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# Rate of Convergence of Space Time Approximations for Stochastic Evolution Equations

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**Abstract** Stochastic evolution equations in Banach spaces with unbounded nonlinear drift and diffusion operators driven by a finite dimensional Brownian motion are considered. Under some regularity condition assumed for the solution, the rates of convergence of various numerical approximations are estimated under strong monotonicity and Lipschitz conditions. The abstract setting involves general consistency conditions and is then applied to a class of quasilinear stochastic PDEs of parabolic type.

**Keywords** Stochastic evolution equations · Monotone operators · Coercivity · Space time approximations · Galerkin method · Wavelets · Finite elements

**Mathematics Subject Classifications (2000)** Primary 60H15 · Secondary 65M60

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## 1 Introduction

Let $V \hookrightarrow H \hookrightarrow V^*$ be a *normal triple* of spaces with dense and continuous embeddings, where $V$ is a separable and reflexive Banach space, $H$ is a Hilbert space, identified with its dual by means of the inner product in $H$, and $V^*$ is the dual of $V$.

Let $W = \{W(t) : t \geq 0\}$ be a $d_1$-dimensional Brownian motion carried by a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Consider the stochastic evolution equation

$$u(t) = u_0 + \int_0^t A(s, u(s))\, ds + \sum_{k=1}^{d_1} \int_0^t B_k(s, u(s))\, dW^k(s)\,, \quad t \in [0, T] \tag{1.1}$$

in the triple $V \hookrightarrow H \hookrightarrow V^*$, with a given $H$-valued $\mathcal{F}_0$-measurable random variable $u_0$, and given operators $A$ and $B = (B_k)$, mapping $[0, \infty) \times \Omega \times V$ into $V^*$ and $H^{d_1} := H \times \cdots \times H$, respectively. Let $\mathcal{P}$ denote the $\sigma$-algebra of the predictable subsets of $[0, \infty) \times \Omega$, and let $\mathcal{B}(V)$, $\mathcal{B}(H)$ and $\mathcal{B}(V^*)$ be the Borel $\sigma$-algebras of $V$, $H$ and $V^*$, respectively. Assume that $A$ and $B_k$ are $\mathcal{P} \otimes \mathcal{B}(V)$-measurable with respect to the $\sigma$-algebras $\mathcal{B}(V^*)$ and $\mathcal{B}(H)$, respectively.

It is well-known that for any $T > 0$ Eq. 1.1 admits a unique solution $u$ if $A$ is *hemicontinuous* in $v \in V$, and $(A, B)$ satisfies a *monotonicity*, *coercivity* and a *linear growth* condition (see [10, 13] and [16]). In [7] it is shown that under these conditions the solutions of various implicit and explicit schemes converge to $u$. In [8] the rate of convergence of implicit Euler approximations is estimated under more restrictive hypotheses: $A$ and $B$ satisfy a strong monotonicity condition, $A$ is Lipschitz continuous in $v \in V$, and the solution $u$ satisfies some regularity conditions. Then Theorem 3.4 from [8] in the case of time independent operators $A$ and $B$ reads as follows. For the implicit Euler approximation $u^\tau$, corresponding to the mesh size $\tau = T/m$ of the partition of $[0, T]$, one has

$$E \max_{i\leq m} |u(i\tau) - u^\tau(i\tau)|_H^2 + \tau E \sum_{i\leq m} \|u(i\tau) - u^\tau(i\tau)\|_V^2 \leq C\tau^{2\nu},$$

where $C$ is a constant, independent of $\tau$, and $\nu \in ]0, \frac{1}{2}]$ is a constant from the regularity condition imposed on $u$.

In this paper, we study space and space-time approximations schemes for Eq. 1.1 in a general framework. In order to obtain rate of convergence estimates we need to require more regularity from the solution $u$ of Eq. 1.1 than what we can express in terms of the spaces $V$ and $H$. Therefore in our setup we introduce additional Hilbert spaces $\mathcal{V}$ and $\mathcal{H}$ such that $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow V$, where $\hookrightarrow$ denotes continuous embeddings. In examples these are Sobolev spaces such that $\mathcal{H}$ and $\mathcal{V}$ satisfy stronger differentiability conditions than $V$ and $\mathcal{H}$, respectively. Our regularity conditions on the solution $u$ are introduced in Section 2 and labeled as **(R1)** and **(R2)**. In connection with these, we introduce also condition **(R3)**, requiring more regularity from $A$ and $B$. Furthermore, condition **(R4)** on Hölder continuity in time of $A$ and $B$ is needed for schemes involving time discretization. We collect these conditions in Assumption 3 and call them *regularity conditions*.

In order to formulate 'space discretizations', we consider for any integer $n \geq 1$ a normal triple

$$V_n \hookrightarrow H_n \hookrightarrow V_n^*, \tag{1.2}$$

the 'discrete' counterpart of $V \hookrightarrow H \hookrightarrow V^*$, and a bounded linear operator

$$\Pi_n : V \to V_n,$$

connecting $V$ to $V_n$. We have in mind discrete Sobolev spaces, wavelets and finite elements spaces, as examples for $V_n$.

The space discretization scheme for Eq. 1.1 is a stochastic evolutional equation in the triple (1.2). We define it by replacing the operators $A$, $B$ and the initial value $u_0$ in Eq. 1.1 by some $\mathcal{P} \otimes \mathcal{B}(V_n)$-measurable operators

$$A^n : [0, \infty[\times \Omega \times V_n \to V_n^*, \quad B^n : [0, \infty[\times \Omega \times V_n \to H_n^{d_1}$$

and by an $H_n$-valued $\mathcal{F}_0$-measurable random variable $u_0^n$, respectively, such that $A^n$ and $B^n$ satisfy in the triple (1.2) the strong monotonicity condition, the linear growth condition, $A^n$ is hemicontinuous and $B^n$ is Lipschitz continuous in $v \in V_n$. These are the conditions **(S1)–(S4)** in Assumption 3.1, which imply, in particular, the existence and uniqueness of a solution $u^n$ to our scheme. We relate $A_n$ and $B_n$ to $A$ and $B$ via a *consistency condition*, **(Cn)** below. Then assuming **(S1)–(S4)**, under the regularity and consistency conditions **(R1)**, **(R3)** and **(Cn)** we have

$$E \sup_{0 \le t \le T} |\Pi_n u(t) - u^n(t)|^2_{H_n} + E \int_0^T \|\Pi_n u(t) - u^n(t)\|^2_{V_n} \, dt \le CE|\Pi_n u_0 - u_0^n|^2_{H_n} + C\varepsilon_n^2,$$

where $C$ is a constant, independent of $n$, and $\varepsilon_n > 0$ is a constant from **(Cn)**. This is Theorem 3.1 below, our main result on the accuracy of approximations by space discretizations.

For an integer $m \ge 1$ we consider the grid $\{t_i = i\tau : 0 \le i \le m\}$ with mesh-size $\tau = T/m$. We define on this grid the space-time implicit and the space-time explicit approximations, $\{u_i^{n,\tau}\}_{i=0}^m$ and $\{u_{\tau,i}^n\}_{i=0}^m$, respectively, by

$$u_{i+1}^{n,\tau} = u_i^{n,\tau} + \tau A_{i+1}^{n,\tau}\big(u_{i+1}^{n,\tau}\big) + \sum_k B_{k,i}^{n,\tau}\big(u_i^{n,\tau}\big)\big(W^k(t_{i+1}) - W^k(t_i)\big),$$

$$u_{\tau,i+1}^n = u_{\tau,i}^n + \tau\, A_i^{n,\tau}(u_{\tau,i}^n) + \sum_k B_{k,i}^{n,\tau}\big(u_{\tau,i}^n\big)\big(W^k(t_{i+1}) - W_k(t_i)\big),$$

for $i = 0, \dots, m-1$ with some $V_n$-valued $\mathcal{F}_0$-measurable random variables $u_0^{n,\tau}$ and $u_{0,\tau}^n$, and with some $\mathcal{F}_{t_i} \otimes \mathcal{B}(V)$-measurable operators

$$A_i^{n,\tau} : \Omega \times V_n \to V_n^*, \quad B_{k,i}^{n,\tau} : \Omega \times V_n \to H_n^{d_1},$$

such that $A_i^{n,\tau}$, $B_{k,i}^{n,\tau}$ satisfy strong monotonicity and linear growth conditions and $A_i^{n,\tau}$ is Lipschitz continuous in $v \in V_n$. These conditions, listed as **(ST1)–(ST3)** in Assumption 4.1 below, are similar to conditions **(S1)–(S3)**, except that instead of the hemicontinuity, the much stronger assumption of Lipschitz continuity is assumed on $A_i^{n,\tau}$. The operators $A_i^{n,\tau}$ and $B_{k,i}^{n,\tau}$ are related to $A$ and $B$ by a consistency condition **(Cn$\tau$)** stated below. Then if $\sup_{n,m} E|u_0^{n,\tau}|^2_{H_n} < \infty$ and Eq. 1.1 satisfies the regularity conditions **(R1)–(R4)** from Assumption 2.3, we have the estimate

$$\begin{aligned} &E \sup_{0 \le i \le m} \big|\Pi_n u(t_i) - u_i^{n,\tau}\big|^2_{H_n} + E \sum_{0 \le i \le m} \big\|\Pi_n u(t_i) - u_i^{n,\tau}\big\|^2_{V_n} \tau \\ &\quad \le CE|\Pi_n u_0 - u_0^{n,\tau}|^2_{H_n} + C\big(\tau^{2\nu} + \varepsilon_n^2\big), \end{aligned}$$

with a constant $C$, independent of $n$ and $\tau$, where $\nu \in ]0, \frac{1}{2}]$ is the Hölder exponent from condition **(R4)** on the regularity of the operators $A$ and $B$ in time, and $\varepsilon_n$ is from **(Cn$\tau$)**. This is Theorem 4.4, our main result on implicit space-time approximations. In our main result, Theorem 5.2, on the explicit space-time approximations we have the same estimate for $u^n_{\tau,i}$ in place of $u^{n,\tau}_i$ if, in addition to the conditions of Theorem 4.4, as in [7], a *stability relation* between the time mesh $\tau$ and a space approximation parameter is satisfied.

Finally, we present as examples a class of quasi-linear stochastic partial differential equations (SPDEs) and linear SPDEs of parabolic type. We show that they satisfy the conditions of the abstract results, Theorems 3.1, 4.4 and 5.2, when we use wavelets, or finite differences. In particular, we obtain rate of convergence results for space and space-time approximations of linear parabolic SPDEs, among them for the Zakai equation of nonlinear filtering. We would like to mention that as far as we know, discrete Sobolev spaces are applied first in [18] to space discretizations and explicit space-time discretizations of linear SPDEs, and it inspired our approach to finite difference schemes. Our abstract results can also be applied to finite elements approximations. To keep down the size of the paper we will consider such applications elsewhere.

We denote by $K$, $L$, $M$ and $r$ some fixed constant, and by $C$ some constants which, as usual, can change from line to line. For given constants $a \in \mathbb{R}^k$ the notation $C(a)$ means that the constant depends on $a$. Finally, when $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ denote two Banach spaces such that $X$ is continuously embedded in $Y$, given $y \in Y$ the inequality $|y|_X < +\infty$ means that $y \in X$.

## 2 Conditions on Eq. 1.1 and on the Approximation Spaces

### 2.1 Conditions on Eq. 1.1

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis, satisfying the usual conditions, i.e., $(\mathcal{F}_t)_{t\geq 0}$ is an increasing right-continuous family of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains every $P$-null set. Let $W = \{W(t) : t \geq 0\}$ be a $d_1$-dimensional Wiener martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, i.e., $W$ is an $\mathcal{F}_t$-adapted Wiener process with values in $\mathbb{R}^{d_1}$ such that $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t$. We use the notation $\mathcal{P}$ for the sigma-algebra of predictable subsets of $[0, \infty) \times \Omega$. If $V$ is a Banach space then $\mathcal{B}(V)$ denotes the sigma-algebra generated by the (closed) balls in $V$.

Let $V$ be a separable reflexive Banach space embedded densely and continuously into a Hilbert space $H$, which is identified with its dual $H^*$ by means of the inner product $(\cdot, \cdot)$ in $H$. Thus we have a *normal triple*

$$V \hookrightarrow H \hookrightarrow V^*,$$

where $H \hookrightarrow V^*$ is the adjoint of the embedding $V \hookrightarrow H$. Thus $\langle v, h\rangle = (v, h)$ for all $v \in V$ and $h \in H^* = H$, where $\langle v, v^*\rangle = \langle v^*, v\rangle$ denotes the duality product of $v \in V$, $v^* \in V^*$, and $(h_1, h_2)$ denotes the inner product of $h_1, h_2 \in H$. We assume, without loss of generality, that $|v|_H \leq \|v\|_H$ for all $v \in V$, where $|\cdot|_H$ and $\|\cdot\|_V$ denote the norms in $H$ and $V$, respectively. For elements $u$ from a normed space $\mathbb{U}$ the notation $|u|_{\mathbb{U}}$ means the norm of $u$ in $\mathbb{U}$.

Let $A$ and $B = (B_k)_{k=1}^{d_1}$ be $\mathcal{P} \otimes \mathcal{B}(V)$-measurable mappings from $[0, \infty) \times \Omega \times V$ into $V^*$ and $H^{d_1}$, respectively. Given an $H$-valued $\mathcal{F}_0$-measurable random variable $u_0$ consider the initial value problem

$$du(t) = A(t, u(t))\, dt + \sum_k B_k(t, u(t))\, dW^k(t), \quad u(0) = u_0 \tag{2.1}$$

on a fixed time interval $[0, T]$.

**Assumption 2.1** *The operators $A$ and $B$ satisfy the following conditions.*

**(i)** ($Monotonicity$ *of* $(A, B)$) *Almost surely for all $t \in [0, T]$ and $u, v \in V$,*

$$2\langle u - v, A(t, u) - A(t, v)\rangle + \sum_k \left|B_k(t, u) - B_k(t, v)\right|_H^2 \le K\left|u - v\right|_H^2,$$

**(ii)** (*Coercivity of* $(A, B)$) *Almost surely for all $t \in [0, T]$ and $u, v \in V$,*

$$2\langle u, A(t, u)\rangle + \sum_k \left|B_k(t, u)\right|_H^2 + \mu\|u\|_V^2 \le K|u|_H^2 + f(t), \tag{2.2}$$

**(iii)** (*Growth conditions on $A$ and $B$*) *Almost surely for all $t \in [0, T]$ and $u \in V$,*

$$|A(t, u)|_{V^*}^2 \le K_1\|u\|_V^2 + f(t), \quad \sum_k \left|B_k(t, u)\right|_H^2 \le K_2\|u\|_V^2 + f(t),$$

**(iv)** (*Hemicontinuity of $A$*) *Almost surely for all $t \in [0, T]$ and $u, v, w \in V$,*

$$\lim_{\varepsilon \to 0}\langle w, A(t, u + \varepsilon v)\rangle = \langle w, A(t, u)\rangle, \tag{2.3}$$

*where $\mu > 0$, $K \ge 0$, $K_1 \ge 0$ and $K_2 \ge 0$ are some constants, and $f$ is a non-negative $(\mathcal{F}_t)$-adapted stochastic process such that*

$$E\int_0^T f(t)\, dt < \infty. \tag{2.4}$$

The following definition of solution is classical.

**Definition 2.1** An $H$-valued adapted continuous process $u = \{u(t) : t \in [0, T]\}$ is a solution to Eq. 1.1 on $[0, T]$ if almost surely $u(t) \in V$ for almost every $t \in [0, T]$,

$$\int_0^T \|u(t)\|_V^2\, dt < \infty,$$

and

$$(u(t), v) = (u_0, v) + \int_0^t \langle A(s, u(s)), v\rangle\, ds + \sum_k \int_0^t (B_k(s, u(s)), v)\, dW^k(s)$$

holds for all $t \in [0, T]$ and $v \in V$. We say that the solution to Eq. 2.1 on $[0, T]$ is unique if for any solutions $u$ and $v$ to Eq. 2.1 on $[0, T]$ we have

$$P\left(\sup_{t\in[0,T]} |u(t) - v(t)|_H > 0\right) = 0.$$

The following result is well known, see [10, 13, 16].

**Theorem 2.2** *Let Assumption* 2.1 *hold. Then Eq.* 2.1 *has a unique solution u. Moreover, if $E|u_0|_H^2 < \infty$, then*

$$E \sup_{t\in[0,T]} |u(t)|_H^2 + E\int_0^T \|u(s)\|_V^2\,ds$$

$$\le CE|u_0|_H^2 + CE\int_0^T \big(f(t)+g(t)\big)\,dt < \infty, \tag{2.5}$$

*where C is a constant depending only on the constants $\lambda$, $K$ and $K_2$.*

If Assumption 2.1 is satisfied then one can also show the convergence of approximations, obtained by various discretization schemes, to the solution $u$ (see [7]). To estimate the rate of convergence of implicit time discretization schemes the following stronger assumptions on $A$ and $B$ are used in [8]

**Assumption 2.2** *The operators $A$, $B$ satisfy the following conditions almost surely.*

**(1)** (*Strong monotonicity*) *For all $t \in [0,T]$, $u, v \in V$,*

$$2\langle u-v, A(t,u)-A(t,v)\rangle + \sum_k |B_k(t,u)-B_k(t,v)|_H^2$$

$$\le -\lambda\|u-v\|_V^2 + L|u-v|_H^2,$$

**(2)** (*Growth conditions on $A$ and $B$*) *For all $t \in [0,T]$, $u \in V$,*

$$|A(t,u)|_{V^*}^2 \le K_1\|u\|_V^2 + f(t), \quad \sum_k |B_k(t,u)|_H^2 \le K_2\|u\|_V^2 + g(t). \tag{2.6}$$

**(3)** (*Lipschitz condition on $A$*) *For all $t \in [0,T]$, $u, v \in V$,*

$$|A(t,u)-A(t,v)|_{V^*}^2 \le L_1\|u-v\|_V^2, \tag{2.7}$$

*where $\lambda > 0$, $K \ge 0$, $K_1 \ge 0$, $K_2 \ge 0$ are constants, and $f$ and $g$ are non-negative adapted processes satisfying Eq.* 2.4

*Remark 2.3* It is easy to see that due to **(1)–(2)**, the coercivity condition (2.2) holds with $\mu = \lambda/2$ and a constant $K = K(\lambda, L, K_2)$.

*Remark 2.4* It is easy to show that **(1)** and **(3)** imply that $B = (B_k)$ is Lipschitz continuous in $u \in V$, i.e., almost surely

$$\sum_k |B_k(t,u)-B_k(t,v)|_H^2 \le L_2\|u-v\|_V^2 \quad \text{for all } u, v \in V, t \in [0,T] \tag{2.8}$$

where $L_2$ is a constant depending on $\lambda$, $L$ and $L_1$.

In order to prove rate of convergence estimates for the approximation schemes presented in this paper, we need to impose additional *regularity conditions* on Eq. 2.1

and on the solution $u$. Therefore we assume that there exist some separable Hilbert spaces $\mathcal{V}$ and $\mathcal{H}$ such that

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow V,$$

where $\hookrightarrow$ means continuous embedding, and introduce the following conditions.

Let $K$, $M$ denote some constants, fixed throughout the paper.

**Assumption 2.3** (*Regularity conditions*)

**(R1)** *There is a unique solution $u$ of Eq. 2.1, it takes values in $\mathcal{V}$ for $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$, $u_0 \in V$ and*

$$E\|u_0\|_V^2 < \infty, \quad E\int_0^T |u(t)|_{\mathcal{V}}^2\, dt =: r_1 < \infty. \tag{2.9}$$

**(R2)** *There is a unique solution $u$ of Eq. 2.1, it has an $\mathcal{H}$-valued stochastic modification, denoted also by $u$, such that*

$$\sup_{t\in[0,T]} E|u(t)|_{\mathcal{H}}^2 =: r_2 < \infty.$$

**(R3)** *Almost surely $A(t, v) \in V$, $B_k(t, u) \in V$ and*

$$\|A(t, v)\|_V^2 \le K|v|_{\mathcal{V}}^2 + \xi(t), \quad \sum_k \|B_k(t, u)\|_V^2 \le K|u|_{\mathcal{H}}^2 + \eta(t) \tag{2.10}$$

*for all $t \in [0, T]$ $v \in \mathcal{V}$ and $u \in \mathcal{H}$, where $\xi$ and $\eta$ are non-negative processes such that for some constant $M$*

$$E\int_0^T \xi(t)\, dt \le M, \quad \sup_{t\in[0,T]} E\eta(t) \le M.$$

**(R4)** (*Time regularity of $A$, $B$*) *There exists a constant $\nu \in ]0, \frac{1}{2}]$ and a non-negative random variable $\eta$ such that $E\eta \le M$, and almost surely*

(i)

$$\|A(s, v) - A(t, v)\|_V^2 \le (K\,|v|_{\mathcal{V}}^2 + \eta)\,|t - s|^{2\nu} \quad \text{for } v \in \mathcal{V}, \tag{2.11}$$

(ii)

$$\sum_k |B_k(s, u) - B_k(t, u)|_V^2 \le (K\,|u|_{\mathcal{V}}^2 + \eta)\,|t - s|^{2\nu} \quad \text{for } u \in \mathcal{V},$$

*for all* $0 \le s < t \le T$.

*Remark 2.5* Assume conditions **(R1)** and **(R3)** from Assumption 2.3. Then the following statements hold.

(i) $u$ has a $V$-valued continuous stochastic modification, denoted also by $u$, such that

$$E\sup_{t\in[0,T]} \|u(t)\|^2 \le 3E\|u_0\|_V^2 + C(r_1 + M);$$

(ii) If condition **(R2)** from Assumption 2.3 also holds, then for $s, t \in [0, T]$,

$$E\|u(t) - u(s)\|_V^2 \le C|t-s|(r_1 + r_2 + M), \tag{2.12}$$

where $C$ is a constant depending only on $T$ and on the constant $K$ from Eq. 2.10.

*Proof* Define

$$F(t) = \int_0^t A(s, u(s))\, ds \quad \text{and} \quad G(t) = \sum_k \int_0^t B_k(s, u(s))\, dW^k(s).$$

Notice that

$$E\int_0^T \|A(s,u(s))\|^2\, ds \le KE\int_0^T |u(s)|_{\mathcal{V}}^2\, ds + E\int_0^T \xi(s)\, ds =: M_1 < \infty,$$

$$\sum_k \int_0^T E\|B_k(s,u(s))\|_V^2\, ds \le K\, E\int_0^T |u(s)|_{\mathcal{H}}^2\, ds + E\int_0^T \eta(s)\, ds =: M_2 < \infty.$$

Hence $F$ and $G$ are $V$-valued continuous processes, and by Jensen's and Doob's inequalities

$$E\sup_{t\le T}\|F(t)\|_V^2 \le TM_1, \qquad E\sup_{t\le T}\|G(t)\|_V^2 \le 4\sum_k E\int_0^T \|B_k(s,u(s))\|_V^2\, ds \le 4M_2.$$

Consequently, the process $u_0 + F(t) + G(t)$ is a $V$-valued continuous modification of $u$, and statement (i) holds. Moreover, if **(R2)** also holds, then

$$\sup_{t\in[0,T]} \sum_k E\|B_k(s,u(s))\|_V^2 \le K \sup_{t\in[0,T]} E|u(t)|_{\mathcal{H}}^2 + \sup_{t\in[0,T]} E\eta(t) := M_3 < +\infty,$$

and

$$E\|F(t) - F(s)\|_V^2 \le |t-s|M_1,$$

$$E\|G(t) - G(s)\|_V^2 = \sum_k \int_s^t E\|B_k(r, u((r))\|_V^2\, dr \le |t-s|M_3$$

for any $0 \le s \le t \le T$, which proves (ii). □

## 2.2 Approximation Spaces and Operators $\Pi_n$

Let $V_n \hookrightarrow H_n \hookrightarrow V_n^*$ be a normal triple and $\Pi_n : V \to V_n$ be a bounded linear operator for each integer $n \ge 0$ such that for all $v \in H$ and $n \ge 0$

$$\|\Pi_n v\|_{V_n} \le p|v|_V \tag{2.13}$$

with some constant $p$ independent of $v \in V$ and $n$. Note that we do not require that the maps $\Pi_n$ be orthogonal projections on the Hilbert space $H$.

We denote by $\langle v, w\rangle_n$ the duality between $v \in V_n$ and $w \in V_n^*$ and similarly by $(h,k)_n$ the inner product of $h, k \in H_n$. To lighten the notation, let $\|v\| := \|v\|_V$ denote the norm of $v$ in $V$, $\|v\|_n := \|v\|_{V_n}$ the norm of $v$ in $V_n$, $|u| := |u|_H$ the norm of $u$ in $H$, $|u|_n = |u|_{H_n}$ the norm of $u$ in $H_n$, and finally $|w|_* := |w|_{V^*}$ and $|y|_{n^*} := |y|_{V_n^*}$ the norm of $w \in V^*$ in $V^*$ and the norm of $y$ in $V_n^*$, respectively.

For $r \geq 0$ let $H^r = W_2^r(\mathbb{R}^d)$ denote the closure of $C_0^\infty(\mathbb{R}^d)$ in the norm defined by

$$|\varphi|_{H^r}^2 = \sum_{|\gamma| \leq r} \int_{\mathbb{R}^d} |D^\gamma \varphi(x)|^2 \, dx.$$

In particular, $H^0 = L_2(\mathbb{R}^d)$.

The following basic examples will be used in the sequel. It describes spaces $V_n$, $H_n$ and $V_n^*$ and operators $\Pi_n$ such that condition (2.13) is satisfied.

*Example 2.6* Wavelet approximation. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an orthonormal scaling function, i.e., a real-valued, compactly supported function, such that:

(i) there exists a sequence $(h_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$ for which $\varphi(x) = \sum_k h_k \varphi(2x-k)$ in $L^2(\mathbb{R})$ ,
(ii) $\int \varphi(x-k)\varphi(x-l)dx = \delta_{k,l}$ for any $k, l \in \mathbb{Z}$.

We assume that the scaling function $\varphi$ belongs to the Sobolev space $H^s(\mathbb{R}) := W_2^s(\mathbb{R})$ for sufficiently large integer $s > 0$.

For $d > 1$, $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, set $\phi(x) = \varphi(x_1) \cdots \varphi(x_d)$ and for $j \geq 0$ and $k \in \mathbb{Z}^d$, set $\phi_{j,k}(x) = 2^{\frac{jd}{2}} \phi(2^j x - k) \in H^s := W_2^s(\mathbb{R}^d)$. For any integer $j \geq 0$, let $H_j$ denote the closure in $L^2(\mathbb{R}^d)$ of the vector space generated by $(\phi_{j,k}, k \in \mathbb{Z}^d)$ and define the operator $\Pi_j$ by

$$\Pi_j f = \sum_{k\in\mathbb{Z}^d} \left(f \, , \, \phi_{j,k}\right)\phi_{j,k} \, , \quad f \in L^2\left(\mathbb{R}^d\right),$$

$\left( \, , \, \right)$ denotes the scalar product in $L^2(\mathbb{R}^d)$.

Thus we have a sequence $H_j \subset H_{j+1}$ of closed subspaces of $L^2(\mathbb{R}^d)$ and orthogonal projections $\Pi_j : L^2(\mathbb{R}^d) \to H_j$ for $j \geq 0$. Assume, moreover that $\cup_{j=0}^\infty H_j$ is dense in $L^2(\mathbb{R}^d)$ and that $\varphi$ is sufficiently regular, such that the inequalities

$$(Direct) \qquad \|f - \Pi_j f\|_{H^r} \leq C\, 2^{-j(s-r)} \, \|f\|_{H^s} \, , \quad \forall f \in H^s, \tag{2.14}$$

$$(Converse) \qquad \|\Pi_j f\|_{H^s} \leq C\, 2^{j(s-r)} \, \|f\|_{H^r}, \qquad \forall f \in H^r \tag{2.15}$$

holds for fixed integers $0 \leq r \leq s$. The proof of these inequalities and more information on wavelets can be found, e.g., in [2].

Fix $r > 0$, set $H := L^2(R^d)$, $V := H^r = W_2^r\left(\mathbb{R}^d\right)$, and identify $H$ with its dual $H^*$ by the help of the inner product in $H$. Then $V \hookrightarrow H^* \hookrightarrow V^*$ is a normal triple, where $H \equiv H^* \hookrightarrow V^*$ is the adjoint of the embedding $V \hookrightarrow H$. We define $V_n$ as the normed space we get by taking the $H^r$ norm on $H_n$. Since the $H^r$ and $H^0$ norms are equivalent on $H_n$, the space $V_n$ is complete, and obviously $V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*$ is a normal triple, where $H_n$ is identified with $H_n^*$ via the inner product $(\, , \,)_n = (\, , \,)$ in $H_n$. Note that due to Eq. 2.15 we have Eq. 2.13 assuming that $\varphi$ is sufficiently smooth.

*Example 2.7* Finite differences – Discrete Sobolev spaces. Consider for fixed $h \in (0, 1)$ the grid

$$\mathbb{G} = h\mathbb{Z}^d = \left\{(k_1 h, k_2 h, \ldots, k_d h) : k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d\right\},$$

where $\mathbb{Z}$ denotes the set of integers. Use the notation $\{e_1, e_2, ..., e_d\}$ for the standard basis in $\mathbb{R}^d$. For any integer $m \geq 0$, let $W^m_{h,2}$ be the set of real valued functions $v$ on $\mathbb{G}$ with

$$|v|^2_{h,m} := \sum_{|\alpha|\leq m} \sum_{z\in\mathbb{G}} |\delta^\alpha_+ v(z)|^2 h^d < \infty,$$

where $\delta^0_{\pm i}$ is the identity and $\delta^\alpha_\pm = \delta^{\alpha_1}_{\pm 1}\delta^{\alpha_2}_{\pm 2} \ldots \delta^{\alpha_d}_{\pm d}$ for multi-indices $\alpha = (\alpha_1, \alpha_2 \ldots, \alpha_d) \in \{0, 1, 2, \ldots\}^d$ of length $|\alpha| := \alpha_1 + \cdots + \alpha_d \geq 1$ is defined for $z \in \mathbb{G}$ by

$$\delta_{\pm i} v(z) := \pm\frac{1}{h}(v(z \pm he_i) - v(z)).$$

We write also $\delta^\alpha$ and $\delta_i$ in place of $\delta^\alpha_+$ and $\delta_{+i}$, respectively. Then $W^m_{h,2}$ with the norm $|\cdot|_{h,m}$ is a separable Hilbert space. It is the discrete counterpart of the Sobolev space $W^m_2(\mathbb{R}^d)$. Set $W^{-1}_{h,2} = (W^1_{h,2})^*$, the adjoint of $W^1_{h,2}$, with its norm denoted by $|\cdot|_{h,-1}$. It is easy to see that $W^m_{h,2} \hookrightarrow W^{m-1}_{h,2}$ is a dense and continuous embedding,

$$|v|_{h,m-1} \leq |v|_{h,m},$$
$$|v|_{h,m} \leq \frac{\kappa}{h}|v|_{h,m-1}, \tag{2.16}$$

for all $v \in W^m_{h,2}$, $m \geq 0$ and $h \in (0, 1)$, where $\kappa$ is a constant depending only on $d$. Notice that for $m \geq 1$

$$\langle v, u\rangle := \sum_{|\alpha|\leq m} \sum_{z\in\mathbb{G}} \delta^\alpha v \delta^\alpha u \leq C|v|_{h,m-1}|u|_{h,m+1} \quad \text{for all } v, u \in W^{m+1}_{h,2}$$

extends to a duality product between $W^{m-1}_{h,2}$ and $W^{m+1}_{h,2}$, which makes it possible to identify $W^{m-1}_{h,2}$ with $(W^{m+1}_{h,2})^*$.

Assume that $m > \frac{d}{2}$. Then by Sobolev's theorem on embedding $W^m_2 := W^m_2(\mathbb{R}^d)$ into $\mathcal{C}(\mathbb{R}^d)$, there is a bounded linear operator $I : W^m_2(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$, such that $Iu = u$ almost everywhere on $\mathbb{R}^d$. Thus, identifying $u$ with $Iu$, we can define the operator $R_h : W^m_2(\mathbb{R}^d) \to W^m_{h,2}$ by restricting the functions $u \in W^m_2$ onto $\mathbb{G} \subset \mathbb{R}^d$. Moreover, due to Sobolev's theorem,

$$\sum_{z\in\mathbb{G}} \sup_{x\in\mathcal{I}(z)} |u(x)|^2 h^d \leq p^2|u|^2_{W^m_2},$$

where $\mathcal{I}(z) := \{x \in \mathbb{R}^d : z_k \leq x_k \leq z_k + h, \quad k = 1, 2 \ldots d\}$ and $p$ is a constant depending only on $m$ and $d$. Hence obviously

$$|R_h u|^2_{h,0} \leq p|u|^2_{W^m_2} \quad \text{for all } u \in W^m_2. \tag{2.17}$$

Moreover, for every integer $l \geq 0$

$$|R_h u|_{h,l} \leq p|u|_{W^{m+l}_2} \quad \text{for all } u \in W^{m+l}_2, \tag{2.18}$$

with a constant $p$ depending only on $m$, $l$ and $d$. Thus setting

$$V_n := W^{m+l}_{h_n,2}, \quad H_n := W^{m+l-1}_{h_n,2}, \quad V_n^* \equiv W^{m+l-2}_{h_n,2},$$
$$\Pi_n := R_{h_n}$$

for any sequence $\{h_n\}_{n=0}^{\infty} \subset (0, 1)$ and any integers $m > \frac{d}{2}$, $l \geq 0$ we get examples of approximation spaces.

When approximating differential operators by finite differences we need to estimate $D_i u - \delta_{\pm i} u$ in discrete Sobolev norms. For $d = 1$ we can estimate this as follows. Let $l \geq 0$ be an integer and set $z_k := kh$ for $k \in \mathbb{Z}$. By the mean value theorem there exist $z'_k$ and $z''_k$ in $[z_k, z_k + lh]$ such that $\delta^l Du(z_k) = D^{l+1}u(z'_k)$ and $\delta^l \delta u(z_k) = D^{l+1}u(z''_k)$, where $D := \frac{d}{dx}$. Hence

$$|\delta^l(Du(z_k) - \delta u(z_k))|^2 = |D^{l+1}u(z'_k) - D^{l+1}u(z''_k)|^2 = \left| \int_{z'_k}^{z''_k} D^{l+2}u(y)\, dy \right|^2$$
$$\leq lh \int_{z_k}^{z_k+lh} |D^{l+2}u(y)|^2\, dy$$

for $u \in C_0^{\infty}(\mathbb{R})$. Consequently,

$$|Du - \delta^l_{\pm} u|_{h,l} \leq l\, h\, |u|_{W_2^{l+2}(\mathbb{R})} \tag{2.19}$$

for $u \in C_0^{\infty}(\mathbb{R})$, and hence for all $u \in W_2^{l+2}(\mathbb{R})$. For $d > 1$ by similar calculation combined with Sobolev's embedding, we get that for $m > l + 2 + \frac{d-1}{2}$

$$|D_i u - \delta_{\pm i} u|_{h,l} \leq Ch|u|_{W_2^m} \tag{2.20}$$

for all $u \in W_2^m$, $h \in (0, 1)$, where $C$ is a constant depending on $l$, $m$ and $d$.

## 3 Space Discretization

### 3.1 Description of the Scheme

Consider for each integer $n \geq 1$ the problem

$$du^n(t) = A^n(t, u^n(t))\, dt + \sum_k B^n(t, u^n(t))\, dW^k(t), \quad u^n(0) = u_0^n, \tag{3.1}$$

in a normal triple $V_n \hookrightarrow H_n \hookrightarrow V_n^*$, satisfying the conditions of Section 2.2, where $u_0^n$ is an $H_n$-valued $\mathcal{F}_0$-measurable random variable, $A^n$ and $B^n = (B_k^n)$ are $\mathcal{P} \otimes \mathcal{B}(V_n)$-measurable mappings from $[0, \infty) \times \Omega \times V_n$ into $V_n^*$ and $H_n^{d_1}$, respectively.

**Assumption 3.1** *The operators $A^n$ and $B^n$ satisfy the following conditions.*

**(S1)** (*Strong monotonicity*) *There exist constants $\lambda > 0$ and $L$ such that for all $n \geq 1$ almost surely*

$$2\langle u - v, A^n(t, u) - A^n(t, v)\rangle_n + \sum_k |B_k^n(t, u) - B_k^n(t, v)|^2_{H_n} + \lambda \|u - v\|^2_{V_n}$$
$$\leq L|u - v|^2_{H_n} \quad \text{for all } t \in [0, T], u, v \in V_n.$$

**(S2)** (*Growth condition*) *Almost surely*

$$|A^n(t,v)|^2_{V_n^*} \le K_1 \|v\|^2_{V_n} + f^n(t), \quad |B^n(t,v)|^2_{H_n} \le K_2 \|v\|^2_{V_n} + g^n(t)$$

*for all* $t \in [0,T]$, $v \in V_n$ *and* $n \ge 1$, *where* $K_1$, $K_2$ *are constants, independent of* $n$, *and* $f^n$ *and* $g^n$ *are non-negative stochastic processes such that*

$$\sup_n E\int_0^T f^n(t)\,dt =: M_1 < \infty\,, \quad \sup_n E\int_0^T g^n(t)\,dt =: M_2 < \infty.$$

**(S3)** (*Hemicontinuity of* $A^n$) *For every* $n \ge 1$, *the operators* $A^n$ *are hemicontinuous in* $v \in V_n$, *i.e., almost surely*

$$\lim_{\varepsilon\to 0}\langle A^n(t, v+\varepsilon u)\,,\, w\rangle_n = \langle A^n(t,v)\,,\, w\rangle_n$$

*for all* $t \in [0,T]$, $v, u, w \in V_n$.

**(S4)** (*Lipschitz condition on* $B^n$) *Almost surely*

$$\sum_k |B^n_k(t,u) - B^n_k(t,v)|^2_{H_n} \le L_B \|u-v\|^2_{V_n}$$

*for all* $t \in [0,T]$ *and* $u, v \in V_n$.

The solution to Eq. 3.1 is understood in the sense of Definition 2.1. Notice that **(S1)–(S2)** imply the coercivity condition

$$2\langle v, A^n(t,v)\rangle_n + \sum_k |B^n_k(t,v)|^2_{H_n} + \frac{\lambda}{2}\|v\|^2_{V_n} \le C\left(|v|^2_{H_n} + f^n(t) + g^n(t)\right)$$

with a constant $C$ depending on $\lambda$, $L$ and $K_2$.

Thus by Theorem 2.2 the conditions **(S1)–(S3)** ensure the existence of a unique solution $u^n$ to Eq. 3.1, and if

$$\sup_n E|u^n_0|^2_{H_n} < \infty, \tag{3.2}$$

then

$$E\sup_{0\le t\le T}|u^n(t)|^2_{H_n} + E\int_0^T \|u^n(t)\|^2_{V_n}\,dt \\ \le C\sup_n\left(E|u^n_0|^2_{H_n} + E\int_0^T \left(f^n(t) + g^n(t)\right)dt\right) < \infty, \tag{3.3}$$

where $C$ is a constant depending only on $\lambda$, $L$ and $K_2$.

## 3.2 Rate of Convergence of the Scheme

We want to approximate $\Pi_n u$ by $u^n$. In order to estimate the accuracy of this approximation we need to relate the operators $A$ and $B$ to $A^n$ and $B^n$, respectively. Therefore we assume the regularity condition **(R3)** from Assumption 2.3 and make the following *consistency* assumption.

**Condition (Cn)** (*Consistency*) *There exist a sequence* $(\varepsilon_n)_{n\geq 1}$ *of positive numbers and a sequence* $(\xi^n)_{n\geq 1}$ *of non-negative adapted processes such that*

$$\sup_n E\int_0^T \xi^n(t)\,dt \leq M < +\infty,$$

*and almost surely* $(t,\omega)\in[0,T]\times\Omega$

$$|\Pi_n A(t,v) - A^n\,(t,\Pi_n v)\,|^2_{V_n^*} + \sum_k |\Pi_n B_k(t,v) - B_k^n\,(t,\Pi_n v)\,|^2_{H_n}$$
$$\leq \varepsilon_n^2\big(|v|^2_{\mathcal{V}} + \xi^n(t)\big)$$

*for all* $v\in\mathcal{V}$.

**Theorem 3.1** *Let Assumption* 3.1*, the regularity conditions* **(R1)** *and* **(R3)** *from Assumption* 2.3*, and the consistency condition* **(Cn)** *hold. Assume furthermore* $\sup_n E|u_0^n|^2_{H_n} < +\infty$. *Then for* $e^n(t) := \Pi_n u(t) - u^n(t)$,

$$E\sup_{0\leq t\leq T}|e^n(t)|^2_{H_n} + E\int_0^T \|e^n(t)\|^2_{V_n}dt \leq C_1 E|e^n(0)|^2_{H_n} + C_2(r_1+M)\varepsilon_n^2 \tag{3.4}$$

*holds for all* $n\geq 1$, *where* $C_1 = C_1(\lambda, L, T)$ *and* $C_2 = C_2(\lambda, L, L_B, T)$ *are constants.*

*Proof* From Eq. 1.1 we deduce that for every $n\geq 1$,

$$\Pi_n u(t) = \Pi_n u_0 + \int_0^t \Pi_n A\,(s,u(s))\;ds + \sum_k\int_0^t \Pi_n B_k\,(s,u(s))\,dW^k(s).$$

Using Itô's formula

$$|e^n(t)|_n^2 = |e^n(0)|_n^2 + \sum_{i\leq 3} I_i(t), \tag{3.5}$$

where

$$I_1(t) = 2\int_0^t \langle e^n(s)\,,\,\Pi_n A(s,u(s)) - A^n(s,u^n(s))\rangle_n ds\,,$$

$$I_2(t) = 2\sum_k\int_0^t \big(e^n(s)\,,\,\Pi_n B_k(s,u(s)) - B_k^n(s,u^n(s))\big)_n dW^k(s),$$

$$I_3(t) = \sum_k\int_0^t \big|\Pi_n B_k(s,u(s)) - B_k^n(s,u^n(s))\big|_n^2 ds\,.$$

We first prove

$$\sup_{0\leq t\leq T} E|e^n(t)|_n^2 + E\int_0^T \|e^n(t)\|_n^2 dt \leq C_1 E|e^n(0)|_n^2 + C_2(r_1+M)\varepsilon_n^2, \tag{3.6}$$

where $C_1 = C_1(\lambda, L, T)$ and $C_2 = C_2(\lambda, L, L_B, T)$ are constants. The strong monotonicity condition **(S1)** from Assumption 3.1 implies

$$I_1(t) + I_3(t) \leq -\lambda\int_0^t \|e^n(s)\|_n^2\,ds + L\int_0^t |e^n(s)|_n^2\,ds + \sum_{i=1,2} R_i(t), \tag{3.7}$$

where

$$R_1(t) = \int_0^t 2\langle e^n(s)\,,\ \Pi_n A(s, u(s)) - A^n(s, \Pi_n u(s))\rangle_n\, ds,$$

$$R_2(t) = \sum_k \int_0^t \Big[ |\Pi_n B_k(s, u(s)) - B_k^n(s, u^n(s))|_n^2 \\ - |B_k^n(s, \Pi_n u(s)) - B_k^n(s, u^n(s))|_n^2 \Big]\, ds.$$

Schwarz's inequality and the consistency condition **(Cn)** imply that for every $n \geq 1$ and $t \in [0, T]$,

$$|R_1(t)| \leq \frac{\lambda}{3} \int_0^t \|e^n(s)\|_n^2\, ds + \frac{3}{\lambda} \int_0^t |\Pi_n A\big(s, u(s)\big) - A^n\big(s, \Pi_n u(s)\big)|_{n*}^2\, ds \\ \leq \frac{\lambda}{3} \int_0^t \|e^n(s)\|_n^2\, ds + \frac{3}{\lambda}\, \varepsilon_n^2 \int_0^t \big(|u(s)|_{\mathcal{V}}^2 + \xi^n(s)\big)\, ds. \tag{3.8}$$

Schwarz's inequality, the consistency condition **(Cn)**, and the Lipschitz condition **(S4)** from Assumption 3.1 yield that for every $\alpha > 0$,

$$|R_2(t)| = \sum_k \int_0^t \Big[ |\Pi_n B_k\big(s, u(s)\big) - B_k^n\big(s, \Pi_n u(s)\big)|_n^2 + 2\Big(\Pi_n B_k\big(s, u(s)\big) \\ - B_k^n\big(s, \Pi_n u(s)\big)\,,\ B_k^n\big(s, \Pi_n u(s)\big) - B_k^n\big(s, u^n(s)\big)\Big)_n \Big]\, ds \\ \leq \Big(1 + \frac{1}{\alpha}\Big) \int_0^t \sum_k \Big|\Pi_n B_k\big(s, u(s)\big) - B_k^n\big(s, \Pi_n u(s)\big)\Big|_n^2 ds \\ + \alpha \int_0^t \sum_k \Big|B_k^n\big(s, \Pi_n u(s)\big) - B_k^n\big(s, u^n(s)\big)\Big|_n^2\, ds \\ \leq \Big(1 + \frac{1}{\alpha}\Big)\, \varepsilon_n^2 \int_0^t \big(|u(s)|_{\mathcal{V}}^2 + \xi^n(s)\big)\, ds + \alpha L_B \int_0^t \|e^n(s)\|_n^2\, ds. \tag{3.9}$$

Thus, for $\alpha L_B \leq \frac{\lambda}{3}$, taking expectations in Eq. 3.5 and Eqs. 3.7–3.9 and using **(S1)** again, we deduce that

$$E|e^n(t)|_n^2 + \frac{\lambda}{3} E \int_0^t \|e^n(s)\|_n^2\, ds \leq L E \int_0^t |e^n(s)|_n^2 ds + E|e^n(0)|_n^2 + C(r_1 + M)\varepsilon_n^2,$$

where $C = C(\lambda, L_B)$ is a constant. Since by Eqs. 2.5 and 3.3

$$\sup_{0 \leq t \leq T} E|e^n(t)|_n^2 < +\infty,$$

Gronwall's lemma gives

$$\sup_{0 \leq t \leq T} E|e^n(t)|_n^2 \leq e^{LT} \big(C(r_1 + M)\varepsilon_n^2 + E|e^n(0)|_n^2\big),$$

which in turn yields Eq. 3.6. We now prove Eq. 3.4. From Eqs. 3.6–3.9 we deduce

$$E \sup_{0\le t\le T} \big(I_1(t) + I_3(t)\big) \le LE\int_0^T |e^n(s)|_n^2\, ds + \frac{2\lambda}{3} E\int_0^T \|e^n(s)\|_n^2 ds + C_2(r_1 + M)\,\varepsilon_n^2$$
$$\le C_1 E|e^n(0)|_n^2 + C_2(r_1 + M)\varepsilon_n^2. \tag{3.10}$$

(Notice that by taking the supremum in both sides of Eq. 3.7 we cannot make use of the term with coefficient $-\lambda$ in the right-hand side of Eq. 3.7. This is why $2\lambda/3$ appears here as the sum of $\lambda/3$ from Eq. 3.8 and $\alpha L_B \le \lambda/3$ from Eq. 3.9.) By Davies' inequality, Eq. 2.5, the Lipschitz condition **(S4)** on $B^n$, the consistency condition **(Cn)** and by the strong monotonicity condition **(S1)**,

$$E \sup_{0\le t\le T} |I_2(t)| \le 6E\left(\int_0^T \sum_k \left|\left(e^n, \Pi_n B_k(u) - B_k^n(u^n)\right)_n\right|^2 ds\right)^{\frac12}$$
$$\le 6E\left\{\sup_{0\le t\le T} |e^n(t)|_n \left(\int_0^T \sum_k |\Pi_n B_k(u) - B_k^n\big(u^n\big)|_n^2\, ds\right)^{\frac12}\right\}$$
$$\le \frac12 E \sup_{0\le t\le T} |e^n(t)|_n^2$$
$$+ 36E\sum_k \int_0^T \Big[|\Pi_n B_k\big(u\big) - B_k^n\big(\Pi_n u\big)|_n^2 + |B_k^n\big(\Pi_n u\big) - B_k^n(u^n)|_n^2\Big] ds$$
$$\le \frac12 E \sup_{0\le t\le T} |e^n(t)|_n^2 + 36L_B E\int_0^T \|e^n(s)\|_n^2 ds + 36(r_1 + M)\varepsilon_n^2, \tag{3.11}$$

where the argument $s$ is omitted from most integrands. Thus inequalities (3.5), (3.10), (3.11) and (3.6) yield

$$\frac12 E \sup_{0\le t\le T} |e^n(t)|_n^2 \le C_1 E|e^n(0)|_n^2 + C_2(r_1 + M)\varepsilon_n^2,$$

with some constants $C_1 = C_1(L, T)$ and $C_2 = C_2(\lambda, L, L_B, T)$, which completes the proof of Eq. 3.4. □

## 3.3 Example

Consider the normal triples

$$V \hookrightarrow H^* \hookrightarrow V^*, \quad V_n \hookrightarrow H_n^* \hookrightarrow V_n^*$$

with the orthogonal projection $\Pi_n : H = L^2(\mathbb{R}^d) \to H_n$ from Example 2.6, where $V = W_2^r(\mathbb{R}^d)$ with $r > 0$. Set $\mathcal{H} = W_2^{r+\rho}(\mathbb{R}^d)$ and $\mathcal{V} = W_2^{r+l}(\mathbb{R}^d)$ for some $l > \rho \ge 0$.

Let $A$ and $B = (B_k)$ be $\mathcal{P} \otimes \mathcal{B}(V)$-measurable mappings from $[0, \infty[\times\Omega \times V$ into $V^*$ and $H^{d_1}$, respectively, satisfying Assumptions 2.2 and 2.3. For $(t, \omega) \in [0, T] \times \Omega$ let $A^n(t, \omega, \cdot) : V^n \to V_n^*$ and $B^n(t, \omega, \cdot) : V^n \to H_n^{d_1}$ be defined by

$$\langle A^n(t, u, \omega), v\rangle_n = \langle A(t, u, \omega), v\rangle \quad \text{and } B_k^n(t, \omega, u) = \Pi_n B_k(t, \omega, u) \tag{3.12}$$

for all $u, v \in V_n$, where $\langle\, , \rangle_n$ denotes the duality between $V_n$ and $V_n^*$. Then it is easy to see that due to conditions **(1)**, **(2)** and **(3)** in Assumption 2.2, the operators $A^n$ and $B^n$ satisfy **(S1)**, **(S2)** and **(S3)** in Assumption 3.1, respectively. Furthermore, taking into account Remark 2.4 it is obvious that **(S4)** holds. Assume the regularity condition **(R3)** from Assumption 2.3. Then by virtue of the definition of $\Pi_n$, $A^n$ and $B^n$, due to Lipschitz conditions **(3)** in Assumption 2.2 and Eq. 2.8 in Remark 2.4, we have, recalling the direct inequality (2.14),

$$\begin{aligned}|\Pi_n A(t,u) - A^n(t,\Pi_n u)|_{V_n^*}^2 + \sum_k |\Pi_n B_k(t,u) - B_k^n(t,\Pi_n u)|_{H_n}^2 \\ \leq |A(t,u) - A^n(t,\Pi_n u)|_{V*}^2 + \sum_k |B_k(t,u) - B_k^n(t,\Pi_n u)|_H^2 \\ \leq C(L_1+L_2)\, 2^{-2nl}\, |u|_{\mathcal{V}}^2\end{aligned}$$

almost surely for all $t \in [0,T]$ and $u \in \mathcal{V}$, which yields **(Cn)** with $\xi^n := 0$ and $\varepsilon_n := C(L_1+L_2)2^{-nl}$. In the last section we will give examples of operators such that Assumption 2.3 holds.

## 4 Implicit Space-time Discretizations

### 4.1 Description of the Scheme

For a fixed integer $m \geq 1$ set $\tau := T/m$ and $t_i = i\tau$ for $i = 0, \cdots, m$. Let $V_n \hookrightarrow H_n \hookrightarrow V_n^*$ satisfy the conditions in Section 2.2. Given a $V_n$-valued $\mathcal{F}_0$-measurable random variable $u_0^{n,\tau}$ and $\mathcal{F}_{t_i} \otimes \mathcal{B}(V_n)$-measurable mappings

$$A_j^{n,\tau} : \Omega \times V_n \to V_n^* \quad \text{and} \quad B_{k,i}^{n,\tau} : \Omega \times V_n \to H_n, \text{ for } k = 1, \cdots, d_1,$$

$j = 1, \ldots, m$ and $i = 0, \ldots m-1$, consider for each $n$ the system of equations

$$u_{i+1}^{n,\tau} = u_i^{n,\tau} + \tau\, A_{i+1}^{n,\tau}\big(u_{i+1}^{n,\tau}\big) + \sum_k B_{k,i}^{n,\tau}\big(u_i^{n,\tau}\big)\big(W^k(t_{i+1}) - W^k(t_i)\big), \tag{4.1}$$

$i = 0, \ldots, m-1$, for $V_n$-valued $\mathcal{F}_{t_i}$-measurable random variables $u_i^{n,\tau}$, $i = 1, \ldots, m$.

**Assumption 4.1** *For almost all $\omega \in \Omega$ the operators $A_j^{n,\tau}$ and $B_{k,i}^{n,\tau}$ satisfy the following conditions for all $j = 1, \ldots, m$, $i = 0, \ldots, m-1$,*

**(ST1)** (*Strong monotonicity*) *There exist constants $\lambda > 0$ and $L \geq 0$ such that a.s.*

$$\begin{aligned}2\big\langle u - v\,,\, A_j^{n,\tau}(u) - A_j^{n,\tau}(v)\big\rangle_n + \sum_k \big|B_{k,j}^{n,\tau}(u) - B_{k,j}^{n,\tau}(v)\big|_{H_n}^2 \\ \leq -\lambda \|u-v\|_{V_n}^2 + L\, |u-v|_{H_n}^2\end{aligned} \tag{4.2}$$

*for all $u, v \in V_n$, $m \geq 1$, $n \geq 0$.*

**(ST2)** (*Growth condition on $A_i^{n,\tau}$ and $B_i^{n,\tau}$*) *There is a constant $K$ such that a.s.*

$$\big|A_j^{n,\tau}(u)\big|_{V_n^*}^2 \leq K\|u\|_{V_n}^2 + f_j^{n,\tau}, \quad \sum_k \big|B_{k,i}^{n,\tau}(u)\big|_{H_n}^2 \leq K\|u\|_{V_n}^2 + g_i^{n,\tau}$$

*for all* $u \in V_n$, $m \geq 1$, $n \geq 0$, *where* $f_j^{n,\tau}$ *and* $g_i^{n,\tau}$ *are non-negative random variables, such that*

$$\sup_{n,m} \sum_j \tau E f_j^{n,\tau} \leq M < +\infty, \quad \sup_{n,m} \max_i E g_i^{n,\tau} \leq M < +\infty.$$

**(ST3)** (*Lipschitz condition on* $A_j^{n,\tau}$) *There exists a constants* $L_1$ *such that a.s.*

$$\left| A_j^{n,\tau}(u) - A_j^{n,\tau}(v) \right|_{V_n^*}^2 \leq L_1 \|u - v\|_{V_n}^2 \tag{4.3}$$

*for all* $u, v \in V_n$, $m \geq 1$, $n \geq 0$.

*Remark 4.1* Clearly, conditions **(ST1)** and **(ST3)** imply the Lipschitz continuity of $B_{k,i}^{n,\tau}$ in $v \in V_n$, i.e., there is a constant $L_2 = L_2(L, \lambda, L_1)$ such that almost surely

$$\sum_k \left| B_{k,j}^{n,\tau}(u) - B_{k,j}^{n,\tau}(v) \right|_{H_n}^2 \leq L_2 \|u - v\|_{V_n}^2 \tag{4.4}$$

for all $u, v \in V_n, n \geq 1, m \geq 0$ and $j = 1, \cdots, m$.

*Remark 4.2* Conditions **(ST1)–(ST2)** imply that almost surely

$$2\langle u, A_j^{n,\tau}(u) \rangle_n + \sum_k |B_{k,j}^{n,\tau}(u)|_{H_n}^2 \leq -\frac{\lambda}{2} \|u\|_{V_n}^2 + L|u|_{H_n}^2 + C\left( f_j^{n,\tau} + g_j^{n,\tau} \right)$$

for all $u \in V_n, n \geq 0, m \geq 1$ and $j = 1, \ldots, m$, where $C = C(\lambda, K)$ is a constant. The Lipschitz condition **(ST3)** obviously implies that $A_i^{n,\tau}$ is hemicontinuous.

**Proposition 4.3** *Let Assumption* 4.1 *hold. Assume* $E\|u_0^{n,\tau}\|_{V_n}^2 < \infty$ *for all* $n \geq 0$ *and* $m \geq 1$. *Then for* $\tau < 1/L$ *Eq.* 4.1 *has a unique* $V_n$*-valued solution* $(u_j^{n,\tau})_{j=1}^m$, *such that* $u_j^{n,\tau}$ *is* $\mathcal{F}_{t_j}$*-measurable and* $E\|u_j^{n,\tau}\|_{V_n}^2$ *is finite for each* $j, n$. (*Here* $1/L := \infty$ *if* $L = 0$.)

*Proof* Equation 4.1 can be rewritten as

$$D_{i+1}\left(u_{i+1}^{n,\tau}\right) = u_i^{n,\tau} + \sum_k B_{k,i}^{n,\tau}\left(u_i^{n,\tau}\right)\left(W^k(t_{i+1}) - W^k(t_i)\right), \tag{4.5}$$

where $D_i : V_n \to V_n^*$ is defined by $D_i(v) = v - \tau A_i^{n,\tau}(v)$ for each $i = 1, 2, \cdots m$. Due to Assumption 4.1 and Remark 4.2 the operator $D_i$ satisfies the assumptions (monotonicity, coercivity, linear growth and hemicontinuity) of Proposition 3.4 in [7] with $p = 2$. By virtue of this proposition, for $\tau < 1/L$, Eq. 4.5 has a unique $V_n$-valued $\mathcal{F}_{t_{i+1}}$-measurable solution $u_{i+1}^{n,\tau}$ for every given $V$-valued $\mathcal{F}_{t_i}$-measurable random variable $u_i^{n,\tau}$, and

$$\begin{aligned} E\|u_{i+1}^{n,\tau}\|_{V_n}^2 &\leq C\, E\left( 1 + f_i^{n,\tau} + g_i^{n,\tau} + \left| \sum_k B_{k,i}^{n,\tau}(u_i^{n,\tau})\,(W^k(t_{i+1}) - W^k(t_i)) \right|^2 \right) \\ &\leq C\left( 1 + Ef_i^{n,\tau} + Eg_i^{n,\tau} + \sum_k \tau\, E|B_{k,i}^{n,\tau}(u_i^{n,\tau})|_n^2 \right) \\ &\leq C\left(1 + Ef_i^{n,\tau} + Eg_i^{n,\tau} + K\tau\, E\|u_i^{n,\tau}\|_{V_n}^2 + \tau\, Eg_i^{n,\tau}\right), \end{aligned}$$

where $C = C(\lambda, \tau)$ is a constant. Hence induction on $i$ concludes the proof. □

4.2 Rate of Convergence of the Implicit Scheme

Let Assumption 2.3 on the regularity of equation 2.1 and its solution $u$ hold. We relate the operators $A(t_i,.)$ and $A_i^{n,\tau}$ as well as the operators $B_k(t_i,.)$ and $B_{k,i}^{n,\tau}$ by the following consistency assumption.

**Condition (Cn$\tau$)** (*Consistency*) *There exist constants* $\nu \in ]0,\frac{1}{2}]$, $c \geq 0$, *a sequence of numbers* $\varepsilon_n \to 0$, *such that almost surely*

$$|\Pi_n A(t_j,u) - A_j^{n,\tau}(\Pi_n u)|_{V_n^*}^2 \leq c\big(|u|_{\mathcal{V}}^2 + \xi_j^{n,\tau}\big)\big(\tau^{2\nu} + \varepsilon_n^2\big),$$

$$\sum_k \big|\Pi_n B_k(t_i,u) - B_{k,i}^{n,\tau}(\Pi_n u)\big|_{H_n}^2 \leq c\big(|u|_{\mathcal{V}}^2 + \eta_i^{n,\tau}\big)\big(\tau^{2\nu} + \varepsilon_n^2\big)$$

*for all* $j = 1, \ldots m$, $i = 0, \cdots, m-1$ *and* $u \in \mathcal{V}$, *where* $\xi_j^{n,\tau}$ *and* $\eta_i^{n,\tau}$ *are non-negative random variables such that*

$$\sup_{n,m} \sum_j \tau E\xi_j^{n,\tau} \leq M, \quad \sup_{n,m} \sum_i \tau E\eta_i^{n,\tau} \leq M.$$

**Theorem 4.4** *Let Assumptions* 2.3 *and* 4.1 *as well as condition* **(Cn$\tau$)** *hold. Assume*

$$\sup_{n,m} E\|u_0^{n,\tau}\|_{V_n}^2 \leq M. \tag{4.6}$$

*Set* $e_i^{n,\tau} = \Pi_n u(t_i) - u_i^{n,\tau}$. *Then for* $\tau < 1/L$ *and* $n \geq 0$

$$\begin{aligned} E \max_{0\leq i\leq m} |e_i^{n,\tau}|_{H_n}^2 &+ \sum_{1\leq i\leq m} \tau\, E\|e_i^{n,\tau}\|_{V_n}^2 \\ &\leq C_1 E|e_0^{n,\tau}|_{H_n}^2 + C_2(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M), \end{aligned} \tag{4.7}$$

*where* $C_1 = C_1(\lambda, L, T)$ *and* $C_2 = C_2(\lambda, L, K, T, p, c, L_1, L_2)$ *are constants.*

*Proof* We fix $n$, $\tau$, and to ease notation we write $e_i$, $A_i$ and $B_{k,i}$ in place of $e_i^{n,\tau}$, $A_i^{n,\tau}$ and $B_{k,i}^{n,\tau}$, respectively. Similarly, we often use $u_i$ in place of $u_i^{n,\tau}$ for $i = 1, 2, \cdots, m$. Then for any $i = 0, \cdots, m-1$,

$$\begin{aligned} |e_{i+1}|_n^2 - |e_i|_n^2 &= 2\int_{t_i}^{t_{i+1}} \big\langle e_{i+1}\,,\, \Pi_n A(s,u(s)) - A_{i+1}(u_{i+1})\big\rangle_n\, ds \\ &\quad + 2\sum_k \int_{t_i}^{t_{i+1}} \big(e_{i+1}\,,\, F_k(s)\big)_n\, dW^k(s) \\ &\quad - \Big|\int_{t_i}^{t_{i+1}} \big[\Pi_n A(s,u(s)) - A_{i+1}(u_{i+1})\big]\, ds + \sum_k \int_{t_i}^{t_{i+1}} F_k(s)\, dW^k(s)\Big|_n^2 \\ &= 2\int_{t_i}^{t_{i+1}} \big\langle e_{i+1}\,,\, \Pi_n A(s,u(s)) - A_{i+1}(u_{i+1})\big\rangle_n\, ds \\ &\quad + \Big|\sum_k \int_{t_i}^{t_{i+1}} F_k(s)\, dW^k(s)\Big|_n^2 + 2\sum_k \int_{t_i}^{t_{i+1}} \big(e_i\,,\, F_k(s)\big)_n\, dW^k(s) \\ &\quad - \Big|\int_{t_i}^{t_{i+1}} \big[\Pi_n A(s,u(s)) - A_{i+1}(u_{i+1})\big]\, ds\Big|_n^2, \end{aligned}$$

where for $k = 1, \cdots, d_1$ one sets

$$F_k(s) = \Pi_n B_k(s, u(s)) - B_{k,i}\big(u_i^{n,\tau}\big), \quad s \in ]t_i, t_{i+1}], \quad i = 0, 1, \cdots, m-1.$$

Summing up for $i = 0, \cdots, l-1$, we obtain

$$|e_l|_n^2 \le |e_0|_n^2 + 2 \sum_{0 \le i < l} \int_{t_i}^{t_{i+1}} \langle e_{i+1}, \Pi_n A(s, u(s)) - A_{i+1}(u_{i+1})\rangle_n \, ds + Q(t_l) + I(t_l), \quad (4.8)$$

where

$$Q(t_l) = \sum_{0 \le i < l} \Big| \sum_k \int_{t_i}^{t_{i+1}} F_k(s)\, dW^k(s) \Big|_n^2,$$

$$I(t_l) = 2 \sum_k \int_0^{t_l} \big(e(s),\ F_k(s)\big)_n \, dW^k(s),\ \ e(s) := e_i \text{ for } s \in ]t_i, t_{i+1}], i = 0, \cdots, m.$$

First we show

$$\sup_{0 \le l \le m} E|e_l|_n^2 + E \sum_{1 \le i \le m} \tau \|e_i\|_n^2 \le C_1 E|e_0|_n^2 + C_2(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M), \quad (4.9)$$

where $C_1 = C_1(\lambda, L, T)$ and $C_2 = C_2(\lambda, L, K, T, p, c, L_1, L_2)$ are constants. To this end we take expectation in both sides of Eq. 4.8 and use the strong monotonicity condition **(ST1)** from Assumption 4.1 to get

$$\begin{aligned} E|e_l|_n^2 \le\ & E|e_0|_n^2 + 2E \sum_{0 \le i < l} \tau \big\langle e_{i+1}, A_{i+1}(\Pi_n u(t_{i+1})) - A_{i+1}(u_{i+1}) \big\rangle_n \\ & + E \sum_{0 \le i < l-1} \sum_k \tau \, \big| B_{k,i+1}(\Pi_n u(t_{i+1})) - B_{k,i+1}(u_{i+1}) \big|_n^2 + \sum_{1 \le j \le 3} S_j \\ \le\ & E|e_0|_n^2 - \lambda \sum_{1 \le i \le l} \tau E\|e_i\|_n^2 + L \sum_{1 \le i \le l} \tau E|e_i|_n^2 + \sum_{1 \le j \le 3} S_j \end{aligned} \quad (4.10)$$

for $l = 1, \cdots, m$, where

$$S_1 = 2 \sum_{1 \le i \le l} E \int_{t_{i-1}}^{t_i} \langle e_i, \Pi_n A(s, u(s)) - A_i(\Pi_n u(t_i))\rangle_n \, ds,$$

$$S_2 = \sum_k \sum_{1 \le i < l} E \int_{t_i}^{t_{i+1}} \big[ |F_k(s)|_n^2 - |B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2 \big] \, ds,$$

$$S_3 = \sum_k E \int_0^\tau |F_k(s)|_n^2 \, ds.$$

For any $\varepsilon > 0$

$$S_1 \le \varepsilon \sum_{1 \le i \le l} \tau\, E\|e_i\|_n^2 + \frac{1}{\varepsilon} R,$$

where

$$R = R(t_l) = \sum_{1\le i\le l} E\int_{t_{i-1}}^{t_i} |\Pi_n A(s,u(s)) - A_i(\Pi_n u(t_i))|_{n^*}^2\, ds \le 3\sum_{1\le j\le 3} R_j,$$

$$R_1 = \sum_{1\le i\le l} E\int_{t_{i-1}}^{t_i} |\Pi_n A(s,u(s)) - \Pi_n A(t_i,u(s))|_{n^*}^2\, ds,$$

$$R_2 = \sum_{1\le i\le l} E\int_{t_{i-1}}^{t_i} |\Pi_n A(t_i,u(s)) - A_i(\Pi_n u(s))|_{n^*}^2\, ds,$$

$$R_3 = \sum_{1\le i\le l} E\int_{t_{i-1}}^{t_i} |A_i(\Pi_n u(s)) - A_i(\Pi_n u(t_i))|_{n^*}^2\, ds. \tag{4.11}$$

Due to condition (2.11) on the time regularity of $A$ in Assumption 2.3, Eq. 2.13, **(Cn$\tau$)**, the Lipschitz condition (4.3) in Assumption 4.1 and inequality (2.12) from Remark 2.5, we deduce

$$R_1 \le \tau^{2\nu} p^2 E\int_0^T (K|u(s)|_{\mathcal{V}}^2 + \eta)\, ds, \tag{4.12}$$

$$R_2 \le c(\tau^{2\nu} + \varepsilon_n^2)\left(E\int_0^T |u(s)|_{\mathcal{V}}^2\, ds + \sum_{1\le i\le m} \tau\, E\xi_i^{n,\tau}\right), \tag{4.13}$$

$$R_3 \le L_1 p^2 \sum_{1\le i\le l}\int_{t_{i-1}}^{t_i} E\|u(s)-u(t_i)\|^2\, ds \le T L_1 p^2 M_1 \tau, \tag{4.14}$$

with $M_1 := C(r_1 + r_2 + M)$. By Eq. 2.13, the regularity condition **(R3)** on $B$ from Assumption 2.3, the growth condition **(ST2)** on $B_{i,k}$ from Assumption 4.1, and by condition (4.6) on the initial values we have

$$\begin{aligned} S_3 &\le 2\sum_k \int_0^\tau E|\Pi_n B_k(s,u(s))|_n^2\, ds + 2\sum_k \tau\, E|B_{k,0}(u_0^{n,\tau})|_n^2 \\ &\le 2\tau p^2\Big(K \sup_{t\in[0,T]} E\|u(t)\|_{\mathcal{H}}^2 + \sup_{t\in[0,T]} E\eta(t)\Big) \\ &\quad +2\tau\Big(K\sup_{n,m} E\|u_0^{n,\tau}\|_n^2 + \sup_{n,m} Eg_0^{n,\tau}\Big). \end{aligned} \tag{4.15}$$

Using the simple inequality $|b|_n^2 - |a|_n^2 \le \varepsilon|a|_n^2 + \left(1+\frac{1}{\varepsilon}\right)|b-a|_n^2$ with

$$a := B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i), \quad b := F_k(s),$$

for any $\varepsilon > 0$ we have $S_2 \le \varepsilon P_1 + \left(1+\frac{1}{\varepsilon}\right) P_2$ with

$$P_1 = P_{1l} = \sum_{1\le i\le l} E\sum_k \tau|B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2,$$

$$P_2 = P_{2l} = \sum_{1\le i<l} E\int_{t_i}^{t_{i+1}} \sum_k |\Pi_n B_k(s,u(s)) - B_{k,i}(\Pi_n u(t_i))|_n^2\, ds. \tag{4.16}$$

By Remark 4.1 on the Lipschitz continuity of $B_{k,i}$ we get $P_1 \le L_2 E \sum_{1\le i\le l} \tau \|e_i\|_n^2$. Clearly, $P_2 \le 3(Q_1 + Q_2 + Q_3)$ with

$$Q_1 = \sum_{1\le i<l} E \int_{t_i}^{t_{i+1}} \sum_k |\Pi_n B_k(s, u(s)) - \Pi_n B_k(t_i, u(s))|_n^2 \, ds,$$

$$Q_2 = \sum_{1\le i<l} E \int_{t_i}^{t_{i+1}} \sum_k |\Pi_n B_k(t_i, u(s)) - B_{k,i}(\Pi_n u(s))|_n^2 \, ds,$$

$$Q_3 = \sum_{1\le i<l} E \int_{t_i}^{t_{i+1}} \sum_k |B_{k,i}(\Pi_n u(s)) - B_{k,i}(\Pi_n u(t_i))|_n^2 \, ds.$$

Due to **(R4) (ii)** in Assumption 2.3 on the time regularity of $B$, consistency **(Cn$\tau$)**, the Lipschitz continuity of $B_{k,i}$ proved in Remark 4.1 , Eq. 2.12 proved in Remark 2.5 and Eq. 2.13,

$$Q_1 \le \tau^{2\nu} p^2 \left( K\, E \int_0^T |u(s)|_{\mathcal{V}}^2 \, ds + T E\eta \right),$$

$$Q_2 \le c(\tau^{2\nu} + \varepsilon_n^2) \left( E \int_0^T |u(s)|_{\mathcal{V}}^2 \, ds + \sup_{n,m} \sum_{0\le i<l} \tau \, E\eta_i^{n,\tau} \right),$$

$$Q_3 \le L_2 p^2 T \sup_{|t-s|\le\tau} E\|u(t) - u(s)\|^2 \le \tau L_2 p^2 T M_1.$$

Hence

$$S_2 \le \varepsilon L_2 \, E \sum_{1\le i\le l} \tau \, \|e_i\|_n^2 + C \left(1 + \frac{1}{\varepsilon}\right) (\tau^{2\nu} + \varepsilon_n^2) \times \left( E \int_0^T |u(s)|_{\mathcal{V}}^2 \, ds + T E\eta + \sup_{n,m} \sum_i \tau \, E\eta_i^{n,\tau} + M_1 \right), \tag{4.17}$$

where $C = C(p, K, L_2, c)$. Choosing $\varepsilon > 0$ sufficiently small, from Eq. 4.10 and Eqs. 4.12–4.17 we obtain for $l = 1, \cdots, m$,

$$E|e_l|_n^2 + \frac{\lambda}{2} E \sum_{1\le i\le l} \tau \|e_i\|_n^2 \le E|e_0|_n^2 + L \sum_{1\le i\le l} \tau \, E|e_i|_n^2 + C(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M), \tag{4.18}$$

where $C = C(K, \lambda, p, T, c, L_1, L_2)$ is a constant. Since $\sup_m \sum_{i=1}^m \tau = T < +\infty$, if $L\tau < 1$ a discrete version of Gronwall's lemma yields the existence of constants $C_1 = C_1(L, \lambda, T)$ and $C_2 = C_2(L, K, \lambda, p, T, c, L_1, L_2)$ such that for sufficiently large $m$

$$\max_{1\le l\le m} E|e_l|_n^2 \le C_1 E|e_0|_n^2 + C_2(r_1 + r_2 + M)(\tau^{2\nu} + \varepsilon_n^2)$$

holds for all $n$. This together with Eq. 4.18 concludes the proof of Eq. 4.9. To prove Eq. 4.7 notice that from Eq. 4.8 by the same calculations as above, but taking first max in $l$ and then expectation, we get

$$E \max_{1\le i\le m} |e_i|_n^2 \le E|e_0|_n^2 + E \sum_{1\le i\le m} \tau \|e_i\|_n^2 + L \sum_{1\le i\le l} \tau E|e_i|_n^2 + R(T) + R_0 + E \max_{0\le i\le m} I(t_i) + EQ(T). \tag{4.19}$$

where $R(T) = R(t_m)$ is defined by Eq. 4.11 and

$$R_0 = P_{1m} + 2P_{2m} + S_3. \tag{4.20}$$

The terms $R(T)$, $P_{1m}$ and $P_{2m}$ have already been estimated above by the right-hand side of Eq. 4.18 and $S_3$ has been estimated by Eq. 4.15. Notice that

$$EQ(T) = E\int_0^T \sum_k |F_k(s)|_n^2\, ds \le 2P_{1m} + 2P_{2m},$$

and by Davis' inequality

$$E \max_{0\le i\le m} I(t_i) \le 6E\left\{\int_0^T \sum_k |(e(s)\,,\ F_k(s))_n|^2\, ds\right\}^{1/2} \le \frac{1}{2} E \max_{0\le i<m} |e_i|_n^2 + 18E\int_0^T \sum_k |F_k(s)|_n^2\, ds.$$

Thus from Eq. 4.19 we obtain Eq. 4.7. □

*Remark 4.5* One can show, like it is observed in [8], that if instead of the Lipschitz condition (4.3) we assume that $A_i^{n,\tau}$ are hemicontinuous and $B_{k,i}^{n,\tau}$ satisfy the Lipschitz condition (4.4), then the order of the speed of convergence is divided by two.

4.3 Examples

(i) Consider from Example 3.3 the normal triples

$$V \hookrightarrow H^* \hookrightarrow V^*, \quad V_n \hookrightarrow H_n^* \hookrightarrow V_n^*$$

with the orthogonal projection $\Pi_n : H = L^2(\mathbb{R}^d) \to H_n$ and auxiliary spaces $\mathcal{H} = W_2^{r+\rho}(\mathbb{R}^d)$ and $\mathcal{V} = W_2^{r+l}(\mathbb{R}^d)$ for some $l > \rho \ge 0$.
Let $A$ and $B = (B_k)$ be $\mathcal{P} \otimes \mathcal{B}(V)$-measurable mappings from $[0, \infty[\times\Omega \times V$ into $V^*$ and $H^{d_1}$, respectively, satisfying Assumptions 2.2 and 2.3 such that $f$ and $g$ in Eq. 2.6 satisfy

$$\sup_{t\in[0,T]} f(t) \le M, \quad \sup_{t\in[0,T]} g(t) \le M.$$

For $\omega \in \Omega$, $j = 1, \ldots, m$ and $i = 0, \ldots, m-1$ let $A_j^{n,\tau}(\omega, \cdot) : V^n \to V_n^*$ and $B_{k,i}^{n,\tau}(\omega, \cdot) : V^n \to H_n$ be defined by

$$\langle A_j^{n,\tau}(\omega, u)\,, v\rangle_n = \langle A(t_j, \omega, u, )\,, v\rangle \quad \text{and } B_{k,i}^{n,\tau}(\omega, u) = \Pi_n B_k(t_i, \omega, u) \tag{4.21}$$

for all $u, v \in V_n$, where $\langle\ ,\ \rangle_n$ denotes the duality between $V_n$ and $V_n^*$. Then it is easy to see, like in Example 3.3, that due to **(1)**, **(2)** and **(3)** in Assumption 2.2, **(ST1)**, **(ST2)** and **(ST3)** in Assumption 2.3 hold respectively. In the same way as **(Cn)** is verified in Example 3.3, one can also easily show that the consistency assumption **(Cn$\tau$)** holds.

(ii) Another choice for $A_j^{n\tau}$ and $B_{k,i}^{n\tau}$ can be defined by

$$\langle A_j^{n,\tau}(u)\,,\,v\rangle_n = \frac{1}{\tau}\int_{t_{j-1}}^{t_j} \langle A(s,u)\,,\,v\rangle ds, \quad u, v \in V_n,$$

$$B_{k,0}^{n,\tau}(u) = \Pi_n B_k(0,u), \quad B_{k,j}^{n,\tau}(u) = \frac{1}{\tau}\int_{t_{j-1}}^{t_j} \Pi_n B_k(s,u)\,ds, \quad u \in V_n, \tag{4.22}$$

instead of Eq. 4.21. One can show by a similar computation as before, combined with the use of Jensen's inequality, that Assumption 2.2 and **(R3)–(R4)** in Assumption 2.3 imply Assumption 4.1 and condition **(Cn$\tau$)**.

(iii) Finally, let $V_n = V$, $H_n = H$ and let $\Pi_n$ be the identity operator for every $n$. Let Assumptions 2.2 and 2.3 hold. Then one recovers the conclusions of Theorems 3.2 and 3.4 in [8] concerning the rate of convergence of the implicit time discretization scheme with $\varepsilon_n = 0$.

## 5 Explicit Space-time Discretization Scheme

### 5.1 Description of the Scheme

Let $V_n$, $H_n$ and $V_n^*$ be a normal triple and $\Pi_n$ be continuous linear operators which satisfy the condition (2.13). Assume moreover that for each $n \geq 0$ as sets

$$V_n = H_n = V_n^*,$$

and there is a constant $\vartheta(n)$ such that

$$\|u\|_{V_n}^2 \leq \vartheta(n)\,|u|_{H_n}^2\,, \quad \forall u \in H_n. \tag{5.1}$$

Then by duality we also have

$$|u|_{H_n}^2 \leq \vartheta(n)\,|u|_{V_n^*}^2\,, \quad \forall u \in V_n^*.$$

Consider for each $n$ and $i = 0, 1, \cdots, m-1$ the equations

$$u_{\tau,i+1}^n = u_{\tau,i}^n + \tau\, A_i^{n,\tau}\big(u_{\tau,i}^n\big) + \sum_k B_{k,i}^{n,\tau}\big(u_{\tau,i}^n\big)\big(W^k(t_{i+1}) - W^k(t_i)\big), \tag{5.2}$$

for $V_n$-valued $\mathcal{F}_{t_i}$-measurable random variables $u_{\tau,i}^n$ for $i = 1, \cdots, m$, where $u_{\tau,0}^n$ is a given $V_n$-valued $\mathcal{F}_0$-measurable random variable, and

$$A_i^{n,\tau} : \Omega \times V_n \to V_n^* \quad \text{and} \quad B_{k,i}^{n,\tau} : \Omega \times V_n \to H_n$$

are given $\mathcal{F}_{t_i} \otimes \mathcal{B}(V_n)$-measurable mappings such that Assumption 4.1 holds.

**Proposition 5.1** *Let Assumption* 4.1 *hold. Then for any $V$-valued $\mathcal{F}_0$-measurable random variable $u^n_{\tau,0}$ such that $E\|u^n_{\tau,0}\|^2_{V_n} < \infty$, the system of Eq.* 5.2 *has a unique solution $(u^n_{\tau,i})^m_{i=1}$ such that $u^n_{\tau,i}$ is $\mathcal{F}_{t_i}$-measurable and $E\|u^n_{\tau,i}\|^2_{V_n} < \infty$ for all i, m and n.*

*Proof* By Eq. 5.1 we have $\|u^n_{\tau,i+1}\|^2_n \le \vartheta(n)|u^n_{\tau,i+1}|^2_n$, and by Eq. 5.2

$$E|u^n_{\tau,i+1}|^2_n \le 3E|u^n_{\tau,i}|^2_n + 3\tau E|A^{n,\tau}_i(u^n_{\tau,i})|^2_n + 3\tau \sum_k E|B^{n\tau}_i\big(u^n_{\tau,i}\big)|^2_n$$

$$\le 3\big(\vartheta(n) + \vartheta(n)\tau K + \tau K\big) E\|u^n_{\tau,i}\|^2_n + 3\tau\big(\vartheta(n)+1\big)M.$$

Hence we get the proposition by induction on $i$. □

## 5.2 Rate of Convergence of the Scheme

The following theorem gives the rate of convergence of $e^n_{\tau,i} := \Pi_n u(t_i) - u^n_{\tau,i}$.

**Theorem 5.2** *Let Assumption* 2.3, *Assumption* 4.1 *with index $j = i = 0, \dots, m-1$ in its formulation, and the consistency condition* **(Cn$\tau$)** *hold. Let n and $\tau$ satisfy*

$$L_1\tau\vartheta(n) + 2\sqrt{L_1L_2\tau\vartheta(n)} \le q \tag{5.3}$$

*for some constant $q < \lambda$, where $L_1$ and $L_2$ are the Lipschitz constants in Eqs.* 4.3 *and* 4.4, *respectively. Then*

$$E \max_{0\le i\le m} |e^n_{\tau,i}|^2_{H_n} + \sum_{0\le i<m} \tau E\|e^n_{\tau,i}\|^2_{V_n} \le C_1 E|e^n_{\tau,0}|^2_{H_n} + C_2\big(\tau^{2\nu} + \varepsilon^2_n\big)(r_1 + r_2 + M), \tag{5.4}$$

*where $C_1 = C_1(\lambda, q, L, T)$ and $C_2 = C_2(\lambda, q, L, K, T, p, c, L_1, L_2)$ are constants.*

*Proof* Note that when we refer to any condition in Assumption 4.1 then we mean it with the index $j$ replaced in its formulation with $i$ running through $0, \cdots, m-1$. To ease notation we omit the indices $n$ and $\tau$ from $e^n_{\tau,i}$, $u^n_{\tau,i}$, $A^{n,\tau}_i$ and $B^{n,\tau}_i$ when this does not cause ambiguity. For any $i = 0, \cdots, m-1$

$$\begin{aligned}|e_{i+1}|^2_n - |e_i|^2_n &= 2\int_{t_i}^{t_{i+1}} \langle e_i\,,\ \Pi_n A(s,u(s)) - A_i(u_i)\rangle_n\,ds \\ &\quad + 2\sum_k \int_{t_i}^{t_{i+1}} (e_i\,,\ F_k(s))_n\,dW^k(s) + \sum_k \Big|\int_{t_i}^{t_{i+1}} F_k(s)\,dW^k(s)\Big|^2_n \\ &\quad + \Big|\int_{t_i}^{t_{i+1}} \big[\Pi_n A(s,u(s)) - A_i(u_i)\big]\,ds\Big|^2_n \\ &\quad + 2\sum_k \Big(\int_{t_i}^{t_{i+1}} \big[\Pi_n A(s,u(s)) - A_i(u_i)\big]\,ds\,,\ \int_{t_i}^{t_{i+1}} F_k(s)\,dW^k(s)\Big)_n\,,\end{aligned}$$

where

$$F_k(s) = \Pi_n B_k(s,u(s)) - B_{k,i}(u_i), \quad s \in ]t_i, t_{i+1}], \quad i = 0, 1, \cdots, m-1.$$

Hence for $l = 1, \cdots, m$ and every $\delta > 0$,

$$|e_l|_n^2 \le |e_0|_n^2 + 2\sum_{0\le i<l}\int_{t_i}^{t_{i+1}} \langle e_i\,,\ \Pi_n A(s,u(s)) - A_i(u_i))\rangle_n\,ds + 2\,I(t_l) + Q(t_l)$$
$$+\left(1+\frac{1}{\delta}\right)S(t_l) + \delta\,Q(t_l), \tag{5.5}$$

where

$$I(t_l) = \int_0^{t_l} (e(s)\,,\,F_k(s))\,dW^k(s), \quad e(s) := e_i \quad \text{for } s \in ]t_i, t_{i+1}],\ i \ge 0,$$
$$S(t_l) = \sum_{0\le i<l}\left|\int_{t_i}^{t_{i+1}} \left[\Pi_n A(s,u(s)) - A_i(u_i)\right] ds\right|_n^2,$$
$$Q(t_l) = \sum_{0\le i<l}\sum_k \left|\int_{t_i}^{t_{i+1}} F_k(s)\,dW^k(s)\right|_n^2.$$

First we prove

$$\max_{1\le i\le m} E|e_i|_n^2 + \sum_{0\le i<m} \tau\,E\|e_i\|_n^2 \le C_1\,E|e_0|_n^2 + C_2\big(\tau^{2\nu} + \varepsilon_n^2\big)(r_1 + r_2 + M), \tag{5.6}$$

with some constants $C_1 = C_1(\lambda, q, L, T)$ and $C_2 = C_2(\lambda, q, L, K, T, p, c, L_1, L_2)$. To this end we take expectation in both sides of Eq. 5.5 and use the strong monotonicity condition (4.2) in Assumption 4.1, to get

$$E|e_l|_n^2 \le E|e_0|_n^2 + 2E\sum_{0\le i<l}\tau\,\langle e_i\,,\ A_i(\Pi_n u(t_i)) - A_i(u_i)\rangle_n$$
$$+E\sum_{0\le i<l}\tau\,|B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2 + \sum_{i=1,2} S_i + \left(1+\frac{1}{\delta}\right)ES(t_l) + \delta\,EQ(t_l)$$
$$\le E|e_0|_n^2 - \lambda E\sum_{0\le i<l}\tau\,\|e_i\|_n^2 + LE\sum_{0\le i<l}\tau\,|e_i|_n^2 + \sum_{i=1,2} S_i + \left(1+\frac{1}{\delta}\right)ES(t_l) + \delta\,EQ(t_l),$$

for any $\delta > 0$, where

$$S_1 = 2\sum_{0\le i<l} E\int_{t_i}^{t_{i+1}} \langle e_i\,,\ \Pi_n A(s,u(s)) - A_i(\Pi_n u(t_i))\rangle_n\,ds,$$
$$S_2 = \sum_k\sum_{0\le i<l} E\int_{t_i}^{t_{i+1}} \left[|F_k(s)|_n^2 - |B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2\right] ds.$$

As in the proof of Theorem 4.4 we get for any $\varepsilon > 0$,

$$S_1 \le \varepsilon\sum_{0\le i<l}\tau\,E\|e_i\|_n^2 + \frac{1}{\varepsilon}C(r_1 + r_2 + M)\big(\tau^{2\nu} + \varepsilon_n^2\big),$$
$$S_2 \le L_2\,\varepsilon\sum_{0\le i<l}\tau\,E\|e_i\|_n^2 + \frac{1}{\varepsilon}C(r_1 + r_2 + M)\big(\tau^{2\nu} + \varepsilon_n^2\big)$$

with a constant $C = C(K, p, T, L_1, L_2, c)$. Notice that for any $\varepsilon > 0$,

$$ES(t_l) \leq \tau\vartheta(n)J(t_l),$$
$$J(t_l) := \sum_{0\leq i<l} E\int_{t_i}^{t_{i+1}} |\Pi_n A(s,u(s)) - A_i(u_i)|_{n^*}^2\, ds \leq (1+\varepsilon)R_0 + \left(1+\frac{1}{\varepsilon}\right)R, \quad (5.7)$$

$$EQ(t_l) \leq (1+\varepsilon)P_1 + \left(1+\frac{1}{\varepsilon}\right)P_2, \quad (5.8)$$

where

$$P_1 := \sum_{0\leq i<l} E\sum_k |B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2 \tau \leq L_2 E\sum_{0\leq i<l} \tau\|e_i\|_n^2$$
$$P_2 := \sum_{0\leq i<l} E\int_{t_i}^{t_{i+1}} \sum_k |\Pi_n B_k(s,u(s)) - B_{k,i}(\Pi_n u(t_i))|_n^2\, ds,$$
$$R_0 := E\sum_{0\leq i<l} \tau|A_i(\Pi_n u(t_i)) - A_i(u_i)|_{n^*}^2 \leq L_1 E\sum_{0\leq i<l} \tau\|e_i\|_n^2,$$
$$R := E\sum_{0\leq i<l}\int_{t_i}^{t_{i+1}} |\Pi_n A(s,u(s)) - A_i(\Pi_n u(t_i))|_{n^*}^2\, ds,$$

for any $l = 1, 2, \cdots, m$. In the same way as in the proof of Theorem 4.4 we obtain

$$R \leq C\big(\tau^{2\nu} + \varepsilon_n^2\big)(r_1 + r_2 + M), \quad (5.9)$$

and that

$$P_2 \leq C'\big(\tau^{2\nu} + \varepsilon_n^2\big)(r_1 + r_2 + M), \quad (5.10)$$

where $C = C(K, p, c, L_1, T)$ and $C' = C'(K, p, c, L_2, T)$ are constants. Consequently,

$$E|e_l|_n^2 \leq E|e_0|_n^2 + (\mu-\lambda)E\sum_{0\leq i<l}\tau\|e_i\|_n^2 + LE\sum_{0\leq i<l}\tau|e_i|_n^2$$
$$+ \big(1+\tau\vartheta(n)\big)\left(1+\frac{1}{\delta}+\frac{1}{\varepsilon}\right)C\big(\tau^{2\nu}+\varepsilon_n^2\big)(r_1+r_2+M) \quad (5.11)$$

for any $\delta > 0$ and $\varepsilon > 0$, where

$$\mu = (1+\varepsilon)\left[\left(1+\frac{1}{\delta}\right)\tau\vartheta(n)L_1 + \delta L_2\right] + \varepsilon(1+L_2),$$

and $C = C(K, p, c, T, L_1, L_2)$ is a constant. It is easy to see that due to Eq. 5.3

$$\inf_{\delta>0}\left(1+\frac{1}{\delta}\right)\tau\vartheta(n)L_1 + \delta L_2 = \tau\vartheta(n)L_1 + 2\sqrt{\tau\vartheta(n)L_1L_2} \leq q.$$

Therefore we can take $\delta > 0$ and $\varepsilon > 0$ such that $\mu \leq (q+\lambda)/2$. Thus from Eq. 5.11 we can get

$$E|e_l|_n^2 \leq E|e_0|_n^2 - \frac{1}{2}(\lambda - q)E\sum_{0\leq i<l}\tau\|e_i\|_n^2 + LE\sum_{0\leq i<l}\tau|e_i|_n^2$$
$$+ C\big(\tau^{2\nu} + \varepsilon_n^2\big)(r_1 + r_2 + M),$$

with a constant $C = C(K, \lambda, q, p, c, T, L_1, L_2)$. Hence by a discrete version of Gronwall's lemma we obtain Eq. 5.6. To prove Eq. 5.4 note that Eq. 5.5 yields

$$E\max_{1\leq l\leq m}|e_l|_n^2 \leq |e_0|_n^2 + E\sum_{0\leq i<l}\tau\|e_i\|_n^2 + 2ES(T) + 2E\max_{1\leq l\leq m}I(t_l) + 2EQ(T), \quad (5.12)$$

where by Eqs. 5.7–5.9 $ES(T) \leq \tau\vartheta(n)J(T)$, and

$$J(T) \leq 2L_1E\sum_{0\leq i<m}\tau\|e_i\|_n^2 + 2C\big(\tau^{2\nu} + \varepsilon_n^2\big)(r_1 + r_2 + M).$$

By Eqs. 5.8, 5.9 and 5.10

$$EQ(T) \leq 2L_2\sum_{0\leq i<m}\tau\|e_i\|_n^2 + 2C'\big(\tau^{2\nu} + \varepsilon_n^2\big)(r_1 + r_2 + M).$$

Finally, in the same way as Eq. 4.21 is obtained, we get

$$E\max_{1\leq i\leq m}I(t_i) \leq 6E\left\{\int_0^T\sum_k|(e(s)\,,\,F_k(s))_n|^2\,ds\right\}^{1/2}$$
$$\leq \frac{1}{2}E\max_{0\leq i<m}|e_i|_n^2 + 18E\int_0^T F_k^2(s)\,ds \leq \frac{1}{2}E\max_{0\leq i<m}|e_i|_n^2 + 18EQ(T).$$

Consequently, from Eq. 5.12 we obtain Eq. 5.4 by Eq. 5.6. □

## 5.3 Example

Consider again the spaces

$$\mathcal{V} \subset \mathcal{H} \subset V \hookrightarrow H \hookrightarrow V^*, \quad V_n \hookrightarrow H_n \hookrightarrow V_n^*$$

from Examples 3.3 and 4.3. Notice that $V_n$, $H_n$ and $V_n^*$ are identified as sets and that due to the converse inequality (2.15) we have Eq. 5.1 with $\vartheta(n) = C2^{nr}$.

Let $A$ and $B$ satisfy the same conditions as in Example 4.3. Define $A_j^{n,\tau}$ and $B_{k,i}^{n,\tau}$ for $j = i = 0, \ldots, m-1$ by Eq. 4.21 or define $A_j^{n,\tau}$ by Eq. 4.21 for $j = 0$ and $A_j^{n,\tau}$, $B_{k,i}^{n,\tau}$ by Eq. 4.22 for $j = 1, \ldots, m-1$ and $i = 0, \ldots, m-1$. Then, as shown in Section 4.3, $A_j^{n,\tau}$ and $B_{k,i}^{n,\tau}$ satisfy the conditions in Assumption 4.1 as well as **(Cn$\tau$)**. Hence, if the solution $u$ satisfies **(R1)–(R2)** in Assumption 2.3, and $L_1\tau\vartheta_n + 2\sqrt{L_1L_2\tau\vartheta_n} \leq q < \lambda$, then the conditions of Theorem 5.2 hold.

## 6 Examples of Approximations of Stochastic PDEs

In this section we present some examples of stochastic PDEs for which the previous theorems provide rates of convergence for the above space and space-time discretization schemes. We refer to Section 5 in [8] for more details. In this section for integers $l$ the notation $|u|_l = |u|_{W^l}$ means the norm of $u$ in $H^l = W^l(\mathbb{R}^d)$.

### 6.1 Quasilinear Equations

Let us consider the stochastic partial differential equation

$$du(t,x) = \big(Lu(t) + F(t,x,\nabla u(t,x), u(t,x)\big)\,dt \\ + \sum_k \big(M_k u(t,x) + g_k(t,x)\big)\,dW^k(t), \quad t \in (0,T],\ x \in \mathbb{R}^d, \tag{6.1}$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \tag{6.2}$$

where $F$ and $g_k$ are Borel functions of $(\omega,t,x,p,r) \in \Omega \times [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ and of $(\omega,t,x) \in \Omega \times [0,\infty) \times \mathbb{R}^d$, respectively, and $L$, $M_k$ are differential operators of the form

$$L(t)v(x) = \sum_{|\alpha|\le 1, |\beta|\le 1} D^\alpha\big(a^{\alpha\beta}(t,x)D^\beta v(x)\big), \ \ M_k(t)v(x) = \sum_{|\alpha|\le 1} b^\alpha_k(t,x)D^\alpha v(x),$$

with functions $a^{\alpha\beta}$ and $b^\alpha_k$ of $(\omega,t,x) \in \Omega \times [0,\infty) \times \mathbb{R}^d$, for all multi-indices $\alpha = (\alpha_1, ..., \alpha_d)$, $\beta = (\beta_1, ..., \beta_d)$ of length $|\alpha| = \sum_i \alpha_i \le 1$, $|\beta| \le 1$. Here, and later on $D^\alpha$ denotes $D_1^{\alpha_1}...D_d^{\alpha_d}$ for any multi-indices $\alpha = (\alpha_1, ..., \alpha_d) \in \{0,1,2,...\}^d$, where $D_i = \frac{\partial}{\partial x_i}$ and $D_i^0$ is the identity operator. We use the notation $\nabla_p := (\partial/\partial p_1, ..., \partial/\partial p_d)$.

Let $K$ and $M$ denote some non-negative numbers. Fix an integer $l \ge 0$ and suppose that the following conditions hold:

**Assumption (A1)** (Stochastic parabolicity). *There exists a constant $\lambda > 0$ such that*

$$\sum_{|\alpha|=1,|\beta|=1} \left( a^{\alpha\beta}(t,x) - \frac{1}{2}\sum_k \big(b^\alpha_k b^\beta_k\big)(t,x) \right) z^\alpha\, z^\beta \ge \lambda \sum_{|\alpha|=1} |z^\alpha|^2$$

*for all $\omega \in \Omega$, $t \in [0,T]$, $x \in \mathbb{R}^d$ and $z = (z^1, ..., z^d) \in \mathbb{R}^d$, where $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} ... z_d^{\alpha_d}$ for $z \in \mathbb{R}^d$ and multi-indices $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$.*

**Assumption (A2)** (Smoothness of the initial condition). *Let $u_0$ be $W^2_l$-valued $\mathcal{F}_0$-measurable random variable such that $E|u_0|_l^2 \le M$.*

**Assumption (A3)** (Smoothness of the linear term). *The derivatives of $a^{\alpha\beta}$ and $b^\alpha_k$ up to order $l$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable real functions such that almost surely*

$$|D^\gamma a^{\alpha\beta}(t,x)| + |D^\gamma b^\alpha_k(t,x)| \le K, \quad \text{for all } |\alpha| \le 1, |\beta| \le 1, k = 1, \cdots, d_1,$$

*$t \in [0,T]$, $x \in \mathbb{R}^d$ and multi-indices $\gamma$ with $|\gamma| \le 2$.*

**Assumption (A4)** (Smoothness of the nonlinear term). *The function $F$ and its first order partial derivatives in $p$, $x$ and $r$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R})$-measurable functions. The function $g_k$ and its derivatives in $x$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions for every $k=1,..,d_1$. There exists a constant $K$ and a $\mathcal{P}\otimes\mathcal{B}$-measurable function $\xi$ of $(\omega,t,x)$ such that almost surely*

$$|\nabla_p F(t,x,p,r)| + |\frac{\partial}{\partial r}F(t,x,p,r)| \le K\,,$$

$$|F(t,\cdot,0,0)|_0^2 + \sum_k |g_k(t,\cdot)|_2^2 \le \eta\,,$$

$$|\nabla_x F(t,x,p,r)| \le L(|p|+|r|)+\xi(t,x), \quad |\xi(t)|_0^2 \le \eta$$

*for all $t,x,p,r$, where $\eta$ is a random variable such that $E\eta\le M$.*

Set $H=L^2(\mathbb{R}^d)=W_2^0$, $V=W_2^1$, $\mathcal{H}=W_2^2$ and $\mathcal{V}=W_2^3$ and suppose that the assumptions **(A1)–(A4)** hold with $l=2$. Then the operators

$$A(t,\varphi)=L(t)\varphi+F(t,.,\nabla\varphi,\varphi), \quad B_k(t,\varphi)=M_k(t)\varphi+g_k(t,.), \quad \varphi\in V$$

and $u_0$ satisfy the conditions of Theorem 2.2. Hence Eqs. 6.1–6.2 has a unique solution $u$ on $[0,T]$. Furthermore, $u$ has a $W_2^2$-valued continuous modification such that

$$E\sup_{0\le t\le T}|u(t)|_2^2 + E\int_0^T |u(t)|_3^2\,dt<\infty.$$

Consequently the regularity conditions **(R1)** and **(R2)** in Assumption 2.3 hold. It is easy to check that $A$ and $B_k$ verify condition **(R3)**.

**Assumption (A5)** (Time regularity of $A$ and $B$) *Almost surely*

(i)

$$\sum_k |D^\gamma(b_k^\alpha(t,x)-b_k^\alpha(s,x))|^2 \le K|t-s|,$$

$$\sum_k |g_k(s,.)-g_k(t,.)|_1^2 \le \eta\,|t-s|.$$

(ii)

$$|D^\gamma(a^{\alpha,\beta}(t,x)-a^{\alpha,\beta}(s,x))|^2 \le K|t-s|,$$

$$|F(t,x,p,r)-F(s,x,p,r)|^2 \le K\,|t-s|\,(|p|^2+|r|^2),$$

$$|\nabla_x F(t,x,p,r)-\nabla_x F(s,x,p,r)|^2 \le K\,|t-s|\,(|p|^2+|r|^2),$$

$$|\nabla_p F(t,x,p,r)-\nabla_p F(s,x,p,r)|^2 \le K\,|t-s|,$$

$$|\frac{\partial}{\partial r}F(t,x,p,r)-\frac{\partial}{\partial r}F(s,x,p,r)|^2 \le K\,|t-s|.$$

*for all $|\alpha|\le 1$, $|\beta|\le 1$, $|\gamma|\le 1$, $s,t\in[0,T]$ and $x\in\mathbb{R}^d$, where $K$ is a constant and $\eta$ is a random variable such that $E\eta\le M$.*

Clearly, Assumptions (i) and (ii) of **(A5)** imply conditions (i) and (ii) of **(R4)** in Assumption 2.3, respectively with $\nu = 1/2$.

Let $H_n$, $V_n$ and $\Pi_n$ be defined as in Example 2.6 and let $A^n(t,u)$ and $B^n_k(t,u)$ be defined by Eq. 3.12. Let $u_0 \in W^2_2 = \mathcal{H}$ and $u^n_0 = \Pi_n u_0$. Recall Example 3.3 and notice that we can apply Theorem 3.1, and by making use of Eq. 2.14 we get the estimate

$$E \sup_{0\le t\le T} |u^n(t) - u(t)|_0^2 + E\int_0^T |u^n(t) - u(t)|_1^2\, dt \le C\, 2^{-2n},$$

with a constant $C$ independent of $n$. Assume now also **(A5)**, recall Example 4.3 and define $A^{n,\tau}$ and $B^{n,\tau}$ by Eq. 4.21. Notice that we can apply Theorem 4.4. Hence if $u^{n,\tau}_0 = \Pi_n u(0)$ we get the estimate

$$E \max_{0\le i\le m} |u^{n,\tau}_i - u(i\tau)|_0^2 + \tau E \sum_{0\le i\le m} |u^{n,\tau}_i - u(i\tau)|_1^2 \le C\left(\tau + 2^{-2n}\right). \tag{6.3}$$

Finally recall Example 5.3 and define $A^{n,\tau}$ and $B^{n,\tau}$ as in Example 5.3. Then we can apply Theorem 5.2, and if $u^n_{\tau,0} := \Pi_n u(0)$ and $T2^{2n}/m \le \gamma$ for some constant $\gamma < c\lambda$, then we get estimate Eq. 6.3 for the explicit space-time approximations $u^n_{\tau,i}$, in place of $u^{n,\tau}_i$, with some constant $C$.

Let us now recall Example 2.7 and approximate Eqs. 6.1–6.2 by finite difference schemes. Consider first the following system of SDEs, corresponding to the space discretization with finite differences for fixed $h \in (0,1)$:

$$dv(t) = \big(L_h(t)v(t) + F_h(t, \nabla_h v(t), v(t))\big)\,dt$$

$$+ \sum_k \big(M_{k,h}(t)v(t) + g_{k,h}(t)\big)\,dW^k(t), \quad z \in \mathbb{G} = hZ^d, \tag{6.4}$$

$$v(0) = (u_0(z))_{z\in\mathbb{G}}, \tag{6.5}$$

where $g_{k,h}(t) = (g_k(t,z))_{z\in\mathbb{G}}$, $F_h(t,p,r) = (F(t,z,p,r)_{z\in\mathbb{G}})$ and

$$L_h(t)\varphi := \sum_{|\alpha|\le 1, |\beta|\le 1} \delta^\alpha_-\big(a^{\alpha\beta}(t,\cdot)\delta^\beta_+\varphi\big), \quad \nabla_h\varphi := (\delta_1\varphi, \delta_2\varphi, \ldots, \delta_d\varphi), \tag{6.6}$$

$$M_{k,h}(t)\varphi := \sum_{|\alpha|\le 1} b^\alpha_k(t)\delta^\alpha\varphi, \tag{6.7}$$

for functions $\varphi$ defined on $\mathbb{G}$. It is not difficult to see that taking the triple $V_n := W^1_{h,2}$, $H_n := W^0_{h,2}$, $V^*_n = (W^1_{h,2})^*$, problem (6.4)–(6.5) can be cast into Eq. 3.1, and we can easily check that Assumption 3.1 and Eq. 3.2 hold. Thus Eqs. 6.4–6.5 has a unique continuous $W^0_{h,2}$-valued solution $v = v^h$ such that for every $h \in (0,1)$,

$$E \sup_{t\in[0,T]} |v^h(t)|^2_{h,0} + E\int_0^T |v^h(t)|^2_{h,1}\, dt \le M < \infty.$$

Assume now that $d = 1$. Consider the normal triple $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ with $V := W_2^1(\mathbb{R})$, $H := W_2^0(\mathbb{R})$ and $V^* \equiv W_2^{-1}(\mathbb{R})$. Notice that Using Eq. 2.19 we can see that there is a constant $C$ such that almost surely for all $t \in [0, T]$

$$|D^\alpha\big(a^{\alpha\beta}(t)D^\beta\varphi\big) - \delta_-^\alpha(a^{\alpha\beta}(t)\delta_+^\beta\varphi)|_{h,0} \le Ch|\varphi|_{W_2^3(\mathbb{R})},$$
$$|b_k^\alpha(t)D^\alpha\varphi - b_k^\alpha(t)\delta^\alpha\varphi|_{h,0} \le Ch|\varphi|_{W_2^2(\mathbb{R})},$$
$$|F_h(t, D\varphi, \varphi) - F_h(t, \delta\varphi, \varphi)|_{h,0} \le C|h|_{W_2^2(\mathbb{R})}$$

for all $\varphi \in W_2^3(\mathbb{R})$ and $h \in (0, 1)$. Hence the consistency condition (**Cn**) holds with $\mathcal{V} = W_2^3(\mathbb{R})$ and $\varepsilon_n = h$. Set $\mathcal{H} = W_2^2(\mathbb{R})$. Assume (A1)–(A4) with $l = 2$. Then the assumptions of Theorem 3.1 are satisfied. Thus there is a constant $C$ such that

$$E\sup_{t\in[0,T]}|u(t) - v^h(t)|_{h,0}^2 + E\int_0^T|u(t) - v^h(t)|_{h,1}^2\,dt \le Ch^2$$

for all $h \in (0, 1)$. Now we approximate Eq. 6.5 by the following Euler approximation schemes:

$$\begin{aligned} w_{i+1} &= w_i + \big(L_h(t_{i+1})w_{i+1} + F_h(t_{i+1}, \nabla_h w_{i+1}, w_{i+1})\big)\tau \\ &\quad + \sum_k\big(M_{k,h}(t_i)w_i + g_{k,h}(t_i)\big)\big(W^k(t_{i+1}) - W^k(t_i)\big), \quad w_0 = u_0, \qquad (6.8) \\ u_{i+1} &= u_i + \big(L_h(t_i)u_i + F_h(t_{i+1}, \nabla_h u_{i+1}, u_i)\big)\tau \\ &\quad + \sum_k\big(M_{k,h}(t_i)u_i + g_{k,h}(t_i)\big)\big(W^k(t_{i+1}) - W^k(t_i)\big), \quad v_0 = u_0. \end{aligned}$$

for $i = 0, 1, 2, \ldots, m-1$, $\tau = T/m$, $t_i = i\tau$. Then by Proposition 4.3 we get the existence of a unique $W_{h,2}^1$-valued solution $w_i$ of Eq. 6.8, such that $w_i$ is $\mathcal{F}_{t_i}$-measurable for $i = 1, 2, \ldots, m$, if $\tau$ is sufficiently small. By Theorem 4.4 for $e_i^{h,\tau} = (u(t_i, z) - w_i(z))_{z\in\mathbb{G}}$, we get

$$E\max_{0\le i\le m}|e_i^{h,\tau}|_{W_{h,2}^0}^2 + \tau\sum_{1\le i\le m}E|e_i^{h,\tau}|_{W_{h,2}^1}^2 \le C(\tau + h^2)$$

with a constant $C$ independent of $\tau$ and $h$. Recall that $\vartheta(n) = \kappa^2/h_n^2$ for any sequence $h_n \in (0, 1)$ by Eq. 2.16. Set $e_{i,\tau}^h = (u(t_i, z) - u_i(z))_{z\in\mathbb{G}}$. Then applying Theorem 5.2 we get

$$E\max_{0\le i\le m}|e_{\tau,i}^h|_{W_{h,2}^0}^2 + \tau\sum_{0\le i<m}E|e_{\tau,i}^h|_{W_{h,2}^1}^2 \le Ch^2,$$

with a constant $C$ independent of $\tau$ and $h$, provided Eq. 5.3 holds with $\kappa^2/h^2$ in place of $\vartheta(n)$. To obtain the corresponding results when $d > 1$ we need more regularity in the space variable from the solution $u$ of Eqs. 6.1–6.2. Assuming more regularity on the data, it is possible to get the required regularity of $u$. We do not want to prove in this paper further results on regularity of the solutions to Eq. 6.1. Instead of that we consider the case of linear equations, i.e., when $F$ does not depend on $p$ and $r$, since in this case the necessary results on regularity of the solutions are well known in the literature. (See e.g. [9] and [16].)

6.2 Linear Stochastic PDEs

We consider again Eqs. 6.1–6.2 and assume that $F = F(t, x, p, r)$ does not depend on $p$ and $r$. We fix and integer $l \geq 0$. Instead of **(A4)** we assume the following.

**Assumption (A*4)** *$F(t, x, p, r) = f(t, x)$ and $g_k(t, x)$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions of $(t, \omega, x)$, and their derivatives in $x$ up to order $l$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable functions such that*

$$|f(t,.)|_l^2 + \sum_k |g_k(t,.)|_l^2 \leq \eta,$$

*where $\eta$ is a random variable such that $E\eta \leq M$.*

Instead of **(A5)** we make the following assumption.

**Assumption (A*5)** *Almost surely*

(i) $\sum_k |D^\gamma(b_k^\alpha(t) - b_k^\alpha(s))| \leq K|t-s|^{\frac{1}{2}}, \quad \sum_k |g_k(s) - g_k(t)|_l^2 \leq \eta\,|t-s|.$
(ii) $|D^\gamma(a^{\alpha,\beta}(t) - a^{\alpha,\beta}(s))| \leq K|t-s|^{\frac{1}{2}}, \quad |f(t) - f(s)|_l^2 \leq \eta\,|t-s|$

*for all $|\gamma| \leq l$, $s, t \in [0, T]$, $x \in \mathbb{R}^d$ and multi-indices $\alpha$ and $\beta$ with $|\alpha| \leq 1$ and $|\beta| \leq 1$, where $K$ is a constant and $\eta$ is a random variable such that $E\eta \leq M$.*

Consider the space-time discretizations with finite differences. The implicit and the explicit approximations, $v^{h,\tau}$ and $v_\tau^h$ are given by the systems of equations defined for $i = 0, \cdots, m-1$ by

$$v^{h,\tau}(t_{i+1}) = v^{h,\tau}(t_i) + \tau\big(L_h(t_{i+1})v^{h,\tau}(t_{i+1}) + f(t_{i+1})\big) + \sum_k \big(M_{k,h}v^{h,\tau}(t_i) + g(t_i)\big)\big(W^k(t_{i+1}) - W^k(t_i)\big), \tag{6.9}$$

$$v^{h,\tau}(0,z) = u(0,z), \quad z \in \mathbb{G}, \tag{6.10}$$

and

$$v_\tau^h(t_{i+1}) = v_\tau^h(t_i) + \tau\big(L_h(t_i)v_\tau^h(t_i) + f(t_i)\big) + \sum_k \big(M_{k,h}v_\tau^h(t_i) + g(t_i)\big)\big(W^k(t_{i+1}) - W^k(t_i)\big), \tag{6.11}$$

$$v_\tau^h(0,z) = u(0,z), \quad z \in \mathbb{G}, \tag{6.12}$$

respectively, where $t_i = i\tau = iT/m$, $v^{h,\tau}(t_i)$ and $v_\tau^h(t_i)$ are functions on $\mathbb{G}$, $L_h(t)$ and $M_{k,h}(t)$ are defined by Eqs. 6.6 and 6.7.

Take $H_n := W_{h,2}^0$ and the normal triple $V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*$ with $V_n := W_{h,2}^1$. Then it is easy to see that

$$(L_h(t_i)\varphi, \psi)_n \leq C|\varphi|_{V_n}||\psi|_{V_n}, \quad (M_{k,h}(t_i)\varphi, \psi)_n \leq C|\varphi|_{V_n}||\psi|_{H_n} \tag{6.13}$$

for all $\varphi, \psi \in V_n$, where $(\cdot,\cdot)_n$ denotes the inner product in $H_n$, and $C$ is a constant depending only on $d$ and the constant $K$ from Assumption **(A3)**. Thus we can define $L_h(t_i)$ and $M_{h,k}(t_i)$ as bounded linear operators from $V_n$ into $V_n^*$ and $H_n$ respectively.

Due to Eqs. 2.17 and 2.18, the restriction of $u_0$, $f(t_i)$ and $g_k(t_i)$ onto $\mathbb{G}$ are $H_n$-valued random variables such that

$$E|f(t_i)|^2_{H_n} \le p^2 E|f(t_i)|^2_l, \quad E|g_k(t_i)|^2_{H_n} \le p^2 E|g(t_i)|^2_l,$$
$$E|u_0|^2_{H_n} \le p^2 E|u_0|^2_l,$$

where $p$ is the constant from Eq. 2.17. Moreover,

$$2(L_h(t_i)\varphi, \varphi)_n + \sum_k |M_{h,k}\varphi|^2_{H_n} \le -\frac{\lambda}{2}|\varphi|^2_{V_n} + C|\varphi|^2_{H_n} \tag{6.14}$$

for all $\varphi \in V_n$, where $C$ is a constant depending only on $d$ and on the constant $K$ from Assumption **(A2)**. Thus using the notation $u_i^{n,\tau} = v^{h,\tau}(t_i)$, $u^n_{\tau,i} = v^{h,\tau}(t_i)$ and defining

$$A_i^{n,\tau}(\varphi) = L_h(t_i)\varphi + f(t_i), \quad B_{k,i}^{n,\tau}(\varphi) = M_{k,h}(t_i)\varphi + g_k(t_i)$$

for $\varphi \in W^1_{h,2}$, we can cast Eqs. 6.9–6.10 and Eqs. 6.11–6.12 into Eq. 4.1 and into Eq. 5.2, respectively, and we can see that Assumption 4.1 and condition (4.6) hold. Consequently, by virtue of Proposition 4.3, for sufficiently small $\tau$ Eqs. 6.9–6.10 has a unique solution $\{v^{h,\tau}(t_i)\}_{i=0}^m$, such that $v^{h,\tau}(t_i)$ is a $W^1_{h,2}$-valued $\mathcal{F}_{t_i}$-measurable random variable and $E|v_i^{h,\tau}|^2_{h,2} < \infty$. Furthermore, by virtue of Proposition 5.1, Eqs. 6.11–6.12 has a unique solution $\{v^h_\tau(t_i)\}_{i=0}^m$, such that $v^h_\tau(t_i)$ is a $W^1_{h,2}$-valued $\mathcal{F}_{t_i}$-measurable random variable and $E|v^h_\tau(t_i)|^2_{h,2} < \infty$.

Let $r \ge 0$ be an integer, and assume that

$$l > r + 2 + \frac{d}{2}. \tag{6.15}$$

Then Theorem 4.4 gives the following result.

**Theorem 6.1** *Let Assumptions* **(A1)**, **(A2)**, **(A3)**, **(A*4)** *and* **(A*5)** *hold with $l$ satisfying Eq. 6.15. Then for sufficiently small $\tau$*

$$E \max_{1\le i\le m} |v^{h,\tau}(t_i) - u(t_i)|^2_{h,r} + E \sum_{1\le i\le m} \tau\, |v^{h,\tau}(t_i) - u(t_i)|^2_{h,r+1} \le C(h^2 + \tau)$$

*for all $h \in (0, 1)$, where $C = C(r, l, p, \lambda, T, K, M, d, d_1)$ is a constant.*

*Proof* Take $H_n := W^r_{h,2}$, $H := W_2^{l-2}(\mathbb{R}^d)$, $\mathcal{H} := W_2^l(\mathbb{R}^d)$ and the normal triples

$$V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*, \quad V \hookrightarrow H \equiv H^* \hookrightarrow V^*, \quad \mathcal{V} \hookrightarrow \mathcal{H} \equiv \mathcal{H}^* \hookrightarrow \mathcal{V}^*$$

where $V_n := W^{r+1}_{h,2}$, $V_n^* \equiv W^{r-1}_{h,2}$, $V := W_2^{l-1}(\mathbb{R}^d)$, $V^* \equiv W_2^{l-3}(\mathbb{R}^d)$, $\mathcal{V} := W_2^{l+1}(\mathbb{R}^d)$ and $\mathcal{V}^* \equiv W_2^{l-1}(\mathbb{R}^d) = V$. Then due to Eq. 6.15 there is a constant $p$ such that for $\Pi_n := R_h$,

$$|\Pi_n\varphi|_{V_n} \le p|\varphi|_V,$$

for all $\varphi \in V$, by virtue of Eq. 2.18. It is easy to check that Eqs. 6.13–6.14 still hold, and hence Eqs. 6.9–6.10, written as Eq. 4.1, satisfies Assumption 4.1 and condition (4.6) in the new triple as well. Using Eq. 2.20 it is easy to show that due to Assumption **(A3)**

$$|L(t_i)\varphi - L_h(t_i)\varphi|_{V_n^*} \le |L(t_i)\varphi - L_h(t_i)\varphi|_{H_n} \le Ch|\varphi|_{W_2^{l+1}(\mathbb{R}^d)},$$
$$\sum_k |M_k(t_i)\varphi - M_{k,h}(t_i)\varphi|_{H_n} \le Ch|\varphi|_{W_2^l(\mathbb{R}^d)}$$

for all $\varphi \in W_2^{l+1}(\mathbb{R}^d)$, where $C$ is a constant depending on $d, l, r$ and on the constant $K$ from Assumption **(A3)**. Hence we can see that **(Cn$\tau$)** holds with $\varepsilon_n = h$. Due to Assumption **(A*5)** we have

$$|L(t)\varphi - L(s)\varphi|_V^2 \le C|t-s|, \quad |f(t) - f(s)|_V^2 \le \eta|t-s|,$$

$$\sum_k |M_k(t)\varphi - M_k(s)\varphi|_V^2 \le C|t-s|, \quad \sum_k |g_k(t) - g_k(s)|_V^2 \le \eta|t-s|,$$

where $\eta$ is the random variable from Assumption **(A*5)**, and $C$ is a constant depending on $d$, $d_1$, $l$ and on the constant $K$ from Assumption **(A*5)**. It is an easy exercise to show that due to Assumptions **(A3)** and **(A*4)** condition **(R3)** from Assumption 2.3 holds. From [9] it is known that under the Assumptions **(A1)–(A3)** and **(A*4)** the problem (6.4)–(6.5) has a unique solution $u$ on $[0, T]$, and that $u$ is a continuous $W_2^l(\mathbb{R}^d)$-valued $(\mathcal{F}_t)$-adapted stochastic process such that

$$E \sup_{t\in[0,T]} |u(t)|_l^2 + E\int_0^T |u(t)|_{l+1}^2\, dt$$
$$\le CE|u_0|_l^2 + CE\int_0^T \left( |f(t)|_{l-1}^2 + \sum_k |g_k(t)|_l^2 \right) dt,$$

where $C$ is a constant depending on $d$, $d_1$ and the constants $\lambda$ and $K$ from Assumptions **(A1)**, **(A3)** and **(A*4)**. Hence the regularity conditions **(R1)** and **(R2)** in Assumption 2.3 clearly hold. Now we can conclude the proof by applying Theorem 4.4. □

Let us now investigate the rate of convergence of the explicit space-time approximations. Take the normal triple $V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*$ with $V_n := W_{h,2}^{r+1}$, $H_n := W_{h,2}^r$, and notice that due to Assumption **(A3)**

$$(L(t_i)\varphi, \psi)_n \le C_1|\varphi|_{V_n}|\psi|_{V_n}, \quad (M_{k,h}(t_i)\varphi, \psi)_n \le C_{2k}|\varphi|_{V_n}|\psi|_{H_n} \tag{6.16}$$

with some constants $C_1$ and $C_{2k}$ depending only on $d$, $r$ and the constant $K$ from Assumption **(A3)**. Set $L_1 = C_1^2$ and $L_2 = \sum_k C_{2k}^2$. Then Theorem 5.2 yields the following theorem, which improves a result from [18].

**Theorem 6.2** *Let Assumptions* **(A1)**, **(A2)**, **(A3)**, **(A*4)** *and* **(A*5)** *hold with l satisfying Eq. 6.15. Let h and $\tau$ satisfy*

$$L_1\kappa^2\frac{\tau}{h^2} + 2\kappa(L_1L_2)^{1/2}\frac{\sqrt{\tau}}{h} \le q \tag{6.17}$$

*for a constant* $q < \lambda$. *Then*

$$E \max_{1\le i\le m} |v_\tau^h(t_i) - u(t_i)|_{h,r}^2 + E \sum_{0\le i<m} \tau\, |v_\tau^h(t_i) - u(t_i)|_{h,r+1}^2 \le C(h^2 + \tau)$$

*for all* $h \in (0, 1)$, *where* $C = C(r, l, p, \lambda, q, T, K, M, d, d_1)$ *is a constant.*

*Proof* As in the proof of Theorem 6.1 we take $H_n := W_{h,2}^r$, $H := W_2^{l-2}(\mathbb{R}^d)$, $\mathcal{H} := W_2^l(\mathbb{R}^d)$ and the normal triples

$$V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*, \quad V \hookrightarrow H \equiv H^* \hookrightarrow V^*, \quad \mathcal{V} \hookrightarrow \mathcal{H} \equiv \mathcal{H}^* \hookrightarrow \mathcal{V}^*$$

with $V_n := W_{h,2}^{r+1}$, $V := W_2^{l-1}(\mathbb{R}^d)$, $\mathcal{V} = W_2^{l+1}(\mathbb{R}^d)$, we cast Eqs. 6.11–6.12 into Eq. 5.2, and see that Assumptions 2.3 and 4.1, conditions **(Cn$\tau$)** and Eq. 4.6 of Theorem 5.2 hold. Furthermore, $\vartheta(n) = \frac{\kappa^2}{h^2}$. We can easily check that by virtue of Eqs. 6.16 and 2.16, condition (6.17) yields condition (5.3). Hence applying Theorem 5.2 we finish the proof. □

**Corollary 6.3** *Let* $k \ge 0$ *be an integer and let Assumptions* **(A1)**, **(A2)**, **(A3)**, **(A*4)** *and* **(A*5)** *hold with* $l$ *satisfying* $l > k + 2 + d$. *Then the following statements are valid for all multi-indices* $\alpha$ *with* $|\alpha| \le k$*:*

(i) *For sufficiently small* $\tau$

$$E \max_{1\le i\le m} \sup_{z\in\mathbb{G}} |\delta^\alpha \big(v^{h,\tau}(t_i, z) - u(t_i, z)\big)| \le C(h + \sqrt{\tau})$$

*holds for all* $h \in (0, 1)$, *where* $C = C(l, p, \lambda, T, K, M, d, d_1)$ *is a constant.*

(ii) *Assume also that* $\tau$ *and* $h$ *satisfy Eq.* 6.17. *Then*

$$E \max_{1\le i\le m} \sup_{z\in\mathbb{G}} |\delta^\alpha \big(v_\tau^h(t_i, z) - u(t_i, z)\big)| \le C(h + \sqrt{\tau}) \le C\left(1 + \kappa^{-1}\sqrt{\lambda/L_1}\right) h$$

*for all* $h \in (0, 1)$, *where* $C = C(r, l, p, \lambda, q, T, K, M, d, d_1)$ *is a constant.*

*Proof* By the discrete version of Sobolev's theorem on embedding $W_m^2(\mathbb{R}^d)$ into $\mathcal{C}^k\big(\mathbb{R}^d\big)$ one knows that if $m > k + \dfrac{d}{2}$, then

$$\sup_{z\in\mathbb{G}} |\delta^\alpha \varphi(z)| \le C|\varphi|_{W_{h,2}^m}$$

for all $h \in (0, 1)$, $\varphi \in W_{h,2}^m$ and $|\alpha| \le k$, where $C = C(d, m, k)$ is a constant (see e.g. [18]). Hence the above statements follow immediately from Theorems 6.1 and 6.2. □

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