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# Uniqueness of Invariant Measures of Infinite Dimensional Stochastic Differential Equations Driven by Lévy Noises

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**Abstract** In this paper, we are attempting to study the uniqueness of invariant measures of a stochastic differential equation driven by a Lévy type noise in a real separable Hilbert space. To investigate this problem, we study the strong Feller property and irreducibility of the corresponding Markov transition semigroup respectively. To show the strong Feller property, we generalize a Bismut–Elworthy–Li type formula to our Markov transition semigroup under a non-degeneracy condition of the coefficient of the Wiener process.

**Keywords** Invariant measure · Lévy noise · Bismut–Elworthy–Li formula · Strong Feller · Irreducibility · Uniqueness

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## 1 Introduction

In this paper, we will mainly study the uniqueness of invariant measures of the Markov process $X(t)$ determined by the following infinite dimensional stochastic

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differential equation driven by a Lévy type noise with an initial datum $\xi$ in a real separable Hilbert space $H$:

$$dX(t) = \big(AX(t) + F(X(t))\big)dt + B(X(t))dW(t) + \int_H G(X(t), u)q(dt, du). \quad (1.1)$$

Here $A$ is an unbounded linear operator, the coefficients $F$, $B$ and $G$ satisfy some conditions which will be specified later, $W(t)$ denotes a cylindrical Wiener process on $H$ and $q$ denotes a compensated Poisson random measure corresponding to a Poisson random measure $N$ with intensity measure $dtd\nu$ on the product space $[0, \infty) \times H$. Such stochastic differential equations have attracted many authors' attention recently and have been actively applied in statistical mechanics, control engineering, financial market, electrical and physical sciences and so on, for instance, see [3, 4, 9, 10, 14, 28, 30] and references therein.

Infinite dimensional stochastic differential equations with respect to Lévy noises were initially investigated by A. Chojnowski-Michalik in 1987 [7], who studied Ornstein–Uhlenbeck processes corresponding to additive Lévy noises and their invariant measures. Then, after a period of 22 years, a large number of papers about infinite dimensional stochastic differential equations driven by Lévy noises (1.1), especially, the existence and uniqueness of solutions appeared. For recently deep studies of Ornstein–Uhlenbeck processes with jumps, we refer readers to the works of D. Applebaum [6] and M. Röckner and his collaborators [13, 23]. For general case, please see the research papers [1, 18, 20, 25] for that driven by compensated Poisson random measures and the monograph [28] for general Lévy noises by semigroup approaches.

It is well known that the ergodicity is very important for the study of stochastic dynamic systems and such problem relative to the solution of Eq. 1.1 with $G = 0$ has been studied by many authors, see research papers [8, 17, 27], the monograph [10] and references therein for details. For the Ornstein–Uhlenbeck process corresponding to an additive Lévy noise, the existence and uniqueness of invariant measures has been studied by several papers, for example, [6, 7, 13]. In these papers, the invariant measures are proved to be a strict subclass (*they are called operator self-decomposable measure with respect to* $S(t)$*, see* [5, 6, 22] *and references therein for more information.*) of the family of infinite divisible measures in $H$. In addition, a new method to the construction of a stochastic heat equation driven by a singular noise with the distributions of real Lévy processes as its invariant measures was introduced in [15] by T. Funaki and the author.

We know that the uniqueness of invariant measures is vital for ergodicity; recalling that if an invariant measure is unique, then it is ergodic. However, to the best of our knowledge, besides some works to study the existence of invariant measures (see [16, 24, 28]), there exist few results on the uniqueness of invariant measures of Eq. 1.1 relative to a general Lévy noise. Therefore, in this paper, our main goal is to investigate the uniqueness of the invariant measures for the general system (1.1) based on the following celebrated theorem (see Theorem 4.2.1 [10]): for any stochastically continuous Markov semigroup $\{P_t\}_{t\geq 0}$, if it is strong Feller and irreducible, then there exists at most one invariant measure. We recall that a measure $\mu$ on $H$ is said to be an invariant (probability) measure with respect to $\{P_t\}_{t\geq 0}$ if $\mu(H) = 1$ and $\mu(P_t f) = \mu(f)$ for each bounded and measurable function $f$ on $H$.

Therefore, to study the uniqueness, we will consider the strong Feller property and irreducibility respectively.

The strong Feller property is very important and a lot of methods have been introduced to study that corresponding to stochastic differential equations driven only by Wiener noises, see [26] for introduction. However, as we know, there exist few works on the strong Feller property corresponding to Eq. 1.1 with a general Lévy noise. In this paper, to achieve the strong Feller property, we first generalize a Bismut–Elworthy–Li type formula, which describes the gradient estimate of the Markov transition semigroup and is very effective in applications, see [11, 12] for finite dimensional diffusion processes. G. Da Prato et al. [8] and S. Peszat et al. [27] (see also [10]) restudied such formula relative to infinite dimensional stochastic differential equations driven by additive and multiplicative Wiener processes, respectively. In this paper, we generalize the Bismut–Elworthy–Li type formula to the Markov transition semigroup corresponding to the Markov process $X(t)$ determined by Eq. 1.1 under some regular conditions, see Theorem 3.3 below, which may be of independent interest. It is interesting to see that by our approach the strong Feller property mainly depends on the continuous diffusion coefficient $B$.

Based on the irreducibility of Eq. 1.1 with $G = 0$ (see Theorem 3.1 [10]), we impose some mild restrictions on the coefficient $G$ and the Lévy measure $\nu$ to achieve the irreducibility of the Markov process $X(t)$, see Section 5.

This paper is organized as follows. In the next section, some notations and main results are formulated and then in Section 3, under some more regular restrictions on the coefficients and Lévy measures, a Bismut–Elworthy–Li type formula is formulated. In Section 4, the strong Feller property of the Markov transition semigroup related to Eq. 1.1 is established and then the irreducibility is considered in Section 5. In Section 6, we discuss a stochastic heat equation driven by a Lévy noise to illustrate our main results and then we conclude this paper with appendices.

## 2 Framework and Main Results

Let $(H, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space with the associated norm $|\cdot|$. We will denote by $\mathcal{L}(H)$ the class of all linear bounded operators on $H$ and by $\mathcal{L}_2(H)$ the space of all Hilbert–Schmidt operators. We will endow $\mathcal{L}(H)$ and $\mathcal{L}_2(H)$ with the usual operator norms $\|\cdot\|$ and $\|\cdot\|_{HS}$ respectively. Let $B_b(H)$ be the collection of all bounded measurable functions on $H$ endowed with the classical supermum norm $\|\cdot\|_\infty$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a right continuous increasing family $\{\mathcal{F}_t\}_{t\geq 0}$, which satisfies the usual conditions. Let $\{W(t)\}_{t\geq 0}$ and $\{N(t,\cdot)\}_{t\geq 0}$ be a cylindrical Wiener process on $H$ and a Poisson random measure on the product space $\mathbb{R}_+ \times H$ with a jump intensity measure $dtd\nu$ with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ respectively. Here $\mathbb{R}_+ = [0, \infty)$, $dt$ denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R}_+)$ and $\nu$ is a Lévy measure on the Borel $\sigma$-algebra $\mathcal{B}(H)$, i.e., it is a Borel measure on $H$ with $\nu(\{0\}) = 0$ and $\int_H \min\{1, |u|^2\}\nu(du) < \infty$.

Let us consider the compensated Poisson random measure

$$q(dt, du) := N(dt, du) - dt\nu(du)$$

on $\mathbb{R}_+ \times H$, which is associated with the pure jump part of an $H$-valued Lévy process by the Lévy–Itô decomposition theorem for infinite dimensional Lévy processes, see Theorem 4.2 [2]. Throughout this paper, we will suppose that the initial value $\xi$, the processes $W(t)$ and $N(t, \cdot)$ are mutually independent.

For the definition of the cylindrical Wiener process and the stochastic integral with respect to $W(t)$, we refer the reader to [10] and references therein for details. Here we will briefly recall the stochastic integral with respect to the compensated Poisson random measure $q(t, \cdot)$ and some properties, for details we refer the reader to [1, 25, 28]. Suppose $f : \mathbb{R}_+ \times H \times \Omega \to H$ is a measurable and $\mathcal{F}_t$-adapted progress satisfying

$$\mathbb{E}\left[\int_0^t \int_H |f(s,u)|^2 ds\nu(du)\right] < \infty, \ t > 0.$$

Then we can define the Itô integral

$$M(t) = \int_0^t \int_H f(s,u) q(ds, du)$$

in a usual way. It is known that the process $M(t)$ is an $H$-valued martingale with the Meyer process

$$\langle M\rangle(t) = \int_0^t \int_H |f(s,u)|^2 ds\nu(du),$$

i.e., the stochastic process $|M(t)|^2 - \langle M\rangle(t)$ is a real valued martingale. Moreover, the following Itô isometry is fulfilled:

$$\mathbb{E}\left[\left|\int_0^t \int_H f(s,u) q(ds, du)\right|^2\right] = \int_0^t \int_H \mathbb{E}[|f(s,u)|^2] ds\nu(du). \tag{2.1}$$

Now we turn to consider the equation (1.1). Here we state some conditions, which will be used throughout this paper. We say that the assumption **A** is satisfied if the following holds:

**A0**) The operator $A$ is the infinitesimal generator of a pseudo-contractive $C_0$-semigroup $S(t)$ on $H$, i.e., there exists $\alpha > 0$ such that $\|S(t)\| \le e^{\alpha t}$. Moreover, for each $t > 0$

$$\int_0^t \|S(s)\|_{HS}^2 ds < \infty.$$

**A1**) The coefficients $F : H \to H$, $B : H \to \mathcal{L}(H)$ and $G : H \times H \to H$ are measurable and there exists a constant $C > 0$ such that for any $x, \ y \in H$

$$|F(x)|^2 + \|B(x)\|^2 + \int_H |G(x,u)|^2 \nu(du) \le C(1 + |x|^2),$$

$$|F(x) - F(y)|^2 + \|B(x) - B(y)\|^2 + \int_H |G(x,u) - G(y,u)|^2 \nu(du) \le C|x - y|^2.$$

Before stating the theorem of the existence and uniqueness of the solutions of Eq. 1.1, let us make the mathematical meaning of it precise.

**Definition 2.1** A predictable $H$-valued process $X(t)$ is said to be a mild solution of Eq. 1.1 with initial value $\xi$, if the following stochastic integral equation holds:

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s)$$
$$+ \int_0^t \int_H S(t-s)G(X(s),u)q(ds,du) \ a.s. \tag{2.2}$$

We remark that in this definition we implicitly assume that all terms appeared in Eq. 2.2 are well defined. Although we only require that the solution is predictable, we can show that the mild solution has a càdlàg modification, see Theorem 2.1 below. From now we will assume that $\xi$ is $\mathcal{F}_0$-measurable and $\mathbb{E}[|\xi|^2] < \infty$. Let $T > 0$ be fixed and let $\mathcal{H}_{2,T}$ denote the collection of all predictable processes $X(t)$ satisfying $\mathbb{E}\left[\sup_{t\in[0,T]} |X(t)|^2\right] < \infty$.

Now let us formulate the existence and uniqueness of solutions of the stochastic differential equation (1.1).

**Theorem 2.1** *Under the assumption* **A**, *there exists a unique mild solution $X(t)$ of Eq. 1.1 such that for each fixed $T$, $X(t) \in \mathcal{H}_{2,T}$. Moreover, the process $X(t)$ has a càdlàg modification and is a homogeneous Markov process.*

We will postpone our proof to Appendix C, see Theorem C.1. If we only want to show the existence and uniqueness of the solution $X(t)$ satisfying a weak condition $\sup_{t\in[0,T]} \mathbb{E}\left[|X(t)|^2\right] < \infty$, then we can easily do by fixed point theorems, see [9, 10] for that relative to Wiener noises and [1, 18, 20] for Poisson random measures. However, since in the above theorem we require $X(t) \in \mathcal{H}_{2,T}$, we have to consider maximal inequalities of stochastic convolutions in mean square sense, see Theorems B.1 and B.2. In addition, we point out that, to assure the regularity of the trajectory of the solution, the pseudo-contractive property of the semigroup $S(t)$ in **A0**) is needed because of the Lévy noise. Although, the càdlàg modification of the solution $X(t)$ can be shown, for simplicity of notation, we will write $X(s)$ instead of $X(s-)$ in relative parts in this paper.

In the following, to emphasize its initial value, we will denote by $X(t,x)$ the unique mild solution $X(t)$ of Eq. 1.1 with initial value $x$. Then the corresponding Markov transition semigroup $P_t$ can be defined as below:

$$P_t f(x) = \mathbb{E}[f(X(t,x))], \ x \in H, \ f \in B_b(H).$$

Now let us formulate our main result of this paper. Before doing it, the following assumption will be stated:

**A2**) The mapping $B(x)$ is invertible and $\sup_{x\in H} \|B^{-1}(x)\| < \infty$.

**Theorem 2.2** *If the assumptions* **A** *and* **A2**) *are satisfied, then the following holds:*

(i) *For any $f \in B_b(H)$, there exists $C > 0$ such that for any $x, y \in H$*

$$|P_t f(x) - P_t f(y)| \le \frac{C}{\sqrt{t}}|x-y|.$$

*In particular, the Markov process $X(t,x)$ is strong Feller, i.e., for arbitrary $f \in B_b(H)$ and each $t > 0$, $P_t f$ is a bounded and continuous function on $H$.*

(ii) *If further*

$$\nu(B(\gamma)) = o(\log \gamma), \ \gamma \to \infty, \tag{2.3}$$

*then the irreducibility of the Markov process $X(t,x)$ holds, i.e., for any non-empty open set $A \subset H$,*

$$P_t(x, A) = P_t 1_A(x) > 0, \ t > 0, x \in H,$$

*where $B(\gamma)$ denotes the closure of*

$$\{u \in H : |G(x,u)| > 0, |x| < \gamma\}.$$

This theorem will be proved in Section 4 for the strong Feller property and in Section 5 for the irreducibility respectively.

*Remark 2.1*

(i) If $\nu$ is bounded, then it is easy to see that Eq. 2.3 holds. A simple proof will be given in Remark 5.2 below.
(ii) Let $G_1$ be a real valued global Lipschitz continuous function defined on $\mathbb{R}$ and let $G_2$ be a mapping from $H$ to $H$. Then $G(x,u) := G_1(|x|)G_2(u)$ satisfies the assumption **A1**) under $\int_H |G_2(u)|^2 \nu(du) < \infty$.
(iii) Let us further assume that $|G_1(r)| = 0$ for $|r| \leq 1$ and $|G_1(r)| \neq 0$ for $|r| > 1$ and $\int_H |u||G_2(u)|^2 \nu(du) < \infty$. Then we have $G(x,u) := G_1(|x|\sqrt{|u|})G_2(u)$ satisfies **A1**) and for each $\gamma$, $B(\gamma) = \{u \in H : |u| \geq \frac{1}{\gamma^2}\}$. Therefore, if for small enough positive $\epsilon$, $\nu(\{u \in H : |u| \geq \epsilon\}) \leq C(\log \epsilon)^2$, then Eq. 2.3 holds. Hence, our result holds for some unbounded Lévy measure, which is more interesting in applications.

*Remark 2.2*

(i) Although for simplicity of notations Theorems 2.1 and 2.2 are established for a Poisson random measure $N$ on $[0,\infty) \times H$, the results will still hold if we consider $N$ on $[0,\infty) \times U$ for another real separable Hilbert space $U$ with some obvious modifications.
(ii) According to Khas'minskii's theorem (see Proposition 4.1.1 [10]) and Doob's theorem, we know that the Markov transition semigroup $P_t$ is $t$-regular, i.e., the Markov transition probability measures $P_t(x,\cdot)$, $x \in H$ are mutually equivalent. Moreover, if $\mu$ is an invariant measure for $P_t$, then $\mu$ is ergodic and even strong mixing in the following sense: for each $f$ with $\mu(|f|^2) < \infty$,

$$\lim_{t\to\infty} \mu(|P_t(f) - \mu(f)|^2) = 0.$$

### 3 Bismut–Elworthy–Li Type Formula

In this section, we devote to investigating a Bismut–Elworthy–Li type formula under our framework. We will denote by $F_x$ (or $DF$) and $F_{xx}$ (or $D^2F$) the first and second order derivatives for a suitable mapping $F$ respectively according to the occasion. Let $C_b^k(H)$ be the set of all the functions defined on $H$ which are $k$-times continuously Fréchet differentiable with bounded derivatives until $k$th order. For simplicity, we will denote by $C_b(H)$ instead of $C_b^0(H)$ for the collection of all bounded and continuous functions from $H$ to $\mathbb{R}$.

We will first consider the differential dependence of the solution $X(t,x)$ of Eq. 1.1 on its initial value $x$. Such problem has been studied by J. Zabczyk and his collaborators [10] for $W(t)$ and by S. Albeverio et al. [1] for pure jump Lévy noises. Here we are going to generalize their results to our framework. We will further use the following assumption:

**A3**) $F$ and $B$ are bounded and continuously Fréchet differentiable in $x$ and for each $u \in H$, $G(x,u)$ is continuously Fréchet differentiable in $x$ and

$$\sup_{x\in H}\int_H \|G_x(x,u)\|^2\nu(du) < \infty. \tag{3.1}$$

**Theorem 3.1** *If the assumptions* **A** *and* **A3**)*are satisfied, then the solution $X(t,x)$ of Eq. 1.1 has mean square directional derivative $X_x(t,x)y$ at $x$ in the direction $y \in H$ and $X_x(t,x)y$ is the unique mild solution of the following linear stochastic differential equation with initial value y:*

$$d\eta(t) = \big(A\eta(t) + DF(X(t,x))\eta(t)\big)dt + DB(X(t,x))\eta(t)dW(t) + \int_H DG(X(t,x),u)\eta(t)q(dt,du). \tag{3.2}$$

*In addition, for each fixed $T > 0$, there exists a positive constant C depending only on T, such that*

$$\mathbb{E}[|X_x(t,x)y|^2] \le C|y|^2,\ \ y \in H. \tag{3.3}$$

*Proof*

Step 1 We first show Eq. 3.3. For brevity, we write $X(t)$ for $X(t,x)$ in the following. Since $\eta(t) = X_x(t,x)y$ is the unique mild solution of Eq. 3.2, by our assumptions and the Itô isometric property, we have

$$\begin{aligned}\mathbb{E}[|\eta(t)|^2] \le 4|S(t)y|^2 &+ 4\mathbb{E}\left[\left|\int_0^t S(t-s)DF(X(s))\eta(s)ds\right|^2\right] \\ &+ 4\mathbb{E}\left[\left|\int_0^t S(t-s)DB(X(s))\eta(s)dW(s)\right|^2\right] \\ &+ 4\mathbb{E}\left[\left|\int_0^t\int_H S(t-s)DG(X(s),u)\eta(s)q(ds,du)\right|^2\right]\end{aligned}$$

$$= 4|S(t)y|^2 + 4t\int_0^t \mathbb{E}[|S(t-s)DF(X(s))\eta(s)|^2]ds$$
$$+ 4\int_0^t \mathbb{E}[\|S(t-s)DB(X(s))\eta(s)\|_{HS}^2]ds$$
$$+ 4\int_0^t\int_H \mathbb{E}[|S(t-s)DG(X(s),u)\eta(s)|^2]ds\nu(du). \tag{3.4}$$

Recalling the pseudo-contractivity of the semigroup $S(t)$ and by **A3**), we can easily obtain the following estimates,

$$|S(t)y|^2 \le e^{2\alpha t}|y|^2,$$

$$\int_0^t \mathbb{E}[|S(t-s)DF(X(s))\eta(s)|^2]ds \le C\int_0^t e^{2\alpha(t-s)}\mathbb{E}[|\eta(s)|^2]ds,$$

$$\int_0^t \mathbb{E}[\|S(t-s)DB(X(s))\eta(s)\|_{HS}^2]ds \le C\int_0^t \|S(t-s)\|_{HS}^2\mathbb{E}[|\eta(s)|^2]ds.$$

Moreover, from **A3**), it follows that

$$\int_0^t\int_H \mathbb{E}[|S(t-s)DG(X(s),u)\eta(s)|^2]ds\nu(du)$$
$$\le \mathbb{E}\left[\int_0^t\int_H \|S(t-s)\|^2\|DG(X(s),u)\|^2|\eta(s)|^2ds\nu(du)\right]$$
$$\le \mathbb{E}\left[\int_0^t e^{2\alpha(t-s)}|\eta(s)|^2\int_H \|DG(X(s),u)\|^2\nu(du)ds\right]$$
$$\le C\int_0^t e^{2\alpha(t-s)}\mathbb{E}[|\eta(s)|^2]ds.$$

Therefore, based on the above estimates, Eq. 3.4 implies

$$\sup_{0\le s\le t}\mathbb{E}[|\eta(s)|^2]$$
$$\le 4e^{2\alpha t}|y|^2 + Ce^{2\alpha t}\left(t + \int_0^t \|S(s)\|_{HS}^2ds\right)\sup_{0\le s\le t}\mathbb{E}[|\eta(s)|^2]. \tag{3.5}$$

By virtue of **A0**), there exists a strict positive constant $t_0 \le T$ such that

$$C_0 = Ce^{2\alpha t_0}\left(t_0 + \int_0^{t_0}\|S(s)\|_{HS}^2ds\right) < 1.$$

Then, Eq. 3.5 implies that

$$\sup_{0\le s\le t_0}\mathbb{E}[|\eta(s)|^2] \le \frac{4e^{2\alpha t_0}}{1-C_0}|y|^2.$$

Now by iteration for the general $T > 0$, we can consider the stochastic differential equation (3.2) in intervals $[0, t_0]$, $[t_0, 2t_0]$, $\cdots$, $[n_0t_0, T]$ to obtain the desired estimate (3.3), where $n_0 = \max\{n \in \mathbb{N} : nt_0 \le T\}$.

Step 2 From now we intend to show the differentiability. Let us define the mapping $\mathcal{K}$ from $H \times \mathcal{H}_{2,T}$ to $\mathcal{H}_{2,T}$ by

$$\mathcal{K}(x, X) = S(t)x + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s) + \int_0^t \int_H S(t-s)G(X(s), u)q(ds, du).$$

Without loss of generality, we can assume that $T$ is small enough in this proof and then by the similar approach used in the proof of Theorem 2.1 (see Appendix C), we can easily show that $\mathcal{K}$ is continuous and contractive. For fixed $X \in \mathcal{H}_{2,T}$, by the Fréchet differentiability of $S(t)$, we can easily show that $\mathcal{K}(\cdot, X)$ is continuously differentiable and for any $y \in H$, $\mathcal{K}_x(x, X)y = S(t)y$.

Step 3 We claim that the mapping $\mathbb{R} \to \mathcal{H}_{2,T}$, $h \to \mathcal{K}(x, X + hY)$ is continuously differentiable in $h$ for any $x \in H$, $X, Y \in \mathcal{H}_{2,T}$. In fact, we can deduce that

$$\begin{aligned}\frac{\partial}{\partial h}\mathcal{K}(x, X + hY)\,|_{h=0} = &\int_0^t S(t-s)DF(X)Y(s)dt \\ &+ \int_0^t S(t-s)DB(X)Y(s)dW(s) \\ &+ \int_0^t \int_H S(t-s)DG(X, u)Y(s)q(ds, du).\end{aligned} \tag{3.6}$$

In addition, if we denote by $D_X\mathcal{K}(x, X)Y$ the right hand of Eq. 3.6, then $D_X\mathcal{K}(x, X)$ is a bounded linear operator from $\mathcal{H}_{2,T}$ to $\mathcal{H}_{2,T}$ for fixed $x \in H$, $X \in \mathcal{H}_{2,T}$ and is continuous in $x$ and $X$. In fact, under our assumptions, our claim can be easily shown as in the proofs of Theorems C.1 and C.2, see Appendices. For the convenience of readers, we state the proof of the continuity of $D_X\mathcal{K}(x, X)Y$. Let the sequence $(x_n, X_n)$ converge to $(x, X)$ in $H \times \mathcal{H}_{2,T}$. We can easily show that

$$\begin{aligned}&\|D_X\mathcal{K}(x_n, X_n)Y - D_X\mathcal{K}(x, X)Y\|^2_{\mathcal{H}_{2,T}} \\ &\le C\mathbb{E}\left[\sup_{t\in[0,T]} |DF(X_n)Y - DF(X)Y|^2\right] \\ &+ C\mathbb{E}\left[\sup_{t\in[0,T]} \|DB(X_n)Y - DB(X)Y\|^2\right] \\ &+ C\mathbb{E}\left[\sup_{t\in[0,T]} \int_H |DG(X_n, u)Y - DG(X, u)Y|^2 \nu(du)\right].\end{aligned}$$

Then by taking an almost surely convergence subsequence of $X_n$, we can conclude the proof by the dominated convergence theorem and **A3**). Now we can deduce that the directional derivative $X_x(t, x)y$ of $X(t, x)$ in the direction $y$ exists and satisfies Eq. 3.2 from the above steps and a local inversion theorem in Appendix C [10].

□

Let $A_n = nA(n-A)^{-1}$, the Yosida approximation of $A$ and let $\{e_n\}$ be a complete orthogonal normal system of $H$. Let $\Gamma_n$ denote the orthogonal projector from $H$ to its subspace $span\{e_i, i = 1, \ldots, n\}$. Consider the following stochastic differential equation with initial value $x \in H$:

$$dX_n(t) = (A_n X_n(t) + F(X_n(t)))\,dt + B_n(X_n(t))dW(t) + \int_H G(X_n(t), u)q(dt, du), \tag{3.7}$$

where $B_n = B\Gamma_n$. Then it is easily to know that Eq. 3.7 has a unique $H$-valued strong solution $X_n(t)$, i.e., the following holds:

$$X_n(t) = x + \int_0^t (A_n X_n(s) + F(X_n(s)))ds + \int_0^t B(X_n(s))dW(s) + \int_0^t \int_H G(X_n(s), u)q(ds, du) \ a.s.$$

and the solution $X_n(t)$ is a homogeneous Markov process. We additionally assume the following holds:

**A4)** $F$ and $B$ have bounded and continuous second Fréchet derivatives in $x$ satisfying

$$|B_{xx}(x)(y, z)w| \le C|y||z||w|, \ \ y, z, w \in H.$$

and for each $u \in H$, $G(x, u)$ has continuous Fréchet derivative in $x$ satisfying

$$\sup_{x\in H} \int_H |G_{xx}(x, u)(y, z)|^2 \nu(du) \le C|y|^2|z|^2, \ \ y, z \in H. \tag{3.8}$$

Then we have the following theorem:

**Theorem 3.2** *Assume assumptions in Theorem* 3.1 *are fulfilled. If further* **A4**) *holds, then for each* $f \in C_b^2(H)$*, the following holds:*

(1) *The function*

$$v_n(t, x) = \mathbb{E}[f(X_n(t, x))]$$

*is the unique solution of the following Kolmogorov equation:*

$$\begin{cases} \frac{\partial}{\partial t} v(t, x) = \mathcal{L}_n v(t, x), & t > 0, \\ v(0, x) = f(x), & x \in H, \end{cases} \tag{3.9}$$

*where* $X_n(t, x)$ *denotes the solution of Eq.* 3.7 *and*

$$\mathcal{L}_n v(x) = \langle A_n x + F(x), v_x(x)\rangle + \frac{1}{2} Tr\big(B_n^*(x) v_{xx}(x) B_n(x)\big) + \int_H \big(v(x + G(x, u)) - v(x) - \langle G(x, u), v_x(x)\rangle\big)\nu(du).$$

(2) *Assume that $X(t,x)$ is the unique mild solution of Eq. 1.1. Then, the following holds almost surely:*

$$f(X(t,x)) = v(t,x) + \int_0^t \big\langle v_x(t-s, X(s,x)), B(X(s,x))dW(s)\big\rangle \tag{3.10}$$
$$+ \int_0^t\int_H \big(v(t-s, X(s,x)+G(X(s,x),u)) - v(t-s, X(s,x))\big)q(ds,du),$$

*where $v(t,x) = P_t f(x)$.*

*Proof* The proof will consist of three steps.

Step 1 Since $f \in C_b^2(H)$, we can apply the Itô formula (see Theorem **A.1** in Appendices) to $f(X_n(t))$ and then deduce that

$$\begin{aligned}
f(X_n(t)) = {} & f(x) + \int_0^t \big\langle A_n + F(X_n(s)), f_x(X_n(s))\big\rangle ds \\
& + \int_0^t \big\langle f_x(X_n(s)), B_n(X_n(s))dW(s)\big\rangle \\
& + \frac{1}{2}\int_0^t Tr\big(B_n^*(X_n(s)) f_{xx}(X_n(s)) B_n(X_n(s))\big)ds \\
& + \int_0^t\int_H \big(f(X_n(s) + G(X_n(s),u)) - f(X_n(s)) \\
& - \big\langle G(X_n(s),u), f_x(X_n(s))\big\rangle\big)\nu(du)ds \\
& + \int_0^t\int_H \big(f(X_n(s) + G(X_n(s),u)) - f(X_n(s))\big)q(ds,du) \\
= {} & f(x) + \int_0^t \mathcal{L}_n f(X_n(s))ds + \int_0^t \big\langle f_x(X_n(s)), B_n(X_n(s))dW(s)\big\rangle \\
& + \int_0^t\int_H \big(f(X_n(s) + G(X_n(s),u)) - f(X_n(s))\big)q(ds,du).
\end{aligned} \tag{3.11}$$

By our assumptions, we can know that

$$\int_0^t \big\langle f_x(X_n(s)), B_n(X_n(s))dW(s)\big\rangle$$

and

$$\int_0^t\int_H \big(f(X_n(s) + G(X_n(s),u)) - f(X_n(s))\big)q(ds,du)$$

are real valued martingales and are uncorrelated. Therefore, taking expectation of both sides of Eq. 3.11, we get

$$\mathbb{E}[f(X_n(t))] = f(x) + \mathbb{E}\left[\int_0^t \mathcal{L}_n f(X_n(s))ds\right],$$

which implies that

$$\lim_{t\downarrow 0}\frac{\mathbb{E}[f(X_n(t))]-f(x)}{t}=\mathcal{L}_n\mathbb{E}[f(X_n(t))].$$

Applying Theorem 3.1 two times, we can prove that $X_n(t,x)$ is twice Fréchet differentiable in $x$. Then we know that $\mathbb{E}[f(X_n(t,x))]$ is also twice differentiable with respect to $x$ and for any $y,z\in H$

$$D_x^2\mathbb{E}[f(X_n(t,x))](y,z)=\mathbb{E}\left[f_{xx}(X_n(t,x))(DX_n(t,x)y,DX_n(t,x)z)\right].$$

Noting that $DX_n(0,x)=I$ and $D^2X_n(0,x)=0$, we have

$$\frac{\partial}{\partial t}v_n(t,x)|_{t=0}=\mathcal{L}_n v_n(0,x).$$

Let us now fix $s>0$, by the homogenous Markov property of the solution $X_n(t)$, we have

$$v_n(t+s,x)=v_n(t,v_n(s,x)).$$

Therefore by using the previous argument with $x$ replaced by $v_n(s,x)$, we can show that $v_n(t,x)=\mathbb{E}[f(X_n(t,x))]$ is a solution of Eq. 3.9.

Step 2 To show uniqueness. Assume that $\tilde{v}(t,x)$ is an arbitrary solution of the Kolmogorov equation (3.9). For each fixed $t>0$, applying Itô's formula to $\tilde{v}(t-s,X_n(s)),s\in[0,t]$, we have

$$\begin{aligned}
d\tilde{v}(t-s,X_n(s)) &= \frac{\partial}{\partial s}\tilde{v}(t-s,X_n(s))ds\\
&\quad+\langle\tilde{v}_x(t-s,X_n(s)),A_nX_n(s)+F_n(X_n(s))\rangle ds\\
&\quad+\frac{1}{2}Tr\Big(B_n^*(s)\tilde{v}_{xx}(t-s,X_n(s))B_n(s)\Big)ds\\
&\quad+\langle\tilde{v}_x(t-s,X_n(s)),B_n(X_n(s))dW(s)\rangle\\
&\quad+\int_H\Big(\tilde{v}\big(t-s,X_n(s)+G(X_n(s),u)\big)\\
&\qquad\quad-\tilde{v}(t-s,X_n(s-))\Big)q(ds,du)\\
&\quad+\int_H\Big(\tilde{v}\big(t-s,X_n(s)+G(X_n(s),u)\big)-\tilde{v}(t-s,X_n(s))\\
&\qquad\quad-\langle\tilde{v}_x(t-s,X_n(s)),G(X_n(s),u)\rangle\Big)ds\nu(du)\\
&=\left(\frac{\partial}{\partial s}+\mathcal{L}_n\right)\tilde{v}(t-s,X_n(s))ds\\
&\quad+\langle\tilde{v}_x(t-s,X_n(s)),B_n(X_n(s))dW(s)\rangle\\
&\quad+\int_H\Big(\tilde{v}\big(t-s,X_n(s)+G(X_n(s),u)\big)\\
&\qquad\quad-\tilde{v}(t-s,X_n(s-))\Big)q(ds,du).
\end{aligned}$$

Since $\tilde{v}(0, X_n(s)) = f(X_n(s))$, $\tilde{v}(t, X_n(0)) = \tilde{v}_n(t, x)$ and by integrating both sides of the above equation from 0 to $t$ and then taking expectation, we have that

$$\tilde{v}(t, x) = \mathbb{E}[f(X_n(t, x))];$$

recalling that under our assumptions $(\frac{\partial}{\partial s} + \mathcal{L}_n)\tilde{v}(t-s, x) = 0$ and

$$\int_0^t \langle \tilde{v}_x(t-s, X_n(s)), B_n(X_n(s))dW(s)\rangle,$$

$$\int_0^t \int_H \Big(\tilde{v}\big(t-s, X_n(s) + G(X_n(s), u)\big) - \tilde{v}(t-s, X_n(s-))\Big)q(ds, du)$$

are martingales. Therefore, the arbitrariness of $\tilde{v}$ implies the uniqueness by *Step 1*.

Step 3 By *Step 1* and tracing the method in *Step 2*, we can obtain

$$\begin{aligned} v_n(0, X_n(t, x)) = v_n(t, X_n(0, x)) &+ \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}_n\right) v_n(t-s, X_n(s, x))ds \\ &+ \int_0^t \big\langle v_{n,x}(t-s, X_n(s, x)), B_n(X_n(s, x))dW(s)\big\rangle \\ &+ \int_0^t \int_H \big(v_n(t-s, X_n(s, x) + G(X_n(s, x), u)) \\ &\qquad - v_n(t-s, X_n(s, x))\big)q(ds, du). \end{aligned}$$

Therefore, noting that $v_n(0, X_n(t, x)) = f(X_n(t, x))$, $v_n(t, X_n(0, x)) = v_n(t, x)$ and

$$\int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}_n\right) v_n(t-s, X_n(s, x))ds = 0,$$

we obtain that

$$\begin{aligned} f(X_n(t, x)) = v_n(t, x) &+ \int_0^t \big\langle v_{n,x}(t-s, X(s, x)), B_n(X_n(s, x))dW(s)\big\rangle \\ &+ \int_0^t \int_H \big(v_n(t-s, X_n(s, x) + G(X_n(s, x), u)) \\ &\qquad - v_n(t-s, X_n(s, x))\big)q(ds, du), \end{aligned} \tag{3.12}$$

and then we know that Eq. 3.10 is satisfied by $X_n(t, x)$.

Now let us turn to consider $X(t, x)$. By the similar method used in Theorem 3.1, we can show that

$$\lim_{n\to\infty} X_n(t, x) = X(t, x), \ \lim_{n\to\infty} DX_n(t, x)y = DX(t, x)y,$$

uniformly in $x$, $y$ in bounded sets. On the other hand, noting that

$$v_{n,x}(t,x)y = \mathbb{E}[\langle f_x(X_n(t,x)), DX_n(t,x)y\rangle],$$

$$v_x(t,x)y = \mathbb{E}[\langle f_x(X(t,x)), DX(t,x)y\rangle],$$

we can obtain Eq. 3.10 holds for $X(t,x)$ by letting $n \to \infty$ in Eq. 3.12.

□

Now we can establish the main result of this section by tracing the approach used in the proof of Lemma 2.4 [27]. We will see that the Bismut–Elworthy–Li type formula for $P_t$ relative to Eq. 1.1 is established and the derivative of $P_t$ is essentially determined by the diffusion coefficient $B(X(t))$ and the Wiener process $W(t)$, i.e., the continuous part of the multiplicative noise.

**Theorem 3.3** *If the assumptions* **A**, **A2**, **A3**) *and* **A4**) *are satisfied, then for each* $f \in C_b^2(H)$, *the semigroup* $P_t$ *is Gateaux differentiable and its directional derivative with respect to* $h \in H$ *has the following representation:*

$$t\big\langle DP_t f(x), h\big\rangle = \mathbb{E}\left[ f(X(t,x)) \int_0^t \big\langle B^{-1}(X(s,x))DX(s,x)h, dW(s)\big\rangle \right]. \tag{3.13}$$

*Proof* This proof follows easily from (2) in Theorem 3.2. In fact, from Eq. 3.10, we obtain that

$$\begin{aligned}
&f(X(t,x)) \int_0^t \big\langle B^{-1}(X(s))X_x(s,x)h, dW(s)\big\rangle \\
&\quad = \int_0^t \big\langle B^{-1}(X(s))X_x(s,x)h, dW(s)\big\rangle \Big( v(t,x) \\
&\qquad + \int_0^t \big\langle v_x(t-s, X(s,x)), B(X(s,x))dW(s)\big\rangle \\
&\qquad + \int_0^t \int_H (v(t-s, X(s,x) + G(X(s,x),u)) - v(t-s, X(s,x)))\, q(ds,du) \Big).
\end{aligned}$$

Because of independence of $W(t)$ and $q(t,\cdot)$, we know that

$$\int_0^t \big\langle B^{-1}(X(s))X_x(s,x)h, dW(s)\big\rangle$$

and

$$\int_0^t \int_H (v(t-s, X(s,x) + G(X(s,x),u)) - v(t-s, X(s,x)))\, q(ds,du)$$

are uncorrelated martingales. Therefore, taking expectation of both sides of the above equation, we obtain

$$\begin{aligned}
&\mathbb{E}\left[ f(X(t,x)) \int_0^t \langle B^{-1}(X(s)) X_x(s,x)h, dW(s)\rangle \right] \\
&= \int_0^t \mathbb{E}\left[\langle B^*(X(s,x)) v_x(t-s, X(s,x)), B^{-1}(X(s,x)) X_x(s,x)h\rangle\right] ds \\
&= \int_0^t \mathbb{E}\left[\langle v_x(t-s, X(s,x)), X_x(s,x)h\rangle\right] ds \\
&= \int_0^t D\mathbb{E}\left[\langle P_{t-s} f(X(s,x)), h\rangle\right] ds \\
&= t\langle DP_t f(x), h\rangle,
\end{aligned}$$

where the martingale property of

$$\int_0^t \langle B^{-1}(X(s)) X_x(s,x)h, dW(s)\rangle$$

and the Markov property of $X(t,x)$ are used for the first and the last equalities respectively. Hence the proof is completed. □

*Remark 3.1* We can show that the above theorem holds for each $f \in C_b(H)$. In fact, for each $f \in C_b(H)$, there exists a sequence of $f_n$ in $C_b^2(H)$ such that $\lim_{n\to\infty} |f_n(t) - f(x)| = 0$, $x \in H$. Therefore, $P_t f_n(x)$ pointwise converges to $P_t f(x)$ for all $t \geq 0$ and then letting $n \to \infty$, we can see that Eq. 3.13 holds for $f \in C_b(H)$; recalling that $DP_t f_n(x)y$ is uniformly bounded for $y$ in bounded subsets of $H$.

## 4 Strong Feller Property

In this part, we will prove the strong Feller property of the Markov transition semigroup $P_t$ based on the probabilistic representation formula of Bismut–Elworthy–Li's type obtained in Theorem 3.3. We will first consider an approximation problem. Let $\{\rho_n\}$ be a sequence of non-negative smooth functions on $\mathbb{R}^n$ with support $\{\theta : |\theta| \leq 1/n\}$ and

$$\int_{\mathbb{R}^n} \rho_n(\theta) d\theta = 1.$$

Set

$$F_n(x) = \int_{\mathbb{R}^n} \rho_n(\theta - \Gamma_n x) F\left(\sum_{i=1}^n \theta_i e_i\right) d\theta,$$

$$B_n(x) = \int_{\mathbb{R}^n} \rho_n(\theta - \Gamma_n x) B\left(\sum_{i=1}^n \theta_i e_i\right) d\theta,$$

$$G_n(x,u)=\int_{\mathbb{R}^n}\rho_n(\theta-\Gamma_n x)G\left(\sum_{i=1}^n\theta_i e_i,u\right)d\theta,$$

and consider the following infinite dimensional stochastic differential equation with the coefficients $F_n$, $B_n$, $G_n$ and the initial value $\xi$:

$$dY_n(t)=(AY_n(t)+F_n(Y_n(t)))\,dt+B_n(Y_n(t))dW(t)+\int_H G_n(Y_n(t),u)q(dt,du). \tag{4.1}$$

Then we have the following result:

**Lemma 4.1** *Under the hypotheses of Theorem* 2.1*, the conditions* **A1**)–**A4**) *are satisfied by* $F_n$, $B_n$ *and* $G_n$*. In addition, the following holds:*

1) *$F_n$ and $B_n$ pointwise converge to $F$ and $B$ respectively as $n\to\infty$. Moreover,*

$$\sup_{n\in\mathbb{N}}\sup_{x\in H}|B_n^{-1}(x)|<\infty.$$

2) *For each $x\in H$, $G_n(x,\cdot)$ converges to $G(x,\cdot)$ in $L^2(\nu)$, i.e.,*

$$\lim_{n\to\infty}\int_H|G_n(x,u)-G(x,u)|^2\nu(du)=0.$$

3) *The equation* (4.1) *has a unique mild solution $Y_n(t)$ and the sequence $Y_n(t), n\in\mathbb{N}$ converges to the solution $X(t)$ of Eq.* 1.1 *in square mean, i.e.,*

$$\lim_{n\to\infty}\mathbb{E}[|Y_n(t)-X(t)|^2]=0.$$

*Proof* To show this lemma, we divide the proof into three steps.

Step 1 For the results relative to $F_n$ and $B_n$, see Lemma 2.5 [10]. In the following, we will only give some proofs relative to $G_n(x,u)$. In fact, for $x,y\in H$, by the definition of $G_n(x,u)$ and the change of variables,

$$\begin{aligned}&\int_H|G_n(x,u)-G_n(y,u)|^2\nu(du)\\&=\int_H\left|\int_{\mathbb{R}^n}(\rho_n(\theta-\Gamma_n x)-\rho_n(\theta-\Gamma_n y))G\left(\sum_{i=1}^n\theta_i e_i,u\right)d\theta\right|^2\nu(du)\\&=\int_H\left|\int_{\mathbb{R}^n}\rho_n(\theta)\left(G\left(\sum_{i=1}^n\theta_i e_i+\Gamma_n x,u\right)\right.\right.\\&\qquad\left.\left.-G\left(\sum_{i=1}^n\theta_i e_i+\Gamma_n y,u\right)\right)d\theta\right|^2\nu(du).\end{aligned}$$

Then by the property of $\rho_n$ and the Cauchy-Schwarz inequality, it follows from the above estimate that

$$\int_H |G_n(x,u) - G_n(y,u)|^2 \nu(du) \le \int_{\mathbb{R}^n} \int_H \rho_n(\theta) \left| G\left(\sum_{i=1}^n \theta_i e_i + \Gamma_n x, u\right) - G\left(\sum_{i=1}^n \theta_i e_i + \Gamma_n y, u\right) \right|^2 \nu(du) d\theta. \tag{4.2}$$

By **A1**) and Eq. 4.2, we can deduce that there exists a constant $C$ independent of $n$ such that

$$\int_H |G_n(x,u) - G_n(y,u)|^2 \nu(du) \le C|x-y|^2.$$

By the similar method, we can easily deduce that

$$\int_H |G_n(x,u)|^2 \nu(du) \le C(1+|x|^2).$$

We can also testify that $G_n(x,u)$ is twice continuously Fréchet differentiable in $x$ and for each fixed $u$

$$D_x G_n(x,u) y = \int_{\mathbb{R}^n} \langle \nabla \rho_n(\theta - \Gamma_n x), \Gamma_n y \rangle G\left(\sum_{i=1}^n \theta_i e_i, u\right) d\theta, \;\; y \in H,$$

and

$$D_x^2 G_n(x,u)(y,z) = \int_{\mathbb{R}^n} \langle Hess\rho_n(\theta - \Gamma_n x)\Gamma_n z, \Gamma_n y \rangle G\left(\sum_{i=1}^n \theta_i e_i, u\right) d\theta, \;\; y,z \in H,$$

where $\nabla$ and $Hess$ denote the gradient and the Hessian matrix of $\rho_n$ respectively. Then under the assumption of **A1**), we can also show that Eqs. 3.1 and 3.8 are fulfilled. Therefore, for the approximative coefficients $F_n$, $B_n$ and $G_n$, the conditions **A1**)–**A4**) are satisfied.

Step 2 According to the definition of $G_n(x,u)$, it is easy to see

$$\begin{aligned}
&\int_H |G_n(x,u) - G(x,u)|^2 \nu(du) \\
&= \int_H |G_n(x,u) - G(\Gamma_n x, u) + G(\Gamma_n x, u) - G(x,u)|^2 \nu(du) \\
&\le 2 \int_H \left| \int_{\mathbb{R}^n} \rho_n(\theta) \left( G\left(\sum_{i=1}^n \theta_i e_i + \Gamma_n x, u\right) - G(\Gamma_n x, u) \right) d\theta \right|^2 \nu(du) \\
&\quad + 2 \int_H |G(\Gamma_n x, u) - G(x,u)|^2 \nu(du).
\end{aligned}$$

By **A1**) and the similar technique to Eq. 4.2, the above estimate implies that

$$\int_H |G_n(x,u) - G(x,u)|^2 \nu(du) \le C \int_{\mathbb{R}^n} \rho_n(\theta)|\theta|^2 d\theta + C|\Gamma_n x - x|$$
$$\le C\left(\frac{1}{n^2} + |\Gamma_n x - x|\right);$$

recalling that the support of $\rho_n$ is a subset of $\left\{\theta \in \mathbb{R}^d : |\theta| \le \frac{1}{n}\right\}$. Therefore, we can obtain 2) by letting $n \to \infty$.

Step 3 To show the third assertion. From the above, we know that the assumptions in Theorem **C.2** below are satisfied. Therefore, 3) is true.

□

The following is going to deal with the first result $(i)$ in Theorem 2.2 based on the Bismut–Elworthy–Li type formula formulated in Section 3.

**Theorem 4.1** *Assume* **A** *and* **A2**) *are satisfied. Then the Markov semigroup $P_t$ corresponding to Eq. 1.1 is strong Feller. To be precise, for each $f \in B_b(H)$, we have*

$$|P_t f(x) - P_t f(y)| \le C t^{-1/2}|x-y|, \ x, y \in H. \tag{4.3}$$

*Proof* We first suppose $f \in C_b^2(H)$ and try to prove Eq. 4.3 is true for the Markov transition semigroup $P_{n,t}f(x) = \mathbb{E}[f(Y_n(t,x))]$ corresponding to the Markov process $Y_n(t)$, the solution of Eq. 4.1. From Theorem 3.3 and Lemma 4.1, it follows that, there exists $z \in H$, such that

$$|P_{n,t}f(x) - P_{n,t}f(y)|^2 = |\langle DP_{n,t}f(z), x-y\rangle|^2$$
$$\le t^{-2}\mathbb{E}\left[\left| f(Y_n(s,z)) \int_0^t \left\langle B^{-1}(Y_n(s,z)DY_n(s,z)(x-y), dW(s)\right\rangle\right|^2\right]$$
$$\le t^{-2}\|f\|_\infty^2 \mathbb{E}\left[\int_0^t |B^{-1}(Y_n(s,z))DY_n(s,z)(x-y)|^2 ds\right].$$

Hence, by the uniform boundedness of $(B_n(x))^{-1}$ and Eq. 3.3 in Theorem 3.1, there exists a positive constant $C$ such that

$$|P_{n,t}f(x) - P_{n,t}f(y)|^2 \le C t^{-1}|x-y|^2. \tag{4.4}$$

Then by the continuity of $f$, 3) in Lemma 4.1 and letting $n \to \infty$ in Eq. 4.4, we can obtain that Eq. 4.3 is true for $f \in C_b^2(H)$.

From now we try to complete our proof. Assume that $f \in B_b(H)$. Then we have

$$|P_t f(x) - P_t f(y)| = \left|\int_H f(z)(P_t(x,dz) - P_t(y,dz))\right|$$
$$\le \|f\|_\infty Var(P_t(x,\cdot) - P_t(y,\cdot)), \tag{4.5}$$

where $Var$ denotes the total variation of a signed measure. On the other hand, since as we used in Remark 3.1 for each function in $C_b(H)$ can be pointwise approximated by a sequence of functions in $C_b^2(H)$ and by the application of the Hahn decomposition theorem, we have

$$Var(P_t(x,\cdot) - P_t(y,\cdot)) = \sup_{\{f \in C_b^2(H): \|f\|_\infty \le 1\}} |P_t f(x) - P_t f(y)|, \ x, y \in H. \tag{4.6}$$

Therefore, we can prove the desired result by Eqs. 4.5 and 4.6. □

*Remark 4.1*

(i) To the best knowledge of the author, there is no paper to study the strong Feller property for the infinite dimensional stochastic differential equations driven by Lévy noises except that authored by M. Röckner and F. Y. Wang [29] in 2003 using analytic approaches. In their case, $F = 0$ and the noise is additive, i.e., $B(x) = B$ is a trace class and $G(x, u) = u$, see Corollary 1.2 [29]. They proved the strong Feller property and the density of the Markov transition probability by studying the generalized Mehler semigroup as the application of dimension-free Harnack inequality initially introduced by F. Y. Wang [31]. In our framework, the relation (4.3) can be proved, which may be important in applications and is more general.
(ii) By our approach, we know that the strong Feller property of Eq. 1.1 is principally determined by the continuous part $B(X(t))W(t)$. In fact, as we have said that the proof of this theorem is based on the Bismut–Elworthy–Li type formula given in Theorem 3.3.

## 5 Irreducibility

In this section, the irreducibility of the solution $X(t)$ of Eq. 1.1 will be mainly investigated. Let us first formulate a criterion of the irreducibility for an $H$-valued Markov process with jumps.

**Lemma 5.1** *Let $\{X(t)\}_{t\ge 0}$ be a càdlàg and strong Markov process with values in $H$ and let $\tau$ be the first jumping time of it, i.e.,*

$$\tau = \inf\{t \ge 0 : |\triangle X(t)| = |X(t) - X(t-)| \ne 0\}.$$

*If the process $\{X(t \wedge \tau)\}_{t\ge 0}$ is irreducible and for each $t$*

$$\mathbb{P}^x(\tau > t) > 0, \tag{5.1}$$

*then the Markov process $X(t)$ is irreducible, where $\mathbb{P}^x$ denotes the distribution of the Markov process $X(t)$ with initial law $\delta_x$.*

*Proof* In this proof, the same notation of the Markov transition probability as before will be used for simplicity. For any $x \in H$ and any non-empty open subset $A \subset H$,

$$\begin{aligned} P_t(x, A) &= \mathbb{P}^x(X(t) \in A) \\ &= \mathbb{P}^x(X(t) \in A, \tau \le t) + \mathbb{P}(X(t) \in A, \tau > t) \\ &\ge \mathbb{P}^x(X(t) \in A, \tau > t), \end{aligned}$$

which completes the proof by virtue of the assumption (5.1) and the irreducibility of the process $\{X(t \wedge \tau), t \geq 0\}$. □

The following shows that the solution $X(t)$ of Eq. 1.1 is a strong Markov process.

**Lemma 5.2** *Suppose the assumption* **A** *is satisfied. Then $X(t)$ is a strong Markov process, i.e., for any stopping time $\tau$ and $f \in B_b(H)$,*

$$\mathbb{E}[f(X(t+\tau, x))|\mathcal{F}_\tau] = P_t f(X(\tau, x)),$$

*where $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t,\ t \geq 0\}$.*

*Proof* It suffices to show that for each $A \in \mathcal{F}_\tau$ and each bounded and continuous function $f$ on $H$, the following holds:

$$\mathbb{E}[f(X(t+\tau, x)) I_{A\cap\{\tau<\infty\}}] = \mathbb{E}[P_t f(X(\tau, x)) I_{A\cap\{\tau<\infty\}}]. \tag{5.2}$$

If the stopping time $\tau$ takes on a finite number of values, then by the Markov property of $X(t)$, see Theorem 2.1, Eq. 5.2 holds. Now let us take a sequence of monotonic decreasing stopping time $\tau_n$ such that $\tau_n$ converges to $\tau$ as $n \to \infty$. Since $\mathcal{F}_\tau \subset \mathcal{F}_{\tau_n}$, it is clear that

$$\mathbb{E}[f(X(t+\tau_n, x)) I_{A\cap\{\tau<\infty\}}] = \mathbb{E}[P_t f(X(\tau_n, x)) I_{A\cap\{\tau<\infty\}}].$$

Taking into account right continuity of the trajectory of $X(t)$ and the continuity of $f$, we obtain that Eq. 5.2 is satisfied for arbitrary stopping time $\tau$. □

**Lemma 5.3** *If the assumption* **A** *holds, then for any fixed $T > 0$, there exists a positive constant $K > 0$ depending on $T$ such that,*

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |X(s, x)|^2\right] \leq K(1+|x|^2),\ t \in [0, T]. \tag{5.3}$$

*Proof* If we can show that there exists a constant $C$ such that for any $t \in [0, T]$

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |X(s, x)|^2\right]$$
$$\leq C(1+|x|^2) + C\int_0^t (1+\|S(s)\|_{HS}^2)\mathbb{E}\left[\sup_{0\leq r\leq s} |X(r, x)|^2\right] ds, \tag{5.4}$$

then we can complete our proof by the classical Gronwall inequality. In fact, we can obtain such estimate by virtue of maximal inequalities proved in Theorem **B.1** and Theorem **B.2** (see Appendices). For example, from the proof of Theorem **B.2**, we can easily know

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|\int_0^s S(s-r)B(X(r))dW(r)\right|^2\right] \leq Ce^{2\alpha t}\int_0^t \|S(s)\|_{HS}^2 \left(1+\mathbb{E}\left[\sup_{0\leq r\leq s}|X(r)|^2\right]\right) ds.$$

Therefore, by the assumption **A**, we can obtain that

$$\mathbb{E}\left[\sup_{0\le s\le t}|X(s,x)|^2\right] \le Ce^{2\alpha t}\left(|x|^2 + t + \int_0^t \|S(s)\|_{HS}^2 ds\right)$$
$$+ Ce^{2\alpha t}\int_0^t (1+\|S(s)\|_{HS}^2)\mathbb{E}\left[\sup_{0\le r\le s}|X(r,x)|^2\right] ds. \quad (5.5)$$

By **A0**), we know that

$$\int_0^T \|S(s)\|_{HS}^2 ds < \infty.$$

Therefore, by Eq. 5.5, we can find a constant $C$ such that Eq. 5.4 holds for fixed $T$.

□

*Remark 5.1* Our proof essentially depends on the maximal inequalities of stochastic convolutions proved in Appendices. Even if we only consider the pure jump noise $q$ and use a known result in Theorem 9.24 [28] to show Eq. 5.3, we must impose the uniform boundedness on the coefficients.

Now let us turn to prove the irreducibility of the solution $X(t)$ of Eq. 1.1, which partially proves the second assertion in Theorem 2.2.

**Theorem 5.1** *Under the assumptions* **A**, **A2**) *and*

$$\lim_{\gamma\to\infty}\frac{\nu(B(\gamma))}{\log\gamma} = 0, \quad (5.6)$$

*the mild solution $X(t)$ of Eq. 1.1 is irreducible, where $B(\gamma)$ is the support of the coefficient $G(x,u)$ in $u \in H$ for all $|x| < \gamma$, i.e., it is the closure of the set*

$$\{u \in H : |G(x,u)| > 0, |x| < \gamma\}.$$

*Proof* By the Markov property of $X(t,x)$, it is sufficient to show that the irreducibility of $X(t,x)$ for a fixed time $t > 0$. Let $\tau$ be defined as in Lemma 5.1 for the solution $X(t,x)$ for each $x \in H$. Then $\tau$ is the first jumping time of $X(t,x)$ and according to Theorem 1.3 [27], $\{X(t\wedge\tau)\}_{t\ge 0}$ is irreducible under our assumptions. Therefore, by virtue of Lemmas 5.1 and 5.2, it is enough for us to show that the probability of $\{\tau > t\}$ is positive. Noting that

$$\mathbb{P}(\tau > t) \ge \mathbb{P}\left(\tau > t, \sup_{0\le s\le t}|X(s,x)| \le \gamma\right), \quad (5.7)$$

we will devote to considering the right hand side of Eq. 5.7. To estimate it, it is enough to consider

$$\mathbb{P}\left(\tau \le t, \sup_{0\le s\le t}|X(s,x)| \le \gamma\right).$$

We assume that $|x| < \gamma$ in the following. According to the definition of $\tau$, we have

$$\mathbb{P}\left(\tau \le t, \sup_{0\le s\le t}|X(s,x)| \le \gamma\right) \le \mathbb{P}(N(t, B(\gamma)) \ge 1).$$

For the set $B(\gamma)$, we know that $N(t, B(\gamma))$ is a Poisson random variable and then we have

$$\mathbb{P}(N(t, B(\gamma)) \geq 1) = 1 - \exp(-t\nu(B(\gamma)))$$

As a consequence of Lemma 5.3 and the Chebyshev inequality, there exists $C > 0$ such that for arbitrary $t \in [0, T]$ and initial value $x$

$$\begin{aligned} \mathbb{P}(\tau > t) &\geq \exp(-t\nu(B(\gamma))) - \mathbb{P}\left(\sup_{0\leq s\leq t} |X(s,x)| > \gamma\right) \\ &\geq \exp(-t\nu(B(\gamma))) - \frac{C}{\gamma^2}(1 + |x|^2) \end{aligned} \tag{5.8}$$

On the other hand, by Eq. 5.6, we can choose a small enough $\delta > 0$ with $t\delta < 2$ such that

$$\nu(B(\gamma)) \leq \delta \log \gamma,$$

Then by Eq. 5.8, we deduce that

$$\mathbb{P}(\tau > t) \geq \frac{1}{\gamma^{t\delta}} - \frac{C}{\gamma^2}(1 + |x|^2).$$

Therefore, if $\gamma$ is large enough, then

$$\mathbb{P}(\tau > t) > 0, \ t > 0,$$

which completes the proof. □

*Remark 5.2* If we assume that the Lévy measure $\nu$ is bounded, i.e., $\nu(H) < \infty$, then the irreducibility of the solution $X(t)$ follows easily. In fact, define

$$L(t) = \int_H uN(t, du)$$

and

$$\sigma = \inf\{t \geq 0 : |L(t) - L(t-)| \neq 0\}.$$

Then it is well known that $L(t)$ is a compound Poisson process with parameter $\nu(H) < \infty$ and $\sigma$ is the first jumping time of $L(t)$, which is distributed as an exponential random variable with parameter $t\nu(H)$. It is clear that the jump of $X(t)$ is determined by the last term in Eq. 1.1. Noting that $\tau > t$ means the first jump does not occur before time $t$, we have that for arbitrary $t > 0$

$$\mathbb{P}(\tau > t) \geq \mathbb{P}(\sigma > t) = e^{-t\nu(H)} > 0,$$

which implies that $X(t)$ is irreducible by Lemmas 5.1 and 5.2.

## 6 Stochastic Heat Equations Driven by Lévy Noises

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a usual filtration $\{\mathcal{F}_t\}_{t\geq 0}$ as before. In this section we are interested in the following stochastic heat equation on a bounded domain driven by a Lévy type noise,

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) = \dfrac{\partial^2 v}{\partial^2 x}(t,\xi) + \phi(v(t,\xi)) + \varphi(v(t,\xi))\dfrac{\partial^2 W}{\partial t \partial \xi}(t,\xi) & \\ \qquad\qquad + \displaystyle\int_{L^2(0,1)} \psi(v(t,\xi),u)\frac{\partial q}{\partial t}(t,du), & t \geq 0,\ \xi \in (0,1), \\ v(t,0) = v(t,1) = 0, & t \geq 0, \\ v(0,\xi) = v_0(\xi), & \xi \in (0,1), \end{cases} \tag{6.1}$$

where $\{W(t,\xi),\ (t,\xi) \in [0,\infty) \times [0,1]\}$ is a Brownian sheet on $[0,\infty) \times [0,1]$, $q(t,u)$ is a compensated Poisson random measure corresponding to an $\mathcal{F}_t$-Poisson random measure $N(t,u)$ on $[0,\infty) \times L^2(0,1)$ with intensity measure $dtdv$ and the coefficients $\phi, \varphi$ on $\mathbb{R}$, $\psi$ on $\mathbb{R} \times L^2(0,1)$ are measurable functions. For the definition of a Brownian sheet and the stochastic Itô integral with respect to it, we refer the reader to [30]. On the other hand, a Brownian sheet can be regarded as a cylindrical Wiener process on $L^2(0,1)$ as below, see the monograph [9] for details.

We additionally impose the following assumption **B** on the coefficients of Eq. 6.1 for our purpose:

**B1)** The coefficients $\phi, \varphi, \psi$ are Lipschitz continuous with linear growth in the following meanings: for arbitrary real numbers $\xi$ and $\tilde{\xi}$, there exists $C > 0$ such that

$$|\phi(\xi)|^2 + |\varphi(\xi)|^2 + \int_{L^2(0,1)} |\psi(\xi,u)|^2 \nu(du) \leq C(1 + |\xi|^2),$$

$$|\phi(\xi) - \phi(\tilde{\xi})|^2 + |\varphi(\xi) - \varphi(\tilde{\xi})|^2 + \int_{L^2(0,1)} |\psi(\xi,u) - \psi(\tilde{\xi},u)|^2 \nu(du) \leq C|\xi - \tilde{\xi}|^2.$$

**B2)** The diffusion coefficient $\varphi$ is uniformly bounded from below and above, i.e., there exist positive constants $C_1$ and $C_2$ such that for all $\xi \in \mathbb{R}$

$$C_1 \leq |\varphi(\xi)| \leq C_2.$$

If **B1**) holds, then by the similar method used in the proof of Theorem 2.1, we can show that Eq. 6.1 has a unique càdlàg mild solution $v(t,\xi)$ satisfying

$$\mathbb{E}\left[\sup_{t \in [0,T]} |v(t)|^2_{L^2(0,1)}\right] < \infty.$$

For a mild solution, we mean that there exists a predictable process $\{v(t,\xi), t \geq 0, \xi \in [0,1]\}$ such that

$$v(t,\xi) = \int_0^1 p(t,\xi,\tilde{\xi})v_0(\tilde{\xi})d\tilde{\xi} + \int_0^t \int_0^1 p(t-s,\xi,\tilde{\xi})\phi(v(s,\tilde{\xi}))dsd\tilde{\xi}$$
$$+ \int_0^t \int_0^1 p(t-s,\xi,\tilde{\xi})\varphi(v(s,\tilde{\xi}))W(ds,d\tilde{\xi})$$
$$+ \int_0^t \int_{L^2(0,1)} \int_0^1 p(t-s,\xi,\tilde{\xi})\psi(v(s,\tilde{\xi}),u)d\tilde{\xi}q(ds,du),$$

where $p(t,\xi,\tilde{\xi})$ is the fundamental solution of the homogenous part of the stochastic heat equation (6.1).

Now let us try to write the stochastic heat equation (6.1) in its abstract form. Let $H = L^2(0,1)$ and let $A = \frac{\partial^2}{\partial^2 x}$ with the domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$. We denote by $\{e_n\}_{n\in\mathbb{N}}$ the complete orthonormal system in $H$ consisting of the eigenfunctions of A,

$$e_n(\xi) = \sqrt{2}\sin(n\pi\xi)$$

so that $Ae_n = -n^2\pi^2 e_n$. Then the Brownian sheet $W(t,\xi)$ can be rewritten as

$$W(t)(\xi) := W(t,\xi) = \sum_{n=1}^{\infty} e_n(\xi) \int_0^t \int_0^1 e_n(\theta)W(ds,d\theta),$$

which is a cylindrical Wiener process on $H$.

In the end, for $\xi \in (0,1)$ and $u, v \in H$, let us define

$$X(t)(\xi) := v(t,x),\ \ F(v)(\xi) := \phi\left(v(\xi)\right),$$

$$B(u)(v)(\xi) := \varphi\left(u(\xi)\right)v(\xi),\ \ G(v,u)(\xi) := \psi(v(\xi),u).$$

Now we can rewrite the stochastic heat equation (6.1) as below:

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))\,dt + B(X(t))dW(t) + \int_H G(X(t),u)q(dt,du), \\ X(0) = v_0, \end{cases}$$

which is just the form of Eq. 1.1.

Hence according to Theorem 2.2, we have the following theorem:

**Theorem 6.1** *Under the assumption* **B**, *the following holds:*

(1) *The Markov process $v(t,\cdot)$ corresponding to Eq. 6.1 has the strong Feller property.*
(2) *Assume $\int_{L^2(0,1)} |u|^2 \nu(du) < \infty$. If furthermore $\psi(\theta,u) := G_1(\theta|u|)u$ and $\nu(\{u \in L^2(0,1) : |u| \geq \epsilon\}) \leq C(\log\epsilon)^2$ holds for all small enough $\epsilon$, then there exists at most one invariant measure for the stochastic heat equation (6.1), where $G_1$ is the function defined in Remark* 2.1 *(iii).*

*Proof* To show (1), it is enough for us to verify that the assumptions **A0**)–**A2**) are fulfilled. It is clear that

$$\|S(t)\|_{HS}^2 = \sum_{n=1}^{\infty} |S(t)e_n|^2 = \sum_{n=1}^{\infty} e^{-2\pi^2 n^2 t},$$

$$\int_0^T \|S(t)\|_{HS}^2 ds = \sum_{n=1}^{\infty} \int_0^T e^{-2\pi^2 n^2 t} dt \leq \frac{1}{12}.$$

Hence, **A0**) is fulfilled. On the other hand, we can easily check that the assumptions **A1**) and **A2**) are satisfied. In fact, by the definition of $G$ and **B1**), we see

$$\begin{aligned}\int_H |G(v,u) - G(v',u)|^2 \nu(du) &= \int_H \int_0^1 |\psi(v(\xi),u) - \psi(v'(\xi),u)|^2 d\xi\, \nu(du)\\ &= \int_0^1 \int_H |\psi(v(\xi),u) - \psi(v'(\xi),u)|^2 \nu(du) d\xi\\ &\leq C \int_0^1 |v(\xi) - v'(\xi)|^2 d\xi.\end{aligned}$$

Then, by the similar methods, we can show **A1**) are also satisfied by $F$ and $B$. It is easy to see that **A2**) can be guaranteed by **B2**). Therefore according to Theorem 2.2, (1) can be proved.

According to Remark 2.1 (*iii*), we can easily show that Eq. 5.6 is fulfilled. Therefore, the proof is completed. □

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## Appendices

### A Itô's Formula

Here we summarize the Itô formula for an $H$-valued semi-martingale with jumps as below, see [19] and references therein for its proof. Suppose that $F:[0,T]\to H$ is a predictable process and $B:[0,T]\to \mathcal{L}_2(H)$ and $G:[0,T]\times H\to H$ are stochastically integrable with respect to $W$ and $q$ respectively. We assume for each fixed $T>0$ the following holds:

$$\int_0^T \mathbb{E}[|F(s)|^2 + \|B(s)\|_{HS}^2] ds + \int_0^T \int_H \mathbb{E}[|G(s,u)|^2] ds \nu(du) < \infty.$$

Let $Y(t)$ be defined as below:

$$Y(t) = \xi + \int_0^t F(s)ds + \int_0^t B(s)dW(s) + \int_0^t \int_H G(s,u) q(ds,du),$$

where $\xi$ is an $\mathcal{F}_0$-measurable $H$-valued random variable. It is known that $Y(t)$ has a càdlàg modification.

**Theorem A.1** (Itô's Formula) *For each function $\Phi \in C_b^{1,2}([0,T] \times H)$, the following holds with probability* 1*:*

$$\begin{aligned}
&\Phi(t, Y(t)) \\
&= \Phi(0, Y(0)) + \int_0^t \Phi_s(s, Y(s))ds + \int_0^t \langle \Phi_x(s, Y(s-)), B(s)dW(s)\rangle \\
&+ \int_0^t \int_H \big(\Phi\big(s, Y(s-) + G(s,u)\big) - \Phi(s, Y(s-))\big)\, q(ds, du) \\
&+ \int_0^t \Big( \langle \Phi_x(s, X(s)), F(X(s))\rangle + \frac{1}{2} Tr\big(B^*(s)\Phi_{xx}(s, X(s-))B(s)\big)\Big) ds \\
&+ \int_0^t \int_H \Big(\Phi\big(s, Y(s-) + G(s,u)\big) - \Phi(s, Y(s-)) - \langle G(s,u), D(Y(s-))\rangle\Big) ds\nu(du).
\end{aligned}$$

B Maximal Inequalities for Stochastic Convolutions

In this part, we will discuss maximal inequalities for stochastic convolutions corresponding to a compensated Poisson random measure $q$ and a cylindrical Wiener process $W$ respectively based on a dilation theorem.

**Theorem B.1** *Suppose the assumption* **A0**) *and* $\int_0^T \int_H \mathbb{E}[|G(s,u)|^2] ds\nu(du) < \infty$. *Then the stochastic convolution*

$$\mathcal{Q}(t) := \int_0^t \int_H S(t-s)G(s,u)q(ds,du)$$

*has a càdlàg modification in $H$ and there exists a constant $C$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\mathcal{Q}(t)|^2\right] \le Ce^{2\alpha T} \int_0^T \int_H \mathbb{E}[|G(s,u)|^2] ds\nu(du).$$

*Proof* Under our assumptions, the càdlàg modification in $H$ has been proved, see [1]. Here we adopt the method initially introduced in [21] to complete our proof. By Sz.-Nagy's theorem, we know that there exists a Hilbert space $\hat{H}$ and a group $\hat{S}(t)$ of linear operators on $\hat{H}$ such that $\mathcal{P}\hat{S}(t) = e^{-\alpha t}S(t)$, where $\mathcal{P}$ is an orthogonal projection from $\hat{H}$ to $H$. Hence we see

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} |\mathcal{Q}(t)|^2\right] &\le e^{2\alpha T}\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_H \mathcal{P}\hat{S}(t)\hat{S}(-s)G(s,u)q(ds,du)\right|_{\hat{H}}^2\right] \\
&\le e^{2\alpha T}\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_H \hat{S}(-s)G(s,u)q(ds,du)\right|_{\hat{H}}^2\right].
\end{aligned}$$

Since the process $\int_0^t \int_H \hat{S}(-s)G(s,u)q(ds,du)$ is an $\hat{H}$-valued martingale and then by Doob's inequality and Itô's isometry (2.1), we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\mathcal{Q}(t)|^2\right] \le Ce^{2\alpha T}\int_0^T \mathbb{E}[\|\hat{S}(-s)\|_{\hat{H}}^2 |G(s,u)|^2] ds\nu(du),$$

which completes our proof. □

*Remark B.1* Our result generalizes the estimate (9.23) in Theorem 9.24 [28] from $\alpha$-th ($\alpha \in (0,2)$) moment to mean square case.

**Theorem B.2** *If the assumption* **A0**) *is satisfied and*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|B(t)\|^2\right] < \infty,$$

*then the stochastic convolution*

$$\mathcal{W}(t) := \int_0^t S(t-s)B(s)dW(s)$$

*has a continuous modification in* $H$ *and*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\mathcal{W}(t)|^2\right] \le Ce^{2\alpha T}\int_0^T \|S(t)\|_{HS}^2 ds \mathbb{E}\left[\sup_{t\in[0,T]} \|B(t)\|^2\right].$$

*Proof* We will trace the method used in the above theorem and utilize the similar notations. By virtue of Theorem 9.23 [28], we can also know that if $s < 0$, then $\mathcal{P}\hat{S}(s) = e^{-\alpha s}S(-s)^*$, where $S(-s)^*$ denotes the dual operator of $S(-s)$. Then by the maximal inequality for martingales, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} |\mathcal{W}(t)|^2\right] &\le e^{2\alpha T}\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \mathcal{P}\hat{S}(t)\hat{S}(-s)B(s)dW(s))\right|_{\hat{H}}^2\right] \\
&\le e^{2\alpha T}\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \hat{S}(-s)B(s)W(ds)\right|_{\hat{H}}^2\right] \\
&\le Ce^{2\alpha T}\mathbb{E}\left[\int_0^T \|\hat{S}(-s)B(s)\|_{\mathcal{L}^2(\hat{H})}^2 ds\right] \\
&\le Ce^{2\alpha T}\mathbb{E}\left[\int_0^T \|\hat{S}(-s)\|_{\mathcal{L}^2(\hat{H})}^2 \|B(s)\|_{\mathcal{L}(\hat{H})}^2 ds\right] \\
&\le Ce^{2\alpha T}\mathbb{E}\left[\int_0^T \|S(s)^*\|_{HS}^2 \|B(s)\|^2 ds\right],
\end{aligned}$$

where $\|\cdot\|_{\mathcal{L}^2(\hat{H})}$ denotes the Hilbert–Schmidt norm of an operator. Then the proof can be completed. □

*Remark B.2* The maximal inequalities for stochastic convolutions especially corresponding to $H$-valued Wiener processes have been considered by many authors, see

[21] and references therein. However, there are few authors to study such inequality corresponding to a cylindrical Wiener process satisfying our condition. We recall that by factorization method Proposition 7.9 [9] shows a maximal inequality relative to a cylindrical Wiener process with a strong condition:

$$\int_0^T t^{-\beta} \|S(t)\|_{HS}^2 dt < \infty, \ \exists\, \beta \in (0, 1/2).$$

C Some Results Relative to Eq. 1.1

In this part, we first state the proof of Theorem 2.1. For the convenience of readers, we now restate this theorem.

**Theorem C.1** *Suppose the assumption* **A** *is satisfied. Then there exists a uniqueness solution* $X(t)$ *such that for each* $T > 0$, $X(t) \in \mathcal{H}_{2,T}$. *Moreover, the process* $X(t)$ *has a càdlàg modification and is a homogeneous Markov process.*

*Proof* We will sketch the proof. We first construct a solution by successive approximation of the following sequence $\{X_n(t), n = 0, 1, 2, \cdots\}$:

$$\begin{aligned} X_0(t) &= S(t)\xi, \\ X_{n+1}(t) &= S(t)x + \int_0^t S(t-s)F(X_n(s))ds + \int_0^t S(t-s)B(X_n(s))dW(s) \\ &\quad + \int_0^t \int_H S(t-s)G(X_n(s), u)q(ds, du)\ a.s., n = 0, 1, 2, \cdots \end{aligned}$$

We first show that $X_n(t) \in \mathcal{H}_{2,T}$ for each $n = 0, 1, 2 \ldots$ In fact, since $\mathbb{E}[|\xi|^2] < \infty$ and $\|S(t)\| \le e^{\alpha t}$, it is easy to know $X_0 \in \mathcal{H}_{2,T}$. On the other hand, by the assumption **A**, Theorem B.1 and B.2, we can show that

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in T} |X_{n+1}(t)|^2\right] \\ &\le e^{2\alpha T}\mathbb{E}[|\xi|^2] + \frac{e^{2\alpha T}}{2\alpha}\int_0^T \mathbb{E}\left[\sup_{t\in[0,T]} |F(X_n(t))|^2\right] dt \\ &+ Ce^{2\alpha T}\int_0^T \|S(t)\|_{HS}^2 dt\mathbb{E}\left[\sup_{t\in[0,T]} \|B(X_n(t))\|^2\right] \\ &+ Ce^{2\alpha T}\mathbb{E}\left[\int_0^T \int_H |G(X_n(s), u)|^2 ds\nu(du)\right] \\ &\le Ce^{2\alpha T} + Ce^{2\alpha T}\left(T + \int_0^T \|S(t)\|_{HS}^2 ds\right)\mathbb{E}\left[1 + \sup_{t\in[0,T]} |X_n(t)|^2\right]. \end{aligned}$$

Then by induction we can obtain that $X_n(t) \in \mathcal{H}_{2,T}, n = 0, 1, 2, \ldots$ By a similar method, we can see that for any $t_0 \leq T$

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,t_0]} |X_{n+1}(t) - X_n(t)|^2\right] &\leq C(t_0)\mathbb{E}\left[\sup_{t\in[0,t_0]} |X_n(t) - X_{n-1}(t)|^2\right] \\ &\leq C(t_0)^n\mathbb{E}\left[\sup_{t\in[0,t_0]} |X_1(t) - X_0(t)|^2\right] \\ &\leq C(t_0)^n \left(1 + \mathbb{E}[|\xi|^2]\right)\end{aligned}$$

where $C(t_0) = Ce^{2\alpha t_0}\left(t_0 + \int_0^{t_0} \|S(s)\|_{HS}^2 ds\right)$ for a constant $C$ independent of $n$ and $t_0$. Hence we have that for any $m \leq n$

$$\left(\mathbb{E}\left[\sup_{t\in[0,t_0]} |X_m(t) - X_n(t)|^2\right]\right)^{1/2} \leq \left(1 + \mathbb{E}[|\xi|^2]\right)^{1/2} \sum_{k=m+1}^{n} C(t_0)^{n/2}.$$

We can choose a small enough $t_0$, such that $C(t_0) < 1$ and then we know that the sequence of $X_n(t)$ is a Cauchy sequence in $\mathcal{H}_{2,t_0}$. Let us denote by $X(t)$ the unique limit of $X_n(t)$. Then by the classical method, we can show that $X(t)$ is the unique mild solution of Eq. 1.1 for $t \in [0, t_0]$. For the fixed $T$, it can be established by considering the equation in intervals $[0, t_0], [t_0, 2t_0], \ldots$

On the other hand, by our construction of the solution, we can easily know that the solution has a càdlàg modification.

From now we intend to show the Markov property of the solution. By the definition of the mild solution and the property of semigroup $S(t)$, it is easy to say that for $s, t \geq 0$,

$$\begin{aligned}X(s+t, x) = S(t)X(s, x) &+ \int_s^{s+t} S(t+s-r)F(X(r))dr \\ &+ \int_s^{s+t} S(t+s-r)B(X(r))dW(r) \\ &+ \int_s^{s+t}\int_H S(t+s-r)G(X(r), u)q(dr, du).\end{aligned} \tag{6.2}$$

Since $W(t)$ has stationary independent increments and $q$ is a stationary compensated Poisson random measure, by the uniqueness of the mild solution and Eq. 6.2, we know that

$$X(s+t, x) = X(t, X(s, x)).$$

On the other hand, we can easily know that the last three terms in Eq. 6.2 are independent of $\mathcal{F}_s$. Therefore, for each $f \in B_b(H)$, we have

$$\begin{aligned}\mathbb{E}[f(X(t+s, x))|\mathcal{F}_s] &= \mathbb{E}[f(X(t, X(s, x))|\mathcal{F}_s] \\ &= \mathbb{E}[f(X(t, y)|\mathcal{F}_s]|_{y=X(s,x)} \\ &= P_t f(X(s, x)),\end{aligned}$$

which shows that $X(t)$ is a homogeneous Markov process. □

In the following, we will see that the solution of Eq. 1.1 continuously depends on its initial value and its coefficients. Suppose the coefficients $F_n$, $B_n$, $G_n$ satisfy the assumption **A** with a common constant $C$ and consider the following equation with initial value $\xi_n \in \mathcal{F}_0$:

$$dX_n(t) = (AX_n(t) + F_n(X_n(t)))dt + B_n(Y_n(t))dW(t) + \int_H G_n(Y_n(t), u)q(dt, du). \tag{6.3}$$

Then we have the following theorem:

**Theorem C.2** *Assume further*

$$\lim_{n\to\infty} \mathbb{E}[|\xi_n - \xi|^2] = 0,$$

$$\lim_{n\to\infty} \{|F_n(x) - F(x)|^2 + \|B_n(x) - B(x)\|^2 + \int_H |G_n(x, u) - G(x, y)|^2 \nu(du)\} = 0.$$

*Then for each n, Eq. 6.3 has a unique mild solution* $X_n(t)$ *and*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |X_n(t) - X(t)|^2\right] = 0.$$

*Proof* Since the assumption **A** is satisfied for each $n$, the existence and uniqueness of the solution $X_n(t)$ is trivial by Theorem C.1. Without loss of generality, we can assume that $T$ is small enough, which will be given later. By a similar approach used in above theorem, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |X_n(t) - X(t)|^2\right] \\
&\leq e^{2\alpha T}\mathbb{E}[|\xi_n - \xi|^2] + C(T)\mathbb{E}\left[\sup_{t\in[0,T]} |X_n(t) - X(t)|^2\right] \\
&+ C(T)\mathbb{E}\left[\sup_{t\in[0,T]} |F_n(X(t)) - F(X(t))|^2 + \sup_{t\in[0,T]} \|B_n(X(t)) - B(X(t)\|^2\right] \\
&+ C(T)\mathbb{E}\left[\int_0^T \int_H |G_n(X(t), u) - G(X(t), u)|^2 dt\nu(du)\right],
\end{aligned}$$

where $C(T) = Ce^{2\alpha T}\left(T + \int_0^T \|S(t)\|_{HS}^2 dt\right)$. If we choose small enough $T$ such that $C(T) < 1$, then we can conclude our proof by our conditions. □

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