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Generalized Bessel and Riesz Potentials on Metric Measure Spaces

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Abstract We introduce generalized Bessel and Riesz potentials on metric measure spaces and the corresponding potential spaces. Estimates of the Bessel and Riesz kernels are given which reflect the intrinsic structure of the spaces. Finally, we state the relationship between Bessel (or Riesz) operators and subordinate semigroups.

Keywords Bessel and Riesz potentials · Heat kernel · Potential space · Subordinate semigroup

Mathematics Subject Classifications (2000) Primary: 47H50 · Secondary: 28A80 · 46E35

1 Introduction

There is a rich literature on the study of Bessel and Riesz potentials on the Euclidean space $\mathbb{R}^n$, see for example the books [1, 16, 20, 23] and the references therein. However, little is known on how to extend the Bessel and Riesz potentials to metric measure spaces in a *reasonable* way. This issue is interesting in that it is closely related with the study of various current topics, such as the definition of Sobolev-type spaces, the study of Markov processes and of PDE's on metric measure spaces. Some classes of fractal sets, in particular, self-similar sets, are nice geometric models of metric measure spaces. In this respect, the reader may refer to the books [2, 8, 17, 24] and the references therein.

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In [28, 29] Riesz potentials of certain fractal subsets of $\mathbb{R}^n$ are introduced as traces of the corresponding Euclidean variants. A related approach for more general quasimetric spaces (X, d, μ) by means of local Euclidean charts can be found in [25, 26]. In particular, the Riesz potentials of order σ on so-called α-sets or spaces are given by

$$I_\mu^\sigma u(x) = \int_X \frac{u(y)}{d(x, y)^{\alpha-\sigma}}\, d\mu(y)\,.$$

In the present paper we will define generalized Bessel potentials and generalized Riesz potentials on metric measure spaces admitting a contractive strongly continuous semigroup of transformations $T_t = e^{At}$, $t \geq 0$, on the space $L^p(\mu)$, $p \geq 1$. In particular, we have in mind $A = \Delta$ for a fractal p-Laplacian Δ. Then we use arbitrary completely monotone functions f in order to introduce associated generalized Bessel and Riesz potential operators (see Definition 2.1 below). They may be interpreted as the operators $f(I - A)$ and $f(-A)$, respectively. Our notions coincide with the classical Bessel and Riesz potentials if the metric space is $\mathbb{R}^n$, $f(s) = s^{-\sigma/2}$, and the semigroup is the Gauss-Weierstrass semigroup. For fractal sets as mentioned above the corresponding Riesz potential is now given by

$$I_\mu^\sigma u(x) = \int_X u(y)\; R_{\alpha,\beta}^\sigma(x, y)\, d\mu(y),$$

where $R_{\alpha,\beta}^\sigma(x, y) \sim d(x, y)^{-(\alpha-\sigma\beta/2)}$, and β denotes the so-called walk dimension (see Example 4.5). This means that our potentials reflect the intrinsic structure of the metric measure spaces.

One of the essential applications of Bessel operators is to define potential spaces. These spaces can be viewed as extensions of their Euclidean variants to metric measure spaces. If $p = 2$ we obtain the corresponding Besov type spaces. (For other approaches see [12, 14, 25] and the references therein.)

The issue of the invertibility of generalized Bessel and Riesz operator is non-trivial. We show that they are invertible on the Hilbert space $L^2(\mu)$ for any non-zero completely monotone function f if the semigroup T_t is μ-symmetric (see Theorem 3.1 which is classical), but invertible on the Banach space $L^p(\mu)$ ($1 \leq p < \infty$) under a certain integrability condition (see Theorems 3.2) and for the case $f = 1/g$ where g is a Bernstein function such that $g(0) = 0$ and $\lim_{s\to\infty} g(s)/s = 0$ (Theorem 5.4).

The generalized Bessel and Riesz kernels are important. We prove that if the heat kernel of the semigroup exists and satisfies two-sided bounds, then the generalized Bessel and Riesz kernels also exist and satisfy upper and lower estimates, see Theorem 4.3 and Propositions 4.6 and 4.7.

Finally, we show that in the case $f = 1/g$ as above the generalized Bessel (or Riesz) operator is the Bochner integral of a certain subordinate semigroup corresponding to g (Theorem 5.2), and its inverse is minus the infinitesimal generator of the subordinate semigroup (Theorem 5.4).

2 Generalized Bessel and Riesz Potentials

Let (X, d, μ) be a *metric measure space*, that is, the (X, d) is a locally compact separable metric space, and μ is a Radon measure supported on X. For $1 \leq p < \infty$, let $L^p(\mu)$ be the space of all real-valued p-integrable functions on X with norm

$$\|u\|_p := \left(\int_X |u(x)|^p \, d\mu(x) \right)^{1/p},$$

and let $L^\infty(\mu)$ be the space of essentially bounded functions on X.

Let $\{T_t\}_{t\geq 0}$ be a strongly continuous semigroup on $L^p(\mu)$, that is, each T_t is a bounded linear operator from $L^p(\mu)$ to itself, and

- $T_0 = I$ (the identity operator).
- $T_{t+s} = T_t \circ T_s$ for any $t, s \geq 0$.
- $\lim_{t\to 0+} \|T_t u - u\|_p = 0$ for any $u \in L^p(\mu)$.

Let A be the *infinitesimal generator* of $\{T_t\}_{t\geq 0}$, that is,

$$\lim_{t\to 0+} \|t^{-1} (T_t u - u) - Au\|_p = 0$$

for any $u \in \mathcal{D}(A)$, the *space* of all functions $u \in L^p(\mu)$ such that the above limit exists. It is known that $\mathcal{D}(A)$ is dense in $L^p(\mu)$ as $\{T_t\}_{t\geq 0}$ is strongly continuous. We may formally write

$$T_t = e^{tA}, \quad t > 0.$$

If $\{T_t\}_{t\geq 0}$ on $L^2(\mu)$ is μ*-symmetric*, that is,

$$(T_t u, v) := \int_X T_t u(x) v(x) \, d\mu(x) = \int_X T_t v(x) u(x) \, d\mu(x)$$

for $u, v \in L^2(\mu)$, then the generator A is self-adjoint. Any self-adjoint A on $L^2(\mu)$ admits a spectral family $\{E_\lambda\}_{-\infty}^{\infty}$, that is,

$$A = -\int_{-\infty}^{\infty} \lambda \, dE_\lambda.$$

The semigroup $\{T_t\}_{t\geq 0}$ can be written as

$$T_t = e^{tA} = \int_{-\infty}^{\infty} e^{-\lambda t} \, dE_\lambda \quad (t > 0).$$

If A is further non-positive definite, that is,

$$-(Au, u) \geq 0 \quad \text{for any } u \in \mathcal{D}(A),$$

then $E_\lambda = 0$ for any $\lambda < 0$, and so

$$A = -\int_0^{\infty} \lambda \, dE_\lambda,$$

$$T_t = \int_0^{\infty} e^{-\lambda t} \, dE_\lambda. \tag{2.1}$$

A semigroup $\{T_t\}_{t\geq 0}$ on $L^p(\mu)$ $(1 \leq p \leq \infty)$ is said to be *contractive* if

$$\|T_t u\|_p \leq \|u\|_p \quad \text{for any } t > 0 \text{ and } u \in L^p(\mu). \tag{2.2}$$

A semigroup $\{T_t\}_{t\geq 0}$ on $L^\infty(\mu)$ is *conservative* if

$$T_t 1 = 1 \quad \text{for any } t > 0. \tag{2.3}$$

A measurable function $h : (0, \infty) \times X \times X \to [0, \infty)$ is called the *heat kernel* of $\{T_t\}_{t\geq 0}$ if

$$T_t u(x) = \int_X h(t, x, y) u(y)\, d\mu(y) \tag{2.4}$$

for all $t > 0$ and μ-almost all $x \in X$. If $\{T_t\}_{t\geq 0}$ is conservative, then

$$\int_X h(t, x, y)\, d\mu(y) = 1 \tag{2.5}$$

for all $t > 0$ and a.a. $x \in X$.

A function $f : (0, \infty) \to \mathbb{R}$ is *completely monotone*, if f has derivatives of all orders and satisfies

$$(-1)^k f^{(k)}(x) \geq 0 \quad \text{for any } x > 0 \text{ and for } k = 0, 1, \ldots .$$

This class of functions was introduced by Hausdorff in 1921 in [13], where such functions were termed "*totally monotone*". By the *Bernstein theorem* [5], a function f is completely monotone if and only if

$$f(x) = \int_0^\infty e^{-sx}\, d\nu(s), \quad x > 0, \tag{2.6}$$

for a (non-negative) measure ν on $[0, \infty)$ for which the above integral converges for any $x > 0$, see also [27, p.161] or the more recent book [16, Theorem 3.8.13, p.164]. By Eq. 2.6, we see that

$$\int_0^1 d\nu(s) \leq e f(1) < \infty. \tag{2.7}$$

A famous example of completely monotone functions is $f(x) = x^{-\sigma}$ for any $\sigma > 0$. Note that

$$\begin{aligned} x^{-\sigma} &= \frac{1}{\Gamma(\sigma)} \int_0^\infty s^{\sigma-1} e^{-sx}\, ds \\ &= \int_0^\infty e^{-sx}\, d\nu(s) \quad (x > 0) \end{aligned} \tag{2.8}$$

for any $\sigma > 0$, where

$$d\nu(s) = \frac{1}{\Gamma(\sigma)} s^{\sigma-1} ds. \tag{2.9}$$

A function $g : (0, \infty) \to \mathbb{R}$ is a *Bernstein function*, if g has derivatives of all orders and satisfies that $g \geq 0$, and

$$(-1)^k g^{(k)}(x) \leq 0 \quad \text{for any } x > 0 \text{ and for } k = 1, 2, \ldots .$$

By definition, we see that a function $f > 0$ on $(0, \infty)$ is completely monotone if $1/f$ is a Bernstein function. A function g is a Bernstein function if and only if

$$g(x) = a + bx + \int_0^\infty \left(1 - e^{-tx}\right) dm(t), \quad x > 0, \tag{2.10}$$

for constants $a, b \geq 0$ and a (non-negative) measure m on $[0, \infty)$ with

$$\int_0^\infty (t \wedge 1)\, dm(t) < \infty, \tag{2.11}$$

see for example [16, Theorem 3.9.4, p.174]. It follows from Eq. 2.10 that

$$a = \lim_{x \to 0} g(x) \quad \text{and} \quad b = \lim_{x \to \infty} \frac{g(x)}{x}.$$

Comparing Eq. 2.7 with Eq. 2.11, we see that the measure m associated with a Bernstein function may have a stronger singularity at 0 than the measure ν associated with a completely monotone function, see for example ν as in Eq. 2.9 and m as in Eq. 2.13 below.

A typical Bernstein function is $g(x) = x^\sigma$ for $0 < \sigma \leq 1$. Observe that

$$\begin{aligned} x^\sigma &= \frac{\sigma}{\Gamma(1-\sigma)} \int_0^\infty t^{-1-\sigma} \left(1 - e^{-tx}\right) dt \\ &= \int_0^\infty \left(1 - e^{-tx}\right) dm(t) \quad (x > 0) \end{aligned} \tag{2.12}$$

for any $0 < \sigma < 1$, where

$$dm(t) = \frac{\sigma}{\Gamma(1-\sigma)} t^{-1-\sigma} dt. \tag{2.13}$$

Recall that a family $\{\mu_t\}_{t \geq 0}$ of Borel measures on $\mathbb{R}^n$ is called a convolution semigroup on $\mathbb{R}^n$ if

- $\mu_t(\mathbb{R}^n) \leq 1$ for all $t \geq 0$, and μ_0 is the Dirac measure at 0.
- $\mu_t * \mu_s = \mu_{t+s}$ for all $s, t \geq 0$, where $*$ means the convolution of measures.
- $\mu_t \rightharpoonup \mu_0$ weakly as $t \to 0$.

A function g on $(0, \infty)$ is a Bernstein function if and only if there is a convolution semigroup $\{\mu_t\}_{t \geq 0}$ on $[0, \infty)$ such that

$$\int_0^\infty e^{-sx}\, d\mu_t(s) = e^{-tg(x)}, \quad x > 0 \text{ and } t > 0, \tag{2.14}$$

see for example [16, Theorem 3.9.7, p.177].

A function $g : (0, \infty) \to \mathbb{R}$ is said to be a *complete Bernstein* function if there exists a Bernstein function g_1 such that

$$g(x) = x^2 \int_0^\infty e^{-sx} g_1(s)\, ds, \quad x > 0. \tag{2.15}$$

Assume that g_1 is given by

$$g_1(x) = a + bx + \int_0^\infty \left(1 - e^{-tx}\right) dm(t).$$

By a straightforward calculation, it follows from Eq. 2.15 that

$$g(x) = b + ax + \int_0^\infty \left(1 - e^{-tx}\right) \eta(t)\, dt$$

where

$$\eta(t) = \int_{(0,\infty)} s^2 e^{-st}\, dm(s), \quad t > 0, \tag{2.16}$$

see for example [16, p.192-193].

Recall that for $\sigma > 0$, the classical Bessel potential J_σ and Riesz potential I_σ on $\mathbb{R}^n$ are respectively determined by

$$\begin{aligned} J_\sigma u &= \frac{1}{\Gamma\left(\frac{\sigma}{2}\right)} \int_0^\infty t^{\frac{\sigma}{2}-1} e^{-t}\, T_t u\, dt = \int_0^\infty e^{-t}\, T_t u\, d\nu(t), \\ I_\sigma u &= \frac{1}{\Gamma\left(\frac{\sigma}{2}\right)} \int_0^\infty t^{\frac{\sigma}{2}-1}\, T_t u\, dt = \int_0^\infty T_t u\, d\nu(t), \end{aligned} \tag{2.17}$$

where ν is given by $d\nu(t) = \frac{1}{\Gamma(\frac{\sigma}{2})} t^{\frac{\sigma}{2}-1} dt$, and $T_t u(x) = \int_{\mathbb{R}^n} G(t,x,y) u(y)\, dy$ is the *Gauss-Weierstrass semigroup* with the *Gauss-Weierstrass heat kernel*

$$G(t,x,y) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{2t}\right), \tag{2.18}$$

see for example [14, 23]. The infinitesimal generator A corresponding to the Gauss-Weierstrass heat kernel G defined as in Eq. 2.18 is the usual Laplacian

$$A = \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

Formally we may write $J_\sigma = (I - A)^{-\frac{\sigma}{2}}$ and $I_\sigma = (-A)^{-\frac{\sigma}{2}}$.

A natural question arises how to generalize the classical Bessel and Riesz potentials on $\mathbb{R}^n$ to the metric measure space in a reasonable way.

Note that there are two notable features in defining the classical Bessel potential and Riesz potential.

- The strongly continuous *semigroup* $\{T_t\}_{t\ge 0}$ that corresponds to the Brownian motion in $\mathbb{R}^n$.
- The *measure* ν that corresponds to the famous completely monotone function $x^{-\frac{\sigma}{2}}$ for $\sigma > 0$.

Based on this observation, we give the generalized Bessel and generalized Riesz potentials on (X, d, μ) as follows.

Definition 2.1 Let f be a completely monotone function on $(0,\infty)$ with associated measure ν as in Eq. 2.6. Let $\mathbf{T} := \{T_t\}_{t\ge 0}$ be a contractive strongly continuous semigroup on $L^p(\mu)$. The corresponding generalized Bessel (potential) operator $J_\mu^{f,\mathbf{T}}$ and generalized Riesz (potential) operator $I_\mu^{f,\mathbf{T}}$ are respectively defined by

$$J_\mu^{f,\mathbf{T}} u = \int_0^\infty e^{-t}\, T_t u \, d\nu(t), \quad u \in L^p(\mu), \tag{2.19}$$

$$I_\mu^{f,\mathbf{T}} u = \int_0^\infty T_t u \, d\nu(t). \tag{2.20}$$

Here the space $L^p(\mu)$ might be replaced by any Banach space.

For convenience, we shall suppress the superscript $\mathbf{T}$, and denote by

$$J_\mu^f := J_\mu^{f,\mathbf{T}} \quad \text{and} \quad I_\mu^f := I_\mu^{f,\mathbf{T}}$$

when no confusion arises. By Eq. 2.19, the Bessel operator J_μ^f is well-defined, and it is linear and bounded on $L^p(\mu)$. In fact, we see from Eqs. 2.19 and 2.6 that

$$\begin{aligned} \left\| J_\mu^f u \right\|_p &\le \int_0^\infty e^{-t} \| T_t u \|_p \, d\nu(t) \\ &\le \|u\|_p \int_0^\infty e^{-t}\, d\nu(t) = f(1) \|u\|_p. \end{aligned} \tag{2.21}$$

However, in general, we do not know whether the Riesz operator I_μ^f is well-defined on $L^p(\mu)$, as we do not know the decay rate of $\|T_t\|_p$ as $t \to \infty$. In other words, the domain of I_μ^f may consist of zero only.

Remark 2.2 The generalized Bessel operator $J_\mu^{f,\mathbf{T}}$ can be viewed as a kind of the generalized Riesz operator in one of the following two ways:

- $J_\mu^{f,\mathbf{T}} = I_\mu^{\widetilde{f},\mathbf{T}}$, where $\widetilde{f}(x) = f(1+x)$. In fact, by Eq. 2.6, we see that

$$\begin{aligned} f(1+x) &= \int_0^\infty e^{-s(1+x)}\, d\nu(s) = \int_0^\infty e^{-sx} e^{-s}\, d\nu(s) \\ &= \int_0^\infty e^{-sx}\, d\widetilde{\nu}(s), \quad x > 0, \end{aligned}$$

  where $d\widetilde{\nu}(s) = e^{-s} d\nu(s)$. Therefore, it follows from Eq. 2.20 that

$$\begin{aligned} I_\mu^{\widetilde{f},\mathbf{T}} u &= \int_0^\infty T_t u \, d\widetilde{\nu}(t) \\ &= \int_0^\infty e^{-t}\, T_t u \, d\nu(t) = J_\mu^{f,\mathbf{T}} u \end{aligned}$$

  for any $u \in L^p(\mu)$, showing that $J_\mu^{f,\mathbf{T}} = I_\mu^{\widetilde{f},\mathbf{T}}$. In this case, we keep the semigroup unchanged but vary the completely monotone functions, and obtain the Bessel operator from the Riesz.
- $J_\mu^{f,\mathbf{T}} = I_\mu^{f,\widetilde{\mathbf{T}}}$, where $\widetilde{\mathbf{T}} = \{e^{-t} T_t\}_{t \ge 0}$. This is easily seen by definition. In this case, we keep the completely monotone function unchanged but vary the semigroups, and can also obtain the Bessel operator from the Riesz operator.

Example 2.3 Let $X = \mathbb{R}^n$ and μ be the Lebesgue measure. Let $\{T_t\}$ be the Gauss-Weierstrass semigroup defined by Eq. 2.18, and let $f(x) = x^{-\frac{\sigma}{2}}$ for $\sigma > 0$. Then the generalized Bessel and Riesz operators J_μ^f and I_μ^f defined as above agree with the classical ones respectively.

Note that for a completely monotone function f defined as in Eq. 2.6,

$$f(1+x) = \int_0^\infty e^{-t(1+x)}\, d\nu(t).$$

Thus, we may formally write

$$\begin{aligned} f(I-A)u &= \int_0^\infty e^{-t(I-A)} u\, d\nu(t) \\ &= \int_0^\infty e^{-t}\, e^{tA} u\, d\nu(t) \\ &= \int_0^\infty e^{-t}\, T_t u\, d\nu(t) = J_\mu^f u. \end{aligned}$$

Therefore, we formally have that

$$J_\mu^f = f(I-A), \tag{2.22}$$

where A is the generator of $\{T_t\}_{t\geq 0}$. In a similar way, we formally have that

$$I_\mu^f = f(-A).$$

For $X = \mathbb{R}^n$ with the Gauss-Weierstrass semigroup $\{T_t\}$ and $0 < \sigma < n$ the author [29] studied such Bessel-type potentials for completely monotone functions of the form $f = (1+g)^{-\sigma}$ for any Bernstein function g, and the Riesz-type potentials for $f = g^{-\sigma}$ under some additional conditions on g.

As in the classical case we now introduce the potential space associated with J_μ^f provided that this operator is injective:

Definition 2.4 (Bessel potential space)

$$H_p^f(\mu) := \left\{u \in L^p(\mu) : \text{ there exists some } \varphi \in L^p(\mu) \text{ such that } u = J_\mu^f \varphi\right\}. \tag{2.23}$$

The norm of $u = J_\mu^f \varphi \in H_p^f(\mu)$ is defined by

$$\|u\|_{H_p^f(\mu)} = \|\varphi\|_p. \tag{2.24}$$

(Since J_μ^f is injective, the norm defined here makes sense.)

Clearly, the space $H_p^f(\mu)$ defined as above is a Banach subspace of $L^p(\mu)$. We call $H_p^f(\mu)$ an *f-Bessel potential space* on (X, d, μ) with respect to $(\{T_t\}, L^p(\mu))$.

Remark 2.5 If $f = 1/g$ for a Bernstein function g such that $g(0) = 0$ and $\lim_{s\to\infty} g(s)/s = 0$ we obtain the interpretation $\left(J_\mu^f\right)^{-1} = g(I - A)$ and the norm equivalence

$$\|u\|_{H_p^f(\mu)} = \|g(I - A)u\|_p \sim \|u\|_p + \|g(-A)u\|_p, \quad 1 \le p < \infty, \tag{2.25}$$

(see Corollary 5.5 below) which is well-known in the classical case.

We now turn to conditions under which the above potential operators are invertible.

3 Invertibility

In this section we investigate the invertibility of the generalized Bessel and Riesz operators. We first show that for μ-symmetric semigroups J_μ^f and I_μ^f are invertible on $L^2(\mu)$ for any completely monotone function f. We then prove an inversion formula for J_μ^f and I_μ^f on the Banach space $L^p(\mu)$ for $1 \le p < \infty$ under some assumptions on f. Finally, we consider the special case $f(s) = s^{-\sigma}$ for $\sigma > 0$ under this point of view. If the generator A of the underlying semigroup $\{T_t\}$ is a Laplace-type operator then the inverses of the Riesz potentials for such f may be interpreted as fractional derivatives. In general, for $0 < \sigma < 1$ this case also fits into the inversion scheme in $L^p(\mu)$ given in Section 5 for $f = 1/g$ with an arbitrary Bernstein functions g as above.

Theorem 3.1 *Let $\{T_t\}$ be a strongly continuous semigroup on $L^2(\mu)$ that is μ-symmetric and contractive, and let $f > 0$ be a completely monotone function defined as in Eq. 2.6. Then the generalized Bessel and Riesz operators J_μ^f and I_μ^f defined as in Eqs. 2.19 and 2.20 are invertible on $L^2(\mu)$, and*

$$\begin{aligned}
\left(J_\mu^f\right)^{-1} u &= \int_0^\infty f(1+\lambda)^{-1}\, dE_\lambda u, \\
\left(I_\mu^f\right)^{-1} u &= \int_0^\infty f(\lambda)^{-1}\, dE_\lambda u,
\end{aligned} \tag{3.1}$$

where $\{E_\lambda\}_{\lambda\ge 0}$ is the spectral family of the generator A of $\{T_t\}$.

Proof This is classical and we sketch the arguments: Since $\{T_t\}$ is μ-symmetric and contractive, its generator A is non-positive definite (see [9]). It follows from Eqs. 2.19 and 2.1 that, using Fubini's theorem,

$$\begin{aligned}
J_\mu^f u &= \int_0^\infty e^{-s} T_s u\, d\nu(s) = \int_0^\infty e^{-s} \left(\int_0^\infty e^{-\lambda s}\, dE_\lambda u\right) d\nu(s) \\
&= \int_0^\infty \left(\int_0^\infty e^{-(1+\lambda)s}\, d\nu(s)\right) dE_\lambda u = \int_0^\infty f(1+\lambda)\, dE_\lambda u.
\end{aligned}$$

Define the operator D_μ^f by

$$D_\mu^f u = \int_0^\infty f(1+\lambda)^{-1}\, dE_\lambda u,$$
$$\mathcal{D}\left(D_\mu^f\right) = \left\{u \in L^2(\mu) : \int_0^\infty f(1+\lambda)^{-2}\, d\left(E_\lambda u, u\right) < \infty\right\}. \tag{3.2}$$

By the functional calculus, we see that $J_\mu^f\left(D_\mu^f u\right) = u$ for any $u \in \mathcal{D}\left(D_\mu^f\right)$, and $D_\mu^f\left(J_\mu^f u\right) = u$ for any $u \in L^2(\mu)$. Thus D_μ^f is the inverse of J_μ^f. In a similar fashion, the Riesz operator I_μ^f is invertible. □

We further investigate the invertibility of the generalized Bessel and Riesz operators on the Banach space $L^p(\mu)$ for $1 \le p < \infty$. The situation is more involved. To do this, we assume that the completely monotone function f is of the form

$$f(x) = \int_0^\infty e^{-sx} \rho(s)\, ds, \quad x > 0 \tag{3.3}$$

for some $\rho : (0, \infty) \to [0, \infty)$, and that

$$f(x)^{-1} = \int_0^\infty P(1 - e^{-tx})\, dm(t), \quad x > 0 \tag{3.4}$$

for a measure m on $[0, \infty)$ with $\int_0^\infty (s \wedge 1)\ dm(s) < \infty$, and for a polynomial $P \ge 0$ on $[0, \infty)$ given by

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{3.5}$$

where $n \ge 1$ is an integer, and constants $a_0 \ge 0$ and $a_i \in \mathbb{R}\,(1 \le i \le n)$. Clearly, for any $\sigma \in (0, 2)$, the function $f(x) = x^{-\sigma/2}$ satisfies conditions (3.3) and (3.4), where $\rho(s) = \frac{1}{\Gamma(\sigma/2)} s^{\sigma/2-1}$, and $P(x) = x$ and $dm(t) = \frac{\sigma}{2\Gamma(\sigma/2)} t^{-\sigma/2-1}\, dt$.

For any $\varepsilon > 0$, define a function q_ε on $(0, \infty)$ by

$$q_\varepsilon(s) := \varepsilon \sum_{k=0}^{n} \sum_{j=0}^{k} a_k (-1)^j \binom{k}{j} \int_\varepsilon^\infty \overline{\rho}(\varepsilon s - jt)\, dm(t), \quad s > 0, \tag{3.6}$$

where $\overline{\rho}(x) = 0$ if $x \le 0$, and $\overline{\rho}(x) = \rho(x)$ if $x > 0$.

Theorem 3.2 *Let $\{T_t\}$ be a contractive strongly continuous semigroup on $L^p(\mu)$ for $1 \le p < \infty$. Assume that f satisfies Eqs. 3.3 and 3.4, and that*

$$|q_\varepsilon(s)| \le q(s) \quad \textit{for any } \varepsilon, s > 0 \tag{3.7}$$

for some function $q \ge 0$ with $\int_0^\infty q(s)\, ds < \infty$. Then the generalized Bessel operator J_μ^f defined as in Eq. 2.19 is invertible on $L^p(\mu)$.

Proof By the monotone convergence theorem and Fubini's theorem, it follows from Eqs. 3.3 and 3.4 that for any $\varepsilon, x > 0$,

$$1 = f(x) \cdot f(x)^{-1} = \lim_{\varepsilon \to 0} \int_0^\infty e^{-sx} \rho(s) \left(\int_\varepsilon^\infty P(1 - e^{-tx})\, dm(t) \right) ds$$
$$= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \left(\int_0^\infty e^{-sx} \rho(s) P(1 - e^{-tx})\, ds \right) dm(t). \tag{3.8}$$

Using the expansion (3.5) of P, we have that

$$\int_0^\infty e^{-sx} \rho(s) P(1 - e^{-tx})\, ds = \sum_{k=0}^n a_k \int_0^\infty e^{-sx} \rho(s) \left(1 - e^{-tx}\right)^k ds. \tag{3.9}$$

We compute that

$$\int_0^\infty e^{-sx} \rho(s) \left(1 - e^{-tx}\right)^k ds = \sum_{j=0}^k (-1)^j \binom{k}{j} \int_0^\infty e^{-(s+jt)x} \rho(s)\, ds$$
$$= \sum_{j=0}^k (-1)^j \binom{k}{j} \int_0^\infty e^{-sx} \overline{\rho}(s - jt)\, ds.$$

Combining this with Eq. 3.9, we see

$$\int_0^\infty e^{-sx} \rho(s) P(1 - e^{-tx})\, ds = \sum_{k=0}^n \sum_{j=0}^k a_k (-1)^j \binom{k}{j} \int_0^\infty e^{-sx} \overline{\rho}(s - jt)\, ds.$$

Therefore, using Fubini's theorem again and then changing variables s by εs, we obtain from Eq. 3.8 that

$$1 = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \left(\sum_{k=0}^n \sum_{j=0}^k a_k (-1)^j \binom{k}{j} \int_0^\infty e^{-sx} \overline{\rho}(s - jt)\, ds \right) dm(t)$$
$$= \lim_{\varepsilon \to 0} \int_0^\infty e^{-s\varepsilon x} q_\varepsilon(s)\, ds \quad \text{for any } x > 0. \tag{3.10}$$

For $\varepsilon > 0$, define the operator D_ε^f by

$$D_\varepsilon^f u = \int_\varepsilon^\infty P\left(I - e^{-t} T_t\right) u\, dm(t)$$
$$= \sum_{k=0}^n \sum_{j=0}^k a_k (-1)^j \binom{k}{j} \int_\varepsilon^\infty e^{-jt} T_{jt} u\, dm(t). \tag{3.11}$$

For each $\varepsilon > 0$, the operator D_ε^f is linear, and its domain is $L^p(\mu)$ since

$$\int_\varepsilon^\infty e^{-jt} \| T_{jt} u \|_p\, dm(t) \le \|u\|_p \int_\varepsilon^\infty dm(t) < \infty$$

for any $u \in L^p(\mu)$, by using that facts that $\|T_{jt}u\|_p \le \|u\|_p$ and that

$$\int_\varepsilon^\infty dm(t) < \infty.$$

Let D_μ^f be the strong limit of D_ε^f, that is,

$$\lim_{\varepsilon\to 0} \left\| D_\varepsilon^f u - D_\mu^f u \right\|_p = 0 \tag{3.12}$$

for $u \in \mathcal{D}\left(D_\mu^f\right)$, the space of all $u \in L^p(\mu)$ such that the above limit exists.

We will show that for any $u \in L^p(\mu)$, the element $J_\mu^f u \in \mathcal{D}\left(D_\mu^f\right)$, and

$$\lim_{\varepsilon\to 0} \left\| D_\varepsilon^f \left(J_\mu^f u\right) - u \right\|_p = 0. \tag{3.13}$$

In fact, by definition,

$$J_\mu^f u = \int_0^\infty e^{-s} T_s u \,\rho(s)\, ds,$$

and so

$$\begin{aligned} e^{-jt} T_{jt}\left(J_\mu^f u\right) &= \int_0^\infty e^{-(jt+s)} T_{jt+s} u \,\rho(s)\, ds \\ &= \int_0^\infty e^{-s} T_s u \,\overline{\rho}(s - jt)\, ds. \end{aligned}$$

Therefore, we see from Eq. 3.11 that

$$\begin{aligned} D_\varepsilon^f\left(J_\mu^f u\right) &= \sum_{k=0}^n \sum_{j=0}^k a_k (-1)^j \binom{k}{j} \int_\varepsilon^\infty e^{-jt} T_{jt}\left(J_\mu^f u\right) dm(t) \\ &= \int_0^\infty e^{-s} T_s u \left(\sum_{k=0}^n \sum_{j=0}^k a_k (-1)^j \binom{k}{j} \int_\varepsilon^\infty \overline{\rho}(s - jt)\, dm(t) \right) ds \\ &= \int_0^\infty T_{\varepsilon s} u \left(e^{-\varepsilon s} q_\varepsilon(s)\right) ds. \end{aligned}$$

From this we infer using Eqs. 3.7 and 3.10 for $x = 1$ and the dominated convergence theorem,

$$\begin{aligned} \left\| D_\varepsilon^f\left(J_\mu^f u\right) - u \right\|_p &\le \int_0^\infty \|T_{\varepsilon s} u - u\|_p \left(e^{-\varepsilon s} q_\varepsilon(s)\right) ds + \left(1 - \int_0^\infty e^{-\varepsilon s} q_\varepsilon(s)\, ds\right) \|u\|_p \\ &\le \int_0^\infty \|T_{\varepsilon s} u - u\|_p \, q(s)\, ds + \left(1 - \int_0^\infty e^{-\varepsilon s} q_\varepsilon(s)\, ds\right) \|u\|_p \\ &\to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Hence, it follows that $J_\mu^f u \in \mathcal{D}\left(D_\mu^f\right)$, and

$$D_\mu^f\left(J_\mu^f u\right) = u \quad \text{for } u \in L^p(\mu). \tag{3.14}$$

Similarly, we can show that $\left\| J_\mu^f\left(D_\mu^f u\right) - u\right\|_p \to 0$ as $\varepsilon \to 0$ for any $u \in \mathcal{D}\left(D_\mu^f\right)$. Thus, the Bessel operator J_μ^f is invertible. □

Remark 3.3 Condition (3.7) can be dropped if $f = 1/g$ for a Bernstein function g on $(0,\infty)$ such that $g(0) = 0$ and $\lim_{s\to\infty} g(s)/s = 0$. In this case the Bessel operator J_μ^f is invertible, and its inverse is minus the generator of a subordinate semigroup, see Theorem 5.4 below.

One can investigate the invertibility of the generalized Riesz operator I_μ^f on the Banach space $L^p(\mu)$ by assuming a similar condition on f. We omit the details.

Remark 3.4 The inversion procedure in the above proof is the analogue of the difference representation for the fractional derivatives arising as inverses of the classical Euclidean Bessel and Riesz potentials (cf. [19, 20]).

In general, the famous completely monotone function mentioned above fits into the approach of Theorem 3.2:

Example 3.5 For $\sigma > 0$, let $f(x) = x^{-\sigma}$. Then

$$f(x)^{-1} = x^\sigma = \frac{1}{\chi(\sigma, l)} \int_0^\infty t^{-\sigma-1}\left(1 - e^{-tx}\right)^l \, dt$$

where $l = [\sigma] + 1$[1] and

$$\chi(\sigma, l) = \int_0^\infty t^{-\sigma-1}\left(1 - e^{-t}\right)^l \, dt.$$

All the hypotheses on f in Theorem 3.2 hold (condition (3.7) is clear as the function q_ε defined as in Eq. 3.6 is independent of ε!), see the detail in [19] or [15]. In particular, if $0 < \sigma < 1$, then

$$x^\sigma = \frac{\sigma}{\Gamma(1-\sigma)} \int_0^\infty t^{-\sigma-1}\left(1 - e^{-tx}\right) dt,$$

and hence $\left(J_\mu^f\right)^{-1} u = (I - A)^\sigma u$ for suitable $u \in L^p(\mu)$, where

$$(I - A)^\sigma u := \frac{\sigma}{\Gamma(1-\sigma)} \int_0^\infty t^{-\sigma-1}\left(u - e^{-t} T_t u\right) dt. \tag{3.15}$$

For $0 < \sigma < 1$, we similarly have that $\left(I_\mu^f\right)^{-1} u = (-A)^\sigma u$ for suitable $u \in L^p(\mu)$, where

$$(-A)^\sigma u := \frac{\sigma}{\Gamma(1-\sigma)} \int_0^\infty t^{-\sigma-1}\left(u - T_t u\right) dt. \tag{3.16}$$

[1] The symbol $[\sigma]$ means the integer part of a real number σ.

Finally, we present an interesting case of how to compute the function ρ in Eq. 3.3 when $f = \frac{1}{g}$ for a Bernstein function g on $(0, \infty)$. In fact, let $\{\mu_t\}$ be the convolution semigroup associated with g as in Eq. 2.14. Assume that μ_t has a density with respect to the Lebesgue measure, that is, there is function $\eta_t : (0, \infty) \to (0, \infty)$ such that $d\mu_t(s) = \eta_t(s)ds$ for any $t > 0$. Integrating Eq. 2.14 in $t \in (0, \infty)$, we see that

$$\begin{aligned} f(x) = g(x)^{-1} &= \int_0^\infty e^{-tg(x)}\, dt \\ &= \int_0^\infty \left(\int_0^\infty e^{-sx} \eta_t(s)\, ds \right) dt \\ &= \int_0^\infty e^{-sx} \left(\int_0^\infty \eta_t(s)\, dt \right) ds. \end{aligned}$$

Therefore, we conclude that

$$\rho(s) = \int_0^\infty \eta_t(s)\, dt, \quad s > 0, \tag{3.17}$$

see also Eq. 5.5 in Section 5 below.

4 Bessel and Riesz Kernels

In this section, we are concerned with whether the generalized Bessel operator or the generalized Riesz operator admits a kernel. It turns our that if the heat kernel of $\{T_t\}$ satisfies two-sided bounds, then the Bessel kernel and the Riesz kernel exist, and both of them decay at a polynomial rate. We present some interesting examples on both $\mathbb{R}^n$ and fractals.

Assume that $\{T_t\}$ is contractive strongly continuous semigroup on $L^p(\mu)$ for $p \geq 1$ that possesses a heat kernel h. By Eq. 2.19 and Fubini's theorem, we have that

$$J_\mu^f u(x) = \int_0^\infty e^{-t}\, T_t u(x)\, d\nu(t) = \int_X u(y) B^f(x, y)\, d\mu(y) \tag{4.1}$$

for any non-negative $u \in L^p(\mu)$, where

$$B^f(x, y) := \int_0^\infty e^{-t} h(t, x, y)\, d\nu(t). \tag{4.2}$$

The function B^f on $X \times X$ is called the *Bessel kernel*. Similarly,

$$I_\mu^f u(x) = \int_X u(y) R^f(x, y)\, d\mu(y) \quad \text{for } u \geq 0, \tag{4.3}$$

where the *Riesz kernel* R^f is defined by

$$R^f(x, y) := \int_0^\infty h(t, x, y)\, d\nu(t) \tag{4.4}$$

if the integral is finite.

Before we proceed, we introduce some notation. Denote by $B(x,r)=\{y\in X: d(y,x)<r\}$ the ball in X with center x and radius r. Set $r_0:=\operatorname{diam}(X)\in(0,\infty]$, the *diameter* of X, and $V(x,r)=\mu(B(x,r))$, the *volume* of the ball. If there is an increasing continuous function $V:(0,\infty)\to[0,\infty)$ such that

$$V(x,r)\sim V(r)$$

for all $x\in X$ and $0<r<r_0$, the set X is called a *V-set*.[2] If there is an $\alpha>0$ such that

$$V(x,r)\sim r^{\alpha}$$

for all $x\in X$ and $0<r<r_0$, we call X an α*-set*. The space $\mathbb{R}^n$ is an n-set for any integer $n\geq 1$. The Sierpinski gasket or the Sierpinski carpet in $\mathbb{R}^n$ is an α-set for some $\alpha>0$, see for example [8].

In the sequel, the letters c, c', c'' and C denote positive constants whose values may change at each occurrence.

The heat kernel h is said to satisfy *condition* (H_Φ), if

$$h(t,x,y)\sim\frac{1}{V(t^{1/\beta})}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right) \tag{4.5}$$

for all $0<t<r_0^{\beta}$ and $x,y\in X$, where $\beta>0$ and $\Phi:[0,\infty)\to(0,\infty)$ is continuous and decreasing, and $V:(0,\infty)\to(0,\infty)$ is continuous and increasing; and moreover,

$$\sup_{X\times X} h(t,x,y)\leq c\,e^{\delta t}\quad\text{for all } t\geq r_0^{\beta} \tag{4.6}$$

when $r_0<\infty$, where $c>0$ and $\delta\in[0,1)$.

Note that if $V(r)\sim r^{\alpha}$ for $\alpha>0$, then Eq. 4.5 implies that X is an α-set, see [10]. We give some examples that the heat kernel h satisfies condition (H_Φ).

Example 4.1 The Gauss-Weierstrass heat kernel satisfies Eq. 4.5 where $\alpha=n$, $\beta=2$, and

$$\Phi(s)=\exp\left(-s^2/2\right)\quad(s\geq 0).$$

The Cauchy-Poisson heat kernel defined by

$$h(t,x,y)=\frac{C_n}{t^n}\left(1+\frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}}$$

where $C_n=\Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$, also satisfies Eq. 4.5 with $\alpha=n$, $\beta=1$, and

$$\Phi(s)=\left(1+s^2\right)^{-\frac{n+1}{2}}.$$

[2]The symbol $f\sim g$ means that there exist constants $c, C>0$ such that $cf\leq g\leq Cf$.

Example 4.2 Let X be the bounded or unbounded Sierpinski gasket in $\mathbb{R}^n$, and let μ be the $\alpha := \log(n+1)/\log 2$-dimensional Haudorff measure on X. In [4], it was shown that there is a heat kernel h satisfying

$$h(t,x,y) \sim t^{-\alpha/\beta} \exp\left(-c\left(\frac{|x-y|}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right), \tag{4.7}$$

for all $0 < t < r_0^\beta$ and all $x, y \in X$, where $\beta = \log(n+3)/\log 2$, termed the walk dimension of the Brownian motion on X.

A similar estimate to Eq. 4.7 also holds on the Sierpinski carpets on $\mathbb{R}^n$ (cf. [3]), and on other fractals (cf. [2, 11] and the references therein).

Theorem 4.3 *Let f be a completely monotone function given as in Eq. 3.3. Assume that $\{T_t\}$ has a heat kernel h satisfying (H_Φ). If $w_1(r) := V(r)/\rho\left(r^\beta\right) \sim r^{\theta_1}$ for some $\theta_1 > \beta$, and if*

$$\int_0^\infty s^{\theta_1-\beta-1}\Phi(s)\,ds < \infty, \tag{4.8}$$

then

$$B^f(x,y) \sim d(x,y)^{-(\theta_1-\beta)} \tag{4.9}$$

for all $x, y \in X$ with $0 < d(x,y) < r_0$.

Proof We only consider $r_0 < \infty$; the case $r_0 = \infty$ is similarly treated. It is enough to estimate the integral in Eq. 4.2 for $x \neq y \in X$. By Eqs. 4.2 and 3.3, we see that

$$\begin{aligned} B^f(x,y) &= \int_0^\infty e^{-t}h(t,x,y)\,d\nu(t) \\ &= \int_0^{r_0^\beta} e^{-t}h(t,x,y)\rho(t)\,dt + \int_{r_0^\beta}^\infty e^{-t}h(t,x,y)\,d\nu(t). \end{aligned} \tag{4.10}$$

We estimate the last two integrals. By Eq. 4.6, we see that

$$\begin{aligned} \int_{r_0^\beta}^\infty e^{-t}h(t,x,y)\,d\nu(t) &\le c\int_{r_0^\beta}^\infty e^{-t}\cdot e^{\delta t}\,d\nu(t) \\ &\le c\int_0^\infty e^{-(1-\delta)t}\,d\nu(t) \\ &= c\,f\big((1-\delta)\big) \le c' d(x,y)^{-(\theta_1-\beta)} \end{aligned} \tag{4.11}$$

for all $x, y \in X$ with $0 < d(x, y) < r_0$, where $c' > 0$, since $\theta_1 > \beta$. On the other hand, it follows from Eqs. 4.5 and 4.8 that

$$\begin{aligned}\int_0^{r_0^\beta} e^{-t} h(t, x, y)\rho(t)\, dt &\sim \int_0^{r_0^\beta} \frac{\rho(t)}{V(t^{1/\beta})}\Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right) dt \\ &= \int_0^{r_0^\beta} \frac{1}{w_1\left(t^{1/\beta}\right)}\Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right) dt \\ &\sim d(x, y)^{-\theta_1} \int_0^{r_0^\beta} \left(\frac{d(x, y)}{t^{1/\beta}}\right)^{\theta_1} \Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right) dt \\ &\sim d(x, y)^{-(\theta_1-\beta)} \int_{d(x,y)r_0^{-1}}^{\infty} s^{\theta_1-\beta-1}\Phi(s)\, ds \\ &\sim d(x, y)^{-(\theta_1-\beta)}\end{aligned}$$

for all $x, y \in X$ with $0 < d(x, y) < r_0$. This combines with Eqs. 4.10 and 4.11 to yield the desired. □

Theorem 4.4 *Let $r_0 = \infty$. Assume that all the hypotheses in Theorem* 4.3 *hold. Then Riesz kernel R^f exists, and satisfies*

$$R^f(x, y) \sim d(x, y)^{-(\theta_1-\beta)}$$

for all $x \neq y \in X$.

Proof We need to estimate the integral

$$\int_0^\infty h(t, x, y)\, d\nu(t).$$

This can be done exactly the same as that in the proof of Theorem 4.3. We omit the details. □

Example 4.5 Let $V(r) \sim r^\alpha$ for $\alpha > 0$, and let $f(x) = x^{-\frac{\sigma}{2}}$ for $\sigma > 0$. Then, we see from Eq. 2.9 that

$$\rho(t) = \frac{1}{\Gamma\left(\frac{\sigma}{2}\right)} t^{\frac{\sigma}{2}-1},$$

and so

$$w_1(r) := V(r)/\rho\left(r^\beta\right) \sim r^{\alpha-\beta(\sigma/2-1)}.$$

Therefore, if the heat kernel h of $\{T_t\}$ satisfies Eqs. 4.5 and 4.8,[3] then

$$B^f(x, y) \sim d(x, y)^{-(\alpha-\sigma\beta/2)},$$
$$R^f(x, y) \sim d(x, y)^{-(\alpha-\sigma\beta/2)}.$$

[3] In this case, note that Eq. 4.8 implies that $\beta\sigma < 2\alpha$.

In particular, if $X = \mathbb{R}^n$ and $\{T_t\}$ is the Gauss-Weierstrass semigroup, then

$$B^f(x, y) \sim |x - y|^{-(n-\sigma)},$$
$$R^f(x, y) \sim |x - y|^{-(n-\sigma)}, \quad (0 < \sigma < n),$$

where $\alpha = n$ and $\beta = 2$.

We further estimate the Riesz kernel in terms of f.

Proposition 4.6 *Let $r_0 = \infty$, and let f be given as in Eq. 3.3 with ρ satisfying*

$$\rho(t) \geq c\, t^{-1} f\left(t^{-1}\right) \quad \text{for all } t > 0 \tag{4.12}$$

for some $c > 0$. Assume that $\{T_t\}$ has a heat kernel h satisfying

$$h(t, x, y) \geq c'\, t^{-\alpha/\beta} \Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right) \tag{4.13}$$

for all $t > 0$ and $x, y \in X$, where $c', \alpha, \beta > 0$ and Φ is continuous decreasing on $[0, \infty)$ with $\Phi(1) < \infty$. Then the Riesz kernel has the lower estimate

$$R^f(x, y) \geq c'' d(x, y)^{-\alpha} f\left(d(x, y)^{-\beta}\right) \tag{4.14}$$

for all $x \neq y \in X$, for some $c'' > 0$.

Proof Let $r := d(x, y) > 0$. It follows from Eqs. 4.4, 4.12 and 4.13 that

$$\begin{aligned} R^f(x, y) &= \int_0^\infty h(t, x, y)\rho(t)\, dt \geq \int_{r^\beta}^\infty h(t, x, y)\rho(t)\, dt \\ &\geq c \int_{r^\beta}^\infty t^{-\alpha/\beta - 1} f\left(t^{-1}\right) \Phi\left(\frac{r}{t^{1/\beta}}\right) dt \\ &\geq c\, \Phi(1) f\left(r^{-\beta}\right) \int_{r^\beta}^\infty t^{-\alpha/\beta - 1}\, dt \\ &= c\, \Phi(1) r^{-\alpha} f\left(r^{-\beta}\right), \end{aligned}$$

where we have used the monotonicity of Φ and f. □

We next derive an upper bound of the Riesz kernel. To do this, we need upper estimates of f, ρ and h.

Proposition 4.7 *Let $r_0 = \infty$, and let f be given as in Eq. 3.3 such that*

$$f(x) \leq \lambda f(\lambda x) \quad \text{for all } x > 0 \text{ and } \lambda \geq 1, \tag{4.15}$$

$$\rho(t) \leq c\, t^{-1} f\left(t^{-1}\right) \quad \text{for all } t > 0, \tag{4.16}$$

where $c > 0$. Assume that $\{T_t\}$ has a heat kernel h satisfying

$$h(t, x, y) \leq c'\, t^{-\alpha/\beta} \Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right) \tag{4.17}$$

for all $t > 0$ and $x, y \in X$, where $c' > 0$ and $\alpha > \beta > 0$, and Φ is continuous decreasing on $[0, \infty)$ with $\Phi(0) < \infty$ and

$$\int_1^{\infty} s^{\alpha-1}\Phi(s)\, ds < \infty. \tag{4.18}$$

Then the Riesz kernel R^f satisfies

$$R^f(x, y) \le c'' d(x, y)^{-\alpha} f\left(d(x, y)^{-\beta}\right) \tag{4.19}$$

for all $x \neq y \in X$, for some $c'' > 0$.

Proof Let $r := d(x, y) > 0$. It follows from Eqs. 4.17 and 4.16 that

$$\begin{aligned} R^f(x, y) &= \int_0^{\infty} h(t, x, y)\rho(t)\, dt \le c \int_0^{\infty} t^{-\alpha/\beta-1} f\left(t^{-1}\right) \Phi\left(\frac{r}{t^{1/\beta}}\right) dt \\ &= c \left\{ \int_0^{r^\beta} t^{-\alpha/\beta-1} f\left(t^{-1}\right) \Phi\left(\frac{r}{t^{1/\beta}}\right) dt \right. \\ &\quad \left. + \int_{r^\beta}^{\infty} t^{-\alpha/\beta-1} f\left(t^{-1}\right) \Phi\left(\frac{r}{t^{1/\beta}}\right) dt \right\}. \end{aligned} \tag{4.20}$$

We compute that, using the monotonicity of f,

$$\begin{aligned} \int_0^{r^\beta} t^{-\alpha/\beta-1} f\left(t^{-1}\right) \Phi\left(\frac{r}{t^{1/\beta}}\right) dt &\le f\left(r^{-\beta}\right) \int_0^{r^\beta} t^{-\alpha/\beta-1} \Phi\left(\frac{r}{t^{1/\beta}}\right) dt \\ &= \beta r^{-\alpha} f\left(r^{-\beta}\right) \int_1^{\infty} s^{\alpha-1}\Phi(s)\, ds \\ &\le c r^{-\alpha} f\left(r^{-\beta}\right) \end{aligned} \tag{4.21}$$

by virtue of Eq. 4.18. On the other hand, if $t \ge r^\beta$, we let $\lambda = tr^{-\beta} \ge 1$ in Eq. 4.15, and obtain that

$$f\left(t^{-1}\right) \le tr^{-\beta} f\left(r^{-\beta}\right) \quad \text{for } t, r > 0.$$

Therefore, using the monotonicity of Φ,

$$\begin{aligned} \int_{r^\beta}^{\infty} t^{-\alpha/\beta-1} f\left(t^{-1}\right) \Phi\left(\frac{r}{t^{1/\beta}}\right) dt &\le r^{-\beta} f\left(r^{-\beta}\right) \Phi(0) \int_{r^\beta}^{\infty} t^{-\alpha/\beta}\, dt \\ &= \frac{\beta}{\beta-\alpha} \Phi(0) r^{-\alpha} f\left(r^{-\beta}\right) \end{aligned} \tag{4.22}$$

since $\alpha > \beta$. Combining Eqs. 4.20, 4.21 and 4.22, we obtain the desired. □

Note that all the heat kernels in Examples 4.1 and 4.2 satisfy the conditions in Propositions 4.6 and 4.7.

We now give some classes of completely monotone functions f for which the assumptions in Propositions 4.6 and 4.7 are also fulfilled.

Example 4.8 Let g be any positive Bernstein function on $(0, \infty)$. Then $f = 1/g$ is completely monotone, and satisfies Eq. 4.15. In fact, since $g > 0$ and its derivative g' is decreasing, we see that

$$g(\lambda x) \leq \lambda g(x) \quad \text{for all } \lambda \geq 1 \text{ and } x > 0.$$

It follows that

$$f(x) = \frac{1}{g(x)} \leq \frac{\lambda}{g(\lambda x)} = \lambda f(\lambda x).$$

Example 4.9 Let f be given as in Eq. 3.3. Then the following is true.

- If ρ is monotone, then

$$\rho(t) \leq \left(1 - e^{-1}\right)^{-1} t^{-1} f\left(t^{-1}\right) \tag{4.23}$$

  for all $t > 0$. See the proof [29], or [6, Chap.III, Sect.1, Prop. 1].
- If ρ is decreasing on $(0, \infty)$ and $f(\lambda x) \leq c\,\lambda^{-\delta} f(x)$ for all $\lambda \geq 1$ and $x > 0$ where $c, \delta > 0$, or if ρ is increasing on $(0, \infty)$ and $f(2x) \geq c\, f(x)$ for all $x > 0$ where $c > 0$, then

$$\rho(t) \geq c' t^{-1} f\left(t^{-1}\right) \tag{4.24}$$

  for all $t > 0$. See [29].

5 Subordination

In this section we will establish relationships between the generalized Bessel or Riesz operators and subordinate semigroups.

Let $\{T_t\}_{t \geq 0}$ be a contractive strongly continuous semigroup on $L^p(\mu)$ with infinitesimal generator $(A, \mathcal{D}(A))$. Let g be a Bernstein function on $(0, \infty)$ with associated convolution semigroup $\{\mu_t\}$ on $[0, \infty)$ as in Eq. 2.14. Define

$$T_t^g u = \int_0^\infty T_s u \, d\mu_t(s) \quad (t \geq 0) \tag{5.1}$$

for $u \in L^p(\mu)$. Then $\{T_t^g\}_{t \geq 0}$ is also a contractive strongly continuous semigroup on $L^p(\mu)$. This new semigroup $\{T_t^g\}_{t \geq 0}$ is termed a *subordinate semigroup*, which was first introduced by Bochner [7] in 1949.

Remark 5.1 For $p = 2$, if the semigroup $\{T_t\}$ is *Markovian*, then the subordinate semigroup $\{T_t^g\}_{t \geq 0}$ is also Markovian. Let X_t and X_t^g be the Markov processes on (X, d, μ) associated with $\{T_t\}$ and $\{T_t^g\}_{t \geq 0}$, respectively. A well-known probabilistic interpretation of X_t^g via time changes is given by

$$X_t^g = X_{S_t^g}, \quad t > 0.$$

Here S_t^g denotes the subordinator corresponding to the Bernstein function g, that is, the non-negative increasing Markov process on $[0, \infty)$ with generating Markov semigroup $\{\mu_t\}$. Moreover, S_t^g is independent of the process X_t. (See for example [9].)

If we replace the semigroup $\{T_t\}$ in Eq. 5.1 by the semigroup $\{e^{-t}T_t\}$, and let

$$P_t^g u := \int_0^\infty e^{-s} T_s u \, d\mu_t(s), \quad t > 0, \tag{5.2}$$

then $\{P_t^g\}_{t\geq 0}$ is also a subordinate semigroup on $L^p(\mu)$.

Theorem 5.2 *Let $\{T_t\}$ be a contractive strongly continuous semigroup on $L^p(\mu)$ for $1 \leq p < \infty$. Let g be a strictly positive Bernstein function on $(0, \infty)$, and let $f(x) = g(x)^{-1}$ for $x > 0$.*[4] *Then*

$$J_\mu^f u = \int_0^\infty P_t^g u \, dt, \tag{5.3}$$

$$I_\mu^f u = \int_0^\infty T_t^g u \, dt \tag{5.4}$$

for suitable $u \in L^p(\mu)$, where $\{T_t^g\}$ and $\{P_t^g\}$ are as in Eqs. 5.1 and 5.2 respectively.

Proof Let $g > 0$ be a Bernstein function on $(0, \infty)$ with the convolution semigroup $\{\mu_t\}$ on $[0, \infty)$ as in Eq. 2.14, and let ν be the measure associated with $f = \frac{1}{g}$ as in Eq. 2.6. Note that for any Borel $B \subset [0, \infty)$,

$$\nu(B) = \int_0^\infty \mu_t(B) \, dt, \tag{5.5}$$

see for example [6, p.74]. It follows from Eqs. 2.19, 5.5, and 5.2 that

$$\begin{aligned} J_\mu^f u &= \int_0^\infty e^{-s} T_s u \, d\nu(s) \\ &= \int_0^\infty \left(\int_0^\infty e^{-s} T_s u \, d\mu_t(s) \right) dt \\ &= \int_0^\infty P_t^g u \, dt, \end{aligned}$$

proving Eq. 5.3. The equality (5.4) can be proved in a similar way. □

Remark 5.3 If the Bernstein function $g(x) = x$, then μ_t is the Dirac measure concentrated at point t, for any $t > 0$, and so $P_t^g = e^{-t} T_t$ and $T_t^g = T_t$. Formula (5.3) and (5.4) coincide with the definition (2.19) and (2.20) respectively, with $d\nu(t) = dt$.

Let $(A^g, \mathcal{D}(A^g))$ be the generator of $\{T_t^g\}_{t\geq 0}$. It was shown by Phillips [18] that $\mathcal{D}(A) \subset \mathcal{D}(A^g)$, and

$$-A^g u = au + b\,Au + \int_0^\infty (u - T_t u) \; dm(t) \tag{5.6}$$

[4] Recall that $f = \frac{1}{g}$ is a completely monotone function if g is a Bernstein function.

if $u \in \mathcal{D}(A)$, where the triple (a, b, m) is uniquely determined by g as in Eq. 2.10. Schilling [21] proved that $\mathcal{D}(A^g) = \mathcal{D}(A)$ if and only if either A is bounded or if $b > 0$. Afterwards, he gave a characterization of $\mathcal{D}(A^g)$ by using the approximation procedure for *complete* Bernstein function g, see [22]. See also [15] for $g(x) = x^\sigma$ for fractal domains X by using the heat kernel.

In the remainder of this section, we assume that g is a Bernstein function with $a = 0$ and $b = \lim_{x\to\infty} \frac{g(x)}{x} = 0$, that is,

$$g(x) = \int_0^\infty \left(1 - e^{-tx}\right) dm(t). \tag{5.7}$$

Note that by Phillips's result, the generator B^g of the subordinate semigroup $\{P_t^g\}$ defined as in Eq. 5.2 is given by

$$-B^g u = \int_0^\infty \left(u - e^{-t} T_t u\right) dm(t) \tag{5.8}$$

for $u \in \mathcal{D}(A)$.

Theorem 5.4 *Let $\{T_t\}$ be a contractive strongly continuous semigroup on $L^p(\mu)$ for $1 \le p < \infty$. Let g be a strictly positive Bernstein function on $(0, \infty)$ given by Eq. 5.7, and let $f = \frac{1}{g}$. Then*

$$\left(J_\mu^f\right)^{-1} = -B^g, \tag{5.9}$$

$$\mathcal{D}\left(B^g\right) = J_\mu^f\left(L^p(\mu)\right) = H_p^f(\mu). \tag{5.10}$$

where B^g is defined as in Eq. 5.8.

Proof We first show that B^g is injective. Assume that there were a function $u_0 \neq 0$ in $\mathcal{D}(B^g)$ such that $B^g(u_0) = 0$. Then we could choose a continuous linear functional Λ on $L^p(\mu)$ with $\Lambda(u_0) = 1$. Letting

$$\phi(t) := \Lambda\left(P_t^g u_0\right), \quad t \ge 0,$$

we obtain

$$\frac{d\phi}{dt}(t) = \Lambda\left(P_t^g B^g u_0\right) = 0,$$

and $\phi(0) = 1$. Thus, $\phi(t) = 1$ for any $t \ge 0$. But this is a contradiction, since

$$|\phi(t)| \le C \left\| P_t^g u_0 \right\|_p \le C||u_0||_p \int_0^\infty e^{-s} d\mu_t(s)$$
$$= C||u_0||_p\, e^{-tg(1)} \to 0 \quad \text{as } t \to \infty,$$

where $C = \|\Lambda\|$. Hence, the operator B^g is injective.

We now show that

$$B^g\left(J^f_\mu u\right) = -u \quad \text{for any } u \in L^p(\mu). \tag{5.11}$$

In fact, it follows from Eq. 5.3 that

$$\begin{aligned} B^g\left(J^f_\mu u\right) &= \lim_{s\to 0} s^{-1}\left(P^g_s - I\right)\int_0^\infty P^g_t u\,dt \\ &= \lim_{s\to 0} s^{-1}\left(\int_0^\infty P^g_{t+s}u\,dt - \int_0^\infty P^g_t u\,dt\right) \\ &= -\lim_{s\to 0} s^{-1}\int_0^s P^g_t u\,dt = -u, \end{aligned}$$

where the limits are taken in the $L^p(\mu)$-norm.

Finally, it follows from Eq. 5.11 that $H^f_p(\mu) = J^f_\mu(L^p(\mu)) \subset \mathcal{D}(B^g)$. On the other hand, for $u \in \mathcal{D}(B^g)$, let $\phi := B^g u \in L^p(\mu)$. We see that $B^g\big(J^f_\mu\phi\big) = -\phi = -B^g u$, showing that $u = -J^f_\mu\phi$ as B^g is injective. Hence, we have that $u \in H^f_p(\mu)$, and so $\mathcal{D}(B^g) = H^f_p(\mu)$ and $-B^g = (J^f_\mu)^{-1}$. This finishes the proof. □

One can obtain a parallel conclusion like Theorem 5.4 for the Riesz operator. We omit the details.

By construction, we get the interpretations

$$-B^g = g(I-A) \quad \text{and} \quad -A^g = g(-A)\,.$$

Corollary 5.5 *Under the conditions of Theorem* 5.4 *we have the norm equivalence*

$$\|u\|_{H^f_p(\mu)} = \|g(I-A)u\|_p \sim \|u\|_p + \|g(-A)u\|_p$$

in the space of f-Bessel potentials.

Proof From the representation formulas for the operators A^g and B^g we infer

$$A^g u - B^g u = \int_0^\infty \left(T_t u - e^{-t}T_t u\right) dm(t)$$

and hence, using the contractivity of the semigroup,

$$\left|\|A^g u\|_p - \|B^g u\|_p\right| \le \|u\|_p \int_0^\infty \left(1-e^{-t}\right)\, dm(t) = g(1)\|u\|_p\,.$$

Furthermore, since $-B^g = \left(J^f_\mu\right)^{-1}$ and $\left\|J^f_\mu u\right\|_p \le g(1)^{-1}\|u\|_p$, we obtain

$$\|u\|_p = \left\|J^f_\mu B^g u\right\|_p \le g(1)^{-1}\|B^g u\|_p$$

which leads to

$$\|B^g u\|_p \sim \|u\|_p + \|A^g u\|_p\,.$$

This finishes the proof. □

Basing on the above formulas we may estimate $\left|\left(J_\mu^f\right)^{-1}u(x)\right|$ for $x \in X$ by using the heat kernel h. To do this, note that by Eqs. 5.9 and 5.8,

$$\begin{aligned}\left(J_\mu^f\right)^{-1}u &= \int_0^\infty \left(u - e^{-t}T_t u\right)\, dm(t)\\ &= u\int_0^\infty \left(1-e^{-t}\right)\, dm(t) + \int_0^\infty e^{-t}\left(u - T_t u\right)\, dm(t)\\ &= c_0 u + Q_1 u, \qquad (5.12)\end{aligned}$$

where $c_0 = f(1)^{-1} = \int_0^\infty \left(1-e^{-t}\right)\, dm(t) < \infty$, and

$$Q_1 u := \int_0^\infty e^{-t}\left(u - T_t u\right)\, dm(t). \qquad (5.13)$$

If $\{T_t\}$ is conservative and has a heat kernel h, we see that

$$u(x) - T_t u(x) = \int_X (u(x)-u(y))h(t,x,y)\, d\mu(y),$$

and so

$$\begin{aligned}|Q_1 u(x)| &= \int_0^\infty e^{-t}\left|u(x) - T_t u(x)\right|\, dm(t)\\ &\le \int_X |u(x)-u(y)|\, k_f(x,y)\, d\mu(y), \qquad (5.14)\end{aligned}$$

where

$$k_f(x,y) = \int_0^\infty e^{-t}h(t,x,y)\, dm(t). \qquad (5.15)$$

Proposition 5.6 *Assume that $\{T_t\}$ is conservative and has a heat kernel h satisfying Eq. 4.5, and that $dm(t) = \eta(t)dt$ for some $\eta : (0,\infty) \to [0,\infty)$. If $w_2(r) := V(r)/\eta\left(r^\beta\right) \sim r^{\theta_2}$ for some $\theta_2 > \beta$, and if*

$$\int_0^\infty s^{\theta_2-\beta-1}\Phi(s)\, ds < \infty, \qquad (5.16)$$

then we have that

$$k_f(x,y) \sim d(x,y)^{-(\theta_2-\beta)} \qquad (5.17)$$

for any $x \neq y \in X$. Consequently,

$$|Q_1 u(x)| \le \int_X \frac{|u(x)-u(y)|}{d(x,y)^{\theta_2-\beta}}\, d\mu(y) \quad (x \in X). \qquad (5.18)$$

Proof The proof is the same as that in Theorem 4.3. We omit the detail. □

Remark 5.7 If $f(x) = x^{-\sigma}$ for $\sigma \in (0, 1)$ and $V(r) \sim r^{\alpha}$ for $\alpha > 0$, then $\eta(t) = \frac{\sigma}{\Gamma(1-\sigma)} t^{-1-\sigma}$, and

$$w_2(r) = V(r)/\eta\left(r^{\beta}\right) \sim r^{\alpha+\beta(1+\sigma)}.$$

Therefore, we have that $k_f(x, y) \sim d(x, y)^{-(\alpha+\beta\sigma)}$, and

$$|Q_1 u(x)| \leq \int_X \frac{|u(x) - u(y)|}{d(x, y)^{\alpha+\beta\sigma}} \, d\mu(y).$$

This makes the inversion formula from the proof of Theorem 3.2 in our special situation more explicit.

In particular, if $X = \mathbb{R}^n$ and μ is the Lebesgue measure, we see that for $\sigma \in (0, 1)$,

$$|Q_1 u(x)| \leq \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x - y|^{n+2\sigma}} \, dy,$$

where $\alpha = n$ and $\beta = 2$ which corresponds with the representation of the inverses of the Bessel potentials by means of hypersingular integrals (see [20, (27.37)]).

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