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A Montel Type Result for Super-Polyharmonic Functions on $\mathbf{R}^N$

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Abstract Our aim in this paper is to discuss a Montel type result for a family $\mathcal{F}$ of super-polyharmonic functions on $\mathbf{R}^N$. We give a condition on spherical means to assure that $\mathcal{F}$ contains a sequence converging outside a set of capacity zero.

Keywords Polyharmonic functions · Super-polyharmonic functions · Spherical means · Riesz decomposition

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1 Introduction

A function u on an open set $\Omega \subset \mathbf{R}^N$ ($N \geq 2$) is called polyharmonic of order m in Ω if $u \in C^{2m}(\Omega)$ and $\Delta^m u = 0$ on Ω, where m is positive integer, Δ denotes the Laplacian and $\Delta^m u = \Delta^{m-1}(\Delta u)$. We denote by $\mathcal{H}^m(\Omega)$ the space of polyharmonic functions of order m on Ω. For fundamental properties of polyharmonic functions,

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we refer the reader to the book by Aronszajn, Creese and Lipkin [3]. We say that a locally integrable function u on Ω is super-polyharmonic of order m in Ω if

(1) $(-\Delta)^m u$ is a nonnegative measure on Ω, that is,

$$\int_\Omega u(x)(-\Delta)^m \varphi(x)dx \geq 0 \qquad \text{for all nonnegative } \varphi \in C_0^\infty(\Omega);$$

(2) u is lower semicontinuous in Ω;
(3) every point of Ω is a Lebesgue point of u

(see [5]); $(-\Delta)^m u$ is referred to as the Riesz measure of u and denoted by μ_u. If $(-\Delta)^m T \geq 0$ on Ω in the sense of distribution, then one can find a super-polyharmonic function u of order m in Ω such that

$$T = u \qquad \text{in the sense of distribution.}$$

We denote by $\mathcal{SH}^m(\Omega)$ the space of super-polyharmonic functions of order m on Ω. In particular, u is biharmonic if $u \in \mathcal{H}^2(\Omega)$ and u is super-biharmonic if $u \in \mathcal{SH}^2(\Omega)$. These spaces are important in connection with the Bergman space.

The open ball and the sphere centered at x with radius r are denoted by $B(x,r)$ and $S(x,r)$, respectively. In particular, the unit ball $B(0,1)$ is written as $\mathbf{B}$. For a Borel measurable function f on $\mathbf{R}^N$ and a Borel measure ν on $\mathbf{R}^N$, we use the standard notation

$$⨍_A f(y)\, d\nu(y) = \frac{1}{\nu(A)} \int_A f(y)\, d\nu(y)$$

for a Borel measurable set A with $0 < \nu(A) < \infty$, if the integral exists. When $d\nu = dS$ and $A = S(x,r)$, we write

$$M(f,x,r) = ⨍_{S(x,r)} f(y)\, dS(y) = \frac{1}{\sigma_N r^{N-1}} \int_{S(x,r)} f(y)\, dS(y),$$

where σ_N is the surface area of a unit sphere. If x is the origin, then we write simply $B(r) = B(0,r)$, $S(r) = S(0,r)$ and $M(f,r) = M(f,0,r)$.

In case $0 < \alpha < N$, we define the capacity of a set $E \subset \mathbf{R}^N$ by

$$\mathrm{Cap}_\alpha(E) = \inf \nu(\mathbf{R}^N),$$

where the infimum is taken over all nonnegative measures ν such that

$$\int_{\mathbf{R}^N} |x-y|^{\alpha-N}\, d\nu(y) \geq 1 \qquad \text{for every } x \in E.$$

In case $\alpha = N$, for $R > 0$, we define the (relative) capacity of a set $E \subset B(R)$ by

$$\mathrm{Cap}_N^{(R)}(E) = \inf \nu(\mathbf{R}^N),$$

where the infimum is taken over all nonnegative measures ν such that

$$\int_{\mathbf{R}^N} \log\left(\frac{2R}{|x-y|}\right) d\nu(y) \geq 1 \qquad \text{for every } x \in E;$$

we write $\mathrm{Cap}_N(E) = 0$ if

$$\mathrm{Cap}_N^{(R)}(E \cap B(R)) = 0$$

for all $R > 0$. In case $\alpha > N$, we write $\mathrm{Cap}_\alpha(E) = 0$ if E is empty. For fundamental properties of capacity, we refer to Hayman-Kennedy [7], Landkof [11] and the last author [14].

The famous Montel's theorem says that a family of analytic functions on complex plane $\mathbf{C}$ which is uniformly bounded on each compact set is normal, see [16, Section 2.2]. Normal families of subharmonic functions were extensively discussed by Anderson and Baernstein [2], Kondratyuk and Tarasyuk [10] and Supper [18]. In this paper, we extend their results to super-polyharmonic functions.

Let φ be a positive continuous function on $[0, \infty)$ such that

$$r^{2-2m}\varphi(r) \text{ is nondecreasing.} \tag{1.1}$$

Our main aim in this paper is to prove the following result.

Theorem 1.1 *Let $\mathcal{S}_m$ be the set of all super-polyharmonic functions of order m in $\mathbf{R}^N$ which are polyharmonic of order m in $\mathbf{B}$. Let $\{u_n\}$ be a sequence in $\mathcal{S}_m$ such that*

$$M(((-1)^m u_n)^+, r) \leq \varphi(r) \qquad \text{for all } r > 0 \text{ and } n. \tag{1.2}$$

If in addition $\{u_n\}$ is uniformly bounded on a neighbourhood of the origin, then there exist a subsequence $\{u_{n_j}\} \subset \{u_n\}$ and $u \in \mathcal{S}_m$ satisfying the following assertions:

(a) *$\{u_{n_j}\}$ converges to u uniformly on each compact set K with $K \cap \overline{\cup_j \mathrm{supp}(\mu_{u_{n_j}})} = \emptyset$;*

(b) *for all $x \in \mathbf{R}^N$,*

$$\liminf_{j\to\infty} u_{n_j}(x) \geq u(x);$$

(c) *there exists a set $E \subset \mathbf{R}^N$ with $\mathrm{Cap}_{2m}(E) = 0$ such that for all $x \in \mathbf{R}^N \setminus E$*

$$\liminf_{j\to\infty} u_{n_j}(x) = u(x);$$

in case $N < 2m$, one may take E as the empty set.

Theorem 1.2 *Let $\{u_n\}$ be a sequence in $\mathcal{SH}^m(\mathbf{R}^N)$ such that*

$$M(|u_n|, r) \leq \varphi(r) \tag{1.3}$$

for all $r > 0$ and n. Then there exist a subsequence $\{u_{n_j}\} \subset \{u_n\}$ and $u \in \mathcal{SH}^m(\mathbf{R}^N)$ satisfying (a), (b) *and* (c) *in Theorem* 1.1.

For proofs of these results, we first show that the family $\{\mu_{u_n}\}$ of Riesz measures is vaguely bounded in $\mathbf{R}^N$, and then apply the local Riesz representation result for $\mathcal{SH}^m(\mathbf{R}^N)$.

2 Normal Family of Polyharmonic Functions

It is well-known that a collection of harmonic functions on a domain Ω which is uniformly bounded on each compact subset of Ω is a normal family, see [4, Theorem 2.6.]. In this section, we extend this result to polyharmonic functions.

Let us begin with the following result.

Lemma 2.1 (cf. [14, Lemma 4.5 of Section 8]) *Let k be a nonnegative integer. Then there is a positive constant $C(N,k,m)$ depending only on N, k and m such that*

$$|\nabla^k h(x)| \le C(N,k,m) r^{-k} \fint_{B(a,2r)} |h(y)|\, dy \tag{2.1}$$

for all $x \in B(a,r)$ and $h \in \mathcal{H}^m(B(a,2r))$, where ∇^k denotes the gradient iterated k times.

As an application of Lemma 2.1, we discuss Montel's type result for a class of polyharmonic functions.

Lemma 2.2 *Let $R > 0$. If $\{h_n\}$ is a sequence in $\mathcal{H}^m(B(2R))$ such that*

$$\sup_n \int_{B(2R)} |h_n(x)| dx < \infty, \tag{2.2}$$

then there exists a subsequence of $\{h_n\}$ which converges uniformly on $\overline{B(R)}$.

Proof Using Lemma 2.1 with $k = 0, 1$, we can find a positive constant $C(R)$ such that

$$\sup_{x \in \overline{B(R)}} |h_n(x)| \le C(R)$$

and

$$\sup_{x \in \overline{B(R)}} |\nabla h_n(x)| \le C(R)$$

for every n. The latter inequality gives

$$|h_n(x) - h_n(y)| \le C(R)|x - y|$$

for each $x, y \in \overline{B(R)}$, which implies that $\{h_n\}$ is equicontinuous on $\overline{B(R)}$. Hence Arzela-Ascoli theorem will give an existence of subsequence converging uniformly on $\overline{B(R)}$. □

Remark 2.3 Let $\{h_n\}$ be a sequence of $\mathcal{H}^m(\Omega)$ such that

$$\sup_n \int_K |h_n(x)|dx < \infty, \tag{2.3}$$

for each compact subset K of Ω. By Lemma 2.2, there exists a subsequence of $\{h_n\}$ which converges uniformly to a function $h \in \mathcal{H}^m(\Omega)$ on each compact subset of Ω.

Proposition 2.4 *Let $R > 0$ and $\{h_n\}$ be a sequence in $\mathcal{H}^m(B(2R))$ such that*

$$\sup_n \int_{B(2R)} h_n^+(x)dx < \infty.$$

Suppose that $\{h_n\}$ is uniformly bounded on a neighbourhood of the origin. Then there exists a subsequence of $\{h_n\}$ which converges uniformly on $\overline{B(R)}$.

Proof In view of the formula of Pizetti [15] for polyharmonic functions of order m, we have

$$M(h_n, r) = \sum_{j=0}^{m-1} a_j r^{2j} \Delta^j h_n(0) \tag{2.4}$$

for $0 < r < 2R$, where $a_0 = 1$ and

$$a_j = \frac{1}{2^j j! N(N+2)\cdots(N+2j-2)} \qquad (j \geq 1). \tag{2.5}$$

By Lemma 2.1, we see that $\left\{\Delta^j h_n(0)\right\}$ are bounded for $0 \leq j \leq m-1$. Since $|h_n| = 2h_n^+ - h_n$, we have

$$\sup_n \int_{B(2R)} |h_n(x)|\, dx < \infty.$$

Hence the proposition follows from Lemma 2.2. □

3 Spherical Means for Super-Polyharmonic Functions

For a multi-index $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and a point $x = (x_1, x_2, \ldots, x_N)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_N,$$

$$\lambda! = \lambda_1!\lambda_2!\cdots\lambda_N!,$$

$$x^\lambda = {x_1}^{\lambda_1}{x_2}^{\lambda_2}\cdots{x_N}^{\lambda_N}$$

and

$$D^\lambda = \left(\frac{\partial}{\partial x}\right)^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1}\left(\frac{\partial}{\partial x_2}\right)^{\lambda_2}\cdots\left(\frac{\partial}{\partial x_N}\right)^{\lambda_N}.$$

Consider the Riesz kernel of order $2m$ defined by

$$\mathcal{R}_{2m}(x) = \begin{cases} \alpha_{N,m}|x|^{2m-N} \log(1/|x|) & \text{if } 2m-N \text{ is an even nonnegative integer,} \\ \alpha_{N,m}|x|^{2m-N} & \text{otherwise.} \end{cases}$$

Here the constant $\alpha_{N,m}$ is chosen such that $(-\Delta)^m \mathcal{R}_{2m}$ is the Dirac measure at the origin, that is,

$$\alpha_{N,m}^{-1} = 2^{m-1}(m-1)!\sigma_N \prod_{\substack{0 \le j \le m-1 \\ j \ne m-N/2}} (N - 2m + 2j)$$

(see [8]).

If $u \in \mathcal{SH}^m(B(R))$, then u is of the form

$$u(x) = \int_{B(R')} \mathcal{R}_{2m}(x-y)\, d\mu_u(y) + h_{R'}(x)$$

for $x \in B(R')$, where $0 < R' < R$ and $h_{R'} \in \mathcal{H}^m(B(R'))$. This is called the Riesz representation of super-polyharmonic functions. For further results of a representation of super-polyharmonic functions, we refer to [1, 5, 6, 9] and [17].

Let us begin with the following result.

Lemma 3.1 *Let $R > 1$. Then*

$$\int_{B(R)} |\mathcal{R}_{2m}(x-y)|\ dx \le CR^{2m} \log(2R)$$

for all $y \in B(R)$, where C is a positive constant depending only on N and m.

Proof Note that

$$\begin{aligned} \int_{B(R)} |x-y|^{2m-N} dx &\le \int_{B(y,2R)} |x-y|^{2m-N} dx \\ &\le CR^{2m}. \end{aligned}$$

If $2m - N$ is a nonnegative even integer, then

$$\begin{aligned} \int_{B(R)} \left| |x-y|^{2m-N} \log(1/|x-y|) \right|\ dx &\le \int_{B(y,1)} |x-y|^{2m-N} \log(1/|x-y|)\, dx \\ &\quad + \int_{B(y,2R)\setminus B(y,1)} |x-y|^{2m-N} \log|x-y|\, dx \\ &\le CR^{2m} \log(2R). \end{aligned}$$

Thus the lemma is proved. □

Following the book by Hayman-Kennedy [7], we consider the remainder term in the Taylor expansion of $\mathcal{R}_{2m}(\cdot - y)$ given by

$$\mathcal{R}_{2m,L}(x,y) = \mathcal{R}_{2m}(x-y) - \sum_{|\lambda| \le L} \frac{x^\lambda}{\lambda!} \left(D^\lambda \mathcal{R}_{2m}\right)(-y),$$

where L is a real number; if $L < 0$, then $\mathcal{R}_{2m,L}(x, y) = \mathcal{R}_{2m}(x - y)$. Further, we consider the function

$$K_{2m,L}(x, y) = \begin{cases} \mathcal{R}_{2m}(x - y) \text{ when } y \in \mathbf{B}, \\ \mathcal{R}_{2m,L}(x, y) \text{ when } y \notin \mathbf{B} . \end{cases}$$

We note that

$$(-\Delta)^m \mathcal{R}_{2m,L}(\cdot, y) = (-\Delta)^m K_{2m,L}(\cdot, y) = \delta_y,$$

where δ_y denotes the Dirac measure at y. By using the kernel function $K_{2m,L}(x, y)$, we see that $u \in \mathcal{SH}^m(\mathbf{R}^N)$ is of the form

$$u(x) = \int_{B(R)} K_{2m,L}(x, y)\, d\mu_u(y) + h_R(x) \tag{3.1}$$

for $x \in B(R)$, where $h_R \in \mathcal{H}^m(B(R))$ (see [7] and [13, 14]).

Since $\Delta^j \mathcal{R}_{2m}(x)$ depends only on $r = |x|$, we write

$$\Delta^j \mathcal{R}_{2m}(r) = \Delta^j \mathcal{R}_{2m}(x)$$

when $r = |x|$.

For $0 < t \leq r$, set

$$g_m(t, r) = \sum_{j=0}^{m-1} a_j \left(t^{2j} \Delta^j \mathcal{R}_{2m}(r) - r^{2j} \Delta^j \mathcal{R}_{2m}(t)\right)$$

with a_j given by (2.5).

Here let us note the following result concerning spherical means for generalized Riesz kernels.

Lemma 3.2 (cf. [6, Lemmas 4.3 and 4.4]) *The following hold:*

(1) $M(\mathcal{R}_{2m,2m-2}(\cdot, y), r) = \begin{cases} g_m(|y|, r) & \text{if } |y| < r, \\ 0 & \text{if } |y| \geq r. \end{cases}$

(2) $(-1)^m g_m(\cdot, r)$ *is positive and strictly decreasing in* $(0, r)$ *for each fixed* $r > 0$.

We next give a growth property for Riesz measures of super-polyharmonic functions satisfying growth conditions on spherical means. Let $u \in \mathcal{S}_m$ and $\mu_u = (-\Delta)^m u$. Note from our assumption $u \in \mathcal{S}_m$ that

$$\mu_u(\mathbf{B}) = 0. \tag{3.2}$$

Then representation (3.1) implies that for each $R > 0$,

$$u(x) = \int_{B(R)} \mathcal{R}_{2m,2m-2}(x, y)\, d\mu_u(y) + h_R(x) \qquad (x \in B(R)), \tag{3.3}$$

where $h_R \in \mathcal{H}^m(B(R))$. Since u is polyharmonic of order m in **B**, we see that $\Delta^j h_R(0) = \Delta^j u(0)$ for $j = 0, 1, \ldots, m-1$. From (2.4) and Lemma 3.2, we have for $0 < r < R$,

$$\begin{aligned}
(-1)^m M(u,r) &= (-1)^m \fint_{S(r)} \left(\int_{B(R)} \mathcal{R}_{2m,2m-2}(x,y)\, d\mu_u(y) \right) dS(x) + (-1)^m M(h_R, r) \\
&= (-1)^m \int_{B(R)} M(\mathcal{R}_{2m,2m-2}(\cdot,y), r)\, d\mu_u(y) + (-1)^m \sum_{j=0}^{m-1} a_j r^{2j} \Delta^j h_R(0) \\
&= \int_{B(r)} (-1)^m g_m(|y|, r)\, d\mu_u(y) + (-1)^m \sum_{j=0}^{m-1} a_j r^{2j} \Delta^j u(0) \\
&\geq \int_{B(r/2)} (-1)^m g_m(r/2, r)\, d\mu_u(y) + (-1)^m \sum_{j=0}^{m-1} a_j r^{2j} \Delta^j u(0) \\
&= r^{2m-N} (-1)^m g_m(1/2, 1) \mu_u(B(r/2)) + (-1)^m \sum_{j=0}^{m-1} a_j r^{2j} \Delta^j u(0). \qquad (3.4)
\end{aligned}$$

Thus, with the aid of (1.1), (3.2) and (3.4), we have the following result.

Lemma 3.3 *Suppose that $u \in \mathcal{S}_m$ satisfies*

$$(-1)^m M(u,r) \leq \varphi(r) \qquad (\text{for all } r > 0) \tag{3.5}$$

and

$$|\Delta^j u(0)| \leq C_1 \qquad (j = 0, 1, \ldots, m-1). \tag{3.6}$$

Then there exists a positive constant C_2 depending only on N, m, C_1 and $\varphi(1)$ such that

$$\mu_u(B(r)) \leq C_2 r^{N-2m} \varphi(2r) \tag{3.7}$$

for all $r > 0$.

4 Proofs of Theorems 1.1 and 1.2

For a proof of Theorem 1.1, we need the following lemma.

Lemma 4.1 (cf [11] and [13, Theorem 4.5, Chapter 2]) *Let ν_n and ν be nonnegative measures on $\mathbf{R}^N$ such that $\{\nu_n\}$ converges vaguely to ν on $\mathbf{R}^N$ and $\cup_n \mathrm{supp}(\nu_n)$ is bounded.*

(1) *In case $N \geq 2m$,*

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \mathcal{R}_{2m}(x-y)\, d\nu_n(y) = \int_{\mathbf{R}^N} \mathcal{R}_{2m}(x-y)\, d\nu(y)$$

uniformly on each compact set K with $K \cap \overline{\cup_n \mathrm{supp}(\nu_n)} = \emptyset$. In case $N < 2m$, the above convergence is uniform on each compact set K.

(2) *In case* $N \geq 2m$,

$$\liminf_{n\to\infty} \int_{\mathbf{R}^N} \mathcal{R}_{2m}(x-y)\, d\nu_n(y) = \int_{\mathbf{R}^N} \mathcal{R}_{2m}(x-y)\, d\nu(y)$$

for each $x \in \mathbf{R}^N$.

(3) *In case* $N \geq 2m$, *there exists a subset* E *of* $\mathbf{R}^N$ *with* $\mathrm{Cap}_{2m}(E) = 0$ *such that*

$$\liminf_{n\to\infty} \int_{\mathbf{R}^N} \mathcal{R}_{2m}(x-y)\, d\nu_n(y) = \int_{\mathbf{R}^N} \mathcal{R}_{2m}(x-y)\, d\nu(y)$$

for each $x \in \mathbf{R}^N \setminus E$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\{u_n\}$ be as in Theorem 1.1. For simplicity we put $\mu_n = \mu_{u_n}$. Throughout this proof, let C denote various constants independent of n.

We see from (1.2) that

$$\int_{B(2R)} ((-1)^m u_n)^+ dx \leq \sigma_N \int_0^{2R} r^{N-1}\varphi(r)\, dr \leq \frac{2^N \sigma_N}{N} R^N \varphi(2R)$$

for $R > 0$.

From our assumption and Lemma 2.1, note that $\{\Delta^j u_n(0)\}$ is bounded for $0 \leq j \leq m-1$. By Lemma 3.3, we have

$$\mu_n(B(r)) \leq C r^{N-2m}\varphi(2r) \tag{4.1}$$

for all $r > 0$. Thus, replacing $\{\mu_n\}$ by a subsequence, we may assume that $\{\mu_n\}$ converges vaguely to a measure μ on $\mathbf{R}^N$; see [12, Theorem 1.23]. Moreover, μ satisfies

$$\mu(\mathbf{B}) = 0 \tag{4.2}$$

and

$$\mu(B(r)) \leq C r^{N-2m}\varphi(2r)$$

for all $r > 0$.

Now represent u_n as

$$u_n(x) = \int_{B(2R)} \mathcal{R}_{2m}(x-y)\, d\mu_n(y) + h_n(x),$$

where $h_n \in \mathcal{H}^m(B(2R))$. In view of Lemma 3.1 and (4.1), we see that

$$\begin{aligned}\int_{B(2R)} \left| \int_{B(2R)} \mathcal{R}_{2m}(x-y)\, d\mu_n(y) \right| dx &\leq \int_{B(2R)} \left(\int_{B(2R)} |\mathcal{R}_{2m}(x-y)|\ dx \right) d\mu_n(y) \\ &\leq C R^{2m} \log(4R) \mu_n(B(2R)) \\ &\leq C R^N \log(4R) \varphi(4R)\end{aligned}$$

for $R > 1$. Hence

$$\int_{B(2R)} ((-1)^m h_n)^+ dx \le \int_{B(2R)} ((-1)^m u_n)^+ dx + CR^N(\log(4R))\varphi(4R)$$
$$\le CR^N(\log(4R))\varphi(4R)$$

for $R > 1$.

Since $\mu_n(\mathbf{B}) = 0$, for $x \in B(1/2)$,

$$\int_{B(2R)} |\mathcal{R}_{2m}(x-y)|\, d\mu_n(y) \le C,$$

so that $\{h_n\}$ is uniformly bounded on $B(1/2)$. Consequently it follows from Proposition 2.4 that there exists a subsequence $\{h_{n_j}\}$ of $\{h_n\}$ which converges to some $h \in \mathcal{H}^m(B(R))$ uniformly on $B(R)$. Thus, with the aid of Lemma 4.1, the proof is completed. □

To show Theorem 1.2, we prepare the following result.

Lemma 4.2 *Suppose that $u \in \mathcal{SH}^m(\mathbf{R}^N)$ satisfies*

$$M(|u|, r) \le \varphi(r)$$

for all $r > 0$. Then there exists a positive constant C depending only on N such that

$$\mu_u(B(r)) \le Cr^{N-2m}\varphi(2r)$$

for all $r > 0$.

Proof Let $\psi \in C_0^\infty(B(2))$ such that $0 \le \psi \le 1$ on $\mathbf{R}^N$ and $\psi = 1$ on $\mathbf{B}$. Set $\psi_r(x) = \psi(x/r)$. Then

$$\begin{aligned}
\mu_u(B(r)) &\le \int_{\mathbf{R}^N} \psi_r(y)\, d\mu_u(y) \\
&= \int_{\mathbf{R}^N} u(y)(-\Delta)^m\psi_r(y)\, dy \\
&\le \int_{B(2r)} |u(y)||(-\Delta)^m\psi_r(y)|\, dy \\
&\le \sigma_N||(-\Delta)^m\psi||_\infty r^{-2m} \int_0^{2r} t^{N-1}\varphi(t)\, dt \\
&= \frac{\sigma_N||(-\Delta)^m\psi||_\infty 2^N}{N} r^{N-2m}\varphi(2r)
\end{aligned}$$

for $r > 0$. Thus the required result follows. □

Proof of Theorem 1.2 Let $\{u_n\}$ be as in Theorem 1.2. By Lemma 4.2, we see that

$$\mu_n(B(r)) \le Cr^{N-2m}\varphi(2r)$$

for all $r > 0$. Thus we may assume that $\{\mu_n\}$ converges vaguely to a measure μ on $\mathbf{R}^N$ satisfying

$$\mu(B(r)) \le Cr^{N-2m}\varphi(2r).$$

Further, u_n can be written as

$$u_n(x) = \int_{B(2R)} \mathcal{R}_{2m}(x-y)\, d\mu_n(y) + h_n(x) \quad (x \in B(2R)),$$

where $h_n \in \mathcal{H}^m(B(2R))$. Hence it follows from Lemma 3.1 that

$$\int_{B(2R)} |h_n(x)|\, dx \le CR^N(\log(4R))\varphi(4R)$$

for $R > 1$. Therefore we see from Lemma 2.2 that there exists a subsequence $\{h_{n_j}\}$ of $\{h_n\}$ which converges to some function $h \in \mathcal{H}^m(B(R))$ uniformly on $B(R)$. Finally, in a way similar to the proof of Theorem 1.1, we can prove the convergence properties of potentials

$$\int_{B(2R)} \mathcal{R}_{2m}(x-y)\, d\mu_n(y),$$

which completes the proof. □

Remark 4.2 If $u \in \mathcal{SH}^m(\mathbf{R}^N)$ satisfies

$$M(|u|, r) \le C(1+r)^{\ell} \tag{4.3}$$

for all $r > 0$, then u is of the form

$$u(x) = \int_{\mathbf{R}^N} K_{2m,\ell}(x, y)\, d\mu_u(y) + h(x) \quad (x \in \mathbf{R}^N),$$

where $h \in \mathcal{H}^m(\mathbf{R}^N)$.

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