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Global Heat Kernel Estimates for Relativistic Stable Processes in Half-space-like Open Sets

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Abstract In this paper, by using probabilistic methods, we establish sharp two-sided large time estimates for the transition densities of relativistic α-stable processes with mass $m \in (0, 1]$ (i.e., for the Dirichlet heat kernels of $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$ with $m \in (0, 1]$) in half-space-like $C^{1,1}$ open sets. The estimates are uniform in m in the sense that the constants are independent of $m \in (0, 1]$. Combining with the sharp two-sided small time estimates, established in Chen et al. (Ann Probab, 2011), valid for all $C^{1,1}$ open sets, we have now sharp two-sided estimates for the transition densities of relativistic α-stable processes with mass $m \in (0, 1]$ in half-space-like $C^{1,1}$ open sets for all times. Integrating the heat kernel estimates with respect to the time variable, one can recover the sharp two-sided Green function estimates for relativistic α-stable processes with mass $m \in (0, 1]$ in half-space-like $C^{1,1}$ open sets established recently in Chen et al. (Stoch Process their Appl, 2011).

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1 Introduction

Throughout this paper we assume that $d \geq 1$ and $\alpha \in (0, 2)$. For any $m > 0$, a relativistic α-stable process X^m on $\mathbb{R}^d$ with mass m is a Lévy process with characteristic function given by

$$\mathbb{E}\left[\exp\left(i\xi \cdot (X_t^m - X_0^m)\right)\right] = \exp\left(-t\left(\left(|\xi|^2 + m^{2/\alpha}\right)^{\alpha/2} - m\right)\right), \qquad \xi \in \mathbb{R}^d. \tag{1.1}$$

The limiting case X^0, corresponding to $m = 0$, is a (rotationally) symmetric α-stable (Lévy) process on $\mathbb{R}^d$, which we will simply denote as X. The infinitesimal generator of X^m is $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$. Note that when $m = 1$, this infinitesimal generator reduces to $1 - (1 - \Delta)^{\alpha/2}$. Thus the 1-resolvent kernel of the relativistic α-stable process X^1 on $\mathbb{R}^d$ is the Bessel potential kernel. (See [3] for more on this connection.) When $\alpha = 1$, the infinitesimal generator reduces to the so-called free relativistic Hamiltonian $m - \sqrt{-\Delta + m^2}$. The operator $m - \sqrt{-\Delta + m^2}$ is very important in mathematical physics due to its correspondence with the kinetic energy of a relativistic particle with mass m, see [22]. Physical models related to this operator have been much studied over the past 30 years and there exists a huge literature on the properties of relativistic Hamiltonians (see [5, 18–20, 22, 28, 29] and the references therein).

Relativistic α-stable processes received intensive study in recent years and various fine properties of these processes have been obtained in [3, 4, 15, 16, 21, 24, 25, 27, 30]. In 2002, Ryznar [30] proved that, when D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$, the Green function G_D^m of the process X^m in D is comparable to the Green function G_D of the symmetric α-stable process X. Soon after that, Chen and Song [16] proved the main result of [30] as a special case of a general perturbation result involving discontinuous Feynman–Kac transforms. In Grzywny and Ryznar [21], sharp two-sided estimates on the Green function G_D^m were established when D is a half-space of $\mathbb{R}^d$. In a recent paper [10], the authors of this paper established sharp two-sided estimates on the Green function G_D^m when D is a half-space like $C^{1,1}$ open set of $\mathbb{R}^d$.

Obtaining sharp two-sided estimates on Dirichlet heat kernels (or equivalently, transition densities of killed processes) is typically much harder than obtaining sharp two-sided estimates on the corresponding Green functions. For example, sharp two-sided estimates on the transition densities of killed Brownian motions in a domain D have been established only recently (see [35]) even though sharp two-sided estimates on the Green functions of killed Brownian motions in D were obtained much earlier (see [34, 36]). The main reason is that the Green function $G_D(x, y)$ is harmonic in $x \in D \setminus \{y\}$ for each fixed $y \in D$, while the transition density function is not. Sharp two-sided short time estimates on the transition densities $p_D^m(t, x, y)$ of X^m in $C^{1,1}$ open sets D were obtained very recently in [9]. To state this result, we first recall the notion of $C^{1,1}$ open set. An open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \ldots, 0)$, $\|\nabla\varphi\|_\infty \le \Lambda_0$, $|\nabla\varphi(x) - \nabla\varphi(z)| \le \Lambda_0|x - z|$, and an orthonormal coordinate system CS_z $y = (y_1, \cdots, y_{d-1}, y_d) := (\widetilde{y},\ y_d)$ with origin at z such that

$$B(z, R) \cap D = \{y = (\widetilde{y},\ y_d) \in B(0, R) \text{ in } CS_z : y_d > \varphi(\widetilde{y})\}.$$

We call (R, Λ_0) the $C^{1,1}$ characteristics of D. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be expressed as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a $C^{1,1}$ open set may be unbounded and disconnected. Here and in the sequel, we use ":=" as a way of definition and, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Theorem 1.1 [9, Theorem 1.1] *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R, Λ_0). Let $\delta_D(x)$ be the distance between x and D^c.*

(i) *For any $M > 0$ and $T > 0$, there exists $C_1 = C_1(d, \alpha, R, \Lambda_0, M, T) > 1$ such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$\frac{1}{C_1}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right) \le p_D^m(t, x, y)$$
$$\le C_1\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|/16)}{|x-y|^{d+\alpha}}\right),$$

where $\phi(r) = e^{-r}(1 + r^{(d+\alpha-1)/2})$.

(ii) *Suppose in addition that D is bounded. For any $M > 0$ and $T > 0$, there exists $C_2 = C_2(d, \alpha, R, \Lambda_0, M, T, diam(D)) > 1$ such that for any $m \in (0, M]$ and $(t, x, y) \in [T, \infty) \times D \times D$,*

$$C_2^{-1}\, e^{-t\lambda_1^{\alpha,m,D}}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2} \le p_D^m(t, x, y) \le C_2\, e^{-t\lambda_1^{\alpha,m,D}}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2},$$

where $\lambda_1^{\alpha,m,D} > 0$ is the smallest eigenvalue of the restriction of $(m^{2/\alpha} - \Delta)^{\alpha/2} - m$ to D with zero exterior condition.

Note that, although the small time two-sided estimates on $p_D^m(t, x, y)$ in Theorem 1.1(i) are valid for all $C^{1,1}$ open sets, the large time two-sided estimates on $p_D^m(t, x, y)$ in Theorem 1.1(ii) are only for bounded $C^{1,1}$ open sets.

The objective of this paper is to establish large time two-sided estimates on $p_D^m(t, x, y)$ for a large class of unbounded $C^{1,1}$ open sets D. The class of unbounded $C^{1,1}$ open sets we are going to work with are the so-called half-space-like $C^{1,1}$ open sets.

To state this result, we first recall the notion of half-space-like open sets. Recall that a half-space is any set which, after isometry, can be written as $\{(x_1, \ldots, x_d) : x_d > 0\}$. An open set D is said to be half-space-like if, after isometry, $H_a \subset D \subset H_b$ for some real numbers $a > b$. Here for any real number a, $H_a := \{(x_1, \ldots, x_d) : x_d > a\}$. H_0 will be simply written as H.

The main result of this paper is the following.

Theorem 1.2 *Suppose that D is a half-space-like $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R, Λ_0) such that $H_a \subset D \subset H_b$. For any $M > 0$, there exist*

$C_i = C_i(M, \alpha, d, R, \Lambda_0, a-b) \geq 1$, $i = 3, 4$, *such that for all* $m \in (0, M]$ *and* $(t, x, y) \in (0, \infty) \times D \times D$,

$$p_D^m(t, x, y) \leq C_3 \begin{cases} \left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|/C_4)}{|x-y|^{d+\alpha}}\right) & \text{for } t \in (0, 1/m]; \\ m^{(d/\alpha)-(d/2)}\left(\frac{m^{(2-\alpha)/2\alpha}\delta_D(x) + \delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) & \\ \times \left(\frac{m^{(2-\alpha)/2\alpha}\delta_D(y) + \delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) & \\ \times t^{-d/2} \exp\left(-C_4^{-1}\left(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}\frac{|x-y|^2}{t}\right)\right) & \text{for } t > 1/m, \end{cases} \tag{1.2}$$

and

$$p_D^m(t, x, y) \geq C_3^{-1} \begin{cases} \left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(C_4 m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right) & \text{for } t \in (0, 1/m]; \\ m^{(d/\alpha)-(d/2)}\left(\frac{m^{(2-\alpha)/2\alpha}\delta_D(x) + \delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) & \\ \times \left(\frac{m^{(2-\alpha)/2\alpha}\delta_D(y) + \delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) & \\ \times t^{-d/2} \exp\left(-C_4\left(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}\frac{|x-y|^2}{t}\right)\right) & \text{for } t > 1/m. \end{cases} \tag{1.3}$$

Remark 1.3

(i) The estimates in Theorem 1.2 are new even when D is the upper half space H. Observe that although H is invariant under scaling, global two-sided estimates on $p_H^m(t, x, y)$ can not be derived through a scaling argument from the short time estimates in Theorem 1.1(i), which holds only for $m \in (0, M]$ and $t \in (0, T]$.

(ii) For a fixed half-space-like $C^{1,1}$ open set D with $C^{1,1}$ characteristics (R, Λ_0) and $H_a \subset D \subset H_b$, mD is still a half-space-like $C^{1,1}$ open set but with $C^{1,1}$-characteristics $(mR, \Lambda_0/m)$ and $H_{ma} \subset mD \subset H_{mb}$. So we can not use the scaling property

$$p_D^m(t, x, y) = m^{d/\alpha} p^1_{m^{1/\alpha}D}(mt, m^{1/\alpha}x, m^{1/\alpha}y) \tag{1.4}$$

to obtain sharp two-sided estimates for $p_D^m(t, x, y)$ that are uniform in $m \in (0, M]$ from that of $p_D^1(t, x, y)$. Nevertheless, the two-sided sharp estimates for $p_D^m(t, x, y)$ obtained in Theorem 1.2 exhibit the scaling property. To see this, for $a > 0$, define

$$\varphi(r) = r + r^{\alpha/2}$$

and

$$\Psi_a(t,r) = \begin{cases} t^{-d/\alpha}\left(1 \wedge \left((t^{1/\alpha}/r)^{d+\alpha}\phi(ar)\right)\right) & \text{if } t \in (0,1], \\ t^{-d/2}\exp\left(-a\left(r \wedge (r^2/t)\right)\right) & \text{if } t > 1. \end{cases}$$

By considering $0 < r \le 1$ and $r \ge 1$ separately, we see that

$$\frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1 \le \frac{\varphi(r)}{\sqrt{t}} \wedge 1 \le 2\left(\frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \quad \text{for } t \in (0,1].$$

Thus the estimates in Theorem 1.2 can be rewritten as

$$\begin{aligned} &(4C_3)^{-1}\left(\frac{\varphi(m^{1/\alpha}\delta_D(x))}{\sqrt{mt}} \wedge 1\right)\left(\frac{\varphi(m^{1/\alpha}\delta_D(y))}{\sqrt{mt}} \wedge 1\right) m^{d/\alpha}\Psi_{C_4}(mt, m^{1/\alpha}|x-y|) \\ &\le p_D^m(t,x,y) \\ &\le C_3\left(\frac{\varphi(m^{1/\alpha}\delta_D(x))}{\sqrt{mt}} \wedge 1\right)\left(\frac{\varphi(m^{1/\alpha}\delta_D(y))}{\sqrt{mt}} \wedge 1\right) m^{d/\alpha}\Psi_{1/C_4}(mt, m^{1/\alpha}|x-y|) \end{aligned} \tag{1.5}$$

for $(t,x,y) \in (0,\infty) \times D \times D$ and $m \in (0, M]$.

(iii) For any open set D, we use τ_D^m to denote the first time X^m exits D, i.e., $\tau_D^m = \inf\{t > 0 : X_t^m \notin D\}$ and τ_D to denote the first time X exits D. By integrating Eq. 1.5 with respect to y, we see that for each fixed $M > 0$,

$$\begin{aligned} \mathbb{P}_x(\tau_D^m > t) &\asymp \frac{\varphi(m^{1/\alpha}\delta_D(x))}{\sqrt{mt}} \wedge 1 \qquad \text{for } m \in (0,M],\ t > 0 \text{ and } x \in D \\ &\asymp \begin{cases} \dfrac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 & \text{for } m \in (0,M] \text{ and } (t,x) \in (0,1/m] \times D; \\ \dfrac{m^{(2-\alpha)/2\alpha}\delta_D(x) + \delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 & \text{for } m \in (0,M] \text{ and } (t,x) \in (1/m,\infty) \times D. \end{cases} \end{aligned} \tag{1.6}$$

Here and in the sequel, for two non-negative functions f, g, $f \asymp g$ means that there is a positive constant $c_0 > 1$ so that $c_0^{-1} f \le g \le c_0 f$ on their common domain of definitions.

Let $p^m(t,x,y)$ denote the transition density function of X^m with respect to the Lebesgue measure on $\mathbb{R}^d$. In view of Eq. 2.2 and the estimates on $p^m(t,x,y)$ to be given below in Theorem 2.1, the estimates in Theorem 1.2 can be restated as follows: There exist constants $c_i = c_i(M, \alpha, d, R, \Lambda_0, a-b) > 1$, $i = 1, 2$, such that for every $(t,x,y) \in (0,\infty) \times D \times D$ and $m \in (0, M]$,

$$\begin{aligned} \frac{1}{c_1}\mathbb{P}_x(\tau_D^m > t)\,\mathbb{P}_y(\tau_D^m > t)\, p^m(t, c_2x, c_2y) &\le p_D^m(t,x,y) \\ &\le c_1\,\mathbb{P}_x(\tau_D^m > t)\,\mathbb{P}_y(\tau_D^m > t)\, p^m(t, x/c_2, y/c_2). \end{aligned} \tag{1.7}$$

In fact, by Theorem 5.1, when D is a half space, Eqs. 1.2–1.3 and 1.5–1.7 are true for all $m > 0$.

(iv) The Lévy exponent for X^m is $\Phi_m(|\xi|)$ with $\Phi_m(r) := (m^{2/\alpha} + r^2)^{\alpha/2} - m$. By considering $r \in (0, m^{1/\alpha}]$ and $r > m^{1/\alpha}$ separately, we see that there is a constant $c \geq 1$ such that

$$c^{-1}(r^\alpha \wedge (m^{1-(2/\alpha)} r^2)) \leq \Phi_m(r) \leq c(r^\alpha \wedge (m^{1-(2/\alpha)} r^2))$$

for all $m > 0$ and $r > 0$. Thus

$$\frac{1}{\Phi_m(1/r)} \asymp \frac{1}{r^{-\alpha} \wedge (m^{1-(2/\alpha)} r^{-2})} = \frac{m^{-1}}{(m^{1/\alpha} r)^{-\alpha} \wedge (m^{1/\alpha} r)^{-2}}$$
$$\asymp \frac{(m^{1/\alpha} r)^\alpha \vee (m^{1/\alpha} r)^2}{m} \asymp \left(\frac{\varphi(m^{1/\alpha} r)}{\sqrt{m}} \right)^2.$$

Thus the factor $\varphi(m^{1/\alpha}\delta_D(x))/\sqrt{mt}$ in Eqs. 1.5 and 1.6 is related to the Lévy exponent Φ_m of X^m as follows:

$$\frac{\varphi(m^{1/\alpha}\delta_D(x))}{\sqrt{mt}} \asymp \frac{1}{\sqrt{t\,\Phi_m(1/\delta_D(x))}}. \tag{1.8}$$

This is not a coincidence. The same phenomenon happens for symmetric stable processes [7, 17], for the mixture of stable processes [11], and for the mixture of Brownian motion and a symmetric stable process [12, 13]; see especially [13, Remark 1.5(i)]. We conjecture that the following holds for a large class of symmetric Lévy process on $\mathbb{R}^d$.

Conjecture *Let X be a rotationally symmetric Lévy process on $\mathbb{R}^d$ and D a $C^{1,1}$ open set in $\mathbb{R}^d$ (D will be assumed to be connected if X has a continuous component). Suppose X has a transition density function $p(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Denote the Lévy exponent of X by $\Phi_m(|\xi|)$. Let $p_D(t, x, y)$ denote the transition density function of the subprocess X^D of X killed upon exiting D. Then for any $T > 0$, there are constants $c_1, c_2 \geq 1$ such that for every $(t, x, y) \in (0, T] \times D \times D$,*

$$\frac{1}{c_1}\left(\frac{1}{\sqrt{t\,\Phi(1/\delta_D(x))}} \wedge 1\right)\left(\frac{1}{\sqrt{t\,\Phi(1/\delta_D(y))}} \wedge 1\right) p(t, c_2 x, c_2 y) \;\leq\; p_D(t, x, y)$$
$$\leq c_1 \left(\frac{1}{\sqrt{t\,\Phi(1/\delta_D(x))}} \wedge 1\right)\left(\frac{1}{\sqrt{t\,\Phi(1/\delta_D(y))}} \wedge 1\right) p(t, x/c_2, y/c_2). \tag{1.9}$$

If, in addition, D is half-space-like, then Eq. 1.9 holds for all $t > 0$.

This paper is a natural continuation of our recent papers [7–9, 12, 17] on heat kernel estimates of discontinuous Markov processes in open subsets of $\mathbb{R}^d$. Our main result Theorem 1.2 covers the main results in [10, 17, 21]. In fact, by integrating the heat kernel estimates in Theorem 1.2 with respect to t one can easily recover the sharp Green function estimates on $G_D^m(x, y)$ of [21] (for half-spaces) and [10] (for half-space-like $C^{1,1}$ open sets). Moreover, by letting $m \to 0$, we recover the heat kernel estimates for α-stable processes on half-space-like $C^{1,1}$ open sets, which are the content of one of the main results in [17] (see [9, Remark 1.2].)

Unlike symmetric stable processes, the Lévy density of X^m does not have a simple form and has exponential decay at infinity as opposed to the polynomial decay of the Lévy density of symmetric stable processes. In particular, relativistic stable processes do not have the stable-scaling property, which is one of the main ingredients used in the approaches of [7, 8]. Consequently, one can not obtain the large time heat kernel estimates for X^m in a half-space from the small time heat kernel estimates through scaling. A major part of this paper is to derive global sharp two-sided heat kernel estimates for X^m in a half-space.

Until very recently, even in $\mathbb{R}^d$, large time sharp two-sided heat kernel estimates were not available for jump processes with Lévy densities decaying exponentially at infinity. In a recent paper [6], the first two named authors, together with Kumagai, succeeded in establishing global sharp two-sided estimates for the heat kernel of a large class of jump processes, including the relativistic α-stable process X^1. This result provides us a guideline in getting the correct interior estimates of the Dirchlet heat kernel for X^1 in a half-space. For symmetric α-stable processes, for each fixed t, the Dirchlet heat kernel decays near the boundary at the rate $\delta_D(x)^{\alpha/2}$. But for large t, the Dirchlet heat kernel of a relativistic α-stable process decays at the rate $\delta_D(x)+\delta_D(x)^{\alpha/2}$. To obtain the lower bound in a half-space, we use the fact that Dirichlet heat kernel of X^m in D is no less than the transition density function of the corresponding subordinated killed Brownian motion in D. Then we use the push-inward technique developed in [17] to extend it to half-space-like $C^{1,1}$ open sets.

For the upper bound estimate in a half-space, we use some results in [21]. Then, to get the full two-sided heat kernel estimates, we adapt the strategy in [2, 7] to deal with large time estimates.

For the sharp estimates in half-space-like $C^{1,1}$ open sets, we follow the general strategy of [17] with several modifications to deal with the difficulties mentioned above and the complicated form of estimates for a half-space. For this, we need the inequalities established in Lemma 2.4 and use the comparison of the global heat kernels (Lemma 2.2 as opposed to the comparison of Dirichlet heat kernels in [17, Lemma 1.6]). The exit distribution estimate in Theorem 2.5 plays an important role in this paper.

In the remainder of this paper, we assume that $m>0$. We will use capital letters $C_1, C_2, \dots$ to denote constants in the statements of results, and their labeling will be fixed. The lower case constants $c_1, c_2, \dots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in every proof. The dependence of the lower case constants on the dimension d will not be mentioned explicitly. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial)=0$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$.

2 Preliminary Estimates

The Lévy measure of the relativistic α-stable process X^m, defined in Eq. 1.1, has a density

$$J^m(x)=j^m(|x|):=\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}\int_0^\infty (4\pi u)^{-d/2}e^{-\frac{|x|^2}{4u}}e^{-m^{2/\alpha}u}u^{-\left(1+\frac{\alpha}{2}\right)}du, \tag{2.1}$$

which is continuous and radially decreasing on $\mathbb{R}^d \setminus \{0\}$ (see [30, Lemma 2]). Here and in the rest of this paper, Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$ for every $\lambda > 0$. Put $J^m(x, y) := j^m(|x-y|)$. Let $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma\left(\frac{d+\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)^{-1}$. Using change of variables twice, first with $u = |x|^2 v$ then with $v = 1/s$, we get from Eq. 2.1 that

$$J^m(x, y) = \mathcal{A}(d, -\alpha)|x-y|^{-d-\alpha}\psi(m^{1/\alpha}|x-y|)$$

where

$$\psi(r) := 2^{-(d+\alpha)}\,\Gamma\left(\frac{d+\alpha}{2}\right)^{-1}\int_0^\infty s^{\frac{d+\alpha}{2}-1}e^{-\frac{s}{4}-\frac{r^2}{s}}\,ds,$$

which is a decreasing smooth function of r^2 satisfying $\psi(0) = 1$, $\psi(r) \le 1$ and

$$c_1^{-1}e^{-r}r^{(d+\alpha-1)/2} \le \psi(r) \le c_1 e^{-r}r^{(d+\alpha-1)/2} \qquad \text{on } [1, \infty) \tag{2.2}$$

for some $c_1 > 1$ (see [16, pp. 276–277] for details). We denote the Lévy density of X by

$$J(x) := J^0(x) = \mathcal{A}(d, -\alpha)|x|^{-(d+\alpha)}$$

and put $J(x, y) = J^0(x, y) = J(y-x)$.

The function $J^m(x, y)$ gives rise to a Lévy system for X^m, which describes the jumps of the process X^m: for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$ and stopping time T (with respect to the filtration of X^m),

$$\mathbb{E}_x\left[\sum_{s\le T} f(s, X^m_{s-}, X^m_s)\right] = \mathbb{E}_x\left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X^m_s, y)J^m(X^m_s, y)dy\right)ds\right]. \tag{2.3}$$

(See, for example, [14, Proof of Lemma 4.7] and [15, Appendix A]).

We will use $p^m(t, x, y) = p^m(t, y-x)$ to denote the transition density of X^m and use $p(t, x, y)$ to denote the transition density of X. It is well known that (cf. [14])

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \qquad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

From Eq. 1.1, one can easily see that X^m has the following scaling property:

$$\{X^m_t - X^m_0, t \ge 0\} \text{ has the same distribution as that of } \{m^{-1/\alpha}(X^1_{mt} - X^1_0), t \ge 0\}.$$

In terms of the transition densities, the scaling property above can be written as

$$p^m(t, x, y) \;=\; m^{d/\alpha}p^1(mt, m^{1/\alpha}x, m^{1/\alpha}y). \tag{2.4}$$

By [6, Theorem 1.2] and [9, Theorem 3.6], for every $T > 0$, there exist $c_1, c_2 > 1$ such that

$$p^1(t, x, y) \le c_1 \begin{cases} t^{-d/\alpha} \wedge tJ^1(x, y) & \text{for } t \in (0, T]; \\ t^{-d/2}\exp\left(-c_2^{-1}\left(|x-y| \wedge \frac{|x-y|^2}{t}\right)\right) & \text{for } t > T \end{cases}$$

and

$$p^1(t, x, y) \ge c_1^{-1} \begin{cases} t^{-d/\alpha} \wedge tJ^1(x, y) & \text{for } t \in (0, T]; \\ t^{-d/2}\exp\left(-c_2\left(|x-y| \wedge \frac{|x-y|^2}{t}\right)\right) & \text{for } t > T. \end{cases}$$

Now using Eq. 2.4 we get

Theorem 2.1 *For every $T>0$ there exist $C_i>1, i=5,6,$ such that for every $m>0$,*

$$p^m(t,x,y)\le C_5\begin{cases} t^{-d/\alpha}\wedge tJ^m(x,y) & \text{for } t\in(0,T/m];\\ m^{d/\alpha-d/2}t^{-d/2}\exp\Big(-C_6^{-1}\Big(m^{1/\alpha}|x-y|\wedge m^{2/\alpha-1}\frac{|x-y|^2}{t}\Big)\Big) & \text{for } t\ge T/m,\end{cases}$$

and

$$p^m(t,x,y)\ge C_5^{-1}\begin{cases} t^{-d/\alpha}\wedge tJ^m(x,y) & \text{for } t\in(0,T/m];\\ m^{d/\alpha-d/2}t^{-d/2}\exp\Big(-C_6\Big(m^{1/\alpha}|x-y|\wedge m^{2/\alpha-1}\frac{|x-y|^2}{t}\Big)\Big) & \text{for } t\ge T/m.\end{cases}$$

We define $X^{m,D}$ by $X_t^{m,D}(\omega)=X_t^m(\omega)$ if $t<\tau_D^m(\omega)$ and $X_t^{m,D}(\omega)=\partial$ if $t\ge\tau_D^m(\omega)$. We define X^D similarly. $X^{m,D}$ is called the subprocess of X^m killed upon exiting D (or, the killed relativistic stable process in D with mass m), and X^D is called the killed symmetric α-stable process in D.

It is known (see [15]) that $X^{m,D}$ has a transition density $p_D^m(t,x,y)$ with respect to the Lebesgue measure, which is continuous on $(0,\infty)\times D\times D$.

We will use $G_D^m(x,y):=\int_0^\infty p_D^m(t,x,y)dt$ to denote the Green function of $X^{m,D}$. The transition density and the Green function of X^D are denoted by $p_D(t,x,y)$ and $G_D(x,y)$, respectively. Recall that the Dirichlet heat kernel $p_D^m(t,x,y)$ enjoys the scaling property (1.4).

The following simple result will be used several times in this paper.

Lemma 2.2 *Let D be an open set in $\mathbb{R}^d$ and $\lambda, t_0, M>0$ be fixed constants. Suppose $x,x_0\in D$ satisfy $|x-x_0|=\lambda t_0^{1/\alpha}$. Then for all $a\in(0,M]$ and $z\in D$,*

$$\left(t_0^{-d/\alpha}\wedge\frac{t_0\phi(a|x-z|)}{|x-z|^{d+\alpha}}\right)\asymp\left(t_0^{-d/\alpha}\wedge\frac{t_0\phi(a|x_0-z|)}{|x_0-z|^{d+\alpha}}\right), \tag{2.5}$$

where the (implicit) comparison constants in Eq. 2.5 depend only on d, M, α, λ and t_0.

Proof By symmetry, it enough to show that

$$\sup_{a\in(0,M]}\left(t_0^{-d/\alpha}\wedge\frac{t_0\phi(a|x-z|)}{|x-z|^{d+\alpha}}\right)\Big/\left(t_0^{-d/\alpha}\wedge\frac{t_0\phi(a|x_0-z|)}{|x_0-z|^{d+\alpha}}\right)\le c_1 \tag{2.6}$$

for some positive constant c_1 depending only on d, M, α, λ and t_0.

Define $\lambda_0=\lambda/4$. Then for $z\in B\big(x_0,\lambda_0 t_0^{1/\alpha}\big)$ we have $|x-z|\asymp t_0^{1/\alpha}$. Thus, in this case,

$$\sup_{a\in(0,M]}\left(t_0^{-d/\alpha}\wedge\frac{t_0\phi(a|x-z|)}{|x-z|^{d+\alpha}}\right)\Big/\left(t_0^{-d/\alpha}\wedge\frac{t_0\phi(a|x_0-z|)}{|x_0-z|^{d+\alpha}}\right)$$

$$\le c_2 t_0^{-d/\alpha}\Big/\left(t_0^{-d/\alpha}\wedge\frac{t_0\phi\big(Mt_0^{1/\alpha}\big)}{t_0^{(d+\alpha)/\alpha}}\right).$$

Similarly, for $z\in B\big(x,\lambda_0 t_0^{1/\alpha}\big)$ we have $|x_0-z|\asymp t_0^{1/\alpha}$ and so Eq. 2.6 is true.

For $z\notin B\big(x,\lambda_0t_0^{1/\alpha}\big)\cup B\big(x_0,\lambda_0t_0^{1/\alpha}\big)$, we have by the triangle inequality,

$$|x-z|-4\lambda_0t_0^{1/\alpha}\le|x_0-z|\le|x-z|+4\lambda_0t_0^{1/\alpha}. \tag{2.7}$$

Thus, if $|z-x| \le 16\lambda_0 t_0^{1/\alpha}$, then $|x-z| \asymp \lambda_0 t_0^{1/\alpha} \asymp |x_0-z|$ and so Eq. 2.6 holds.

Finally, if $z \notin B(x, \lambda_0 t_0^{1/\alpha}) \cup B(x_0, \lambda_0 t_0^{1/\alpha})$ with $|z-x| \ge 16\lambda_0 t_0^{1/\alpha}$, then by Eq. 2.7, $|x_0-z| \asymp |x-z|$. Thus by the explicit expression of ϕ,

$$\begin{aligned}
&\sup_{a\in(0,M]} \left(t_0^{-d/\alpha} \wedge \frac{t_0\phi(a|x-z|)}{|x-z|^{d+\alpha}} \right) \Big/ \left(t_0^{-d/\alpha} \wedge \frac{t_0\phi(a|x_0-z|)}{|x_0-z|^{d+\alpha}} \right) \\
&\le c_3 \sup_{a\in(0,M]} \frac{t_0\phi\big(a|x_0-z| - 4a\lambda_0 t_0^{1/\alpha}\big)}{|x_0-z|^{d+\alpha}} \Big/ \frac{t_0\phi(a|x_0-z|)}{|x_0-z|^{d+\alpha}} \\
&\le c_4 \sup_{a\in(0,M]} \frac{\phi\big(a|x_0-z| - 4a\lambda_0 t_0^{1/\alpha}\big)}{\phi(a|x_0-z|)} \le c_5 e^{4M\lambda_0 t_0^{1/\alpha}}.
\end{aligned}$$

□

Let D be an open set in $\mathbb{R}^d$ such that $H_a \subset D \subset H$ for some $a > 0$. Set

$$t_0 := (1 \vee a)^\alpha \tag{2.8}$$

and $e_d = (\tilde{0}, 1)$. For x and y in D, define two points

$$x_0 := x + 2t_0^{1/\alpha} e_d \qquad \text{and} \qquad y_0 := y + 2t_0^{1/\alpha} e_d. \tag{2.9}$$

Observe that $\delta_D(x_0) \ge \delta_{H_a}(x_0) > t_0^{1/\alpha}$. The following simple lemma is established in [17, Lemma 2.2].

Lemma 2.3 *Suppose that D is an open set in $\mathbb{R}^d$ with $H_a \subset D \subset H$ for some $a > 0$. Let t_0 be defined as in Eq. 2.8. For $x \in D$, let x_0 be defined as in Eq. 2.9. Then there is a constant $C_7 \ge 1$ depending only on d, α, t_0 and a such that*

$$C_7^{-1}(1 \wedge \delta_D(x)^\alpha)\left(1 \wedge \frac{\delta_H(x_0)^\alpha}{t}\right) \le 1 \wedge \frac{\delta_D(x)^\alpha}{t} \le C_7(1 \wedge \delta_D(x)^\alpha)\left(1 \wedge \frac{\delta_{H_a}(x_0)^\alpha}{t}\right)$$

for all $t \ge t_0$.

The following elementary result will play an important role later in this paper.

Lemma 2.4 *Let D be an open set in $\mathbb{R}^d$ such that $H_a \subset D \subset H$ and t_0 be the constant defined in Eq. 2.8. For $x \in D$, let x_0 be defined as in Eq. 2.9. For any $M > 0$, there exists a constant $C_8 = C_8(a, \alpha, M, t_0) > 1$ such that*

(i) *for all $t > 0$, $m \in (0, M]$ and $x \in D$,*

$$\begin{aligned}
&1 \wedge \frac{\delta_D(x)^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}\delta_D(x)}{\sqrt{t}} \\
&\ge C_8^{-1}\left(1 \wedge \delta_D(x)^{\alpha/2}\right)\left(1 \wedge \frac{\delta_H(x_0)^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}\delta_H(x_0)}{\sqrt{t}}\right);
\end{aligned}$$

(ii) *for all* $t > t_0$, $m \in (0, M]$ *and* $x \in D$,

$$1 \wedge \frac{\delta_D(x)^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}\delta_D(x)}{\sqrt{t}} \le C_8 \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \times \left(1 \wedge \frac{\delta_{H_a}(x_0)^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}\delta_{H_a}(x_0)}{\sqrt{t}}\right).$$

Proof Note that

$$\delta_D(x) + t_0^{1/\alpha} \le \delta_{H_a}(x_0) \le \delta_D(x) + 2t_0^{1/\alpha} \quad \text{and} \quad \delta_D(x) + 2t_0^{1/\alpha} \le \delta_H(x_0) \le \delta_D(x) + 3t_0^{1/\alpha}.$$

We consider the two cases $\delta_D(x) \le t_0^{1/\alpha}$ and $\delta_D(x) > t_0^{1/\alpha}$ separately. When $\delta_D(x) > t_0^{1/\alpha}$, we have

$$\delta_D(x) \le \delta_{H_a}(x_0) < \delta_H(x_0) \le 4\delta_D(x).$$

The inequalities in (i)–(ii) are trivial in this case. So from now on we assume that $\delta_D(x) \le t_0^{1/\alpha}$. In this case, $\delta_{H_a}(x_0) \in \left(t_0^{1/\alpha}, 3t_0^{1/\alpha}\right)$ and $\delta_H(x_0) \in \left(2t_0^{1/\alpha}, 4t_0^{1/\alpha}\right)$.

(i) For all $t > 0$, $m \in (0, M]$ and $x \in D$

$$\left(1 \wedge \delta_D(x)^{\alpha/2}\right)\left(1\wedge\frac{\delta_H(x_0)^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}\delta_H(x_0)}{\sqrt{t}}\right) \le c_1 \left(1 \wedge \delta_D(x)^{\alpha/2}\right)\left(1 \wedge \frac{1}{\sqrt{t}}\right)$$
$$\le c_1 \left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \le c_1 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}\delta_D(x)}{\sqrt{t}}\right).$$

(ii) For all $t > t_0$, $m \in (0, M]$ and $x \in D$,

$$\left(1 \wedge \delta_D(x)^{\alpha/2}\right)\left(1\wedge\frac{\delta_{H_a}(x_0)^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}\delta_{H_a}(x_0)}{\sqrt{t}}\right) \ge c_2 \left(1 \wedge \delta_D(x)^{\alpha/2}\right)\left(1\wedge\frac{1}{\sqrt{t}}\right)$$
$$\ge c_2 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \ge c_3 \left(1\wedge\frac{\delta_D(x)^{\alpha/2} + m^{(2-\alpha)/(2\alpha)}\delta_D(x)}{\sqrt{t}}\right).$$

□

The following estimates on harmonic measures will play a crucial role in Section 4.

Theorem 2.5 *For any* $r_0 > 0$, *there exists a constant* $C_9 = C_9(\alpha, r_0) > 0$ *such that for every* $r \ge r_0$ *and open set* $U \subset B(0, r)$

$$\mathbb{P}_x\left(X^1_{\tau^1_U} \in B(0, r)^c\right) \le C_9 r^{-2} \int_U G^1_U(x, y)dy, \quad \textit{for every } x \in U \cap B(0, r/2).$$

Proof Without loss of generality, we assume that $0 < r_0 < 1$. Recall that $C_c^\infty(\mathbb{R}^d)$, the space of continuous functions with compact support, is in the domain of the L_2-generator $\mathcal{L}_1$ of X^1 and

$$\mathcal{L}_1\eta(x) = \int_{\mathbb{R}^d} (\eta(x + y) - \eta(x) - (\nabla\eta(x) \cdot y)\mathbf{1}_{B(0,\varepsilon)}(y))J^1(|y|)dy \qquad \text{for } \eta \in C_c^\infty\left(\mathbb{R}^d\right)$$

(see [31, Section 4.1]). Observe that the right hand side of the above display is independent of the choice of $\varepsilon > 0$. Using the argument in [26, pp.152], one can easily see that the last formula on [26, pp.152] is valid for X^1 for all $d \geq 1$. Thus we have that for every η in $C_c^\infty(\mathbb{R}^d)$ with $\eta(x) = 0$,

$$\mathbb{E}_x\left[\eta\left(X^1_{\tau^1_U}\right)\right] = \int_U G^1_U(x,y)\mathcal{L}_1\eta(y)dy \quad \text{for any } x \in U. \tag{2.10}$$

Take a sequence of radial functions η_k in $C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \eta_k \leq 1$,

$$\eta_k(y) = \begin{cases} 0 \text{ when } |y| < 1/2, \\ 1 \text{ when } 1 \leq |y| \leq k+1, \\ 0 \text{ when } |y| > k+2, \end{cases}$$

and that $\sum_{i,j} |\frac{\partial^2}{\partial y_i \partial y_j}\eta_k|$ is uniformly bounded. Define $\eta_{k,r}(y) = \eta_k\left(\frac{y}{r}\right)$. Then we have $0 \leq \eta_{k,r} \leq 1$,

$$\eta_{k,r}(y) = \begin{cases} 0 \text{ when } |y| < r/2, \\ 1 \text{ when } r \leq |y| \leq r(k+1), \\ 0 \text{ when } |y| > r(k+2), \end{cases}$$

and

$$\sup_{y\in\mathbb{R}^d} \sum_{i,j} \left|\frac{\partial^2}{\partial y_i \partial y_j}\eta_{k,r}(y)\right| < c_0\, r^{-2}.$$

Using this inequality and the fact that

$$j^1(|y|) = j^0(|y|)\psi(|y|) \leq c_1|y|^{-d-\alpha}\mathbf{1}_{\{|y|\leq r_0\}} + c_1 e^{-|y|}|y|^{-(d+\alpha+1)/2}\mathbf{1}_{\{|y|>r_0\}}$$

for some $c_1 = c_1(r_0, \alpha) > 0$ (see Eq. 2.2), we have for $r \geq r_0$

$$\begin{aligned}
&\sup_{k\geq 1}\sup_{z\in\mathbb{R}^d} |\mathcal{L}_1\eta_{k,r}(z)| \\
&= \sup_{k\geq 1}\sup_{z\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(\eta_{k,r}(z+y) - \eta_{k,r}(z) - (\nabla\eta_{k,r}(z)\cdot y)\mathbf{1}_{B(0,r)}(y))\, j^1(|y|)dy\right| \\
&\leq c_2 \sup_{k\geq 1}\sup_{z\in\mathbb{R}^d}\left(\int_{\{|y|\leq r\}}\left|\frac{\eta_{k,r}(z+y) - \eta_{k,r}(z) - (\nabla\eta_{k,r}(z)\cdot y)}{|y|^{d+\alpha}}\right|\psi(|y|)dy\right. \\
&\qquad \left. + \int_{\{r<|y|\}} e^{-|y|}|y|^{-(d+\alpha+1)/2}dy\right) \\
&\leq c_3\left(\frac{1}{r^2}\int_{\{|y|\leq r_0\}}\frac{|y|^2}{|y|^{d+\alpha}}dy + \frac{1}{r^2}\int_{\{r_0<|y|\leq r\}} e^{-|y|}|y|^{-(d+\alpha+1)/2+2}dy\right. \\
&\qquad \left. + \int_{\{r<|y|\}} e^{-|y|}|y|^{-(d+\alpha+1)/2}dy\right) \\
&\leq c_4\left(r^{-2} + \frac{1}{r^2}\int_{\{r_0<|y|<\infty\}} e^{-|y|}|y|^{-(d+\alpha+1)/2+2}dy\right. \\
&\qquad \left. + \int_r^\infty e^{-s}s^{d/2-\alpha/2+3/2}ds\right) \leq c_5 r^{-2}.
\end{aligned} \tag{2.11}$$

When $U \subset B(0,r)$ for some $r \geq r_0$, we get, by combining Eqs. 2.10 and 2.11, that for any $x \in U \cap B(0,r/2)$,

$$\mathbb{P}_x\left(X^1_{\tau^1_U} \in \left\{y \in \mathbb{R}^d : r \leq |y| < (k+1)r\right\}\right) \leq \mathbb{E}_x\left[\eta_{k,r}\left(X^1_{\tau^1_U}\right)\right] \leq c_5 r^{-2} \int_U G^1_U(x,y)dy. \tag{2.12}$$

Therefore, for any $x \in U \cap B(0,r/2)$,

$$\begin{aligned}\mathbb{P}_x\left(X^1_{\tau^1_U} \in B(0,r)^c\right) &= \lim_{k\to\infty} \mathbb{P}_x\left(X^1_{\tau^1_U} \in \left\{y \in \mathbb{R}^d : r \leq |y| < (k+1)r\right\}\right)\\ &\leq c_5 r^{-2} \int_U G^1_U(x,y)dy.\end{aligned}$$

□

3 Lower Bound Heat Kernel Estimate on Half-space

Recall that $H = \{(x_1,\ldots,x_d) : x_d > 0\}$. In this section we will prove the lower bound estimate for $p^1_H(t,x,y)$. One of the tools we will use is a subordinate killed Brownian motion in H. We first recall this concept. Let B_t be a Brownian motion on $\mathbb{R}^d$ such that $\mathbb{E}[\exp(i\xi\cdot(B_t - B_0))] = \exp(-t|\xi|^2)$ for all $\xi \in \mathbb{R}^d$ and let B^H be the Brownian motion killed upon exiting H. Let T_t be an $\frac{\alpha}{2}$-stable subordinator with Laplace exponent $\lambda^{\alpha/2}$ and independent of B. Denote the density of T_t by $\theta_\alpha(t,u)$. Let T^1_t be a subordinator with Laplace exponent $(\lambda+1)^{\alpha/2} - 1$. Then the density of T^1_t is $e^t e^{-u}\theta_\alpha(t,u)$. The process $\{Z^{1,H}_t : t \geq 0\}$ defined by $Z^{1,H}_t = B^H_{T^1_t}$ is called a subordinate killed Brownian motion in H. Let $q^1_H(t,x,y)$ be the transition density of $Z^{1,H}$. Then it follows from [32, Proposition 3.1] that

$$p^1_H(t,z,w) \geq q^1_H(t,z,w) \quad \text{for } (t,z,w) \in (0,\infty)\times H \times H. \tag{3.1}$$

We will use this fact in the next result.

Lemma 3.1 *There exist positive constants $C_i = C_i(\alpha,d)$, $i = 10, 11$, such that for all $t \in [1,\infty)$ and x, y in H,*

$$p^1_H(t,x,y) \geq C_{10} t^{-d/2}\left(1 \wedge \frac{\delta_H(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_H(y)}{\sqrt{t}}\right) e^{-C_{11}|x-y|^2/t}.$$

Proof Let $p_H(t,x,y)$ be the transition density of the killed Brownian motion B^H in H. Using [23, (2.8.4)], one can easily show that there exists $c_1 > 1$ such that

$$c_1^{-1}\left(1 \wedge \frac{\delta_H(x)}{\sqrt{s}}\right) \leq \mathbb{P}_x(\tau_H > s) \leq c_1\left(1 \wedge \frac{\delta_H(x)}{\sqrt{s}}\right), \quad x \in H.$$

Thus it follows from [33] that there exist positive constants c_i, $i = 2, 3$, such that for any $(s,x,y) \in (0,\infty)\times H \times H$,

$$p_H(s,x,y) \geq c_2\left(1 \wedge \frac{\delta_H(x)}{\sqrt{s}}\right)\left(1 \wedge \frac{\delta_H(y)}{\sqrt{s}}\right) s^{-d/2} e^{-c_3|x-y|^2/s}. \tag{3.2}$$

It follows from the definition of $Z_t^{1,H} = B^H_{T_t^1}$ (for example, see [1, page 149]) that for every $t \geq 1$, the transition density $q^1_H(t, x, y)$ of $Z^{1,H}$ is given by the following formula,

$$q^1_H(t, x, y) = \int_0^\infty p_H(s, x, y)\mathbb{P}\big(T_t^1 \in ds\big) = \int_0^\infty p_H(s, x, y)e^t e^{-s}\theta_\alpha(t, s)ds.$$

Note that, since $\mathbb{E}[e^{-bT_t^1}] = e^t e^{-t(b+1)^{\alpha/2}}$, we have

$$\mathbb{P}(bT_t^1 \leq t) \leq \mathbb{P}\big(e^{-bT_t^1} \geq e^{-t}\big) \leq e^t\mathbb{E}\big[e^{-bT_t^1}\big] \leq e^{2t}e^{-t(b+1)^{\alpha/2}} = e^{-t((b+1)^{\alpha/2}-2)}. \quad (3.3)$$

Let $b := 3^{2/\alpha} - 1$ so that $\mathbb{P}(bT_t^1 \leq t) \leq e^{-t}$. Using this fact and Eqs. 3.1–3.3, we get that for every $t \geq 1$,

$$\begin{aligned}
p^1_H(t, x, y) &\geq q^1_H(t, x, y) \\
&\geq \int_{bt}^{2t} p_H(s, x, y)e^t e^{-s}\theta_\alpha(t, s)ds \\
&\geq \frac{c_4}{2}\left(1 \wedge \frac{\delta_H(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_H(y)}{\sqrt{t}}\right)t^{-d/2}e^{-bc_3|x-y|^2/t}\int_{bt}^{2t} e^t e^{-s}\theta_\alpha(t, s)ds \\
&= \frac{c_4}{2}\left(1 \wedge \frac{\delta_H(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_H(y)}{\sqrt{t}}\right)t^{-d/2}e^{-bc_3|x-y|^2/t}\left(1 - \mathbb{P}(bT_t^1 \leq t) - \int_{2t}^\infty e^{t-s}\theta_\alpha(t, s)ds\right) \\
&\geq \frac{c_4}{2}\left(1 \wedge \frac{\delta_H(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_H(y)}{\sqrt{t}}\right)t^{-d/2}e^{-bc_3|x-y|^2/t}\left(1 - e^{-t} - e^{-t}\int_0^\infty \theta_\alpha(t, s)ds\right) \\
&\geq \frac{c_4}{2}\left(1 \wedge \frac{\delta_H(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_H(y)}{\sqrt{t}}\right)t^{-d/2}e^{-bc_3|x-y|^2/t}\left(1 - 2e^{-1}\right).
\end{aligned}$$

□

Lemma 3.2 *There exist positive constants $C_i = C_i(\alpha, d)$, $i = 12, 13$, such that for all $t \in [1, \infty)$ and x, y in H,*

$$p^1_H(t, x, y) \geq C_{12}t^{-d/2}\left(1 \wedge \frac{\delta_H(x) + \delta_H(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_H(y) + \delta_H(y)^{\alpha/2}}{\sqrt{t}}\right)e^{-C_{13}|x-y|^2/t}.$$

Proof It follows from Theorem 1.1(i) and Lemma 3.1 that we only need to consider the case $t \geq 3$. Now in the remainder of this proof we assume that $t \geq 3$. Let x_0 and y_0

be defined by Eq. 2.9 with $t_0 = 1$ ($a = 1$). By the semigroup property, Theorem 1.1(i) and Eq. 2.5 we have

$$\begin{aligned} p_H^1(t,x,y) &= \int_H\int_H p_H^1(1,x,z)p_H^1(t-2,z,w)p_H^1(1,w,y)dzdw \\ &\geq c_1\,(1\wedge\delta_H(x))^{\alpha/2}(1\wedge\delta_H(y))^{\alpha/2}\int_{H\times H}(1\wedge\delta_H(z))^{\alpha/2}\left(1\wedge\frac{\phi(|x-z|)}{|x-z|^{d+\alpha}}\right) \\ &\quad\times p_H^1(t-2,z,w)(1\wedge\delta_H(w))^{\alpha/2}\left(1\wedge\frac{\phi(|w-y|)}{|w-y|^{d+\alpha}}\right)dzdw \\ &\geq c_2\,(1\wedge\delta_H(x))^{\alpha/2}(1\wedge\delta_H(y))^{\alpha/2}\int_{H\times H}(1\wedge\delta_H(z))^{\alpha/2}\left(1\wedge\frac{\phi(|x_0-z|)}{|x_0-z|^{d+\alpha}}\right) \\ &\quad\times p_H^1(t-2,z,w)(1\wedge\delta_H(w))^{\alpha/2}\left(1\wedge\frac{\phi(|w-y_0|)}{|w-y_0|^{d+\alpha}}\right)dzdw. \end{aligned} \tag{3.4}$$

By the domain monotonicity property, we have $p_{H_{1/2}}^1(s,w,z)\leq p_H^1(s,w,x)\leq p^1(s,w,x)$ on $(0,\infty)\times H_{1/2}\times H_{1/2}$. In particular for $z,w\in H_{1/2}$, by Theorem 2.1,

$$p_{H_{1/2}}^1(1,x_0,z)\leq p^1(1,x_0,z)\leq c_3\left(1\wedge\frac{\phi(|x_0-z|)}{|x_0-z|^{d+\alpha}}\right)$$

and

$$p_{H_{1/2}}^1(1,y_0,w)\leq p^1(1,y_0,w)\leq c_3\left(1\wedge\frac{\phi(|w-y_0|)}{|w-y_0|^{d+\alpha}}\right).$$

Thus we have by Eq. 3.4 that $p_H^1(t,x,y)$ is greater than or equal to $c_4\,(1\wedge\delta_H(x))^{\alpha/2}\,(1\wedge\delta_H(y))^{\alpha/2}$ times

$$\begin{aligned} &\int_{H_{1/2}\times H_{1/2}}\left(1\wedge\frac{\phi(|x_0-z|)}{|x_0-z|^{d+\alpha}}\right)p_{H_{1/2}}^1(t-2,z,w)\left(1\wedge\frac{\phi(|w-y_0|)}{|w-y_0|^{d+\alpha}}\right)dzdw \\ &\geq\ c_5\int_{H_{1/2}\times H_{1/2}} p_{H_{1/2}}^1(1,x_0,z)p_{H_{1/2}}^1(t-2,z,w)\,p_{H_{1/2}}^1(1,y_0,w)dzdw \\ &= c_5 p_{H_{1/2}}^1(t,x_0,y_0). \end{aligned} \tag{3.5}$$

Since, by Lemma 3.1,

$$\begin{aligned} p_{H_{1/2}}^1(t,x_0,y_0) &\geq c_6 t^{-d/2}\left(1\wedge\frac{\delta_{H_{1/2}}(x_0)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_{H_{1/2}}(y_0)}{\sqrt{t}}\right)e^{-c_7|x_0-y_0|^2/t} \\ &\geq c_8 t^{-d/2}\left(1\wedge\frac{\delta_H(x_0)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_H(y_0)}{\sqrt{t}}\right)e^{-c_7|x-y|^2/t}, \end{aligned}$$

using Lemma 2.4 and Eq. 3.5 we get

$$p_H^1(t,x,y)\geq c_9 t^{-d/2}\left(1\wedge\frac{\delta_H(x)+\delta_H(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_H(y)+\delta_H(y)^{\alpha/2}}{\sqrt{t}}\right)e^{-c_7|x-y|^2/t}.$$

□

The following result is similar to the second assertion of [2, Lemma 2] and its proof. We give the proof here for the sake of completeness.

Lemma 3.3 *Suppose that* U_1, U_2, E *are open subsets of* $\mathbb{R}^d$ *with* $U_1, U_2 \subset E$ *and* $\mathrm{dist}(U_1, U_2) > 0$. *If* $x \in U_1$ *and* $y \in U_2$, *then for all* $m > 0$ *and* $t > 0$,

$$p_E^m(t, x, y) \geq t\, \mathbb{P}_x\big(\tau_{U_1}^m > t\big)\, \mathbb{P}_y\big(\tau_{U_2}^m > t\big) \inf_{u \in U_1,\, z \in U_2} J^m(u, z)\,. \tag{3.6}$$

Proof Using the strong Markov property, we have

$$p_E^m(t, x, y) \geq \mathbb{E}_x\left[p_E^m\Big(t - \tau_{U_1}^m, X_{\tau_{U_1}^m}^m, y\Big) : \tau_{U_1}^m < t, X_{\tau_{U_1}^m}^m \in U_2\right].$$

Thus by taking $T = \tau_{U_1}^m$ and $f(s, w, z) = \mathbf{1}_{(0,t)}(s) p_E^m(t - s, z, y)\mathbf{1}_{U_1}(w)\mathbf{1}_{U_2}(z)$ in Eq. 2.3, we have

$$\begin{aligned} p_E^m(t, x, y) &\geq \int_0^t \left(\int_{U_1} p_{U_1}^m(s, x, u) \left(\int_{U_2} J^m(u, z) p_E^m(t - s, z, y) dz\right) du\right) ds \\ &\geq \inf_{u \in U_1,\, z \in U_2} J^m(u, z) \int_0^t \int_{U_2} p_E^m(t - s, z, y) \mathbb{P}_x(\tau_{U_1}^m > s) dz ds \end{aligned}$$

and

$$\begin{aligned} \int_0^t \int_{U_2} p_E^m(t - s, z, y) \mathbb{P}_x\big(\tau_{U_1}^m > s\big) dz ds &\geq \mathbb{P}_x\big(\tau_{U_1}^m > t\big) \int_0^t \mathbb{P}_y\big(\tau_{U_2}^m > t - s\big) ds \\ &\geq t\, \mathbb{P}_x\big(\tau_{U_1}^m > t\big) \mathbb{P}_y\big(\tau_{U_2}^m > t\big)\,. \end{aligned}$$

This completes the proof of the lemma. □

The following result from [6] will be used in the proof of Lemma 3.5.

Theorem 3.4 [6, Theorem 4.8] *For each* $t_0 > 0$ *and* $0 < \varepsilon < M < \infty$, *there exists* $C_{14} = C_{14}(t_0, \varepsilon, M) > 0$ *such that for every* $x_0 \in \mathbb{R}^d$, $t \geq t_0$,

$$p_{B(x_0, t^{1/2})}^1(s, x, y) \geq C_{14}\, t^{-d/2} \qquad \text{for every } s \in [\varepsilon t, Mt] \text{ and } x, y \in B(x_0, 3\sqrt{t}/4). \tag{3.7}$$

Lemma 3.5 *Suppose that* $t_0 > 0$. *There exists* $C_{15} = C_{15}(t_0, \alpha) > 0$ *such that for all* $t \geq t_0$ *and* $u, v \in \mathbb{R}^d$ *with* $|u - v| \geq \sqrt{t}/2$,

$$p_{B(u,\sqrt{t}) \cup B(v,\sqrt{t})}^1(t/3, u, v) \geq C_{15}\, t\, j^1(3|u - v|/2).$$

Proof Let $E = B(u, \sqrt{t}) \cup B(v, \sqrt{t})$. With $U_1 = B(u, \sqrt{t}/8)$ and $U_2 = B(v, \sqrt{t}/8)$, we have by Lemma 3.3 that

$$p_E^1(t/3, u, v) \geq \frac{t}{3} \mathbb{P}_u(\tau_{U_1}^1 > t/3) \left(\inf_{w \in U_1,\, z \in U_2} j^1(|w - z|)\right) \mathbb{P}_v(\tau_{U_2}^1 > t/3)\,.$$

Note that for $w \in U_1$ and $z \in U_2$,

$$|w - z| \le |u - v| + |w - u| + |z - v| \le |u - v| + \sqrt{t}/4 \le \frac{3}{2}|u - v|.$$

Thus by Theorem 3.4,

$$\begin{aligned} p^1_{B(u,\sqrt{t})\cup B(v,\sqrt{t})}(t/3, u, v) &\ge \frac{t}{3}\left(\mathbb{P}_0(\tau^1_{B(0,\sqrt{t}/8)} > t/3)\right)^2 \left(\inf_{w\in U_1,\, z\in U_2} j^1(|w - z|)\right) \\ &\ge \frac{t}{3}\left(\int_{B(0,\sqrt{t}/8)} p^1_{B(0,\sqrt{t}/8)}(t/3, 0, w)dw\right)^2 j^1(3|u - v|/2) \\ &\ge c_1\, t \left(\int_{B(0,\sqrt{t}/8)} t^{-d/2} dw\right)^2 j^1(3|u - v|/2) \\ &\ge c_2\, t\, j^1(3|u - v|/2). \end{aligned}$$

□

Now we are in a position to establish the main result of this section.

Theorem 3.6 *There are positive constants $C_i = C_i(\alpha)$, $i = 16, 17$, such that for all $(t, x, y) \in [1, \infty) \times H \times H$*

$$p^1_H(t, x, y) \ge C_{16} t^{-d/2} \left(\frac{\delta_H(x)+\delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_H(y) + \delta_H(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) e^{-C_{17}\left(|x-y|\wedge \frac{|x-y|^2}{t}\right)}.$$

Proof By Lemma 3.2, it suffices to prove the theorem for $|x - y| \ge t \ge 1$. Assume $|x - y| \ge t \ge 1$ and let $\xi_x := x + (\widetilde{0}, \sqrt{t})$ and $\xi_y := y + (\widetilde{0}, \sqrt{t})$. By Lemma 3.2,

$$\begin{aligned} &\int_{B(\xi_x,\sqrt{t}/4)} p^1_H(t/3, x, u)du \\ &\ge c_1 t^{-d/2}\left(1 \wedge \frac{\delta_H(x) + \delta_H(x)^{\alpha/2}}{\sqrt{t}}\right)\int_{B(\xi_x,\sqrt{t}/4)}\left(1 \wedge \frac{\delta_H(u)}{\sqrt{t}}\right) e^{-c_2|x-u|^2/t}du \\ &\ge c_3 t^{-d/2}\left(1 \wedge \frac{\delta_H(x) + \delta_H(x)^{\alpha/2}}{\sqrt{t}}\right)\int_{B(\xi_x,\sqrt{t}/4)} e^{-c_4}du \\ &\ge c_5\left(1 \wedge \frac{\delta_H(x) + \delta_H(x)^{\alpha/2}}{\sqrt{t}}\right). \end{aligned} \tag{3.8}$$

Similarly,

$$\int_{B(\xi_y,\sqrt{t}/4)} p^1_H(t/3, y, u)du \ge c_6\left(1 \wedge \frac{\delta_H(y) + \delta_H(y)^{\alpha/2}}{\sqrt{t}}\right). \tag{3.9}$$

Note that by the semigroup property and Lemma 3.5,

$$
\begin{aligned}
& p^1_H(t,x,y) \\
& \geq \int_{B(\xi_y,\sqrt{t}/4)} \int_{B(\xi_x,\sqrt{t}/4)} p^1_H(t/3,x,u)\, p^1_H(t/3,u,v)\, p^1_H(t/3,v,y)\,dudv \\
& \geq \int_{B(\xi_y,\sqrt{t}/4)} \int_{B(\xi_x,\sqrt{t}/4)} p^1_H(t/3,x,u)\, p^1_{B(u,\sqrt{t}/2)\cup B(v,\sqrt{t}/2)}(t/3,u,v)\, p^1_H(t/3,v,y)\,dudv \\
& \geq c_7 t \int_{B(\xi_y,\sqrt{t}/4)} \int_{B(\xi_x,\sqrt{t}/4)} p^1_H(t/3,x,u)\, j^1\left(\frac{3}{2}|u-v|\right) p^1_H(t/3,v,y)\,dudv \\
& \geq c_7 t \left(\inf_{(u,v)\in B(\xi_x,\sqrt{t}/4)\times B(\xi_y,\sqrt{t}/4)} j^1\left(\frac{3}{2}|u-v|\right)\right) \\
& \quad \times \int_{B(\xi_y,\sqrt{t}/4)} \int_{B(\xi_x,\sqrt{t}/4)} p^1_H(t/3,x,u)\, p^1_H(t/3,v,y)\,dudv \\
& \geq c_8 t \left(\inf_{(u,v)\in B(\xi_x,\sqrt{t}/4)\times B(\xi_y,\sqrt{t}/4)} j^1\left(\frac{3}{2}|u-v|\right)\right) \\
& \quad \times \left(\frac{\delta_H(x)+\delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_H(y)+\delta_H(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right). \qquad (3.10)
\end{aligned}
$$

Since $|x-y| \geq t \geq 1$, for $u \in B(\xi_x, \sqrt{t}/4)$ and $v \in B(\xi_y, \sqrt{t}/4)$,

$$
|u-v| \leq \sqrt{t}/2 + |x-y| \leq \frac{3}{2}|x-y|.
$$

Hence

$$
\inf_{(u,v)\in B(\xi_x,\sqrt{t}/4)\times B(\xi_y,\sqrt{t}/4)} j^1(3|u-v|/2) \geq \ j^1(9|x-y|/4) \geq c_9 e^{-c_{10}|x-y|}. \qquad (3.11)
$$

By Eq. 3.10, we conclude that for $|x-y| \geq t \geq \sqrt{t} \geq 1$

$$
\begin{aligned}
p^1_H(t,x,y) &\geq c_{11} t \left(\frac{\delta_H(x)+\delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_H(y)+\delta_H(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) e^{-c_{10}|x-y|} \\
&\geq c_{11} t^{-d/2} \left(\frac{\delta_H(x)+\delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_H(y)+\delta_H(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) e^{-c_{10}|x-y|}.
\end{aligned}
$$

This completes the proof of the theorem. □

4 Upper Bound Heat Kernel Estimate on Half-space

In this section we will prove the upper bound on $p^1_H(t,x,y)$.

By [21, Theorem 4.4] and its proof, we know that, for every $t \geq t_0$, there exists $c(d,\alpha,t_0) > 1$ such that

$$
c^{-1}\left(\frac{\delta_H(x)^{\alpha/2} \vee \delta_H(x)}{\sqrt{t}} \wedge 1\right) \leq \mathbb{P}_x\big(\tau^1_H > t\big) \leq c\left(\frac{\delta_H(x)^{\alpha/2} \vee \delta_H(x)}{\sqrt{t}} \wedge 1\right). \qquad (4.1)
$$

Only the upper bound in Eq. 4.1 will be used later in this paper.

Lemma 4.1 *For every $t_0 > 0$, there exists $C_{18} = C_{18}(d, \alpha, t_0) > 1$ such that for every $(t, x, y) \in [t_0, \infty) \times H \times H$,*

$$p^1_H(t, x, y) \le C_{18} t^{-d/2} \left(\frac{\delta_H(x)^{\alpha/2} \vee \delta_H(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_H(y)^{\alpha/2} \vee \delta_H(y)}{\sqrt{t}} \wedge 1 \right).$$

Proof By the semigroup property and symmetry,

$$\begin{aligned} p^1_H(t, x, y) &= \int_H \int_H p^1_H(t/3, x, z) p^1_H(t/3, z, w) p^1_H(t/3, w, y) dz dw \\ &\le \left(\sup_{z,w} p^1(t/3, z, w) \right) \int_H \int_H p^1_H(t/3, x, z) p^1_H(t/3, y, w) dz dw \\ &= \left(\sup_{z,w} p^1(t/3, z, w) \right) \mathbb{P}_x\big(\tau^1_H > t/3\big) \mathbb{P}_y\big(\tau^1_H > t/3\big). \end{aligned}$$

Now the theorem follows from Theorem 2.1 and Eq. 4.1. □

The next lemma is similar to [2, Lemma 2] and [12, Lemma 3.4] (see also [7, Lemma 2.2]). We provide the proof here for the sake of completeness.

Lemma 4.2 *Suppose that E_1, E_3, E are open subsets of $\mathbb{R}^d$ with $E_1, E_3 \subset E$ and $\text{dist}(E_1, E_3) > 0$. Let $E_2 := E \setminus (E_1 \cup E_3)$. If $x \in E_1$ and $y \in E_3$, then for all $m > 0$ and $t > 0$,*

$$p^m_E(t, x, y) \le \mathbb{P}_x(X^m_{\tau^m_{E_1}} \in E_2) \left(\sup_{s<t,\, z \in E_2} p^m_E(s, z, y) \right) + \mathbb{E}_x[\tau^m_{E_1}] \left(\sup_{u \in E_1,\, z \in E_3} J^m(u, z) \right). \tag{4.2}$$

Proof Using the strong Markov property, we have

$$\begin{aligned} p^m_E(t, x, y) &= \mathbb{E}_x \left[p^m_E\big(t - \tau^m_{E_1}, X^m_{\tau^m_{E_1}}, y\big) : \tau^m_{E_1} < t \right] \\ &= \mathbb{E}_x \left[p^m_E\big(t - \tau^m_{E_1}, X^m_{\tau^m_{E_1}}, y\big) : \tau^m_{E_1} < t, X^m_{\tau^m_{E_1}} \in E_2 \right] \\ &\quad + \mathbb{E}_x \left[p^m_E\big(t - \tau^m_{E_1}, X^m_{\tau^m_{E_1}}, y\big) : \tau^m_{E_1} < t, X^m_{\tau^m_{E_1}} \in E_3 \right] \\ &=: I + II. \end{aligned}$$

Clearly

$$I \le \mathbb{P}_x \left(\tau^m_{E_1} < t, X^m_{\tau^m_{E_1}} \in E_2 \right) \left(\sup_{s<t,\, z \in E_2} p^m_E(s, z, y) \right) \le \mathbb{P}_x \left(X^m_{\tau^m_{E_1}} \in E_2 \right) \left(\sup_{s<t,\, z \in E_2} p^m_E(s, z, y) \right).$$

On the other hand, by Eq. 2.3,

$$\begin{aligned} II &= \int_0^t \left(\int_{E_1} p^m_{E_1}(s,x,u) \left(\int_{E_3} J^m(u,z) p^m_E(t-s,z,y)dz \right) du \right) ds \\ &\le \left(\sup_{u\in E_1,\, z\in E_3} J^m(u,z) \right) \int_0^t \mathbb{P}_x\big(\tau^m_{E_1} > s\big) \left(\int_{E_3} p^m_E(t-s,z,y)dz \right) ds \\ &\le \int_0^t \mathbb{P}_x\big(\tau^m_{E_1} > s\big)ds \left(\sup_{u\in E_1,\, z\in E_3} J^m(u,z) \right) \le \mathbb{E}_x[\tau^m_{E_1}] \left(\sup_{u\in E_1,\, z\in E_3} J^m(u,z) \right). \end{aligned}$$

This completes the proof of the lemma. □

By [21, Theorem 4.2], there exists $c(d,\alpha) > 1$ such that for every $R > 0$

$$\begin{aligned} &c^{-1}\big(R \vee R^{\alpha/2}\big)\left((R-|x|)^{\alpha/2} \vee (R-|x|)\right) \\ &\quad \le \mathbb{E}_x\big[\tau^1_{B(0,R)}\big] \le c\big(R \vee R^{\alpha/2}\big)\left((R-|x|)^{\alpha/2} \vee (R-|x|)\right). \end{aligned} \tag{4.3}$$

We will only use the upper bound in Eq. 4.3 later in this paper.

Lemma 4.3 *There exist positive constants C_{19} and C_{20} such that for every $(t,x,y) \in [1,\infty) \times H \times H$,*

$$p^1_H(t,x,y) \le C_{19} t^{-d/2} \left(\frac{\delta_H(x) \vee \delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) e^{-C_{20}\left(|x-y| \wedge \frac{|x-y|^2}{t}\right)}.$$

Proof By Theorem 2.1, there exist positive constants c_1 and c_2 such that

$$p^1(t,x,y) \le c_1 \begin{cases} t^{-d/\alpha} \wedge tJ^1(x,y) & \text{for } t \in (0,1]; \\ t^{-d/2} \exp\left(-c_2\left(|x-y| \wedge \frac{|x-y|^2}{t}\right)\right) & \text{for } t > 1. \end{cases} \tag{4.4}$$

Let $c_3 := (2d/c_2)^{1/2}$. By Eq. 4.4 and Lemma 4.1, it suffice to prove the lemma for $\delta_H(x) \le c_3\sqrt{t}/9$ and $|x-y| \ge c_3\sqrt{t}$. Thus we fix $(t,x,y) \in [1,\infty) \times H \times H$ with $\delta_H(x) \le c_3\sqrt{t}/9$ and $|x-y| \ge c_3\sqrt{t}$. Let $x_0 = (\widetilde{x}, 0)$, $E_1 := B(x_0, c_3\sqrt{t}/4) \cap H$, $E_3 := \{z \in H : |z-x| > |x-y|/2\}$ and $E_2 := H \setminus (E_1 \cup E_3)$.

Write $X^1 = (X^{1,1}, \ldots, X^{1,d})$. For any open interval (β,γ) in $\mathbb{R}^1$, let $\widehat{\tau}_{(\beta,\gamma)} := \inf\{t > 0 : X^{1,d} \notin (\beta,\gamma)\}$. Note that $X^{1,d}$ is a 1-dimensional relativistic α-stable process. So by the one-dimensional version of Eq. 4.3 and the assumption that $t \ge 1$ and $\delta_H(x) = x_d \le c_3\sqrt{t}/9$, we have

$$\mathbb{E}_x\big[\tau^1_{E_1}\big] \le \mathbb{E}_{x_d}\big[\widehat{\tau}^1_{(0,c_3\sqrt{t}/4)}\big] \le c_4\sqrt{t}\left(x_d \vee x_d^{\alpha/2}\right) = c_4\sqrt{t}\left(\delta_H(x) \vee \delta_H(x)^{\alpha/2}\right). \tag{4.5}$$

Since

$$|z-x| > \frac{|x-y|}{2} \ge \frac{c_3}{2}\sqrt{t} \qquad \text{for } z \in E_3,$$

$E_1 \cap E_3 = \emptyset$ and, for $u \in E_1$ and $z \in E_3$

$$|u-z| \ge |z-x| - |x-u| \ge |z-x| - \frac{c_3\sqrt{t}}{4} \ge \frac{1}{2}|z-x| \ge \frac{1}{4}|x-y|. \tag{4.6}$$

Thus, using the assumption $|x-y| \geq c_3\sqrt{t} \geq c_3$, we have

$$\sup_{u\in E_1,\, z\in E_3} J^1(u,z) \leq \sup_{(u,z):|u-z|\geq \frac{1}{4}|x-y|} J^1(u,z) \leq j^1(|x-y|/4) \leq c_5 e^{-c_6|x-y|}. \quad (4.7)$$

On the other hand, if $z \in E_2$,

$$\frac{3}{2}|x-y| \geq |x-y| + |x-z| \geq |z-y| \geq |x-y| - |x-z| \geq \frac{|x-y|}{2}. \quad (4.8)$$

Thus, by Eqs. 4.4, 4.8 and the assumption $|x-y| \geq c_3\sqrt{t} \geq c_3$,

$$\sup_{s\leq 1,\, z\in E_2} p^1(s,z,y) \leq c_1 \sup_{s\leq 1,\, z\in E_2} \left(s^{-d/\alpha} \wedge sJ^1(z,y)\right) \leq c_1 \sup_{|z-y|\geq|x-y|/2} J^1(z,y) \leq c_7 e^{-c_8|x-y|}$$

and, by Eqs. 4.4 and 4.8,

$$\begin{aligned}\sup_{1<s<t,\, z\in E_2} p^1(s,z,y) &\leq c_1 \sup_{1<s<t,\, z\in E_2} s^{-d/2}\exp\left(-c_2\left(|z-y| \wedge \frac{|z-y|^2}{s}\right)\right)\\ &\leq c_1\left(\exp\left(-c_2|x-y|/2\right) + \sup_{1<s<t} s^{-d/2}\exp\left(-c_2\frac{|x-y|^2}{4s}\right)\right).\end{aligned} \quad (4.9)$$

Observe that the function $s \to s^{-d/2}e^{-\beta/s}$ is increasing for $s \in (0, 2\beta/d]$. Since $c_2|x-y|^2/(2d) = |x-y|^2/c_3^2 \geq t$, we have

$$\sup_{1<s<t} s^{-d/2}\exp\left(-c_2\frac{|x-y|^2}{4s}\right) \leq t^{-d/2}\exp\left(-c_2\frac{|x-y|^2}{4t}\right).$$

Thus we have

$$\begin{aligned}\sup_{0<s\leq t,\, z\in E_2} p^1(s,z,y) &\leq c_1 e^{-c_2|x-y|/2} + c_1 t^{-d/2} e^{-c_2\frac{|x-y|^2}{4t}}\\ &\leq c_9 t^{-d/2} e^{-c_{10}|x-y|} + c_1 t^{-d/2} e^{-c_2\frac{|x-y|^2}{4t}}\\ &\leq c_{11} t^{-d/2} e^{-c_{12}\left(|x-y| \wedge \frac{|x-y|^2}{t}\right)},\end{aligned} \quad (4.10)$$

where in the second to the last inequality, we used the assumption that $|x-y| \geq c_3\sqrt{t} \geq c_3$.

On the other hand, applying Theorem 2.5 with $U = E_1$ and $r = 4^{-1}c_3\sqrt{t} \geq 4^{-1}c_3$ yields

$$\mathbb{P}_x\left(X^1_{\tau^1_{E_1}} \in E_2\right) \leq \mathbb{P}_x\left(X^1_{\tau^1_{E_1}} \in B\left(x_0, 4^{-1}c_3\sqrt{t}\right)^c\right) \leq \frac{c_{13}}{t}\int_{E_1} G^1_{E_1}(x,y)dy = \frac{c_{13}}{t}\mathbb{E}_x\left[\tau^1_{E_1}\right].$$

Thus we have by Lemma 4.2, Eqs. 4.5, 4.7 and 4.10 that

$$
\begin{aligned}
p_H^1(t,x,y) &\le c_5 \mathbb{E}_x\left[\tau_{E_1}^1\right] e^{-c_6|x-y|} + c_{11}\mathbb{P}_x\left(X^1_{\tau^1_{E_1}} \in E_2\right) t^{-d/2}\, e^{-c_{12}\left(|x-y|\wedge \frac{|x-y|^2}{t}\right)} \\
&\le c_{14}\frac{\delta_H(x)\vee\delta_H(x)^{\alpha/2}}{\sqrt{t}}\left(te^{-c_6|x-y|} + t^{-d/2}e^{-c_{12}\left(|x-y|\wedge\frac{|x-y|^2}{t}\right)}\right) \\
&\le c_{15}\frac{\delta_H(x)\vee\delta_H(x)^{\alpha/2}}{\sqrt{t}}\, t^{-d/2}\left(e^{-c_{16}|x-y|} + e^{-c_{12}\left(|x-y|\wedge\frac{|x-y|^2}{t}\right)}\right) \\
&\le c_{17}\frac{\delta_H(x)\vee\delta_H(x)^{\alpha/2}}{\sqrt{t}}\, t^{-d/2}\exp\left(-c_{18}\left(|x-y|\wedge\frac{|x-y|^2}{t}\right)\right),
\end{aligned}
$$

where in the second to the last inequality we again used the assumption that $|x-y| \ge c_3\sqrt{t}$ and $t \ge 1$. This together Eq. 4.4 yields the desired upper bound estimate of $p_H^1(t,x,y)$. □

Theorem 4.4 *There are positive constants $C_i = C_i(\alpha)$, $i = 21, 22$, such that for all $(t,x,y) \in [1,\infty)\times H\times H$,*

$$
p_H^1(t,x,y) \le C_{21}t^{-d/2}\left(\frac{\delta_H(x)\vee\delta_H(x)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta_H(y)\vee\delta_H(y)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)e^{-C_{22}\left(|x-y|\wedge\frac{|x-y|^2}{t}\right)}.
$$

Proof In view of Lemma 4.3 and Theorem 2.1, it suffices to prove the first inequality in the theorem for $\delta_H(x)\vee\delta_H(y) \le \sqrt{t}$ and $t \ge 1$. By the semigroup property and Lemma 4.3,

$$
\begin{aligned}
p_H^1(t,x,y) &= \int_H p_H^1(t/2,x,z)p_H^1(t/2,z,y)dz \\
&\le c_1\left(\frac{\delta_H(x)\vee\delta_H(x)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta_H(y)\vee\delta_H(y)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)h(t,x,y),
\end{aligned}
$$

where

$$
h(t,x,y) = \int_H t^{-d/2}e^{-c_2\left(|x-z|\wedge\frac{|x-z|^2}{t/2}\right)}t^{-d/2}e^{-c_2\left(|z-y|\wedge\frac{|z-y|^2}{t/2}\right)}dz
$$

for some constant $c_2 \in (0,1]$. We know from Theorem 2.1 that there are constants $c_3 > 0$ and $c_4 > 1$ such that

$$
p^1(t,x,y) \ge c_3\begin{cases} t^{-d/\alpha}\wedge te^{-c_4|x-y|} & \text{for } t\in(0,1]; \\ t^{-d/2}\, e^{-c_4\left(|x-y|\wedge\frac{|x-y|^2}{t}\right)} & \text{for } t \ge 1. \end{cases} \tag{4.11}
$$

Let $c_5 := c_2/c_4$ and $x_0 := c_5 x$ and $y_0 := c_5 y$. Use the change of variable $w = c_5 z$ and Eq. 4.11, we have for $t \geq 1$,

$$\begin{aligned} h(t,x,y) &= \int_H t^{-d/2} e^{-c_4\left(|c_5 x - c_5 z| \wedge \frac{|c_5 x - c_5 z|^2}{c_5 t/2}\right)} t^{-d/2} e^{-c_4\left(|c_5 z - c_5 y| \wedge \frac{|c_5 z - c_5 y|^2}{c_5 t/2}\right)} dz \\ &= c_6 \int_H (c_5 t)^{-d/2} e^{-c_4\left(|x_0 - w| \wedge \frac{|x_0 - w|^2}{c_5 t/2}\right)} (c_5 t)^{-d/2} e^{-c_4\left(|w - y_0| \wedge \frac{|w - y_0|^2}{c_5 t/2}\right)} dw \\ &\leq c_7 \int_H p^1(c_5 t/2, x_0, w) p^1(c_5 t/2, w, y_0) dw \\ &= c_8\, p^1(c_5 t, x_0, y_0) \;\leq\; c_9\, t^{-d/2}\, e^{-c_{10}\left(|x-y| \wedge \frac{|x-y|^2}{t}\right)}, \end{aligned}$$

where in the last inequality we used the upper bound estimate of Theorem 2.1 with $m = 1$ and $T = c_5$. The proof of the theorem is now complete. □

5 Proof of the Main Result

Combining Theorem 1.1(i), Theorems 3.6 and 4.4 with Eq. 2.4, we get

Theorem 5.1 *There exist constants C_i, $i = 23, 24$, depending only on d and α such that for all $m > 0$ and $(t, x, y) \in (0, \infty) \times H \times H$,*

$$p_H^m(t,x,y) \leq C_{23} \begin{cases} \left(\frac{\delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta_H(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|/16)}{|x-y|^{d+\alpha}}\right) & \text{for } t \in (0, 1/m]; \\ m^{d/\alpha - d/2} \left(\frac{m^{(2-\alpha)/2\alpha}\delta_H(x) + \delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) & \\ \times \left(\frac{m^{(2-\alpha)/2\alpha}\delta_H(y) + \delta_H(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) & \\ \times t^{-d/2} \exp\left(-C_{24}^{-1}\left(m^{1/\alpha}|x-y| \wedge m^{2/\alpha - 1}\frac{|x-y|^2}{t}\right)\right) & \text{for } t > 1/m, \end{cases}$$

and

$$p_H^m(t,x,y) \geq C_{23}^{-1} \begin{cases} \left(\frac{\delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta_H(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right) & \text{for } t \in (0, 1/m]; \\ m^{d/\alpha - d/2} \left(\frac{m^{(2-\alpha)/2\alpha}\delta_H(x) + \delta_H(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) & \\ \times \left(\frac{m^{(2-\alpha)/2\alpha}\delta_H(y) + \delta_H(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) & \\ \times t^{-d/2} \exp\left(-C_{24}\left(m^{1/\alpha}|x-y| \wedge m^{2/\alpha - 1}\frac{|x-y|^2}{t}\right)\right) & \text{for } t > 1/m. \end{cases}$$

Now we are in a position to prove the main result of this paper.

Proof of Theorem 1.2 Without loss of generality, we may assume that $H_a \subset D \subset H$ for some constant $a > 0$. Clearly

$$p^m_{H_a}(t,x,y) \le p^m_D(t,x,y) \le p^m_H(t,x,y), \quad m>0, (t,x,y) \in (0,\infty)\times H_a \times H_a. \tag{5.1}$$

Define t_0 as in Eq. 2.8. In view of Theorem 1.1(i) we only need to prove the theorem for $t > 3t_0$. Suppose $t > 3t_0$. For any $x, y \in D$, we define x_0 and y_0 as in Eq. 2.9. By the semigroup property and Theorem 1.1(i),

$$\begin{aligned} p^m_D(t,x,y) &= \int_D\int_D p^m_D(t_0,x,z)p^m_D(t-2t_0,z,w)p^m_D(t_0,w,y)dzdw \\ &\ge c_1\,(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2} f_1(t,x,y), \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} f_1(t,x,y) = &\int_{D\times D}(1\wedge\delta_D(z))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|)}{|x-z|^{d+\alpha}}\right)p^m_D(t-2t_0,z,w)(1\wedge\delta_D(w))^{\alpha/2} \\ &\times\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|)}{|w-y|^{d+\alpha}}\right)dzdw. \end{aligned}$$

Define $A := 16 \vee C_{24}$. Note that by Theorem 5.1, for $z, w \in D$ and $t \ge 3t_0$,

$$p^m_D(t-2t_0,z,w) \ge p^m_{H_{3a/2}}(t-2t_0,z,w) \ge c_2 p^m_{H_{3a/2}}(t-2t_0,A^2z,A^2w).$$

Since $m \le M$, the upper bound estimate in Theorem 1.1(i) and the above display together with Eq. 2.5 imply that

$$\begin{aligned} &f_1(t,x,y) \\ &\ge c_3\int_{D\times D}(1\wedge\delta_D(z))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}A^2|x_0-z|)}{|x_0-z|^{d+\alpha}}\right) \\ &\quad\times p^m_{H_{3a/2}}(t-2t_0,A^2z,A^2w)(1\wedge\delta_D(w))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}A^2|w-y_0|)}{|w-y_0|^{d+\alpha}}\right)dzdw \\ &\ge c_3\int_{H_{3a/2}\times H_{3a/2}}\left(1\wedge\frac{\phi(m^{1/\alpha}A^2|x_0-z|)}{|x_0-z|^{d+\alpha}}\right) \\ &\quad\times p^m_{H_{3a/2}}(t-2t_0,A^2z,A^2w)\left(1\wedge\frac{\phi(m^{1/\alpha}A^2|w-y_0|)}{|w-y_0|^{d+\alpha}}\right)dzdw \\ &\ge c_4\int_{H_{3a/2}\times H_{3a/2}}(1\wedge\delta_{H_{3a/2}}(A^2z))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|A^2x_0-A^2z|)}{|A^2x_0-A^2z|^{d+\alpha}}\right) \\ &\quad\times p^m_{H_{3a/2}}(t-2t_0,A^2z,A^2w)(1\wedge\delta_{H_{3a/2}}(A^2w))^{\alpha/2} \\ &\quad\times\left(1\wedge\frac{\phi(m^{1/\alpha}|A^2w-A^2y_0|)}{|A^2w-A^2y_0|^{d+\alpha}}\right)dzdw. \end{aligned}$$

Thus by the change of variables $\widehat{z} = A^2 z$, $\widehat{w} = A^2 w$, and Theorem 1.1(i), we have that, with $\beta := 3aA$,

$$\begin{aligned} &f_1(t, x, y) \\ &\geq c_5 \int_{H_\beta \times H_\beta} (1 \wedge \delta_{H_\beta}(\widehat{z}))^{\alpha/2} (1 \wedge \delta_{H_\beta}(A^2 x_0))^{\alpha/2} \left(1 \wedge \frac{\phi(m^{1/\alpha}|A^2 x_0 - \widehat{z}|)}{|A^2 x_0 - \widehat{z}|^{d+\alpha}}\right) \\ &\quad \times p^m_{H_\beta}(t - 2t_0, \widehat{z}, \widehat{w})(1 \wedge \delta_{H_\beta}(A^2 y_0))^{\alpha/2} (1 \wedge \delta_{H_\beta}(\widehat{w}))^{\alpha/2} \\ &\quad \times \left(1 \wedge \frac{\phi(m^{1/\alpha}|\widehat{w} - A^2 y_0|)}{|\widehat{w} - A^2 y_0|^{d+\alpha}}\right) d\widehat{z} d\widehat{w} \\ &\geq c_6 \int_{H_\beta \times H_\beta} p^m_{H_\beta}\big(t_0, A^2 x_0, \widehat{z}\big) p^m_{H_\beta}\big(t - 2t_0, \widehat{z}, \widehat{w}\big) p^m_{H_\beta}\big(t_0, \widehat{w}, A^2 y_0\big) dz dw \\ &= c_6\, p^m_{H_\beta}(t, A^2 x_0, A^2 y_0). \end{aligned} \tag{5.3}$$

The desired lower bound estimate on $p^m_D(t, x, y)$ now follows from Eqs. 5.2, 5.3, the lower bound estimate in Theorem 5.1 for $p^m_{H_{3a/2}}(t, x_0, y_0)$, Lemmas 2.3 and 2.4.

On the other hand, by the semigroup property and Theorem 1.1(i), we have for $t \geq 3t_0$ and $x, y \in D$,

$$\begin{aligned} p^m_D(t, x, y) &= \int_D \int_D p^m_D(t_0, x, z) p^m_D(t - 2t_0, z, w) p^m_D(t_0, w, y) dz dw \\ &\leq c_7\, (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} f_2(t, x, y), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} f_2(t, x, y) = &\int_{D \times D} (1 \wedge \delta_D(z))^{\alpha/2} \left(1 \wedge \frac{\phi(m^{1/\alpha}|x - z|/16)}{|x - z|^{d+\alpha}}\right) p^m_D(t - 2t_0, z, w)(1 \wedge \delta_D(w))^{\alpha/2} \\ &\times \left(1 \wedge \frac{\phi(m^{1/\alpha}|w - y|/16)}{|w - y|^{d+\alpha}}\right) dz dw. \end{aligned}$$

Recall $A = 16 \vee C_{24}$. Note that by Theorem 5.1, for $z, w \in D$ and $t \geq 3t_0$,

$$p^m_D(t - 2t_0, z, w) \leq p^m_H(t - 2t_0, z, w) \leq c_8\, p^m_H(t - 2t_0, A^{-2} z, A^{-2} w).$$

Let $b := a/(2A)$. The above display together with Eq. 2.5 implies that

$$\begin{aligned} &f_2(t, x, y) \\ &\leq c_9 \int_{D \times D} (1 \wedge \delta_D(z))^{\alpha/2} \left(1 \wedge \frac{\phi(m^{1/\alpha}|x_0 - z|/A^2)}{|x_0 - z|^{d+\alpha}}\right) \\ &\quad \times p^m_H(t - 2t_0, A^{-2} z, A^{-2} w)(1 \wedge \delta_D(w))^{\alpha/2} \left(1 \wedge \frac{\phi(m^{1/\alpha}|w - y_0|/A^2)}{|w - y_0|^{d+\alpha}}\right) dz dw \\ &\leq c_{10} \int_{H_{-a/2} \times H_{-a/2}} (1 \wedge \delta_H(A^{-2} z))^{\alpha/2} \left(1 \wedge \frac{\phi(m^{1/\alpha}|A^{-2} x_0 - A^{-2} z|)}{|A^{-2} x_0 - A^{-2} z|^{d+\alpha}}\right) \\ &\quad \times p^m_H(t - 2t_0, A^{-2} z, A^{-2} w)(1 \wedge \delta_H(A^{-2} w))^{\alpha/2} \left(1 \wedge \frac{\phi(m^{1/\alpha}|A^{-2} w - A^{-2} y_0|)}{|A^{-2} w - A^{-2} y_0|^{d+\alpha}}\right) dz dw. \end{aligned}$$

Thus by the change of variables $\widehat{z} = A^{-2}z$, $\widehat{w} = A^{-2}w$, and Theorem 1.1(i),

$$
\begin{aligned}
&f_2(t,x,y)\\
&\le c_{11}\int_{H_{-b}\times H_{-b}} (1\wedge\delta_{H_{-b}}(\widehat{z}))^{\alpha/2}(1\wedge\delta_{H_{-b}}(A^{-2}x_0))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|A^{-2}x_0-\widehat{z}|)}{|A^{-2}x_0-\widehat{z}|^{d+\alpha}}\right)\\
&\quad\times p^m_{H_{-b}}(t-2t_0,\widehat{z},\widehat{w})(1\wedge\delta_{H_{-b}}(A^{-2}y_0))^{\alpha/2}(1\wedge\delta_{H_{-b}}(\widehat{w}))^{\alpha/2}\\
&\quad\times\left(1\wedge\frac{\phi(m^{1/\alpha}|\widehat{w}-A^{-2}y_0|)}{|\widehat{w}-A^{-2}y_0|^{d+\alpha}}\right)d\widehat{z}d\widehat{w}\\
&\le c_9\int_{H_{-b}\times H_{-b}} p^m_{H_{-b}}(t_0,A^{-2}x_0,\widehat{z})p^m_{H_{-b}}(t-2t_0,\widehat{z},\widehat{w})p^m_{H_{-b}}(t_0,\widehat{w},A^{-2}y_0)dzdw\\
&= c_{12}\,p^m_{H_{-b}}(t,A^{-2}x_0,A^{-2}y_0).
\end{aligned}
$$

The desired upper bound estimate for $p^m_D(t,x,y)$ now follows from Eq. 5.4, the upper bound estimate in Theorem 5.1 for $p^m_{H_{-b}}(t,x_0/A^2,y_0/A^2)$, Lemmas 2.3 and 2.4. □

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