{M}^{p,\eta}} \le C \|g\|_{\mathcal{M}^{p,\psi}} \left(\|M_{\tilde{\rho}/\psi} f\|_{\mathcal{M}^{p,\eta}} + \|f\|_{\mathcal{M}^{1,\phi}} \right).$$

(2) *Assume that the condition 2.11;*

$$\int_t^\infty \frac{\rho(s)\psi(s)}{s\tilde{\rho}(s)\phi(s)} \, ds \le C \frac{\psi(t)}{\phi(t)} \text{ for all } t > 0.$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{M}^{p,\phi}} \le C \|g\|_{\mathcal{M}^{p,\psi}} \|M_{\tilde{\rho}/\psi} f\|_{\mathcal{M}^{p,\phi}}.$$

Proof Except for some sufficient modifications, the proof of the lemma follows the argument in [19]. Retaining the same notation as the proof of Lemma 2.12, we recall that

$$T_\rho f(x) \le C \sum_{Q \in \mathcal{D}} \hat{\rho}(\ell(Q)) m_{3Q}(f) \chi_Q(x).$$

Noticing Eq. 2.12, for any $Q_0 \in \mathcal{D}$ we wish to estimate

$$\left(\int_{Q_0} \left(g(x) T_\rho f(x) \right)^p \, dx \right)^{1/p}. \tag{3.3}$$

In the same manner as the proof of Lemma 2.12, we set

$$\begin{cases} \mathcal{D}_1(Q_0) := \{Q \in \mathcal{D} : \ Q \subset Q_0\}, \\ \mathcal{D}_2(Q_0) := \{Q \in \mathcal{D} : \ Q \supsetneq Q_0\}. \end{cases}$$

Let us define as before, for $i = 1,\ 2$,

$$F_i(x) := \sum_{Q\in\mathcal{D}_i(Q_0)} \hat{\rho}(\ell(Q)) m_{3Q}(f)\chi_Q(x)$$

and we shall estimate

$$\left(\int_{Q_0} (g(x)F_i(x))^p\ dx\right)^{1/p}.$$

The case $i = 1$ We need the following lemma, the proof of which is straightforward and is omitted (see [13, 19]).

Lemma 3.8 *For a nonnegative function h in $L^\infty(Q_0)$ we let $\gamma_0 := m_{Q_0}(h)$ and $c := 2^{n+1}$. For $k = 1, 2, \ldots$ let*

$$D_k := \bigcup_{Q\in\mathcal{D}_1(Q_0):\, m_Q(h)>\gamma_0 c^k} Q.$$

Considering the maximal cubes with respect to inclusion, we can write

$$D_k = \bigcup_j Q_{k,\ j},$$

where the cubes $\{Q_{k,\ j}\} \subset \mathcal{D}_1(Q_0)$ are nonoverlapping. By virtue of the maximality of $Q_{k,\ j}$ one has

$$\gamma_0 c^k < m_{Q_{k,\ j}}(h) \le 2^n \gamma_0 c^k.$$

Let

$$E_0 := Q_0 \setminus D_1, \quad E_{k,\ j} := Q_{k,\ j} \setminus D_{k+1}.$$

Then $\{E_0\} \cup \{E_{k,\ j}\}$ is a disjoint family of sets which decomposes Q_0 and satisfies

$$|Q_0| \le 2|E_0|, \quad |Q_{k,\ j}| \le 2|E_{k,\ j}|. \tag{3.4}$$

Also, we set

$$\mathcal{D}_0 := \left\{Q \in \mathcal{D}_1(Q_0) :\ m_Q(h) \le \gamma_0 c\right\}$$
$$\mathcal{D}_{k,\ j} := \left\{Q \in \mathcal{D}_1(Q_0) :\ Q \subset Q_{k,\ j},\ \gamma_0 c^k < m_Q(h) \le \gamma_0 c^{k+1}\right\}.$$

Then

$$\mathcal{D}_1(Q_0) = \mathcal{D}_0 \cup \bigcup_{k,\ j} \mathcal{D}_{k,\ j}. \tag{3.5}$$

Let us return to the proof. We need only verify that

$$\int_{Q_0} g(x)^p F_1(x)^p\, dx \le C\|g\|^p_{\mathcal{M}^{p,\psi}} \int_{Q_0} M_{\tilde{\rho}/\psi} f(x)^p\, dx. \tag{3.6}$$

Inserting the definition of F_1, we have

$$\int_{Q_0} g(x)^p F_1(x)^p\, dx = \sum_{Q\in\mathcal{D}_1(Q_0)} \hat{\rho}(\ell(Q)) m_{3Q}(f) \int_Q g(x)^p F_1(x)^{p-1}\, dx.$$

Letting $h = g^p$, we shall apply Lemma 3.8 to estimate this quantity. Retaining the same notation as Lemma 3.8 and noticing Eq. 3.5, we have

$$\int_{Q_0} g(x)^p F_1(x)^p\,dx = \sum_{Q\in\mathcal{D}_0} \hat\rho(\ell(Q)) m_{3Q}(f) \int_Q g(x)^p F_1(x)^{p-1}\,dx$$
$$+ \sum_{k,\,j} \sum_{Q\in\mathcal{D}_{k,\,j}} \hat\rho(\ell(Q)) m_{3Q}(f) \int_Q g(x)^p F_1(x)^{p-1}\,dx.$$

We first evaluate

$$\sum_{Q\in\mathcal{D}_{k,\,j}} \hat\rho(\ell(Q)) m_{3Q}(f) \int_Q g(x)^p F_1(x)^{p-1}\,dx. \tag{3.7}$$

Noticing $p-1 \le 0$ and the definition of F_1, we see that

$$F_1(x)^{p-1} \le \left(\hat\rho(\ell(Q_{k,\,j})) m_{3Q_{k,\,j}}(f)\right)^{p-1} \text{ for all } x \in Q_{k,\,j}.$$

It follows from this inequality and the definition of $\mathcal{D}_{k,\,j}$ that Eq. 3.7 is bounded by

$$\left(\hat\rho(\ell(Q_{k,\,j})) m_{3Q_{k,\,j}}(f)\right)^{p-1} \gamma_0 c^{k+1} \sum_{Q\in\mathcal{D}_{k,\,j}} \hat\rho(\ell(Q)) \int_{3Q} f(y)\,dy.$$

By virtue of the support condition we have

$$\sum_{Q\in\mathcal{D}_{k,\,j}} \hat\rho(\ell(Q)) \int_{3Q} f(y)\,dy = \sum_{\nu=-\infty}^{\log_2 \ell(Q_{k,\,j})} \hat\rho(2^\nu) \left(\sum_{Q\in\mathcal{D}_{k,\,j}:\,\ell(Q)=2^\nu} \int_{3Q} f(y)\,dy \right)$$
$$\le C \int_{3Q_{k,\,j}} f(y)\,dy \left(\sum_{\nu=-\infty}^{\log_2 \ell(Q_{k,\,j})} \hat\rho(2^\nu) \right)$$
$$\le C\tilde\rho(\ell(c_2 Q_{k,\,j})) \int_{3Q_{k,\,j}} f(y)\,dy.$$

This implies

$$\text{Eq. } 3.7 \le C\left(\tilde\rho(\ell(c_2 Q_{k,\,j})) m_{3Q_{k,\,j}}(f)\right)^p \gamma_0 c^{k+1} |Q_{k,\,j}|.$$

If we invoke relations 3.4; $|Q_{k,\,j}| \le 2|E_{k,\,j}|$ and

$$\gamma_0 c^k < m_{Q_{k,\,j}}(g^p) \le \|g\|_{\mathcal{M}^{p,\psi}}^p \psi(\ell(Q_{k,\,j}))^{-p},$$

which follows from the definition of the Morrey norm, then

$$\text{Eq. } 3.7 \le C\tilde\rho(\ell(c_0 Q_{k,\,j}))^p m_{c_0 Q_{k,\,j}}(f)^p \|g\|_{\mathcal{M}^{p,\psi}}^p \psi(\ell(Q_{k,\,j}))^{-p} |E_{k,\,j}|$$
$$\le C\|g\|_{\mathcal{M}^{p,\psi}}^p \int_{E_{k,\,j}} M_{\tilde\rho/\psi} f(x)^p\,dx.$$

Similarly, we have

$$\sum_{Q\in\mathcal{D}_0} \hat\rho(\ell(Q)) m_{3Q}(f) \int_Q g(x)^p F_1(x)^{p-1}\,dx \le C\|g\|_{\mathcal{M}^{p,\psi}}^p \int_{E_0} M_{\tilde\rho/\psi} f(x)^p\,dx.$$

Summing up all factors, we obtain Eq. 3.6, by noticing $\{E_0\} \cup \{E_{k,j}\}$ is a disjoint family of sets which decomposes Q_0.

The case $i = 2$ An estimate cruder than the case $i = 1$ suffices. By a property of the dyadic cubes we have

$$F_2(x) = \sum_{Q \in \mathcal{D}_2(Q_0)} \hat{\rho}(\ell(Q)) m_{3Q}(f) \text{ for all } x \in Q_0.$$

For all $Q \in \mathcal{D}_2(Q_0)$, recalling that $c_0 \equiv \max(c_2, 3)$, it follows from the definition of Morrey norm that

$$\begin{aligned}
\hat{\rho}(\ell(Q)) m_{3Q}(f) &\leq C \frac{\hat{\rho}(\ell(Q)) \psi(\ell(c_0 Q))}{\tilde{\rho}(\ell(c_0 Q))} \inf_{x \in Q} M_{\tilde{\rho}/\psi} f(x) \\
&\leq C \frac{\hat{\rho}(\ell(Q)) \psi(\ell(c_0 Q))}{\tilde{\rho}(\ell(c_0 Q))} \left(\frac{1}{|Q|} \int_Q M_{\tilde{\rho}/\psi} f(x)^p \, dx \right)^{1/p} \\
&\leq C \| M_{\tilde{\rho}/\psi} f \|_{\mathcal{M}^{p,\phi}} \frac{\hat{\rho}(\ell(Q)) \psi(\ell(c_0 Q))}{\tilde{\rho}(\ell(c_0 Q)) \phi(\ell(Q))}.
\end{aligned}$$

Since $\psi(t)$, $\phi(t)$ and $\tilde{\rho}(t)$ are nondecreasing and ψ and ϕ satisfy the doubling condition, we have

$$\begin{aligned}
F_2(x) &\leq C \| M_{\tilde{\rho}/\psi} f \|_{\mathcal{M}^{p,\phi}} \sum_{Q \in \mathcal{D}_2(Q_0)} \frac{\hat{\rho}(\ell(Q)) \psi(\ell(c_0 Q))}{\tilde{\rho}(\ell(c_0 Q)) \phi(\ell(Q))} \\
&\leq C \| M_{\tilde{\rho}/\psi} f \|_{\mathcal{M}^{p,\phi}} \sum_{\nu = 1 + \log_2 \ell(Q_0)}^{\infty} \int_{c_1 2^\nu}^{c_2 2^\nu} \frac{\rho(s) \psi(s)}{s \tilde{\rho}(s) \phi(s)} \, ds \\
&\leq C \| M_{\tilde{\rho}/\psi} f \|_{\mathcal{M}^{p,\phi}} \int_{c_1 \ell(Q_0)}^{\infty} \frac{\rho(s) \psi(s)}{s \tilde{\rho}(s) \phi(s)} \, ds \\
&\leq C \| M_{\tilde{\rho}/\psi} f \|_{\mathcal{M}^{p,\phi}} \frac{\psi(\ell(Q_0))}{\phi(\ell(Q_0))},
\end{aligned}$$

where in the last inequality we have used the condition 2.11 and the doubling condition of ψ and ϕ. This pointwise estimate gives

$$\begin{aligned}
&\left(\int_{Q_0} (g(x) F_2(x))^p \, dx \right)^{1/p} \\
&\quad \leq C \| M_{\tilde{\rho}/\psi} f \|_{\mathcal{M}^{p,\phi}} \left(\int_{Q_0} g(x)^p \, dx \right)^{1/p} \frac{\psi(\ell(Q_0))}{\phi(\ell(Q_0))} \\
&\quad \leq C \| M_{\tilde{\rho}/\psi} f \|_{\mathcal{M}^{p,\phi}} \, \|g\|_{\mathcal{M}^{p,\psi}} \, |Q_0|^{1/p} \, \phi(\ell(Q_0))^{-1}.
\end{aligned}$$

Thus, we have

$$\left(\int_{Q_0} (g(x) F_2(x))^p \, dx \right)^{1/p} \leq C \| M_{\tilde{\rho}/\psi} f \|_{\mathcal{M}^{p,\phi}} \, \|g\|_{\mathcal{M}^{p,\psi}} \, |Q_0|^{1/p} \, \phi(\ell(Q_0))^{-1}. \tag{3.8}$$

It follows from Eqs. 3.3, 3.6 and 3.8 that

$$
\begin{aligned}
&C\phi(\ell(Q_0))\left(\frac{1}{|Q_0|}\int_{Q_0}\left(g(x)T_\rho f(x)\right)^p\,dx\right)^{1/p}\\
&\quad\le \|g\|_{\mathcal{M}^{p,\psi}}\left(\phi(\ell(Q_0))\left(\frac{1}{|Q_0|}\int_{Q_0}M_{\tilde\rho/\psi}f(x)^p\,dx\right)^{1/p}+\|M_{\tilde\rho/\psi}f\|_{\mathcal{M}^{p,\phi}}\right)\\
&\quad\le 2\|g\|_{\mathcal{M}^{p,\psi}}\|M_{\tilde\rho/\psi}f\|_{\mathcal{M}^{p,\phi}}.
\end{aligned}
$$

By taking the supremum over all dyadic cubes $Q_0 \in \mathcal{D}$ in the left side, (2) of Lemma 3.7 is verified.

Lemma 3.7 (1) holds by the same argument using

$$
\begin{aligned}
F_2(x) &\le \|f\|_{\mathcal{M}^{1,\phi}}\sum_{Q\in\mathcal{D}_2(Q_0)}\frac{\hat\rho(\ell(Q))}{\phi(\ell(3Q))}\\
&\le C\|f\|_{\mathcal{M}^{1,\phi}}\int_{c_1\ell(Q_0)}^{\infty}\frac{\rho(s)}{s\phi(s)}\,ds \le C\|f\|_{\mathcal{M}^{1,\phi}}\frac{\psi(\ell(Q_0))}{\eta(\ell(Q_0))},
\end{aligned}
$$

where we have used the condition 2.10. This implies (with the same argument as above)

$$
\left(\int_{Q_0}(g(x)F_2(x))^p\,dx\right)^{1/p} \le C\|g\|_{\mathcal{M}^{p,\psi}}\,\|f\|_{\mathcal{M}^{1,\phi}}\,|Q_0|^{1/p}\,\eta(\ell(Q_0))^{-1}. \tag{3.9}
$$

The desired inequality then follows from Eqs. 3.3, 3.6 and 3.9. The proof of Lemma 3.7 is now complete. □

3.3 Proof of Theorems 3.1–3.3

Combining the previous lemmas, we can prove Theorems 3.1–3.3.

Theorem 3.1 follows from (2) of Lemmas 3.4 and 3.7. Similarly, Theorem 3.2 is a consequence of (2) of Lemmas 3.5 and 3.7. By the same argument as the proof of Theorem 2.10, Theorem 3.3 can be obtained from Lemmas 3.4 and 3.6 and (1) of Lemma 3.7.

4 Some Examples

In this section we see some examples. In the following lemma, we introduce a sufficient condition for which $\phi(t) \equiv \psi(\tilde\rho(t))$ fulfills the condition 2.6.

Lemma 4.1 *Let $\rho \in \mathcal{G}_0$. If a nonnegative, nondecreasing and differentiable function $\psi(t)$ satisfies*

$$
C_0 := \inf_{t>0}\frac{t\psi'(t)}{\psi(t)} > 1, \tag{4.1}
$$

then $\phi(t) \equiv \psi(\tilde\rho(t))$ fulfills the condition 2.6. More precisely, then,

$$
\int_t^\infty \frac{\rho(s)}{s\psi(\tilde\rho(s))}\,ds \le (C_0-1)^{-1}\frac{\tilde\rho(t)}{\psi(\tilde\rho(t))} \text{ for all } t>0.
$$

Proof It follows that

$$\begin{aligned}\int_t^\infty \frac{\rho(s)}{s\psi(\tilde\rho(s))}\,ds &\le (C_0-1)^{-1}\int_t^\infty \left(\frac{\tilde\rho(s)\,\psi'(\tilde\rho(s))}{\psi(\tilde\rho(s))}-1\right)\frac{\rho(s)}{s\psi(\tilde\rho(s))}\,ds\\ &= (C_0-1)^{-1}\int_t^\infty -\left(\frac{\psi(\tilde\rho(s))-\tilde\rho(s)\,\psi'(\tilde\rho(s))}{\psi(\tilde\rho(s))^2}\,\frac{\rho(s)}{s}\right)ds\\ &= (C_0-1)^{-1}\int_t^\infty -\left(\frac{\tilde\rho(s)}{\psi(\tilde\rho(s))}\right)'ds \le (C_0-1)^{-1}\frac{\tilde\rho(t)}{\psi(\tilde\rho(t))},\end{aligned}$$

where we have used $\tilde\rho'(s)=\dfrac{\rho(s)}{s}$. □

Example 4.2 From Lemma 4.1 we see that $\tilde\rho(t)^b$, $b>1$, satisfies the condition 2.6. Letting, for $b>1$,

$$\psi(t)\equiv\begin{cases}\frac{t^b}{\log(e/t)}, & 0<t\le 1,\\ t^b\log(et), & 1\le t<\infty,\end{cases}$$

we see also that $\psi(\tilde\rho(t))$ satisfies the condition 2.6.

Examples 4.3

(1) Let $1<p<\infty$ and $\delta>1$. Set

$$\Phi(t)\equiv t^p,\qquad \Psi(t)\equiv\frac{t^p}{[\log(2+t)]^\delta}.$$

Then, checking the condition posed in (3) of Proposition 2.17, we see that M^Ψ is locally bounded in the norm determined by Φ.

(2) Let $1\le q<p<\infty$ and $\delta\in\mathbb{R}$. Set

$$\Phi(t)\equiv t^p[\log(2+t)]^\delta,\qquad \Psi(t)\equiv t^q.$$

Then M^Ψ is locally bounded in the norm determined by Φ. Indeed, checking the condition posed in (2) of Proposition 2.17, we have

$$\int_1^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\,ds\approx t^q\int_1^t s^{p-q-1}[\log(2+s)]^\delta\,ds\le Ct^p[\log(2+t)]^\delta.$$

Claim 4.4 Let $1<p<\infty$ and $\delta\ge 0$. Then, the complementary function of $\Phi(t)\equiv t^p[\log(2+t)]^{(p-1)\delta}$ is approximately $\Psi(t)\equiv\dfrac{t^{p'}}{[\log(2+t)]^\delta}$.

Proof Let us assume that $t\gg 1$. The derivatives of Φ and Ψ are given by

$$\Phi'(t)\approx t^{p-1}[\log(2+t)]^{(p-1)\delta},\qquad \Psi'(t)\approx\frac{t^{p'-1}}{[\log(2+t)]^\delta}.$$

Therefore, it follows that

$$(\Phi' \circ \Psi')(t) \approx \left(t^{p'-1}[\log(2+t)]^{-\delta}\right)^{p-1} \left[\log\left(2+t^{p-1}[\log(2+t)]^{-\delta}\right)\right]^{(p-1)\delta}$$
$$\approx t[\log(2+t)]^{-(p-1)\delta}[\log(2+t)]^{(p-1)\delta}$$
$$= t.$$

By symmetry the same can be said for $\Psi' \circ \Phi'$. Thus, we have

$$(\Phi' \circ \Psi')(t) \approx t, \quad (\Psi' \circ \Phi')(t) \approx t.$$

Since the functions Φ' and Ψ' are doubling, it follows that

$$(\Phi')^{-1}(t) \approx \Psi'(t), \quad (\Psi')^{-1}(t) \approx \Phi'(t).$$

Now that

$$\bar{\Phi}(t) \approx \int_0^t (\Psi')^{-1}(s)ds, \quad \bar{\Psi}(t) \approx \int_0^t (\Phi')^{-1}(s)ds,$$

we see that

$$\Psi(t) \approx \bar{\Phi}(t), \quad \Phi(t) \approx \bar{\Psi}(t).$$

The proof is now complete. □

Recall the space $\tilde{\mathcal{M}}^{p,\,1/\alpha}$, which is introduced in Section 1. The following claim shows the accurate description of Orlicz–Morrey spaces (see Eq. 1.4 in Section 1).

Claim 4.5 Let $0 < \alpha < 1$, $1 < p < q \le 1/\alpha$ and $\delta > 1$. Set

$$\psi(t) \equiv t^{n\alpha}, \quad \Psi(t) \equiv t^p[\log(2+t)]^{(p-1)\delta}.$$

Then

$$\|f\|_{\mathcal{M}^{p,\,1/\alpha}} \le \|f\|_{\tilde{\mathcal{M}}^{p,\,1/\alpha}} \le C\|f\|_{\mathcal{L}^{\Psi,\,\psi}} \le C\|f\|_{\mathcal{M}^{q,\,1/\alpha}}.$$

Proof We need only verify that

$$\|f\|_{\tilde{\mathcal{M}}^{p,\,1/\alpha}} \le C\|f\|_{\mathcal{L}^{\Psi,\,\psi}}.$$

Using generalized Hölder's inequality 2.5, we have for any $Q \in \mathcal{Q}$

$$\frac{|Q|^\alpha}{|Q|}\int_Q |f(x)|^p\,dx \le C|Q|^\alpha \|f\|_{\Psi,\,Q}\|\,|f|^{p-1}\,\|_{\bar{\Psi},\,Q}.$$

This implies

$$M_\alpha[|f|^p\chi_Q](x) \le \|f\|_{\mathcal{L}^{\Psi,\,\psi}} M^{\bar{\Psi}}[|f|^{p-1}\chi_Q](x).$$

Since, $M^{\bar{\Phi}}$ is locally bounded in the norm determined by $\Phi(t) \equiv t^{p'}$, which holds from Claim 4.4 and Examples 4.3, we obtain

$$\int_Q M_\alpha[|f|^p \chi_Q](x)^{p'}\,dx \le C\|f\|_{\mathcal{L}^{\Psi,\psi}}^{p'} \int_Q |f(x)|^p\,dx < \infty.$$

This is our desired inequality. □

In view of the significant example in [17], we remark that the space $\tilde{\mathcal{M}}^{p,1/\alpha}$ is the proper subset of the space $\mathcal{M}^{p,1/\alpha}$.

Theorem 2.9 (2), Claim 2.13, Examples 4.2, 4.3 and Claim 4.4 yield the following.

Examples 4.6

(1) Let $\rho \in \mathcal{G}_0$, $1 < p < \infty$, $b > 1$ and $\delta > 1$. Set

$$\phi(t) \equiv \tilde{\rho}(t)^b, \quad \Psi(t) \equiv t^p[\log(2+t)]^{(p-1)\delta}.$$

Assume that

$$\sup_{t>1}\left(\frac{\phi(t)}{t^{n/p}} + \frac{\tilde{\rho}(t)}{\Psi^{-1}(t^n)}\right) < \infty.$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{M}^{p,\phi}} \le C\|g\|_{\mathcal{L}^{\Psi,\tilde{\rho}}}\|f\|_{\mathcal{M}^{p,\phi}}.$$

(2) Let $0 < \alpha < 1$, $1 < p \le p_0 < 1/\alpha$ and $\delta > 1$. Set

$$\psi(t) \equiv t^{\alpha n}, \quad \Psi(t) \equiv t^p[\log(2+t)]^{(p-1)\delta}.$$

Assume that

$$\sup_{t>1}\frac{\psi(t)}{\Psi^{-1}(t^n)} < \infty.$$

Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{p,p_0}} \le C\|g\|_{\mathcal{L}^{\Psi,\psi}}\|f\|_{\mathcal{M}^{p,p_0}}.$$

This example sharpens the Olsen inequality 1.5, which is introduced in Section 1.

(3) Let $\rho \in \mathcal{G}_0$, $1 < p < q < \infty$, $b > 1$ and $\delta > 0$. Set

$$\phi(t) \equiv \tilde{\rho}(t)^b, \quad \Phi(t) \equiv t^p[\log(2+t)]^{\delta}.$$

Assume that

$$\sup_{t>1}\left(\frac{\phi(t)}{\Phi^{-1}(t^n)} + \frac{\tilde{\rho}(t)}{t^{n/q}}\right) < \infty.$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\phi}} \le C\|g\|_{\mathcal{M}^{q,\tilde{\rho}}}\|f\|_{\mathcal{L}^{\Phi,\phi}}.$$

(4) Let $0 < \alpha < 1$, $1 < p < p_0 < 1/\alpha$, $1 < p < q \le 1/\alpha$ and $\delta > 0$. Set

$$\phi(t) \equiv t^{n/p_0}, \quad \Phi(t) \equiv t^p[\log(2+t)]^{\delta}.$$

Then

$$\|g \cdot I_\alpha f\|_{\mathcal{L}^{\Phi,\phi}} \le C\|g\|_{\mathcal{M}^{q,1/\alpha}} \|f\|_{\mathcal{L}^{\Phi,\phi}}.$$

We remark that

$$\|g \cdot I_\alpha f\|_{\mathcal{L}^{\Phi,\phi}} \ge \|g \cdot I_\alpha f\|_{\mathcal{M}^{p,p_0}}.$$

Thus, this example also sharpens the Olsen inequality 1.5.

We dare restate Theorems 3.1–3.3 in terms of the fractional integral operator I_α.

Examples 4.7 Let $0 < \alpha < 1$.

(1) Let $0 < p < 1 \le p_0 < 1/\alpha$. Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{p,p_0}} \le C\|g\|_{\mathcal{M}^{p,1/\alpha}} \|f\|_{\mathcal{M}^{1,p_0}}.$$

(2) Let $p = 1 < p_0 < 1/\alpha$. Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{1,p_0}} \le C\|g\|_{\mathcal{M}^{1,1/\alpha}} \|f\|_{\mathcal{M}^{L(\log L),p_0}}.$$

(3) Let $0 < p < r \le 1$, $0 < p \le p_0 < 1/\alpha$, $0 < r \le q_0$, $0 < r \le r_0$, $1/r_0 = 1/q_0 + 1/p_0 - \alpha$ and $p/p_0 = r/r_0$. Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r,r_0}} \le C\|g\|_{\mathcal{M}^{r,q_0}} \|f\|_{\mathcal{M}^{1,p_0}}.$$

Examples 4.7 give us that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{p,p_0}} \le C\|g\|_{\mathcal{M}^{p,1/\alpha}} \|f\|_{\mathcal{M}^{L(\log L),p_0}}, \quad 0 < p \le 1 < p_0 < 1/\alpha.$$

This example sharpens Eq. 1.6, which is introduced in Section 1.

5 Some Additional Results

Finally, we state and verify some additional results.

Claim 5.1 Let $\phi \in \mathcal{G}_1$ and let Φ_1 and Φ_2 be Young functions. If, for all $t > 1$ $\Phi_1(t) = \Phi_2(t)$, then

$$\mathcal{L}^{\Phi_1,\phi} = \mathcal{L}^{\Phi_2,\phi}$$

with norm equivalence.

Proof By symmetry, we wish only verify that

$$\mathcal{L}^{\Phi_1,\phi} \subset \mathcal{L}^{\Phi_2,\phi}.$$

Let $f \in \mathcal{L}^{\Phi_1,\phi}$. We first prove, for any $Q \in \mathcal{Q}$,

$$\|f\|_{\Phi_2,Q} \le C\|f\|_{\Phi_1,Q}.$$

Without loss of generality, we may assume that $\|f\|_{\Phi_1,\,Q} = 1$. Then we have

$$\begin{aligned}\int_Q \Phi_2(|f(x)|)\,dx &= \int_{Q\cap\{|f|\le 1\}} \Phi_2(|f(x)|)\,dx + \int_{Q\cap\{|f|>1\}} \Phi_2(|f(x)|)\,dx\\ &\le \Phi_2(1)|Q| + \int_{Q\cap\{|f|>1\}} \Phi_2(|f(x)|)\,dx.\end{aligned}$$

Since, $\Phi_2(t) = \Phi_1(t)$, $t > 1$, we have

$$\int_{Q\cap\{|f|>1\}} \Phi_2(|f(x)|)\,dx = \int_{Q\cap\{|f|>1\}} \Phi_1(|f(x)|)\,dx \le \int_Q \Phi_1(|f(x)|)\,dx \le |Q|,$$

where in the last inequality we have used the fact that $\|f\|_{\Phi_1,\,Q} = 1$. These imply

$$\frac{1}{|Q|}\int_Q \Phi_2\left(\frac{|f(x)|}{1+\Phi_2(1)}\right)dx \le 1$$

and $\|f\|_{\Phi_2,\,Q} \le 1 + \Phi_2(1)$. Therefore, we have by the definition of the Orlicz–Morrey norm $\|f\|_{\mathcal{L}^{(\Phi_2,\phi)}} \le (1+\Phi_2(1))\|f\|_{\mathcal{L}^{(\Phi_1,\phi)}}$. □

The following theorem covers the outrange of Theorem 3.3.

Theorem 5.2 *Let* $\rho \in \mathcal{G}_0$, $\phi, \psi \in \mathcal{G}_1$ *and* $0 < p \le r < \infty$. *Suppose that the condition* 3.1*;*

$$\frac{\tilde\rho(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le C\frac{\psi(t)}{\phi(t)^{p/r}} \text{ for all } t > 0.$$

(1) *Let* $0 < p \le r < q < \infty$, $p \ne 1$ *and* $r > 1$. *Then*

$$\|g\cdot T_\rho f\|_{\mathcal{M}^{r,\phi^{p/r}}} \le C\|g\|_{\mathcal{M}^{q,\psi}}\|f\|_{\mathcal{M}^{\max(1,p),\phi}}.$$

(2) *Let* $0 < p < r < q < \infty$ *and* $p = 1$. *Then*

$$\|g\cdot T_\rho f\|_{\mathcal{M}^{r,\phi^{1/r}}} \le C\|g\|_{\mathcal{M}^{q,\psi}}\|f\|_{\mathcal{M}^{L(\log L),\phi}}.$$

Proof The special case of Lemma 2.12 gives us that, for $1 < r < q < \infty$,

$$\|g\cdot T_\rho f\|_{\mathcal{M}^{r,\phi^{p/r}}} \le C\|g\|_{\mathcal{M}^{q,\psi}}\left(\|M_{\tilde\rho/\psi} f\|_{\mathcal{M}^{r,\phi^{p/r}}} + \|f\|_{\mathcal{M}^{1,\phi}}\right).$$

Noticing that $\dfrac{\tilde\rho(t)}{\psi(t)} \le C\phi(t)^{1-p/r}$, we have by Lemma 3.6

$$\|M_{\tilde\rho/\psi} f\|_{\mathcal{M}^{r,\phi^{p/r}}} \le \|M_{\phi^{1-p/r}} f\|_{\mathcal{M}^{r,\phi^{p/r}}} \le C\|Mf\|_{\mathcal{M}^{p,\phi}}.$$

Thus, we obtain

$$\|g\cdot T_\rho f\|_{\mathcal{M}^{r,\phi^{p/r}}} \le C\|g\|_{\mathcal{M}^{q,\psi}}\left(\|Mf\|_{\mathcal{M}^{p,\phi}} + \|f\|_{\mathcal{M}^{1,\phi}}\right).$$

Lemmas 3.4 and 3.5 and Eq. 2.20 yield the theorem. □

Letting $g(x) \equiv 1$ and $\psi(t) \equiv 1$ in Theorem 5.2 and using Lemma 4.1, we have the following corollary, which extends the classical theorem due to Adams in [1].

Corollary 5.3 *Suppose that ψ satisfy Eq. 4.1. Let $\rho \in \mathcal{G}_0$ and set $\phi(t) \equiv \psi(\tilde{\rho}(t))$.*

(1) *Let $0 < p < r < \infty$ and $p \neq 1$. Assume that*

$$\sup_{t>0} t\psi(t)^{p/r-1} \leq C.$$

Then

$$\| T_\rho f \|_{\mathcal{M}^{r,\phi^{p/r}}} \leq C \| f \|_{\mathcal{M}^{\max(1,p),\phi}}.$$

(2) *Let $1 < r < \infty$. Assume that*

$$\sup_{t>0} t\psi(t)^{1/r-1} \leq C.$$

Then

$$\| T_\rho f \|_{\mathcal{M}^{r,\phi^{1/r}}} \leq C \| f \|_{\mathcal{M}^{L(\log L),\phi}}.$$

Using the same method developed in the last part of the proof of Lemma 2.22, we can directly reprove Corollary 2.11 without the assumption $\bar{\Psi} \in \nabla_2$.

Theorem 5.4 *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \leq 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that the condition

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\, ds \leq \frac{C}{\eta(t)} \text{ for all } t > 0.$$

Then

$$\| T_\rho f \|_{\mathcal{L}^{\Psi,\eta}} \leq C \| f \|_{\mathcal{L}^{\Phi,\phi}}.$$

Proof Fix $x \in \mathbb{R}^n$. We may assume that f is nonnegative and $T_\rho f(x)$ is finite. Then we see that there exists $R > 0$ such that

$$\int_{\{|x-y|\leq R\}} f(y) \frac{\rho(|x-y|)}{|x-y|^n}\, dy = \frac{T_\rho f(x)}{2}.$$

Retaining the same notation as the proof of Lemma 2.12, we have by Eq. 2.13 that

$$\begin{aligned}
\frac{T_\rho f(x)}{2} &= \sum_{j=-\infty}^{0} \int_{\{2^{j-1}R<|x-y|\leq 2^j R\}} f(y) \frac{\rho(|x-y|)}{|x-y|^n}\, dy \\
&\leq C \sum_{j=-\infty}^{0} \frac{\hat{\rho}(2^j R)}{(2^j R)^n} \int_{\{|x-y|\leq 2^j R\}} f(y)\, dy \\
&\leq CMf(x) \sum_{j=-\infty}^{0} \hat{\rho}(2^j R) \leq CMf(x) \int_0^{c_2 R} \frac{\rho(s)}{s}\, ds = C\tilde{\rho}(c_2 R) Mf(x).
\end{aligned}$$

We have also that

$$\begin{aligned}\frac{T_\rho f(x)}{2} &= \sum_{j=1}^{\infty} \int_{\{2^{j-1}R<|x-y|\le 2^j R\}} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\,dy\\ &\le C\sum_{j=1}^{\infty}\frac{\hat\rho(2^jR)}{(2^jR)^n}\int_{\{|x-y|\le 2^jR\}} f(y)\,dy\\ &= C\sum_{j=1}^{\infty}\frac{\hat\rho(2^jR)}{\phi(2^jR)}\frac{\phi(2^jR)}{(2^jR)^n}\int_{\{|x-y|\le 2^jR\}} f(y)\,dy\\ &\le C\|f\|_{\mathcal{L}^{\Phi,\phi}}\sum_{j=1}^{\infty}\frac{\hat\rho(2^jR)}{\phi(2^jR)}\\ &\le C\|f\|_{\mathcal{L}^{\Phi,\phi}}\int_{c_1R}^{\infty}\frac{\rho(s)}{s\phi(s)}\,ds,\end{aligned}$$

where we have used the fact that $\phi(s)$ is a nondecreasing and doubling function. By the condition for any $t>0$ we have

$$\tilde\rho(t)\le C\phi(t)^{1-a},\quad \int_t^{\infty}\frac{\rho(s)}{s\phi(s)}\,ds\le C\phi(t)^{-a}$$

and by the doubling condition of ϕ

$$\tilde\rho(c_2R)\le C\phi(R)^{1-a},\quad \int_{c_1R}^{\infty}\frac{\rho(s)}{s\phi(s)}\,ds\le C\phi(R)^{-a}.$$

Thus, we obtain

$$\begin{aligned}T_\rho f(x) &\le C\min\left(\phi(R)^{1-a}Mf(x),\ \phi(R)^{-a}\|Mf\|_{\mathcal{L}^{\Phi,\phi}}\right)\\ &\le C\sup_{t>0}\min\left(t^{1-a}Mf(x),\ t^{-a}\|Mf\|_{\mathcal{L}^{\Phi,\phi}}\right)\\ &= C\|Mf\|_{\mathcal{L}^{\Phi,\phi}}^{1-a}Mf(x)^a.\end{aligned}$$

Once we have verified this inequality, the remainder of the proof is the same as the last part of the proof of Lemma 2.22, by noticing Corollary 2.18. □

Theorem 5.4 and Example 4.2 yield the following.

Example 5.5 Let $\rho\in\mathcal{G}_0$, $\Phi\in\nabla_2$ and $b>1$. Set

$$\phi(t)\equiv\tilde\rho(t)^b,\quad \eta(t)\equiv\tilde\rho(t)^{b-1},\quad \Psi(t)\equiv\Phi(t^{b/(b-1)}).$$

Then

$$\|T_\rho f\|_{\mathcal{L}^{\Psi,\eta}}\le C\|f\|_{\mathcal{L}^{\Phi,\phi}}.$$

Example 5.6 Let $\delta \geq 1$ and $b > 1$. Set

$$\rho(t) \equiv \min\left(t,\ (1+|\log t|)^{\delta-1}\right),$$
$$\phi(t) \equiv \min\left(t,\ (1+|\log t|)^{\delta}\right)^{b},$$
$$\eta(t) \equiv \min\left(t,\ (1+|\log t|)^{\delta}\right)^{b-1}$$

and set

$$\Phi(t) \equiv \exp(t^{1/\delta b}) - 1, \quad \Psi(t) \equiv \exp(t^{1/\delta(b-1)}) - 1.$$

Then

$$\|T_\rho f\|_{\mathcal{L}^{\Psi,\eta}} \leq C\|f\|_{\mathcal{L}^{\Phi,\phi}}.$$

We remark that $\tilde{\rho}(t) \approx \min\left(t,\ (1+|\log t|)^{\delta}\right)$ and $\sup_{t>1} \dfrac{\phi(t)}{\Phi^{-1}(t^n)} < \infty$.

We see that this example is valid only after deleting the assumption $\bar{\Psi} \in \nabla_2$, since $t(\log(2+t))^{\delta(b-1)} \notin \nabla_2$.

Proposition 5.7 *Let Φ and Ψ be normalized Young functions. Then the following are equivalent.*

(1) *The Orlicz maximal operator M^Ψ is $\mathcal{L}^\Phi$-bounded;*
(2) *The functions Φ and Ψ satisfy*

$$\int_0^t \Psi\left(\frac{t}{s}\right)\Phi'(s)\,ds \leq \Phi(Ct) \text{ for some } C > 0 \text{ and for all } t > 0;$$

(3) *The functions Φ and Ψ satisfy*

$$\int_0^t \Phi\left(\frac{t}{s}\right)\Psi'(s)\,ds \leq \Phi(Ct) \text{ for some } C > 0 \text{ and for all } t > 0.$$

Proof The proof is obtained by an argument similar to Proposition 2.17. □

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References

1. Adams, D.: A note on Riesz potentials. Duke Math. J. **42**, 765–778 (1975)
2. Adams, D., Xiao, J.: Nonlinear potential analysis on Morrey spaces and their capacities. Indiana Univ. Math. J. **53**, 1629–1663 (2004)
3. Garcia-Cuerva, J., Rubio de Francia, J.L.: Weighted norm inequalities and related topics. In: Mathematics Student, vol. 116. North-Holland, Amsterdam (1985)
4. Gilbarg, D., Trudinger, S.N.: Elliptic Partial Differential Equations of Second Order, 2nd edn. Springer, Berlin (1983)
5. Kerman, R., Sawyer, E.: The trace inequality and eigenvalue estimates for Schrödinger operators. Ann. Inst. Fourier (Grenoble) **36**, 207–228 (1986)
6. Kita, H.: On maximal functions in Orlicz spaces. Proc. Am. Math. Soc. **124**, 3019–3025 (1996)
7. Kita, H.: On Hardy–Littlewood maximal functions in Orlicz spaces. Math. Nachr. **183**, 135–155 (1997)

8. Nakai, E.: On generalized fractional integrals. Taiwan. J. Math. **5**, 587–602 (2001)
9. Nakai, E.: Generalized fractional integrals on Orlicz–Morrey spaces. In: Banach and Function Spaces, pp. 323–333. Yokohama Publishers, Yokohama (2004)
10. Nakai, E.: Orlicz–Morrey spaces and the Hardy–Littlewood maximal function. Stud. Math. **188**, 193–221 (2008)
11. Olsen, P.: Fractional integration, Morrey spaces and Schrödinger equation. Commun. Partial Differ. Equ. **20**, 2005–2055 (1995)
12. Pérez, C.: Two weighted inequalities for potential and fractional type maximal operators. Indiana Univ. Math. J. **43**, 663–683 (1994)
13. Pérez, C.: Sharp L^p-weighted Sobolev inequalities. Ann. Inst. Fourier (Grenoble) **45**, 809–824 (1995)
14. Rao, M.M., Ren, D.Z.: Theory of Orlicz Spaces. Marcel Dekker, New York (1991)
15. Sawano, Y., Sobukawa, T., Tanaka, H.: Limiting case of the boundedness of fractional integral operators on nonhomogeneous space. J. Inequal. Appl. 16 pp. (2006). doi:10.1155/JIA/2006/92470
16. Sawano, Y., Sugano, S., Tanaka, H.: A note on generalized fractional integral operators on generalized Morrey spaces. Boundary Value Problems **2009**, 18 pp. (2009). doi:10.1155/2009/835865
17. Sawano, Y., Sugano, S., Tanaka, H.: Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces. Trans. Am. Math. Soc. (2011, to appear)
18. Stein, M.E.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
19. Tanaka, H.: Morrey spaces and fractional operators. J. Aust. Math. Soc. **88**, 247–259 (2010)