Potential Anal (2011) 35:329–351
DOI 10.1007/s11118-010-9215-8

The Fundamental Convergence Theorem for $p(\cdot)$-Superharmonic Functions

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Received: 4 December 2009 / Accepted: 6 December 2010 / Published online: 23 December 2010

Abstract We study the balayage related to the supersolutions of the variable exponent $p(\cdot)$-Laplace equation. We prove the fundamental convergence theorem for the balayage and apply it for proving the Kellogg property, boundary regularity results for the balayage, and a removability theorem for $p(\cdot)$-solutions.

Keywords Non-standard growth · $p(\cdot)$-Laplacian · Comparison principle · Fundamental convergence theorem · Boundary regularity · Removability

Mathematics Subject Classifications (2010) 31C45 (35J60, 35B60)

1 Introduction

In potential theory, the *balayage* of a given obstacle function ψ on an open set Ω is defined as the lower semicontinuous regularization of the *réduite*, the pointwise infimum of all superharmonic functions that lie above ψ. To develop potential theory based on the notion of balayage, it is crucial to know that the balayage is a superharmonic function which equals the réduite outside a set of capacity zero. This result is known as *the fundamental convergence theorem.*

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Our purpose is to establish the fundamental convergence theorem for balayage related to supersolutions of the elliptic equation

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = 0. \tag{1.1}$$

We use the standard assumptions on the exponent $p(\cdot)$, i.e. log-Hölder continuity, boundedness, and $\inf p > 1$. As an application of the fundamental convergence theorem, we generalize a removability theorem for Hölder continuous solutions to the variable exponent setting. To this end, we first prove that the set of irregular boundary points has zero capacity (the Kellogg property), and a boundary regularity result for the balayage.

The classical theory of balayage, including the fundamental convergence theorem, goes back to Brelot and Cartan, see [7, 8, 10] and the references therein. In nonlinear potential theory, the theorem was first proved in [22]. Apparently, the most recent version of the fundamental convergence theorem appears in [4], which gives a new proof based on the direct methods in the calculus of variations. This proof, which works also in metric spaces, does not require as advanced tools of potential theory as the previous proofs, which is an advantage for us.

To prove the fundamental convergence theorem, we follow the ideas of [4]. The crux of the argument is the fact that quasicontinuity, i.e. continuity outside a set of small capacity, is preserved in the limit processes that turn up. In Newtonian spaces used in [4], quasicontinuity is automatically preserved but this a non-trivial fact, see [6]. The main distinction between the arguments of [4] and our argumentation is that we use the variable exponent variant of standard Sobolev spaces instead of Newtonian spaces. Therefore, in our setting the notion of quasicontinuity requires extra care.

The key difference between the Eq. (1.1) and the ordinary p-Laplace equation is the fact that supersolutions to Eq. (1.1) can not be multiplied by positive constants. This difference has the implication that the existing approaches to the boundary regularity of balayage in [23] and [5] can not be easily extended to the variable exponent setting. Note that both [23] and [5] use advanced tools of potential theory, such as Perron solutions and barriers. We have not succeeded in extending the commonly used barrier characterization of boundary regularity to the setting of this paper; hence it seems that new approaches are needed to overcome the lack of homogeneity. We present such an approach by first identifying the balayage and the solution of obstacle problem for smooth obstacles by using the Kellogg property. Although our result on the boundary regularity is not quite as general as in [5], we find our approach simple, natural, and also sufficient for our present purposes.

In removability results, one is given a solution and an exceptional set, together with extra assumptions on both. The conclusion is then that the solution can be extended as a solution into the exceptional set. The removability theorem of [24] uses an estimate derived from these extra assumptions to show that the balayage of the given solution to be extended is in fact a solution. A comparison argument then shows that the balayage must coincide with the original solution in the whole domain.

The paper is organized as follows. In Section 2, we describe the background material, including variable exponent function spaces, logarithmic Hölder continuity and its implications, and the properties of $p(\cdot)$-solutions and $p(\cdot)$-superharmonic functions. Section 3 contains the proof of the fundamental convergence theorem, and Section 4 the definition and basic properties of balayage. In Sections 5 and 6, we prove the Kellogg property, and use it to study the boundary regularity of balayage. Finally, Section 7 concludes the paper with the proof of the removability theorem.

2 Preliminaries

Variable Exponent Function Spaces We call a measurable function $p : \mathbb{R}^n \to (1, \infty)$, $n \geq 2$, a *variable exponent* and denote

$$p_E^- = \inf_{x \in E} p(x) \quad \text{and} \quad p_E^+ = \sup_{x \in E} p(x),$$

whenever E is a measurable subset of $\mathbb{R}^n$. We assume that $1 < p_{\mathbb{R}^n}^- \leq p_{\mathbb{R}^n}^+ < \infty$.

Throughout this paper, we let Ω be an open and bounded subset of $\mathbb{R}^n$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions f defined on Ω for which the $p(\cdot)$-modular

$$\varrho_{p(\cdot)}(f) = \int_\Omega |f|^{p(x)}\, dx$$

is finite. The Luxemburg norm on this space is defined as

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space, see [28]. For a constant function p the variable exponent Lebesgue space coincides with the standard Lebesgue space.

A version of Hölder's inequality,

$$\int_\Omega fg\, dx \leq C\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)}, \tag{2.1}$$

holds for functions $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, where the conjugate exponent $p'(\cdot)$ of $p(\cdot)$ is defined pointwise. Further, since $1 < p^- \leq p^+ < \infty$, the dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, and the space $L^{p(\cdot)}(\Omega)$ is reflexive.

In estimates, one often needs to pass between the norm and the modular. This is accomplished by the coarse but useful inequalities

$$\begin{aligned}\min\left\{\|u\|_{L^{p(\cdot)}(\Omega)}^{p^+}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}\right\} &\leq \varrho_{p(\cdot)}(u)\\ &\leq \max\left\{\|u\|_{L^{p(\cdot)}(\Omega)}^{p^+}, \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}\right\},\end{aligned} \tag{2.2}$$

which follow from the definition of the norm in a straightforward manner.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions $f \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇f exists and satisfies $|\nabla f| \in L^{p(\cdot)}(\Omega)$. This space is a Banach space with the norm

$$\|f\|_{1,p(\cdot)} = \|f\|_{p(\cdot)} + \|\nabla f\|_{p(\cdot)}.$$

The local Sobolev space $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is now defined in the usual way. For basic properties of the spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, we refer to [28].

Logarithmic Hölder Continuity Smooth functions are not necessarily dense in $W^{1,p(\cdot)}(\Omega)$ without additional assumptions on the exponent $p(\cdot)$. This was observed by Zhikov [33, 34] in the context of the Lavrentiev phenomenon, which means that minimal values of variational integrals may differ depending on whether one minimizes over smooth functions or Sobolev functions. Zhikov has also introduced the *logarithmic Hölder continuity* condition to rectify this. The condition is

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)} \tag{2.3}$$

for all $x, y \in \Omega$ such that $|x - y| \leq 1/2$. If the exponent is bounded and satisfies Eq. (2.3), smooth functions are dense in variable exponent Sobolev spaces. Then we can define the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. We refer to [9, 12, 20, 32] for density results in variable exponent Sobolev spaces.

We will use logarithmic Hölder continuity in the form

$$R^{-(p_B^+ - p_B^-)} \leq C, \quad B = B(x_0, 2R) \Subset \Omega. \tag{2.4}$$

It is well-known that Eq. (2.4) is equivalent to the condition (2.3); a proof of this is given in [9, Lemma 3.2]. An elementary consequence of Eq. (2.4) is the inequality

$$C^{-1} R^{-p(y)} \leq R^{-p(x)} \leq C R^{-p(y)}, \tag{2.5}$$

which holds for any points $x, y \in B(x_0, 2R)$ with a constant depending only on the constant of Eq. (2.4). We use phrases like "by log-Hölder continuity" when applying either Eq. (2.4) or (2.5).

We assume throughout the rest of the paper that p is log-Hölder continuous.

Capacity We recall two approaches to the notion of capacity, see [13, 15, 19]. For $E \subset \mathbb{R}^n$ we denote

$$S_{p(\cdot)}(E) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^n) : u \geq 1 \text{ a.e. in an open neighbourhood of } E\}$$

and define the *Sobolev $p(\cdot)$-capacity* $C_{p(\cdot)}$ to be the number

$$C_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \int_{\mathbb{R}^n} |u|^{p(x)} + |\nabla u|^{p(x)}\, dx.$$

Here we use the convention that $C_{p(\cdot)}(E) = \infty$ if $S_{p(\cdot)}(E) = \emptyset$. In order to use the uniqueness result of [25], notice that the compatibility condition

$$C_{p(\cdot)}(U \setminus E) = C_{p(\cdot)}(U)$$

holds whenever U is open and E has zero measure. This is clear since $S_{p(\cdot)}(U \setminus E) = S_{p(\cdot)}(U)$.

We say that a given property holds $p(\cdot)$-*quasieverywhere* ($p(\cdot)$-*q.e.*) if it holds outside a set of $p(\cdot)$-capacity zero.

A function $u\colon \Omega \to [-\infty, \infty]$ is called $p(\cdot)$-*quasicontinuous* (in Ω) if for every $\varepsilon > 0$ there exists a set E, with $C_{p(\cdot)}(E) \leq \varepsilon$, so that u is continuous when restricted to $\Omega \setminus E$. Since the Sobolev capacity is an outer capacity, we can assume that E is open. It is well-known that every Sobolev function has a representative that is $p(\cdot)$-quasicontinuous, see [15].

In what follows we also use the following version of the *relative $p(\cdot)$-capacity*, see [19, Lemma 6.1]. For any $E \Subset \Omega$ we define

$$\operatorname{cap}_{p(\cdot)}(E, \Omega) = \inf \int_\Omega |\nabla u|^{p(x)}\, dx,$$

where the infimum is taken over all $p(\cdot)$-quasicontinuous $u \in W_0^{1,p(\cdot)}(\Omega)$ satisfying $u \geq 1$ $p(\cdot)$-q.e. in E.

Since smooth functions are dense in the Sobolev space, for $E \Subset \Omega$ it holds that $C_{p(\cdot)}(E) = 0$ if and only if $\operatorname{cap}_{p(\cdot)}(E, \Omega) = 0$. For the proof, see [19, Remark 3.13].

$p(\cdot)$-Supersolutions and $p(\cdot)$-Superharmonic Functions We say that a function $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a *$p(\cdot)$-supersolution* in Ω if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx \geq 0 \tag{2.6}$$

for all nonnegative test functions $\varphi \in C_0^\infty(\Omega)$. Further, u is a *$p(\cdot)$-subsolution* in Ω if $-u$ is a *$p(\cdot)$-supersolution* in Ω, and u is a *$p(\cdot)$-solution* in Ω if u is a $p(\cdot)$-supersolution and a $p(\cdot)$-subsolution in Ω. Equivalently, u is a $p(\cdot)$-solution in Ω if and only if Eq. (2.6) holds with equality for all $\varphi \in C_0^\infty(\Omega)$.

Recall that the dual of $L^{p(\cdot)}(\Omega)$ is the space $L^{p'(\cdot)}(\Omega)$ obtained by conjugating the exponent pointwise, see [28]. Combining this with the definition of $W_0^{1,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ allows us to employ test functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ with compact support in Ω by the usual approximation argument.

For any $f \in W^{1,p(\cdot)}(\Omega)$, there exists a unique $p(\cdot)$-solution $u \in W^{1,p(\cdot)}(\Omega)$ such that $u - f \in W_0^{1,p(\cdot)}(\Omega)$, see [16, Theorem 5.3]. It is well-known that $p(\cdot)$-solutions have a locally Hölder continuous representative, see [1, 2, 11].

For continuity of $p(\cdot)$-solutions up to the boundary, a natural generalization of the Wiener criterion is available, see [3]. More precisely, a boundary point $x_0 \in \partial\Omega$ is *regular*, if for every function $f \in W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$ the continuous $p(\cdot)$-solution u with $u - f \in W_0^{1,p(\cdot)}(\Omega)$ satisfies

$$\lim_{x \to x_0} u(x) = f(x_0).$$

A boundary point x_0 is regular if and only if

$$\int_0^1 \left(\frac{\operatorname{cap}_{p(\cdot)}(B(x_0, r) \cap \Omega,\, B(x_0, 2r))}{r^{n-p(x_0)}} \right)^{1/(p(x_0)-1)} \frac{dr}{r} = \infty,$$

see [3, Theorem 1.1]. We call a domain Ω regular, if every boundary point is regular.

In this paper, we need two consequences of the Wiener criterion. First, the sufficiency of the Wiener criterion implies that "simple" sets, i.e. balls, polyhedra and the like, are regular. A further consequence of this is that we can exhaust general open sets by regular ones. Second, if U is an open set such that $U \subset \Omega$, and x_0 is a boundary point of both U and Ω, then x_0 is regular with respect to U if it is regular with respect to Ω. This fact uses both the necessity and the sufficiency of the Wiener criterion.

We say that a function $u : \Omega \to (-\infty, \infty]$ is *$p(\cdot)$-superharmonic* in Ω if

(1) u is lower semicontinuous,
(2) u belongs to $L^t_{loc}(\Omega)$ for some $t > 0$, and
(3) the comparison principle holds: Let $U \Subset \Omega$ be an open set. If h is a $p(\cdot)$-solution in U, continuous in $\overline{U}$ and $u \geq h$ on ∂U, then $u \geq h$ in U.

Note that our definition is stronger than the one given in [14, 19]. More specifically, we require that u belongs to $L^t_{loc}(\Omega)$ for some $t > 0$, instead of just assuming that u is finite almost everywhere. This way we avoid the repetition of the L^t_{loc}-assumption. This assumption is necessary to carry out the Poisson modification procedure, see [14, Lemma 7.2], which is needed in the proofs of Lemmas 4.5 and 5.3 below.

The ess liminf-regularized representatives (see Eq. (3.1) below) of $p(\cdot)$-supersolutions are $p(\cdot)$-superharmonic, see [18, Theorem 4.1] and [14, Theorem 6.1]. However, the class of $p(\cdot)$-superharmonic functions is stricly larger, as the example considered in [14, Section 6] shows. Further properties of $p(\cdot)$-superharmonic functions can also be found in [14]. Many of these properties are similar to the case of p-superharmonic or $\mathcal{A}$-superharmonic functions considered in, e.g., [21, 23, 29]. We mention the fact that if u is $p(\cdot)$-superharmonic, then $\min(u, k)$ is a $p(\cdot)$-supersolution for any constant k. Indeed, it is easy to see that $\min(u, k)$ is $p(\cdot)$-superharmonic, and bounded $p(\cdot)$-superharmonic functions are $p(\cdot)$-supersolutions, see [14, Corollary 6.6].

The following pasting lemma is true for $p(\cdot)$-superharmonic functions. The proof of [23, Lemma 7.9] works verbatim.

Lemma 2.1 *Assume that D is open, that u is $p(\cdot)$-superharmonic in Ω and v is $p(\cdot)$-superharmonic in D. If the function*

$$s = \begin{cases} \min(u, v), & \text{in } D, \\ u, & \text{in } \Omega \setminus D \end{cases}$$

is lower semicontinuous, it is $p(\cdot)$-superharmonic in Ω.

In what follows, we need different versions of the comparison principle. We first recall two versions which do not deal with an exceptional set.

Lemma 2.2 *Let u be a $p(\cdot)$-supersolution in Ω and let v be a $p(\cdot)$-subsolution in Ω. If either*

(a) $\min(u - v, 0) \in W^{1,p(\cdot)}_0(\Omega)$ *or*
(b) $\limsup_{y\to x} v(y) \leq \liminf_{y\to x} u(y)$ *for all $x \in \partial\Omega$,*

then $u \geq v$ a.e. in Ω.

Proof

(a) It is enough to imitate the proof of [23, Lemma 3.18].
(b) Let $\varepsilon > 0$. By assumption, for each $x \in \partial\Omega$ there is an open ball B_x with the center x such that $v \leq u + \varepsilon$ in $\overline{B}_x$. Since Ω is bounded, we may pick a finite number of balls $B_{x_1}, \ldots, B_{x_n}$ such that $U = \cup_{i=1}^k B_{x_i}$ covers $\partial\Omega$. Denoting

$\Omega' = \Omega \setminus \overline{U}$, we have $\min(u+\varepsilon-v, 0) \in W_0^{1,p}(\Omega')$. By (a), we conclude that $u+\varepsilon \geq v$ a.e. in Ω'. Now it easily follows that $u \geq v$ a.e. in Ω. □

Roughly speaking, $p(\cdot)$-superharmonic functions can be characterized as solutions of the non-homogenous $p(\cdot)$-Laplace equation with a positive measure on the right hand side. In Section 7, we need the following fact, see [30, Theorem 4.2].

Lemma 2.3 *Let u be a $p(\cdot)$-superharmonic function in Ω. Then there is a positive Radon measure μ such that*

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \mu$$

in the sense of distributions.

For our removability result in Section 7, we record the following Caccioppoli and weak Harnack estimates. For the first, see [30, Lemma 3.1], for the second [2, Theorem 1] or [31, Lemma 3.7], and for the third [3, Lemma 6.3].

Lemma 2.4 *Let u be a nonnegative $p(\cdot)$-subsolution in an open ball $B \Subset \Omega$ and let $\eta \in C_0^\infty(B)$ be such that $0 \leq \eta \leq 1$. Then*

$$\int_B |\nabla u|^{p(x)} \eta^{p_B^+}\, dx \leq C \int_B u^{p(x)} |\nabla \eta|^{p(x)}\, dx.$$

Lemma 2.5 *Let u be a nonnegative bounded $p(\cdot)$-subsolution in an open ball $B = B(x_0, 2R) \Subset \Omega$. Then*

$$\operatorname*{ess\,sup}_{B(x_0,R)} u \leq C\left[\left(\fint_{B(x_0,\frac{3}{2}R)} u^{p_B^- -1}\, dx\right)^{1/p_B^- -1} + R\right].$$

Lemma 2.6 *Let u be a nonnegative bounded $p(\cdot)$-supersolution in an open ball $B = B(x_0, 2R) \Subset \Omega$. Then*

$$\left(\fint_{B(x_0,\frac{3}{2}R)} u^s\, dx\right)^{1/s} \leq C\left(\inf_{B(x_0,R)} u + R\right) \tag{2.7}$$

for any $0 < s < \frac{n}{n-1}(p_B^- - 1)$.

3 The Fundamental Convergence Theorem

Recall that we assume throughout this paper that p is log-Hölder continuous. In what follows, $\widetilde{f}$ denotes the *lim inf-regularization* of the function $f : \Omega \to [-\infty, +\infty]$. This is defined by

$$\widetilde{f}(x) := \liminf_{y\to x} f(y) = \lim_{r\to 0} \inf_{B(x,r)} f$$

for $x \in \Omega$. It is clear that $\widetilde{f} \leq f$ and it is elementary that $\widetilde{f}$ is lower semicontinuous. Observe here that we allow the value $-\infty$ for a lower semicontinuous function.

We also denote f^* for the *ess lim inf-regularization* of f defined by

$$f^*(x) := \operatorname{ess}\liminf_{y\to x} f(y) = \lim_{R\to 0} \operatorname*{ess\,inf}_{B(x,R)} f \tag{3.1}$$

for $x \in \Omega$. This regularization is lower semicontinuous as well. Note that the ess lim inf-regularization of a $p(\cdot)$-supersolution is $p(\cdot)$-superharmonic and $p(\cdot)$-quasicontinuous as well, see [14, 18, 19].

Our goal in this section is the following theorem.

Theorem 3.1 (The Fundamental Convergence Theorem) *Let $\mathcal{F}$ be a nonempty family of $p(\cdot)$-superharmonic functions in Ω. Assume that there is a function $f \in W^{1,p(\cdot)}_{loc}(\Omega)$ such that $u \geq f$ a.e. for all $u \in \mathcal{F}$. Let $w := \inf \mathcal{F}$. Then the following are true:*

(a) $\widetilde{w}$ *is $p(\cdot)$-superharmonic;*
(b) $\widetilde{w} = w^*$ *in Ω;*
(c) $\widetilde{w} = w$ *$p(\cdot)$-q.e. in Ω.*

To prove the theorem, we establish several auxiliary results.

Lemma 3.2 *Let (u_i) be a sequence of $p(\cdot)$-quasicontinuous functions in $W^{1,p(\cdot)}_{loc}(\Omega)$ such that (u_i) is bounded in $W^{1,p(\cdot)}(D)$ for any open set $D \Subset \Omega$. If $u = \lim_{i\to\infty} u_i$ $p(\cdot)$-q.e. in Ω, then u is $p(\cdot)$-quasicontinuous.*

Proof Fix open sets $D' \Subset D \Subset \Omega$. It is enough to prove the $p(\cdot)$-quasicontinuity in D'. Because (u_i) is bounded in $W^{1,p(\cdot)}(D)$ and $u = \lim u_i$ $p(\cdot)$-q.e., $u \in W^{1,p(\cdot)}(D)$ and there is a subsequence (u_i) such that (u_i) converges weakly to u in $W^{1,p(\cdot)}(D)$.

By Mazur's lemma we find a sequence (v_j) of convex combinations of u_i, i.e. $v_j = \sum_{i=j}^{N_j} a_{ij}u_i$, $a_{ij} \geq 0$, and $\sum_{i=j}^{N_j} a_{ij} = 1$, such that $v_j \to u$ strongly in $W^{1,p(\cdot)}(D)$. Clearly, v_j are $p(\cdot)$-quasicontinuous and $\lim_{j\to\infty} v_j = u$ $p(\cdot)$-q.e. in D. Hence (v_j) and (∇v_j) are Cauchy sequences in $L^{p(\cdot)}(D)$, and we find a subsequence (v_j), for which

$$\sum_{j=1}^{\infty} \int_D 2^{jp(x)} \left(|v_j - v_{j+1}|^{p(x)} + |\nabla v_j - \nabla v_{j+1}|^{p(x)} \right) dx < \infty.$$

Let $\varepsilon > 0$ and let $\phi \in C_0^\infty(D)$ be a cut-off function with $\phi = 1$ in D'. Then

$$\sum_{j=1}^{\infty} \int_D 2^{jp(x)} |\nabla \left(\phi \left(v_j - v_{j+1} \right) \right)|^{p(x)} \, dx < \infty,$$

and we may fix j_ε such that

$$\sum_{j=j_\varepsilon}^{\infty} \int_D 2^{jp(x)} |\nabla \left(\phi \left(v_j - v_{j+1} \right) \right)|^{p(x)} \, dx < \frac{\varepsilon}{2}.$$

Denoting

$$E_j := \left\{ x \in D' \mid |v_j(x) - v_{j+1}(x)| > 2^{-j} \right\}, \quad E_\varepsilon^1 := \bigcup_{j=j_\varepsilon}^{\infty} E_j,$$

we have

$$\operatorname{cap}_{p(\cdot)}(E_j; D) \le \int_U 2^{jp(x)} |\nabla \left(\phi \left(v_j - v_{j+1}\right)\right)|^{p(x)}\, dx$$

and therefore

$$\operatorname{cap}_{p(\cdot)}\left(E^1_\varepsilon; D\right) \le \sum_{j=j_\varepsilon}^{\infty} \operatorname{cap}_{p(\cdot)}\left(E_j; D\right) < \frac{\varepsilon}{2}.$$

Moreover, by the subadditivity and the $p(\cdot)$-quasicontinuity, we find a set $E^2_\varepsilon \subset D'$ with $\operatorname{cap}_{p(\cdot)}(E^2_\varepsilon; D) < \frac{\varepsilon}{2}$ such that the restrictions of all the functions v_j to $D' \setminus E^2_\varepsilon$ are continuous. Then, for $E_\varepsilon = E^1_\varepsilon \cup E^2_\varepsilon$, we have $\operatorname{cap}_{p(\cdot)}(E_\varepsilon; D) < \varepsilon$ and

$$|v_j - v_k| \le \sum_{l=j}^{k-1} 2^{-l} \le 2^{1-j}$$

in $D' \setminus E_\varepsilon$ whenever $j_\varepsilon \le j \le k$. Hence (v_j) converges to u uniformly in $D' \setminus E_\varepsilon$, so that the restriction of u to $D' \setminus E_\varepsilon$ is continuous. This completes the proof. □

Theorem 3.3 *Let (u_i) be a sequence of $p(\cdot)$-quasicontinuous $p(\cdot)$-supersolutions in Ω such that $u_i \to u$ $p(\cdot)$-q.e. in Ω. If there is a function $f \in W^{1,p(\cdot)}_{loc}(\Omega)$ such that $|u_i| \le f$ a.e. in Ω for all $i = 1, 2, \ldots,$ then $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ and u is $p(\cdot)$-quasicontinuous.*

Proof It is immediate that (u_i) is bounded in $L^{p(\cdot)}_{loc}(\Omega)$. Hence the claim follows from Lemma 3.2 once we prove that (∇u_i) is bounded in $L^{p(\cdot)}_{loc}(\Omega)$.

To establish the required boundedness, we let $\varphi = \eta^{p^+}(f - u_i)$, where $\eta \in C^\infty_0(\Omega)$ is a smooth cutoff function such that $0 \le \eta \le 1$, and p^+ is the maximum of $p(x)$ over the support of η. Then $\varphi \ge 0$ a.e. due to the assumptions on f and u_i, and using it as a test function we obtain

$$\begin{aligned} &p^+ \int_\Omega |\nabla u_i|^{p(x)-2} (\nabla u_i \cdot \nabla \eta) \eta^{p^+-1} (f - u_i)\, dx \\ &\quad + \int_\Omega |\nabla u_i|^{p(x)-2} (\nabla u_i \cdot \nabla f) \eta^{p^+}\, dx - \int_\Omega |\nabla u_i|^{p(x)-2} (\nabla u_i \cdot \nabla u_i) \eta^{p^+}\, dx \ge 0. \end{aligned}$$

By Young's inequality

$$ab \le \left(\frac{1}{\varepsilon}\right)^{p(x)-1} \frac{a^{p(x)}}{p(x)} + \varepsilon \frac{b^{\,p'(x)}}{p'(x)},$$

where $\varepsilon > 0$, we have

$$\begin{aligned} p^+ |\nabla u_i|^{p(x)-1} \eta^{p^+-1} |\nabla \eta| |f - u_i| &\le \frac{1}{4} |\nabla u_i|^{p(x)} \eta^{\frac{p^+-1}{p(x)-1} p(x)} \\ &\quad + C |\nabla \eta|^{p(x)} |f - u_i|^{p(x)} \end{aligned}$$

and similarly

$$|\nabla u_i|^{p(x)-1} \eta^{p^+} |\nabla f| \le \frac{1}{4} |\nabla u_i|^{p(x)} \eta^{\frac{p^+-1}{p(x)-1} p(x)} + C \eta^{p(x)} |\nabla f|^{p(x)}.$$

Since $\eta \leq 1$, we get

$$\eta^{\frac{p^+-1}{p(x)-1}p(x)} \leq \eta^{p^+}.$$

Further, by the assumption $|u_i| \leq f$ a.e., we see that $|f-u_i| \leq 2f$ almost everywhere. We insert the above estimates into Eq. (3.2), and find that

$$\begin{aligned}
0 \leq& \int_\Omega |\nabla u_i|^{p(x)}\eta^{p^+}\,dx \\
\leq& p^+\int_\Omega |\nabla u_i|^{p(x)-1}\eta^{p^+-1}|\nabla\eta||f-u_i|dx + \int_\Omega |\nabla u_i|^{p(x)-1}\eta^{p^+}|\nabla f|dx \\
\leq& \frac{1}{2}\int_\Omega |\nabla u_i|^{p(x)}\eta^{p^+}\,dx + C\int_\Omega |\nabla\eta|^{p(x)} f^{p(x)}dx + C\int_\Omega \eta^{p^+}|\nabla f|^{p(x)}dx.
\end{aligned}$$

Absorbing the matching term from the right hand side, we get the estimate

$$\int_\Omega |\nabla u_i|^{p(x)}\eta^{p^+}\,dx \leq C\left[\int_\Omega |\nabla\eta|^{p(x)} f^{p(x)}dx + \int_\Omega \eta^{p^+}|\nabla f|^{p(x)}dx\right].$$

Since the right hand side is independent of i, this inequality implies the desired boundedness. □

Theorem 3.4 *Let (u_i) be a decreasing sequence of $p(\cdot)$-quasicontinuous $p(\cdot)$-supersolutions in Ω, and assume there is $f \in W^{1,p(\cdot)}_{loc}(\Omega)$ so that $u_i \geq f$ a.e. in Ω for all $i = 1, 2, \dots$. Let $u := \lim_{i\to\infty} u_i$. Then*

(i) *u is a $p(\cdot)$-quasicontinuous $p(\cdot)$-supersolution in Ω;*
(ii) *u^* is $p(\cdot)$-superharmonic in Ω;*
(iii) *$u = u^*$ $p(\cdot)$-q.e. in Ω.*

Proof Since $|u_i| \leq \max\{|u_1|, |f|\}$ and $\max\{|u_1|, |f|\} \in W^{1,p(\cdot)}_{\text{loc}}(\Omega)$, Theorem 3.3 implies that $u \in W^{1,p(\cdot)}_{\text{loc}}(\Omega)$ and u is $p(\cdot)$-quasicontinuous.

Moreover, by the proof of Theorem 3.3, the sequence (u_i) is bounded in $W^{1,p(\cdot)}(D)$ for any open set $D \Subset \Omega$. Hence the proof of [14, Theorem 5.1] implies that u is a $p(\cdot)$-supersolution in Ω, and [18, Theorem 4.1] implies that u^* is $p(\cdot)$-superharmonic in Ω and $u = u^*$ a.e. in Ω. Since every $p(\cdot)$-superharmonic function is $p(\cdot)$-quasicontinuous by [19, Theorem 6.7], the claim $u = u^*$ $p(\cdot)$-q.e. in Ω holds true by [25]. □

Remark 3.5

(a) The argumentation in the proof of Theorem 3.4 also yields the following increasing version: Let (u_i) be an increasing sequence of $p(\cdot)$-quasicontinuous $p(\cdot)$-supersolutions in Ω, and assume there is $f \in W^{1,p(\cdot)}_{\text{loc}}(\Omega)$ so that $u_i \leq f$ a.e. in Ω for all $i = 1, 2, \dots$. Let $u := \lim_{i\to\infty} u_i$. Then

   (i) u is a $p(\cdot)$-quasicontinuous $p(\cdot)$-supersolution in Ω;
   (ii) u^* is $p(\cdot)$-superharmonic in Ω;
   (iii) $u = u^*$ $p(\cdot)$-q.e. in Ω.

(b) Our proof for Theorem 3.4 (iii) differs from the ones in [4, Theorem 5.2] and [26, Theorem 5.1]. It works also in the framework of [4] and [26] since the p-quasicontinuity of p-superharmonic functions in metric spaces is proved in [27, Corollary 2.4].

As the last step we extend Theorem 3.4 to the class of $p(\cdot)$-superharmonic functions.

Theorem 3.6 *Let (u_i) be a decreasing sequence of $p(\cdot)$-superharmonic functions in Ω such that $v := \lim_{i\to\infty} u_i \geq f$ for some $f \in W^{1,p(\cdot)}_{loc}(\Omega)$. Then v^* is $p(\cdot)$-superharmonic in Ω and $v = v^*$ $p(\cdot)$-q.e. in Ω.*

Proof Let $v_k := \min\{v, k\} = \lim_{i\to\infty} \min\{u_i, k\}$. By [19, Theorem 6.7] and [14, Corollary 6.6], each $\min\{u_i, k\}$ is a $p(\cdot)$-quasicontinuous $p(\cdot)$-supersolution in Ω. Since $\min\{u_i, k\} \geq \min\{f, k\} \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ a.e. in Ω, Theorem 3.4 implies that v_k is a $p(\cdot)$-quasicontinuous $p(\cdot)$-supersolution, $v_k = v_k^*$ $p(\cdot)$-q.e. in Ω and v_k^* is $p(\cdot)$-superharmonic in Ω.

Since $v_k^* = \min\{v^*, k\}$ in Ω, it is easy to check that v^* is $p(\cdot)$-superharmonic. Moreover,

$$v^* = \lim_{k\to\infty} v_k^* = \lim_{k\to\infty} v_k = v$$

$p(\cdot)$-q.e. in Ω. □

Proof of the Fundamental Convergence Theorem Once we have established Theorem 3.6, the proof of [4, Theorem 8.3] works verbatim. For proving (b) we only need the definitions and the fact that $p(\cdot)$-superharmonic functions are ess liminf-regularized. For proving (a) and (c) we need Theorem 3.6 together with Choquet's topological lemma. For the proof of the latter, see e.g. [23, Lemma 8.3]. □

4 Balayage

In this section, we define and study the balayage of a function. Roughly speaking, the balayage procedure can be thought of as Perron's method applied to the obstacle problem, allowing the obstacle to be almost any function. Having the fundamental convergence theorem available at this stage makes developing the basic properties of balayage rather effortless.

We call the function $\psi : \Omega \to [-\infty, +\infty]$ in the following definitions an *obstacle function*. We denote

$$\Phi^\psi = \left\{ u \mid u \text{ is } p(\cdot)\text{-superharmonic in } \Omega \text{ and } u \geq \psi \text{ in } \Omega \right\},$$

$$R^\psi = R^\psi(\Omega) = \inf \Phi^\psi,$$

and define the *balayage* $\widetilde{R}^\psi = \widetilde{R}^\psi(\Omega)$ of ψ in Ω as the lim inf-regularization of the *réduite* R^ψ. If the set Φ^ψ is empty, we use the convention $\widetilde{R}^\psi \equiv +\infty$. More generally, for any $E \subset \Omega$, we define

$$\widetilde{R}^\psi_E = \widetilde{R}^\psi_E(\Omega) = \widetilde{R}^{\psi\chi_E}(\Omega),$$

where χ_E is the characteristic function of E.

Remark 4.1 Notice that $\widetilde{R}^{\psi_1} \leq \widetilde{R}^{\psi_2}$ in Ω whenever $\psi_1 \leq \psi_2$ in Ω.

Theorem 4.2 *Suppose that $\psi \geq f$ a.e. in Ω for some $f \in W^{1,p(\cdot)}_{loc}(\Omega)$ and $\Phi^\psi \neq \emptyset$. Then the following assertions hold true:*

(a) $\widetilde{R}^\psi = (R^\psi)^*$ *in* Ω;
(b) $\widetilde{R}^\psi = R^\psi$ $p(\cdot)$*-q.e. in* Ω;
(c) $\widetilde{R}^\psi$ *is* $p(\cdot)$*-superharmonic.*

Proof The claims follow immediately from the fundamental convergence theorem since $R^\psi \geq \psi$ in Ω. □

We need the following simple auxiliary property.

Lemma 4.3 *Suppose that $\psi \geq f$ a.e. in Ω for some $f \in W^{1,p(\cdot)}_{loc}(\Omega)$ and $\Phi^\psi \neq \emptyset$. Then, for all $\alpha \in \mathbb{R}$, we have*

$$\widetilde{R}^{\psi+\alpha} = \widetilde{R}^\psi + \alpha.$$

Proof This is a straightforward consequence of the definition of $\widetilde{R}^\psi$ and the fact that $u + a$ is $p(\cdot)$-superharmonic if u is $p(\cdot)$-superharmonic and $a \in \mathbb{R}$. □

Remark 4.4 In the variable exponent case we do not have the more general form

$$\widetilde{R}^{\lambda\psi+\alpha} = \lambda\widetilde{R}^\psi + \alpha,$$

for $\lambda > 0$ and $\alpha \in \mathbb{R}$.

Lemma 4.5 *If $\psi \geq 0$ and $E \subset \Omega$, then $\widetilde{R}^\psi_E$ is a weak solution in $\Omega \setminus \overline{E}$.*

Proof We may imitate the proof of [23, Lemma 8.4]. Note that the proof uses the Poisson modification procedure, so the L^t_{loc} –condition in the definition of $p(\cdot)$-superharmonic functions is needed here, see [14, Lemma 7.1]. □

5 The Kellogg Property

In this section, we prove the Kellogg property, the fact that the set of irregular boundary points has zero capacity. The balayage is a key tool in this. Recall that a boundary point $x_0 \in \partial\Omega$ is regular, if for every function $f \in W^{1,p(\cdot)}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ the continuous $p(\cdot)$-solution u with $u - f \in W^{1,p(\cdot)}_0(\Omega)$ satisfies

$$\lim_{x \to x_0} u(x) = f(x_0).$$

Theorem 5.1 (The Kellogg Property) *The set of irregular boundary points of an open, bounded set Ω is of $p(\cdot)$-capacity zero.*

The arguments follow the lines of [23, Chapter 8], with two exceptions. First, we already have the fundamental convergence theorem available, and Lemma 5.2 becomes easy in comparison to [23, Lemma 8.7]. Second, a modification to take into account the fact that $p(\cdot)$-solutions can not be multiplied by constants is needed in Lemma 5.4 below.

Lemma 5.2 *Let K be a compact subset of Ω, and let λ be a positive constant. Then $\widetilde{R}_K^{\lambda} = \lambda$ $p(\cdot)$-quasieverywhere in K.*

Proof This follows immediately from the fundamental convergence theorem. □

Lemma 5.3 *Suppose that u is a nonnegative $p(\cdot)$-superharmonic function in a ball B. If K is a compact subset of B, then*

$$\lim_{x\to y} \widetilde{R}_K^u(B)(x) = 0$$

for all $y \in \partial B$.

Proof The proof of [23, Lemma 8.8] works here; for the reader's convenience, we give the details. Let $v \in \Phi_K^u$, let $B_0 \Subset B$ be a ball containing K, and denote $D = B \setminus B_0$. By replacing v by its Poisson modification in a suitable annulus containing ∂B_0, we may assume $v < M < \infty$ on ∂B_0. Note that the remark concerning the Poisson modification in the proof of Lemma 4.5 applies here as well.

Let h be the unique continuous $p(\cdot)$-solution with boundary values 0 on ∂B, and M on ∂B_0. Then the function

$$s = \begin{cases} v, & \text{in } \overline{B}_0, \\ \min(v, h) & \text{in } D \end{cases}$$

is $p(\cdot)$-superharmonic by the pasting Lemma 2.1, and belongs to Φ_K^u. We now have

$$0 \leq \widetilde{R}_K^u \leq s.$$

Since $s \leq h$ near ∂B, the claim follows since D is a regular set by the Wiener criterion. □

Lemma 5.4 *Let x_0 be a boundary point of an open, bounded set Ω. If*

$$\widetilde{R}^{\lambda}_{\overline{B}\setminus\Omega}(2B)(x_0) = \lambda$$

for all balls B with rational centers and radii containing x_0 and positive rational numbers λ, then the point x_0 is regular.

Proof Let $f \in C(\overline{\Omega}) \cap W^{1,p(\cdot)}(\Omega)$, and let $h \in W^{1,p(\cdot)}(\Omega)$ be the unique continuous $p(\cdot)$-solution such that $h - f \in W_0^{1,p(\cdot)}(\Omega)$. Without loss of generality, we may assume that $f(x_0) = 0$ by adding a constant.

For an arbitrary $\varepsilon > 0$, we may find a ball B containing x_0 with rational center and radius such that $\partial(2B) \cap \Omega \neq \emptyset$, and $|f| \leq \varepsilon$ in $2\overline{B} \cap \overline{\Omega}$. For some rational number $\lambda \geq \sup_{\overline{\Omega}} |f|$, set

$$u = \begin{cases} \lambda - \widetilde{R}^{\lambda}_{\overline{B}\setminus\Omega}(2B) + \varepsilon & \text{in } 2B, \\ \lambda + \varepsilon & \text{in } \Omega \setminus 2B. \end{cases}$$

Lemmas 4.5, 5.3, and the pasting Lemma 2.1 imply that u is $p(\cdot)$-superharmonic and continuous in $\Omega \cup 2B$.

Since $u \geq f$ in Ω, it is easy to see that $\min(u - h, 0) \in W_0^{1,p(\cdot)}(\Omega)$. Since u is bounded, u is a $p(\cdot)$-supersolution in Ω by [14, Corollary 6.6]. Hence by Lemma 2.2 and the continuity of u and h we have $u \geq h$ in Ω. By assumption,

$$\limsup_{x \to x_0} h(x) \leq \limsup_{x \to x_0} u(x) \leq u(x_0) = \varepsilon.$$

Similarly $-u \leq f$ in Ω, which implies that $\min(h + u, 0) \in W_0^{1,p(\cdot)}(\Omega)$. Reasoning in the above fashion we see that

$$\liminf_{x \to x_0} h(x) \geq \liminf_{x \to x_0} -u(x) \geq -u(x_0) = -\varepsilon.$$

Since ε was arbitrary, we get

$$\lim_{x \to x_0} h(x) = 0 = f(x_0),$$

as desired. □

Proof of the Kellogg Property Let E be the set of all irregular boundary points. If $x_0 \in E$, Lemma Lemma 5.4 implies the existence of a ball B_i with rational center and radius and a rational number λ_i such that

$$\widetilde{R}^{\lambda_i}_{\overline{B}_i\setminus\Omega}(2B_i)(x_0) < \lambda_i.$$

Hence

$$E \subset \bigcup_i \left\{ x \in \overline{B}_i \setminus \Omega : \widetilde{R}^{\lambda_i}_{\overline{B}_i\setminus\Omega}(2B_i)(x) < \lambda_i \right\},$$

and by Lemma 5.2 each of the sets in the union is of zero $p(\cdot)$-capacity. Thus the claim follows by the subadditivity of the capacity. □

6 Boundary Regularity of Balayage

In this section, we prove that the balayage of a continuous function attains the right boundary values at regular boundary points. To accomplish this, we use the connection between balayage and the obstacle problem.

In the constant exponent case, the property of Lemma 5.4 can be used to characterize regular boundary points, see [23, Theorem 9.17]. In our setting, such a characterization is open, since the closely related barrier characterization of boundary regularity is not available. The lack of barrier characterization can be (at least to some extent) compensated by the Wiener criterion, which was established in [3]. In this section, we need the Wiener criterion only in Theorem 6.1 below to conclude that if $x_0 \in \partial\Omega$ is regular relative to Ω and $U \subset \Omega$ is open with $x_0 \in \partial U$, then x_0 is regular relative to U.

Obstacle Problem Let $\psi : \Omega \to [-\infty, \infty)$ be a function, the *obstacle*, and let $w \in W^{1,p(\cdot)}(\Omega)$ be a function which will give the boundary values. We define

$$\mathcal{K}_{\psi,w} = \left\{ u \in W^{1,p(\cdot)}(\Omega) : u - w \in W_0^{1,p(\cdot)}(\Omega),\ u \geq \psi \text{ a.e.} \right\},$$

and call a function $u \in \mathcal{K}_{\psi,w}$ a *solution to the obstacle problem in* $\mathcal{K}_{\psi,w}$, if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla(u - v)\, dx \leq 0$$

for all $v \in \mathcal{K}_{\psi,w}$. It follows easily from the definition that $p(\cdot)$-solutions to the obstacle problem are $p(\cdot)$-supersolutions. Further, if the obstacle ψ is continuous, then there is a unique continuous solution to the obstacle problem which is a $p(\cdot)$-solution in the open set $\{x \in \Omega : u(x) > \psi(x)\}$, see [14, Theorem 4.11].

We start by proving that solutions to the obstacle problem have the same regular boundary points as $p(\cdot)$-solutions.

Theorem 6.1 *Let $\varphi \in \mathcal{C}(\overline{\Omega}) \cap W^{1,p(\cdot)}(\Omega)$ and let u be the continuous solution to the obstacle problem in $\mathcal{K}_{\varphi,\varphi}$. If $x_0 \in \partial\Omega$ is a regular boundary point, then*

$$\lim_{x \to x_0} u(x) = \varphi(x_0).$$

Proof Let $x_0 \in \Omega$ be regular and let h be the $p(\cdot)$-solution in Ω with boundary values φ in the Sobolev sense. Since u is a $p(\cdot)$-supersolution and $u - h \in W_0^{1,p(\cdot)}(\Omega)$, we have $u \geq h$ in Ω by Lemma 2.2 and by the fact that u is continuous. Thus

$$\liminf_{x \to x_0} u(x) \geq \liminf_{x \to x_0} h(x) = \varphi(x_0).$$

Denote $U = \{u > \varphi\} \subset \Omega$ and assume first that $x_0 \in \partial U$. Since the Wiener criterion characterizes regular boundary points, it is clear that x_0 is also regular for the set U. Observing that $u - \varphi \in W_0^{1,p(\cdot)}(U)$ and u is a $p(\cdot)$-solution in U, we arrive at

$$\lim_{\substack{x \to x_0 \\ x \in U}} u(x) = \varphi(x_0)$$

due to the regularity of the point x_0. Since $u \leq \varphi$ in $\Omega \setminus U$, it follows that

$$\limsup_{x \to x_0} u(x) \leq \varphi(x_0). \tag{6.1}$$

Finally, if $x_0 \notin \partial U$, then Eq. (6.1) holds trivially. Hence the claim follows. □

To relate solutions to obstacle problems and balayage, we need the following improved comparison lemma.

Lemma 6.2 *Let u and $-v$ be bounded $p(\cdot)$-superharmonic functions in Ω such that*

$$\limsup_{y \to x} v(y) \leq \liminf_{y \to x} u(y)$$

for $p(\cdot)$-quasievery $x \in \partial\Omega$. If either $u \in W^{1,p(\cdot)}(\Omega)$ or $v \in W^{1,p(\cdot)}(\Omega)$, then $v \leq u$ in Ω.

Proof The proof is essentially the same as in [23, Lemma 7.37]. For the reader's convenience, we recall the proof with required references. Let

$$E = \left\{ x \in \partial\Omega : \liminf_{y\to x} u(y) < \limsup_{y\to x} v(y) \right\}$$

be the exceptional set with $C_{p(\cdot)}(E) = 0$.

We assume first that E is compact. By symmetry, we may assume that $u \in W^{1,p}(\Omega)$. By [15, Lemma 3.6], we may test the capacity of E by smooth functions. Hence following [23, Lemma 7.37], we find a decreasing sequence (φ_i) of continuous functions in $W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $\varphi_i = M$ in a neighborhood of E, $0 \le \varphi_i \le M$, and $\|\varphi_i\|_{1,p(\cdot)} \to 0$. Here we denote $M = \sup |u| + \sup |v|$. Then write

$$\psi_i = u + \varphi_i$$

and let u_i be the $p(\cdot)$-superharmonic solution to the obstacle problem in $\mathcal{K}_{\psi_i,\psi_i}(\Omega)$. By the ess liminf property $u_i \ge \psi_i$ in Ω. Therefore

$$\limsup_{y\to x} v(y) \le \sup_{\Omega} |v| \le \liminf_{y\to x} \psi_i(y) \le \liminf_{y\to x} u_i(y)$$

for all $x \in E$ and for each i. Since v and u_i are bounded, they are $p(\cdot)$-supersolutions and hence by Lemma 2.2 $u_i \ge v$ in Ω. By virtue of [14, Theorem 5.4] and the uniqueness of the solution to the obstacle problem, we have $\lim_{i\to\infty} u_i = u$ a.e. in Ω. By the ess liminf-property, $u \ge v$ in Ω.

Finally, for any expectional set E and for any positive integer j, we may consider the set

$$U_j := \left\{ x \in \partial\Omega : \liminf_{y\to x} u(y) + \tfrac{1}{j} > \limsup_{y\to x} v(y) \right\},$$

which is open in $\partial\Omega$. The set $E_j = \partial\Omega \setminus U_j$ is a compact subset of E and therefore the first part of the proof yields $u \ge v - \frac{1}{j}$ in Ω. Hence the claim of lemma follows by letting $j \to \infty$. □

With the help of the previous lemma, we show that for smooth functions, solving the obstacle problem gives the same result as taking the balayage.

Theorem 6.3 *Let $\varphi \in \mathcal{C}(\overline{\Omega}) \cap W^{1,p(\cdot)}(\Omega)$ and let u be the continuous solution to the obstacle problem in $\mathcal{K}_{\varphi,\varphi}$. Then*

$$u = \widetilde{R}^{\varphi}(\Omega).$$

Proof Since u is a continuous $p(\cdot)$-supersolution and $u \ge \varphi$, we see that

$$u \ge \widetilde{R}^{\varphi}(\Omega).$$

To prove the opposite inequality, let v be a $p(\cdot)$-superharmonic function such that $v \ge \varphi$. By the fundamental convergence theorem and the continuity of u it is enough to show that $v \ge u$. Since φ is bounded, we are free to assume that v is bounded by considering the function $\min(v, \sup_\Omega \varphi)$.

Observe that the claim $v \geq u$ holds trivially outside $U = \{u > \varphi\}$. To prove the inequality $v \geq u$ in U, let $x_0 \in \partial U$. If $x_0 \notin \partial\Omega$, we have

$$\liminf_{x \to x_0} v(x) \geq \varphi(x_0) = \lim_{x \to x_0} u(x) \tag{6.2}$$

by the continuity of u. If $x_0 \in \partial\Omega$ and x_0 is a regular boundary point for U, then Eq. (6.2) holds true by Theorem 6.1. Hence the inequality (6.2) is true for $p(\cdot)$-quasievery boundary point of U by the Kellogg property of Theorem 5.1. Since u and v are bounded, $u \in W^{1,p(\cdot)}(\Omega)$, and u is a continuous $p(\cdot)$-solution in U, Lemma 6.2 implies the claim $v \geq u$ in U. □

The fact that the balayage of a continuous function attains the right boundary values now follows from Theorem 6.3 by an approximation argument. The following result corresponds to [23, Theorem 9.26].

Corollary 6.5 *Let $\varphi : \overline{\Omega} \to \mathbb{R}$ be continuous and let $x_0 \in \partial\Omega$ be a regular boundary point. Then*

$$\lim_{x \to x_0} \widetilde{R}^{\varphi}(x) = \varphi(x_0).$$

Proof Let $\psi_j \in C^\infty(\mathbb{R}^n)$ be such that the sequence (ψ_j) converges uniformly to φ in $\overline{\Omega}$. For each positive integer i, fix j_i such that $\psi_{j_i} - \frac{1}{i} \leq \varphi \leq \psi_{j_i} + \frac{1}{i}$ in $\overline{\Omega}$. By Remark 4.1 and Lemma 4.3,

$$\widetilde{R}^{\psi_{j_i}} - \frac{1}{i} \leq \widetilde{R}^{\varphi} \leq \widetilde{R}^{\psi_{j_i}} + \frac{1}{i}$$

in Ω. Hence Theorems 6.1 and 6.3 imply that

$$\begin{aligned}\psi_{j_i}(x_0) - \frac{1}{i} &= \limsup_{x \to x_0} \widetilde{R}^{\psi_{j_i}}(x) - \frac{1}{i} \leq \limsup_{x \to x_0} \widetilde{R}^{\varphi}(x) \\ &\leq \limsup_{x \to x_0} \widetilde{R}^{\psi_{j_i}}(x) + \frac{1}{i} = \psi_{j_i}(x_0) + \frac{1}{i}.\end{aligned}$$

By letting $i \to \infty$

$$\varphi(x_0) = \limsup_{x \to x_0} \widetilde{R}^{\varphi}(x).$$

The assertion concerning limes inferior is proved analogously. □

In the proof of Theorem 6.1, we could replace the Wiener criterion by the Kellogg property. The conclusion would then be that the right boundary value is attained at quasievery point, and a similarly modified form of Corollary 6.5 would follow. This approach would imply our Theorem 6.3. However, this weaker approach would not allow to apply any version of our comparison principles in the proof of Theorem 7.2, since Lemma 6.2 requires that either u or v belongs to $W^{1,p(\cdot)}(\Omega)$. Hence we need Corollary 6.5 in the form above to apply Lemma 2.2 in Theorem 7.2.

Remark 6.6 Let $\varphi : \overline{\Omega} \to \mathbb{R}$ be continuous and let $u = \widetilde{R}^{\varphi}$ be its balayage in Ω. Then u is continuous in Ω and u is a $p(\cdot)$-solution in the open set $\{x \in \Omega : u(x) > \varphi(x)\}$.

This can be seen by approximating φ uniformly by functions $\psi_j \in \mathcal{C}^\infty(\mathbb{R}^n)$ in $\overline{\Omega}$ and using Theorem 6.3 together with the fact that the required properties hold for the solutions of the obstacle problem in $\mathcal{K}_{\psi_j,\psi_j}$.

7 Removability Theorem for Hölder Continuous $p(\cdot)$-Solutions

In this section, we generalize a removability theorem of Kilpeläinen and Zhong [24] for equations similar to the p-Laplacian to the variable exponent setting. Balayage is a key tool in this theorem.

In removability theorems, one is given a solution u in $\Omega \setminus E$, and some additional assumptions on the exceptional set $E \subset \Omega$ and the solution u. The conclusion is then that the solution can be extended to the whole domain Ω. In [24], the solution is assumed to be Hölder continuous with an exponent α, and the exceptional set E is assumed to be of zero s-Hausdorff measure, where $s = n - p + \alpha(p-1)$. Thus, our assumption on the removable set E needs to be given in terms of the $s(\cdot)$-Hausdorff measure $\mathcal{H}^{s(\cdot)}$ defined in [17]. We assume that $s(\cdot)$ is a log-Hölder continuous and positive function, and define first

$$\mathcal{H}^{s(\cdot)}_\delta(E) = \inf \sum_j r_j^{s(x_j)}, \tag{7.1}$$

where the infimum is taken over countable coverings of E by balls $B(x_j, r_j)$ such that $r_j \le \delta$. Then the actual $s(\cdot)$-Hausdorff measure is given by

$$\mathcal{H}^{s(\cdot)}(E) = \lim_{\delta \to 0} \mathcal{H}^{s(\cdot)}_\delta(E).$$

This is a special case of a measure construction due to Carathéodory, so that $\mathcal{H}^{s(\cdot)}$ is a Borel regular outer measure.

The following lemma provides the key estimate. In the proof, we will use the fact that $1/p$, p' and $1/p'$ are log-Hölder continuous if p itself is log-Hölder continuous. This follows from the general fact that if f is a Lipschitz continuous function, then the composition $x \mapsto f(p(x))$ is log-Hölder continuous.

Lemma 7.1 *Let $K \subset \Omega$ be compact and let ψ be continuous in $\overline{\Omega}$ such that*

$$|\psi(x) - \psi(y)| \le M|x-y|^\alpha$$

for all $x \in K$ and $y \in \Omega$ with $0 < M < \infty$ and $0 < \alpha < 1$. Set $u = \widetilde{R}^\psi(\Omega)$ and

$$\mu = -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right).$$

Then

$$\mu(B(x,r)) \le C r^{n-p(x)+\alpha(p(x)-1)}$$

for all $x \in K$ and $r < r_0 = \min(1, \frac{1}{64}\operatorname{dist}(K, \partial\Omega))$. Here C depends on n, p, M, and α.

Proof For the existence of μ, see Lemma 2.3. Denote the contact set by

$$I = \{x \in \Omega : \psi(x) = u(x)\},$$

and pick a point $x_0 \in I$. We may assume that $u(x_0) = \psi(x_0) = 0$ by adding a constant. We consider first a radius $r \le \frac{1}{8}\operatorname{dist}(x_0, \partial\Omega)$ and set $\gamma_0 = \operatorname{osc}_{B(x_0,8r)} \psi$. Since $\{u > \gamma_0\}$ is a subset of $\{u > \psi\}$ and u is a $p(\cdot)$-solution in $\{u > \psi\}$ (by Remark 6.6), $(u - \gamma_0)_+$ is a non-negative $p(\cdot)$-subsolution in $B(x_0, 8r)$. To see this, apply the pasting Lemma 2.1 with $D = \{u > \psi\}$ to zero and $-(u - \gamma_0)_+$. On the other hand $u + \gamma_0$ is a non-negative $p(\cdot)$-supersolution in $B(x_0, 8r)$ as a locally bounded $p(\cdot)$-superharmonic function; see Theorem 4.2 and [14, Corollary 6.6]. Hence we may use the weak Harnack estimates (Lemmas 2.5 and 2.6) to obtain that

$$\begin{aligned}\sup_{B(x_0,2r)} (u - \gamma_0) &\le C\left[\left(\fint_{B(x_0,3)} (u - \gamma_0)_+^q\,dx\right)^{1/q} + r\right]\\ &\le C\left[\left(\fint_{B(x_0,3r)} (u + \gamma_0)^q\,dx\right)^{1/q} + r\right]\\ &\le C\left[\inf_{B(x_0,2r)} (u + \gamma_0) + r\right]\\ &\le C\,(\gamma_0 + r)\,.\end{aligned}$$

Here $q = p^-_{B(x_0,4r)} - 1$. Since $u \ge \psi \ge -\gamma_0$ in $B(x_0, 8r)$, we have

$$\operatorname*{osc}_{B(x_0,2r)} u = \sup_{B(x_0,2r)} (u - \gamma_0) + \gamma_0 - \inf_{B(x_0,2r)} u \le C(\gamma_0 + r).$$

Next we assume that $x_0 \in I$ and r satisfy

$$\operatorname{dist}(x_0, K) \le 2r < \min\left(1, \tfrac{1}{32}\operatorname{dist}(K, \partial\Omega)\right).$$

Then $2r \le \frac{1}{31}\operatorname{dist}(x_0, \partial\Omega)$ and we have the estimate

$$\operatorname*{osc}_{B(x_0,2r)} u \le C(\gamma_0 + r) \le C(M + 1)r^\alpha \tag{7.2}$$

by the assumption on ψ.

Next we estimate $\mu(B(x_0, r))$. To simplify the notation, we write $p^+ = p^+_{B(x_0,2r)}$ and $p^- = p^-_{B(x_0,2r)}$ for the rest of the proof. Let $\eta \in C_0^\infty(B(x_0, 2r))$ be a cutoff function with $0 \le \eta \le 1$, $\eta = 1$ in $B(x_0, r)$, and $|\nabla\eta| \le 5/r$. We have

$$\begin{aligned}\mu(B(x_0, r)) &\le \int_{B(x_0,2r)} \eta^{p^+}\,d\mu\\ &= p^+ \int_{B(x_0,2r)} |\nabla u|^{p(x)-2}(\nabla u \cdot \nabla\eta)\eta^{p^+-1}\,dx\\ &\le C\||\nabla u|^{p(x)-1}\eta^{p^+-1}\|_{p'(\cdot)}\|\nabla\eta\|_{p(\cdot)}.\end{aligned} \tag{7.3}$$

Here we have used the variable exponent Hölder inequality (2.1) and the norms are computed over the ball $B(x_0, 2r)$.

For the first factor in Eq. (7.3), we have

$$\begin{aligned}\||\nabla u|^{p(x)-1}\eta^{p^+-1}\|_{p'(\cdot)} \le \max\Big\{&\varrho_{p'(\cdot)}(|\nabla u|^{p(x)-1}\eta^{p^+-1})^{1/(p')^+},\\ &\varrho_{p'(\cdot)}(|\nabla u|^{p(x)-1}\eta^{p^+-1})^{1/(p')^-}\Big\}\end{aligned}$$

by Eq. (2.2). Here the modulars, $(p')^+$ and $(p')^-$ are computed over $B(x_0, 2r)$.

Since $\eta \le 1$, we have $\eta^{\frac{p^+-1}{p(x)-1}p(x)} \le \eta^{p^+}$. Thus

$$\varrho_{p'(\cdot)}\left(|\nabla u|^{p(x)-1}\eta^{p^+-1}\right) \le \int_{B(x_0,2r)} |\nabla u|^{p(x)}\eta^{p^+}\,dx.$$

By applying the Caccioppoli inequality (Lemma 2.4) to $\sup_{B(x_0,2r)} u - u$ we get

$$\int_{B(x_0,2r)} |\nabla u|^{p(x)}\eta^{p^+}\,dx \le \int_{B(x_0,2r)} \left(\sup_{B(x_0,2r)} u - u\right)^{p(x)} |\nabla \eta|^{p(x)}\,dx$$
$$\le C r^{\alpha p_0 + n - p_0},$$

where $p_0 = p(x_0)$. Here we have also used the log-Hölder continuity of $p(\cdot)$ in the form of inequality (2.5) and the fact that Eq. (7.2) implies the estimate $\operatorname{osc}_{B(x_0,2r)} u \le Cr^{\alpha}$. We combine these estimates, and get

$$\| |\nabla u|^{p(x)-1}\eta^{p^+-1}\|_{p'(\cdot)} \le C \max\left\{ r^{(\alpha p_0+n-p_0)/(p')^+}, r^{(\alpha p_0+n-p_0)/(p')^-}\right\}.$$

Appealing to the log-Hölder continuity of $1/p'(\cdot)$, we can replace both $1/(p')^+$ and $1/(p')^-$ by $1/(p_0)' = (p_0-1)/p_0$ by means of Eq. (2.5). Thus we finally get

$$\| |\nabla u|^{p(x)-1}\eta^{p^+-1}\|_{p'(\cdot)} \le C r^{(\alpha p_0+n-p_0)(p_0-1)/p_0}.$$

The second factor in Eq. (7.3) can also be estimated in a similar fashion by Eqs. (2.2) and (2.5). Indeed, we have

$$\|\nabla \eta\|_{p(\cdot)} \le \max\left\{ \varrho_{p(\cdot)}(\nabla\eta)^{1/p^+}, \varrho_{p(\cdot)}(\nabla\eta)^{1/p^-}\right\}$$
$$\le \max\left\{ Cr^{n/p^- -1}, Cr^{n/p^+ -1}\right\}$$
$$\le Cr^{(n-p_0)/p_0}$$

by the log-Hölder continuity of $p(\cdot)$ and $1/p(\cdot)$.

The above considerations give the estimate

$$\mu(B(x_0, r)) \le Cr^{\alpha(p_0-1)+n-p_0}$$

whenever x_0 is a point in the contact set I such that $\operatorname{dist}(x_0, K) \le r < \min(1, \frac{1}{32}\operatorname{dist}(K, \partial\Omega))$.

Finally, for any point $x \in K$ and $r < \min(1, \frac{1}{64}\operatorname{dist}(K, \partial\Omega))$, there are two possibilities. If $B(x,r) \cap I = \emptyset$, then $\mu(B(x,r)) = 0$ because $B(x,r)$ is a subset of $U = \{u > \psi\}$ and u is a $p(\cdot)$-solution in U. Otherwise, there exists a point $x_0 \in B(x,r) \cap I$. We get

$$\mu(B(x,r)) \le \mu(B(x_0,2r)) \le Cr^{\alpha(p_0-1)+n-p_0} \le Cr^{\alpha(p(x)-1)+n-p(x)}$$

by the above estimate and log-Hölder continuity. In either case, the desired estimate holds. □

The removability theorem is now a rather simple consequence of the preceding lemma.

Theorem 7.2 *Let $E \subset \Omega$ be closed and let u be continuous in Ω, $p(\cdot)$-solution in $\Omega \setminus E$, and assume that*

$$|u(x_0) - u(y)| \le M|x_0 - y|^{\alpha}$$

for all $y \in \Omega$ and $x_0 \in E$ for some $0 < \alpha < 1$. If

$$\mathcal{H}^{s(\cdot)}(E) = 0,$$

where

$$s(x) = \alpha(p(x) - 1) + n - p(x),$$

then u is a $p(\cdot)$-solution in Ω.

Let us make some remarks before proving the theorem. First, the result is sharp in the constant exponent case in the following sense: a closed set E is removable for α-Hölder continuous solutions of the p-Laplacian if and only if $\mathcal{H}^s(E) = 0$, where s and α are related by

$$s = \alpha(p - 1) + n - p.$$

See [24, Theorem 1.10]. For the converse, one assumes that $\mathcal{H}^s(E) > 0$, and constructs a function u satisfying the assumptions of the theorem, but not the conclusion, as follows. Frostman's lemma gives a nontrivial measure ν supported in E such that

$$\nu(B(x, r)) \le Cr^s.$$

Then the desired function u is a solution of

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \nu.$$

Indeed, Theorem 1.14 of [24] yields exactly the Hölder exponent α for u. As far as we know, there is currently no variable exponent version of this Hölder regularity theorem in the literature. Hence the converse of Theorem 7.2 remains open.

Second, the case $\alpha = 0$ is included in the removability result in [31] provided that we interpret the assumption on u as boundedness. Indeed, by using the estimate

$$\operatorname{cap}_{p(\cdot)}\left(\overline{B}(x, r),\, B(x, 2r)\right) \le Cr^{n-p(x)}$$

it is not hard to see that if $\mathcal{H}^{n-p(\cdot)}(E) = 0$, then also $\operatorname{cap}_{p(\cdot)}(E, \Omega) = 0$, which is the assumption of Theorem 3.9 of [31] for bounded solutions.

Proof Let $D \Subset \Omega$ be a regular set, let $v = \widetilde{R}^u(D)$, and

$$\mu = -\operatorname{div}(|\nabla v|^{p(x)-2}\nabla v).$$

Pick a compact set $K \subset E \cap D$. By the assumption on u, Lemma 7.1 yields

$$\mu(B(x, r)) \le Cr^{s(x)}$$

for all $x \in K$ and for all sufficiently small radii r. Since $\mathcal{H}^{s(\cdot)}(K) = 0$, for any $\varepsilon > 0$ we find a covering of K by sufficiently small balls, so that

$$\mu(K) \le \sum_j \mu\left(B\left(x_j, r_j\right)\right) \le C\sum_j r^{s(x_j)} < \varepsilon.$$

Thus $\mu(E \cap D) = 0$, and v is a $p(\cdot)$-solution in D such that $u \leq v$.

The same reasoning applies to $\widetilde{R}^{-u}(D)$, so we get a $p(\cdot)$-solution w such that $w \geq u$ by setting $w = -\widetilde{R}^{-u}(D)$. By the boundary regularity of balayage (see Corollary 6.5), $v = u = w$ on ∂D, and thus $v = w$ in D by the comparison principle of Lemma 2.2. We now have

$$w \leq u \leq v = w$$

in D, so that u must be a $p(\cdot)$-solution in D, and then also in the whole domain Ω. □

Acknowledgements Part of the research in this paper was done while the first two authors were visiting the De Giorgi Center in Pisa During September 2009. The authors wish to thank the De Giorgi Center for the hospitality. The second author is supported by the Norwegian Research Council project "Nonlinear Problems in Mathematical Analysis". The authors would also like to thank the anonymous rewiever for carefully reading the manuscript.

References

1. Acerbi, E., Mingione, G.: Regularity results for a class of functionals with nonstandard growth. Arch. Ration. Mech. Anal. **156**, 121–140 (2001)
2. Alkhutov, Yu.A.: The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition. Diff. Equ. **33**(12), 1653–1663 (1997)
3. Alkhutov, Yu.A., Krasheninnikova, O.V.: Continuity at boundary points of solutions of quasi-linear elliptic equations with a nonstandard growth condition. Izv. Ross. Akad. Nauk Ser. Mat. **68**(6), 3–60 (2004)
4. Björn, A., Björn, J., Parviainen, M.: Lebesgue points and the fundamental convergence theorem for superharmonic functions. Rev. Mat. Iberoam. **26**(1), 147–174 (2010)
5. Björn, A., Björn, J., Mäkäläinen, T., Parviainen, M.: Nonlinear balayage on metric spaces. Nonlinear Anal. **71**(5–6), 2153–2171 (2009)
6. Björn, A., Björn, J., Shanmugalingam, N.: Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces. Houston J. Math. **34**(4), 1197–1211 (2008)
7. Brelot, M.: Minorantes sousharmoniques, extrémales et capaciles. J. Math. Pures Appl. **207**(9), 836–839 (1945)
8. Cartan, H.: Theorie du potential newtonian: energie, capacite, suites de potentiels. Bull. Soc. Math. France **73**, 74–106 (1945)
9. Diening, L.: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. Math. Inequal. Appl. **7**(2), 245–253 (2004)
10. Doob, J.L.: Classical potential theory and its probabilistic counterpart. In: Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 262. Springer, New York (1984)
11. Fan, X., Zhao, D.: A class of De Giorgi type and Hölder continuity. Nonlinear Anal. **36**(3), 295–318 (1999)
12. Harjulehto, P.: Variable exponent Sobolev spaces with zero boundary values. Math. Bohem. **132**(2), 125–136 (2007)
13. Harjulehto, P., Hästö, P., Koskenoja, M.: Properties of capacities in variable exponent Sobolev spaces. J. Anal. Appl. **5**(2), 71–92 (2007)
14. Harjulehto, P., Hästö, P., Koskenoja, M., Lukkari, T., Marola, N.: An obstacle problem and superharmonic functions with nonstandard growth. Nonlinear Anal. **67**(12), 3424–3440 (2007)
15. Harjulehto, P., Hästö, P., Koskenoja, M., Varonen, S.: Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$. J. Funct. Spaces Appl. **1**, 17–33 (2003)
16. Harjulehto, P., Hästö, P., Koskenoja, M., Varonen, S.: The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values. Potential Anal. **25**(3), 205–222 (2006)
17. Harjulehto, P., Hästö, P., Latvala, V.: Sobolev embeddings in metric measure spaces with variable dimension. Math. Z. **254**(3), 591–609 (2006)
18. Harjulehto, P., Kinnunen, J., Lukkari, T.: Unbounded supersolutions of nonlinear equations with nonstandard growth. Bound. Value Probl. Article ID 48348, vol. 2007, 20 pp. (2007). Available at http://www.hindawi.com/GetArticle.aspx?doi=10.1155/2007/48348

19. Harjulehto, P., Latvala, V.: Fine topology of variable exponent energy superminimizers. Ann. Acad. Sci. Fenn. Math. **33**(2), 491–510 (2008)
20. Hästö, P.: On the density of continuous functions in variable exponent Sobolev space. Rev. Mat. Iberoamer. **23**(1), 215–237 (2007)
21. Heinonen, J., Kilpeläinen, T.: A-superharmonic functions and supersolutions of degenerate elliptic equations. Ark. Mat. **26**(1), 87–105 (1988)
22. Heinonen, J., Kilpeläinen, T.: Polar sets for supersolutions of degenerate elliptic equations. Math. Scand. **63**, 136–159 (1988)
23. Heinonen, J., Kilpeläinen, T., Martio, O.: Nonlinear Potential Theory of Degenerate Elliptic Equations. Dover, Mineola (2006). Unabridged republication of the 1993 original
24. Kilpeläinen, T., Zhong, X.: Removable sets for continuous solutions of quasilinear elliptic equations. Proc. Am. Math. Soc. **130**(6), 1681–1688. Electronic (2002)
25. Kilpeläinen, T.: A remark on the uniqueness of quasi continuous functions. Ann. Acad. Sci. Fenn. Math. **23**(1), 261–262 (1998)
26. Kinnunen, J., Martio, O.: Nonlinear potential theory on metric spaces. Ill. Math. J. **46**, 857–883 (2002)
27. Kinnunen, J., Shanmugalingam, N.: Polar sets on metric spaces. Trans. Am. Math. Soc. **358**, 11–37 (2006)
28. Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czech. Math. J. **41**(116), 592–618 (1991)
29. Lindqvist, P.: On the definition and properties of p-superharmonic functions. J. Reine Angew. Math. **365**, 67–79 (1986)
30. Lukkari, T.: Elliptic equations with nonstandard growth involving measures. Hiroshima Math. J. **38**(1), 155–176 (2008)
31. Lukkari, T.: Singular solutions of elliptic equations with nonstandard growth. Math. Nachr. **282**(12), 1770–1787 (2009)
32. Samko, S.: Denseness of $C_0^\infty(\mathbf{R}^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(\mathbf{R}^N)$. In: Direct and Inverse Problems of Mathematical Physics (Newark, DE, 1997). Int. Soc. Anal. Appl. Comput., vol. 5, pp. 333–342. Kluwer, Dordrecht (2000)
33. Zhikov, V.V.: On Lavrentiev's phenomenon. Russ. J. Math. Phys. **3**(2), 249–269 (1995)
34. Zhikov, V.V.: On some variational problems. Russ. J. Math. Phys. **5**(1), 105–116 (1997)