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Moser Type Inequalities for Higher-Order Derivatives in Lorentz Spaces

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Abstract Sharp constants are exhibited in exponential inequalities corresponding to the limiting case of the Sobolev inequalities in Lorentz-Sobolev spaces of arbitrary order.

Keywords Moser inequalities · Higher-order derivatives · Lorentz-Sobolev spaces

Mathematics Subject Classifications (2000) 46E35 · 46E30

1 Introduction

In the celebreted paper [17] dealing with the exceptional case of the Sobolev inequality, J. Moser proved that a constant C, depending only on n, exists such that, if Ω is an open domain in $\mathbb{R}^n$ having finite measure, $n \geq 2$, and u is any function from the Sobolev space $W_0^{1,n}(\Omega)$ satisfying $\int_\Omega |\nabla u|^n \, dx \leq 1$, then

$$\int_\Omega \exp\left(n\omega_n^{1/n} |u(x)|\right)^{n'} dx \leq C |\Omega|. \tag{1}$$

Here, $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$, the measure of the unit ball in $\mathbb{R}^n$, Γ is the Gamma function, $n' = n/(n-1)$, the Hölder conjugate of n, and $|E|$ stands for the Lebesgue measure of a subset E of $\mathbb{R}^n$. Moreover, the result is optimal, in the sense that, if $n\omega_n^{1/n}$ is replaced by any larger constant in Eq. 1, the integral is still finite, but not uniformly bounded.

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Inequality Eq. 1 provides a sharp version of the limiting Sobolev embedding

$$W_0^{1,n}(\Omega) \hookrightarrow \exp L^{n'}(\Omega) \tag{2}$$

going back to Yudovich [22], Pohozaev [19], Trudinger [21], where $\exp L^{n'}(\Omega)$ denotes the Orlicz space associated with the Young function $e^{t^{n'}} - 1$, and "$\hookrightarrow$" stands for continuous inclusion.

Embedding Eq. 2 can be regarded as a distinguished first order case of a whole family of embeddings, of arbitrary order, for limiting Sobolev type spaces built upon Lorentz spaces.

Specifically, let us define, for any integer $m \geq 1$ and any $1 \leq p, q \leq \infty$,

$W_0^m L^{p,q}(\Omega) = \{u : u$ is a real-valued function in Ω whose continuation by 0 outside Ω is m-times weakly differentiable in the whole of $\mathbb{R}^n$, and $\| |\nabla^m u| \|_{L^{p,q}} < \infty\}$.

Here, $\nabla^m u$ stands for the m-th order gradient of u, i.e.

$$\nabla^m u = \begin{cases} \Delta^{m/2} u & \text{if } m \text{ is even} \\ \nabla \Delta^{(m-1)/2} u & \text{if } m \text{ is odd,} \end{cases} \tag{3}$$

where ∇ and Δ denote the standard gradient and Laplacian, respectively, and $|\nabla^m u|$ is the Euclidean norm of $\nabla^m u$.

Moreover, $L^{p,q}(\Omega)$ denotes the Lorentz space of those functions u for which the quantity

$$\|u\|_{L^{p,q}(\Omega)} = \left\| s^{\frac{1}{p} - \frac{1}{q}} u^*(s) \right\|_{L^q(0,|\Omega|)}$$

is finite. The function u^* is the decreasing rearrangement of u, namely the unique non-increasing right-continuous function from $[0, +\infty)$ into $[0, +\infty]$ which is equidistributed with u. Then, in particular, $\|u\|_{L^{p,q}(\Omega)} \equiv \|u^*\|_{L^{p,q}(0,|\Omega|)}$. Note that $L^{p,p}(\Omega) = L^p(\Omega)$ for every $p \geq 1$, and hence $W_0^m L^{p,p}(\Omega) = W_0^{m,p}(\Omega)$, the usual Sobolev space, for every $p \geq 1$.

Then we have (see e.g. [9])

$$W_0^m L^{\frac{n}{m},q}(\Omega) \hookrightarrow \exp L^{q'}(\Omega) \tag{4}$$

for every integer m, with $1 \leq m < n$, and every $q \in (1, \infty]$.

A version of inequality Eq. 2, corresponding to Eq. 4, was established in [1] in the case where $q = \dfrac{n}{m}$ and $1 \leq m < n$, namely for the Sobolev space $W_0^{m,\frac{n}{m}}(\Omega)$. On the other hand, the case where q is any exponent in $(1, +\infty]$, but $m = 1$, was treated in [3]. (Other contributions to this subject can be found in several papers including [6–8, 10–13, 15–17, 20].)

The objective of the present paper is to complete this picture, and to prove a Moser type inequality corresponding to the embedding Eq. 4 for general m and q.

Our result reads as follows.

Theorem 1 *Let m be a positive integer satisfying $1 \le m < n$ and let $q \in (1, +\infty]$. Define*

$$\beta_{n,m} = \begin{cases} \dfrac{\pi^{n/2}\, 2^m\, \Gamma\left(\frac{m}{2}\right)}{\omega_n^{\frac{n-m}{n}}\ \Gamma\left(\frac{n-m}{2}\right)} & \text{if } m \text{ is even} \\[2ex] \dfrac{\pi^{n/2}\, 2^m\, \Gamma\left(\frac{m+1}{2}\right)}{\omega_n^{\frac{n-m}{n}}\ \Gamma\left(\frac{n-m+1}{2}\right)} & \text{if } m \text{ is odd}\,. \end{cases} \tag{5}$$

(i) *If $q \in (1, +\infty)$, there exists a constant $C = C(n, m, |\Omega|, q)$ such that*

$$\int_\Omega \exp\left(\beta_{n,m}|u(x)|\right)^{q'} dx \le C \tag{6}$$

for every $u \in W_0^m L^{\frac{n}{m},q}(\Omega)$ fulfilling $\||\nabla^m u|\|_{L^{\frac{n}{m},q}(\Omega)} \le 1$. The result is sharp in the sense that, the left-hand side of Eq. 6, with $\beta_{n,m}$ replaced by any larger constant, cannot be uniformly bounded as u ranges among all functions from $W_0^m L^{\frac{n}{m},q}(\Omega)$ satisfying $\||\nabla^m u|\|_{L^{\frac{n}{m},q}(\Omega)} \le 1$.

(ii) *If $q = +\infty$, for every $\gamma < \beta_{n,m}$, there exists a constant $C = C(n, m, \gamma, |\Omega|)$ such that*

$$\int_\Omega \exp(\gamma|u(x)|)\, dx \le C \tag{7}$$

for every $u \in W_0^m L^{\frac{n}{m},\infty}(\Omega)$ fulfilling $\||\nabla^m u|\|_{L^{\frac{n}{m},\infty}(\Omega)} \le 1$. The result is sharp in the sense that, for each $\gamma > \beta_{n,m}$, there exists a function $u \in W_0^m L^{\frac{n}{m},\infty}(\Omega)$ satisfying $\||\nabla^m u|\|_{L^{\frac{n}{m},\infty}(\Omega)} \le 1$ and $\int_\Omega \exp(\gamma|u(x)|)\, dx = +\infty$.

Note that the case where $q = 1$ is not dealt with Theorem 1 since $W_0^m L^{\frac{n}{m},1}(\Omega) \hookrightarrow L^\infty(\Omega)$ ([3, 4, 9, 14]).

Note also that the case where $q = +\infty$ and $\gamma = \beta_{n,m}$ in Theorem 1 is left open. Inequality Eq. 7 is known not to hold with $\gamma = \beta_{n,m}$ when $m = 1$ ([3]), and can be shown to fail for $m = 2$ as well by the methods of [2]; we conjecture that this should be the case also for $m \ge 3$, but so far we have no counterexample in this connection.

2 Proof

Our approach to Theorem 1 is related to that of [1], and relies on a representation formula for Sobolev functions u in terms of Riesz potentials of their m-th order gradient $\nabla^m u$, which tells us that

$$u(x) = \begin{cases} \dfrac{(-1)^{m/2}}{\omega_n^{(n-m)/n}\beta_{n,m}} \displaystyle\int_{\mathbb{R}^n} |x-y|^{m-n} \nabla^m u(y)\, dy & \text{if } m \text{ is even} \\[2ex] \dfrac{(-1)^{(m-1)/2}}{\omega_n^{(n-m)/n}\beta_{n,m}} \displaystyle\int_{\mathbb{R}^n} |x-y|^{m-1-n} (x-y)\cdot \nabla^m u(y)\, dy & \text{if } m \text{ is odd} \end{cases} \tag{8}$$

for a.e. $x \in \mathbb{R}^n$ (see [1, Lemma 2]).

Obviously, Eq. 8 enables us to estimate u^* in terms of the decreasing rearrangement of a convolution of $|\nabla^m u|$ against the kernel $|x-y|^{m-n}$. A key result relating the rearrangement of a convolution to the rearrangement of the convoluted functions is provided by the following special case of a lemma by O'Neil [18].

Lemma A *Let f, g and h be measurable functions on* $\mathbb{R}^n$ *such that* $h = f \star g$, *then*

$$h^*(t) \le \frac{1}{t}\int_0^t f^*(s)\,ds \int_0^t g^*(s)\,ds + \int_t^{+\infty} f^*(s)g^*(s)\,ds \tag{9}$$

for every $t > 0$.

Equation 8 combined with Lemma A easily reduced the proof of inequalities Eqs. 6 and 7 to one-dimensional problems. The estimates which have to be faced after this rearrangement process require the use of an extension of Moser's original one-dimensional lemma established in [1] and reading as follows.

Lemma B *Let* $a(s,t)$ *be a nonnegative measurable function in* $[0,+\infty)\times[0,+\infty)$ *such that*

$$a(s,t) \le 1 \qquad \text{for a.e. } 0 \le s < t. \tag{10}$$

Suppose that

$$\sup_{t>0}\left(\int_t^{+\infty} a(s,t)^{p'}\,ds\right)^{1/p'} = b < +\infty. \tag{11}$$

Then, there exists a constant $c = c(p,b)$ *such that, for every nonnegative measurable function* ϕ *in* $(0,+\infty)$ *satisfying*

$$\int_0^{+\infty} \phi(s)^p\,ds \le 1, \tag{12}$$

one has

$$\int_0^{+\infty} e^{-F(t)}\,dt \le c \tag{13}$$

where $F(t) = t - \left(\int_0^{+\infty} a(s,t)\,\phi(s)\,ds\right)^{p'}$.

Let us emphasize that specifical difficulties arise in the present setting, especially in dealing with the sharpness of inequalities Eqs. 6–7, owing to the non integral form of Lorentz norms.

We are now in position to prove our result.

Proof of Theorem 1

Part (i) Equation 8 ensures that

$$|u(x)| \le \frac{1}{\omega_n^{(n-m)/n}\,\beta_{n,m}}\big(I_m \star |\nabla^m u|\big)(x) \tag{14}$$

for a.e. $x\in\Omega$. Let us define $g(x)=|x-y|^{-(n-m)}$. Then $g^*(t)=(\omega_n\, t^{-1})^{(n-m)/n}$. From Lemma A, one has, for every $t>0$,

$$
\begin{aligned}
u^*(t) &\le \frac{1}{\omega_n^{(n-m)/n}\ \beta_{n,m}}\ \left(\mathrm{I}_m \star |\nabla^m u|\right)^*(t)\\
&\le \frac{1}{\omega_n^{(n-m)/n}\ \beta_{n,m}}\ \left(\frac{1}{t}\int_0^t g^*(s)\,ds\int_0^t |\nabla^m u|^*(s)\,ds+\int_t^\infty g^*(s)|\nabla^m u|^*(s)\,ds\right)\\
&= \frac{\omega_n^{(n-m)/n}}{\omega_n^{(n-m)/n}\ \beta_{n,m}}\ \left(\frac{n}{m}\,t^{-\frac{n-m}{n}}\ \int_0^t |\nabla^m u|^*(s)\,ds+\int_t^{|\Omega|}|\nabla^m u|^*(s)s^{-\frac{n-m}{n}}\,ds\right)\\
&= \frac{1}{\beta_{n,m}}\ \left(\frac{n}{m}\,t^{-\frac{n-m}{n}}\ \int_0^t |\nabla^m u|^*(s)\,ds+\int_t^{|\Omega|}|\nabla^m u|^*(s)s^{-\frac{n-m}{n}}\,ds\right). \qquad (15)
\end{aligned}
$$

Hence, by a change of variables, we obtain

$$
\begin{aligned}
\beta_{n,m}\,u^*\left(|\Omega|e^{-\tau}\right) \le \frac{n}{m}\ \left(|\Omega|e^{-\tau}\right)^{-\frac{n-m}{n}}\ &\int_\tau^\infty |\nabla^m u|^*(|\Omega|e^{-\sigma})\ \left(|\Omega|e^{-\sigma}\right)\,d\sigma\\
&+\int_0^\tau |\nabla^m u|^*\left(|\Omega|e^{-\sigma}\right)\left(|\Omega|e^{-\sigma}\right)^{-\frac{n-m}{n}}\ \left(|\Omega|e^{-\sigma}\right)\,d\sigma\\
&=\int_0^{+\infty} a(\sigma,\tau)\phi(\sigma)\,d\sigma, \qquad (16)
\end{aligned}
$$

where

$$
\phi(\sigma)=|\nabla^m u|^*\left(|\Omega|e^{-\sigma}\right)\ \left(|\Omega|e^{-\sigma}\right)^{\frac{m}{n}} \qquad \text{if } \sigma>0, \qquad (17)
$$

and

$$
a(\sigma,\tau)=\begin{cases} 1 & \text{if } 0\le\sigma<\tau<+\infty\\ \dfrac{n}{m}\,(|\Omega|e^{-\tau})^{-\frac{n-m}{n}}\ (|\Omega|e^{-\sigma})^{\frac{n-m}{n}} & \text{if } 0\le\tau<\sigma<+\infty. \end{cases} \qquad (18)
$$

Lemma A comes into play at this stage. Assumption Eq. 10 is obviously satisfied if a is given by Eq. 18. As far as Eq. 11 is concerned we have

$$
\begin{aligned}
\int_\tau^\infty a(\sigma,\tau)^{q'}\,d\sigma &= \left(\frac{n}{m}\right)^{q'}\ \left(|\Omega|e^{-\tau}\right)^{-q'\frac{n-m}{n}}\ \int_\tau^\infty \left(|\Omega|e^{-\sigma}\right)^{q'\frac{n-m}{n}}\,d\sigma\\
&= \left(\frac{n}{m}\right)^{q'}\ \frac{1}{q'\left(\frac{n-m}{n}\right)+1} \qquad \text{for } \tau>0,
\end{aligned}
$$

whence $\sup\limits_{\tau>0}\displaystyle\int_\tau^\infty a(\sigma,\tau)^{q'}\,d\sigma<\infty$. By Eq. 17

$$
\begin{aligned}
\|\phi\|^q_{L^q(0,+\infty)} &= \int_0^\infty\left(|\nabla^m u|^*\left(|\Omega|e^{-\sigma}\right)\left(|\Omega|e^{-\sigma}\right)^{\frac{m}{n}}\right)^q\,d\sigma\\
&= \int_0^{|\Omega|}\left(|\nabla^m u|^*(t)\,t^{\frac{m}{n}}\right)^q\frac{dt}{t}\\
&= \||\nabla^m u|\|^q_{L^{\frac{n}{m},q}(\Omega)}\le 1 \qquad \text{for } q\in(1,+\infty).
\end{aligned}
$$

Thus, by Eq. 16 and Lemma A

$$\int_\Omega \exp\left(\beta_{n,m}|u(x)|\right)^{q'}\,dx = \int_0^{|\Omega|} \exp\left(\beta_{n,m}u^*(t)\right)^{q'}\,dt$$
$$= |\Omega| \int_0^\infty \exp\left[\left(\beta_{n,m}u^*(|\Omega|e^{-\tau})\right)^{q'} - \tau\right] d\tau$$
$$\le |\Omega| \int_0^\infty \exp\left[\left(\int_0^{+\infty} a(\sigma,\tau)\phi(\sigma)\,d\sigma\right)^{q'} - \tau\right] d\tau$$
$$= |\Omega| \int_0^\infty e^{-F(\tau)}\,d\tau \le C$$

for some constant $C = C(n, m, |\Omega|, q)$.
In order to prove the optimality of Eq. 6, we construct a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W_0^m L^{\frac{n}{m},q}(\Omega)$ such that

$$\lim_{k\to+\infty} \int_\Omega \exp\left(\frac{\beta|u_k(x)|}{\||\nabla^m u_k|\|_{L^{\frac{n}{m},q}(\Omega)}}\right)^{q'} dx = +\infty \qquad \forall \beta > \beta_{n,m}.$$

Let B be the unit ball of $\mathbb{R}^n$ centered at the origin. Up to rescaling and translating, we may assume, without loss of generality, that $B \subset\subset \Omega$. Let φ be an increasing smooth function (of class $\mathcal{C}^m$, say) defined in $\mathbb{R}$ as equals to zero, if $t \le 0$, and equals to t , if $t \ge 1$. Given $\epsilon \in (0, 1)$, let $H_\epsilon : \mathbb{R} \to [0, 1]$ be defined as

$$H_\epsilon(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ \epsilon\,\varphi\left(\frac{t}{\epsilon}\right) & \text{if} \quad 0 \le t \le \epsilon \\ t & \text{if} \quad \epsilon < t < 1-\epsilon \\ 1 - \epsilon\,\varphi\left(\frac{1-t}{\epsilon}\right) & \text{if} \quad 1-\epsilon < t \le 1 \\ 1 & \text{if} \quad t > 1. \end{cases}$$

Observe that H_ϵ is of class $\mathcal{C}^m$. For $k \in \mathbb{N}$, define $u_{\epsilon,k} : \Omega \to \mathbb{R}$ as

$$u_{\epsilon,k}(x) = \begin{cases} H_\epsilon\left(\dfrac{\log\frac{1}{|x|}}{\log k}\right) & \text{if} \quad x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $u_{\epsilon,k} \in \mathcal{C}_0^m(\Omega)$. Computations show that

$$\nabla^m u_{\epsilon,k}(x) = \begin{cases} \displaystyle\sum_{i=1}^{2h} c_i H_\epsilon^{(i)}\left(\frac{\log\frac{1}{|x|}}{\log k}\right) \frac{1}{|x|^{2h}(\log k)^i} & \text{if} \quad m = 2h \\ \displaystyle\sum_{i=1}^{2h+1} c_i H_\epsilon^{(i)}\left(\frac{\log\frac{1}{|x|}}{\log k}\right) \frac{1}{|x|^{2h+1}(\log k)^i} & \text{if} \quad m = 2h+1, \end{cases}$$

with $h \in \mathbb{N}$, if $x \in B$, and $\nabla^m u_{\epsilon,k}(x) = 0$ otherwise. Since φ is smooth (of class $\mathcal{C}^m$), then a constant $C = C(\varphi, m)$, which be assumed to be larger than

$n-m$, exists such that, on setting $G(\epsilon, m) = C\Big(1 + \frac{1}{\epsilon \log k} + \frac{1}{\epsilon^2 (\log k)^2} + \dots + \frac{1}{\epsilon^{m-1} (\log k)^{m-1}}\Big)$, we have

$$|\nabla^m u_{\epsilon,k}(x)| \begin{cases} = 0 & \text{if} \quad 0 \le |x| < \frac{1}{k} \\ \le \dfrac{G(\epsilon, m)}{|x|^m \log k} & \text{if} \quad \frac{1}{k} \le |x| < \frac{1}{k^{1-\epsilon}} \\ = \dfrac{\beta_{n,m}}{n\,\omega_n^{m/n}} \dfrac{1}{|x|^m \log k} & \text{if} \quad \frac{1}{k^{1-\epsilon}} \le |x| < \frac{1}{k^{\epsilon}} \\ \le \dfrac{G(\epsilon, m)}{|x|^m \log k} & \text{if} \quad \frac{1}{k^{\epsilon}} \le |x| < 1 \end{cases} \tag{19}$$

for $x \in B$. Let $g_{\epsilon,k} : [0, +\infty) \to [0, +\infty)$ be the function defined as

$$g_{\epsilon,k}(s) = \begin{cases} \dfrac{G(\epsilon, m)\omega_n^{m/n}}{s^{m/n} \log k} & \text{if} \quad \frac{1}{k^n} < s < \frac{1}{k^{(1-\epsilon)n}} \\ \dfrac{\beta_{n,m}}{n\, s^{m/n} \log k} & \text{if} \quad \frac{1}{k^{(1-\epsilon)n}} \le s < \frac{1}{k^{\epsilon n}} \\ \dfrac{G(\epsilon, m)\omega_n^{m/n}}{s^{m/n} \log k} & \text{if} \quad \frac{1}{k^{\epsilon n}} \le s < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Eq. 19, $|\nabla^m u_{\epsilon,k}(x)| \le g_{\epsilon,k}(|x|^n)$, for $x \in B$, and hence $(\nabla^m u_{\epsilon,k})^*(s) \le g^*_{\epsilon,k}(s)$, for $s > 0$. If k is sufficiently large, one has

$$g^*_{\epsilon,k}(s) = \begin{cases} \dfrac{G(\epsilon, m)\,\omega_n^{m/n}}{(\log k)(s+k^{-n})^{m/n}} & \text{if} \quad 0 \le s < \frac{1}{k^{(1-\epsilon)n}} - \frac{1}{k^n} \\ \dfrac{\beta_{n,m}}{n\,(\log k)(s+k^{-n})^{m/n}} & \text{if} \quad \frac{1}{k^{(1-\epsilon)n}} - \frac{1}{k^n} \le s < \frac{1}{k^{\epsilon n}}\gamma_{\epsilon,n,m} - \frac{1}{k^n} \\ \left(\dfrac{(\omega_n^{-m/n}\, n^{-1}\, \beta_{n,m})^{n/m} + G(\epsilon, m)^{n/m}}{s + k^{-n} + k^{-\epsilon n}}\right)^{\frac{m}{n}} \dfrac{\omega_n^{m/n}}{\log k} & \text{if} \quad \frac{1}{k^{\epsilon n}}\gamma_{\epsilon,n,m} - \frac{1}{k^n} \le s < \frac{1}{k^{\epsilon n}}\gamma^{-1}_{\epsilon,n,m} - \frac{1}{k^n} \\ \dfrac{G(\epsilon, m)\,\omega_n^{m/n}}{(\log k)(s+k^{-n})^{m/n}} & \text{if} \quad \frac{1}{k^{\epsilon n}}\gamma^{-1}_{\epsilon,n,m} - \frac{1}{k^n} \le s < 1 - \frac{1}{k^n} \\ 0 & \text{if} \quad 1 - \frac{1}{k^n} \le s, \end{cases}$$

where $\gamma_{\epsilon,n,m} = \left(\dfrac{\omega_n^{-m/n}\, n^{-1}\, \beta_{n,m}}{G(\epsilon, m)}\right)^{n/m}$.

Fix $\epsilon \in (0,1)$. Given $\beta > 0$, we have

$$\begin{aligned}\int_\Omega \exp\left(\frac{\beta\,|u_{\epsilon,k}(x)|}{\||\nabla^m u_{\epsilon,k}|\|_{\frac{n}{m},q}}\right)^{q'} dx &\geq \int_{B\cap\{|x|<\frac{1}{k}\}} \exp\left(\frac{\beta\,|u_{\epsilon,k}(x)|}{\||\nabla^m u_{\epsilon,k}|\|_{\frac{n}{m},q}}\right)^{q'} dx\\ &\geq \int_{B\cap\{|x|<\frac{1}{k}\}} \exp\left(\frac{\beta\,|u_{\epsilon,k}(x)|}{\|g^*_{\epsilon,k}\|_{\frac{n}{m},q}}\right)^{q'} dx\\ &= \int_{B\cap\{|x|<\frac{1}{k}\}} \exp\left(\frac{\beta}{\|g^*_{\epsilon,k}\|_{\frac{n}{m},q}}\right)^{q'} dx\\ &= \frac{1}{k^n}\,\exp\left(\frac{\beta}{\|g^*_{\epsilon,k}\|_{\frac{n}{m},q}}\right)^{q'}\\ &= \exp\left[\left(\frac{\beta}{\|g^*_{\epsilon,k}\|_{\frac{n}{m},q}}\right)^{q'} - n\log k\right]\\ &= \exp\left[\left(-n + \frac{\beta^{q'}}{\|g^*_{\epsilon,k}\|^{q'}_{\frac{n}{m},q}}\,\frac{1}{\log k}\right)\log k\right].\end{aligned}$$

Hence, $\lim_{k\to+\infty}\int_\Omega \exp\left(\frac{\beta\,|u_{\epsilon,k}(x)|}{\||\nabla^m u_{\epsilon,k}|\|_{\frac{n}{m},q}}\right)^{q'} dx < +\infty$ only if

$$\lim_{k\to+\infty}\left(-n + \frac{\beta^{q'}}{\|g^*_{\epsilon,k}\|^{q'}_{\frac{n}{m},q}}\,\frac{1}{\log k}\right)\log k < +\infty. \tag{20}$$

Since

$$\begin{aligned}&\lim_{k\to+\infty}\log k\left(-n + \frac{\beta^{q'}}{\|g^*_{\epsilon,k}\|^{q'}_{\frac{n}{m},q}}\,\frac{1}{\log k}\right)\\ &= \lim_{k\to+\infty}\log\,k\Bigg\{-n + \frac{\beta^{q'}\,\omega_n^{-q'm/n}}{\log k\,(\log k)^{-q'}}\Bigg[G(\epsilon,m)^q\int_0^{k^{-(1-\epsilon)n}-k^{-n}} \frac{s^{\frac{mq}{n}-1}}{(s+k^{-n})^{\frac{mq}{n}}}\,ds\\ &\qquad + \left(\frac{\beta_{n,m}}{n\omega_n^{m/n}}\right)^q \int_{k^{-(1-\epsilon)n}-k^{-n}}^{k^{-\epsilon n}\gamma_{\epsilon,n,m}-k^{-n}} \frac{s^{\frac{mq}{n}-1}}{(s+k^{-n})^{\frac{mq}{n}}}\,ds\\ &\qquad + \left((n^{-1}\omega_n^{-m/n}\beta_{n,m}\right)^{n/m}\\ &\qquad + G(\epsilon,m)^{n/m}\Big)^q \int_{k^{-\epsilon n}\gamma_{\epsilon,n,m}-k^{-n}}^{k^{-\epsilon n}\gamma^{-1}_{\epsilon,n,m}-k^{-n}} \frac{s^{\frac{mq}{n}-1}}{(s+k^{-n}+k^{-\epsilon n})^{\frac{mq}{n}}}\,ds\\ &\qquad + G(\epsilon,n)^q \int_{k^{-\epsilon n}\gamma^{-1}_{\epsilon,n,m}-k^{-n}}^{1-k^{-n}} \frac{s^{\frac{mq}{n}-1}}{(s+k^{-n})^{\frac{mq}{n}}}\,ds\Bigg]^{-\frac{q'}{q}}\Bigg\}\end{aligned}$$

$$= \lim_{k\to+\infty} \log k \Bigg\{ -n + \frac{\beta^{q'}}{\omega_n^{q'm/n}} \Bigg[\frac{G(\epsilon,m)^q}{\log \mathrm{k}} \int_0^{k^{\epsilon n}-1} \frac{t^{\frac{mq}{n}-1}}{(t+1)^{\frac{mq}{n}}} dt$$

$$+ \left(\frac{\beta_{n,m}}{n\omega_n^{m/n}}\right)^q \frac{1}{\log k} \int_{k^{\epsilon n-1}}^{k^{(1-\epsilon)}\gamma_{\epsilon,n,m}-1} \frac{t^{\frac{mq}{n}-1}}{(t+1)^{\frac{mq}{n}}} dt$$

$$+ \frac{\left((n^{-1}\omega_n^{-m/n}\beta_{n,m})^{n/m} + G(\epsilon,m)^{n/m}\right)^q}{\log k}$$

$$\times \int_{\frac{k^{-\epsilon n}\gamma_{\epsilon,n,m}-k^{-n}}{(k^{-n}+k^{-\epsilon n})}}^{\frac{k^{-\epsilon n}\gamma_{\epsilon,n,m}^{-1}-k^{-n}}{(k^{-n}+k^{-\epsilon n})}} \frac{t^{\frac{mq}{n}-1}}{(t+1)^{\frac{mq}{n}}} dt$$

$$+ \frac{G(\epsilon,m)^q}{\log k} \int_{k^{(1-\epsilon)n}\gamma_{\epsilon,n,m}-1}^{k^n-1} \frac{t^{\frac{mq}{n}-1}}{(t+1)^{\frac{mq}{n}}} dt \Bigg]^{-1/(q-1)} \Bigg\}$$

$$= \lim_{k\to+\infty} \log k \left\{ -n + \frac{\beta^{q'}}{\omega_n^{q'm/n}} \left[\epsilon n G(\epsilon,m)^q + \left(\frac{\beta_{n,m}}{n\omega_n^{m/n}}\right)^q n(1-2\epsilon) + \epsilon n G(\epsilon,m)^q \right]^{-1/(q-1)} \right\},$$

Eq. 20 holds if and only if

$$-n + \frac{\beta^{q\prime}}{\omega_n^{q'm/n}} \left[2\epsilon n G(\epsilon,m)^q + \left(\frac{\beta_{n,m}}{n\omega_n^{m/n}}\right)^q n(1-2\epsilon) \right]^{-1/(q-1)} \le 0.$$

This condition is in turn equivalent to

$$\beta \le n^{\frac{1}{q'}} \omega_n^{\frac{m}{n}} \left[2n\epsilon G(\epsilon,m)^q + \left(\frac{\beta_{n,m}}{n\omega_n^{m/n}}\right)^q n(1-2\epsilon) \right]^{\frac{1}{(q-1)q'}}. \qquad (21)$$

Thanks to the arbitrarinees of ϵ, the conclusion follows.

Part (ii) Since we are assuming that $\| |\nabla^m u| \|_{L^{\frac{n}{m},\infty}(\Omega)} \le 1$,

$$(\nabla^m u)^*(t) \le t^{-\frac{m}{n}} \qquad \text{for } t \in (0, |\Omega|). \qquad (22)$$

From Eqs. 15 and 22, we infer that

$$u^*(t) \le \frac{1}{\beta_{n,m}} \left[\frac{n}{m} t^{\frac{m}{n}-1} \int_0^t |\nabla^m u|^*(s)\, ds + \int_t^{+\infty} s^{-\frac{n-m}{n}} |\nabla^m u|^*(s)\, ds \right]$$

$$\le \frac{1}{\beta_{n,m}} \left[\frac{n}{m} t^{\frac{m}{n}-1} \int_0^t s^{-\frac{m}{n}}\, ds + \int_t^{|\Omega|} s^{-1}\, ds \right]$$

$$= \frac{1}{\beta_{n,m}} \left[\frac{n^2}{m(n-m)} + \log \frac{|\Omega|}{t} \right] \qquad \text{for } t \in (0, \Omega).$$

Thus a constant $C = C(n, m)$ exists such that

$$\int_{\Omega} \exp\left(\gamma |u(x)|\right) dx = \int_0^{|\Omega|} \exp\left(\gamma u^*(t)\right) dt$$
$$\leq \int_0^{|\Omega|} \exp\left(\gamma \left[C + \frac{1}{\beta_{n,m}} \log \frac{|\Omega|}{t}\right]\right) dt < +\infty$$

for every $\gamma < \beta_{n,m}$.
We conclude by exhibiting a function $u \in W_0^m L^{\frac{n}{m},\infty}(\Omega)$ such that

$$\int_{\Omega} \exp\left(\frac{\gamma |u(x)|}{\| |\nabla^m u| \|_{L^{\frac{n}{m},\infty}(\Omega)}}\right) dx = +\infty \tag{23}$$

for every $\gamma > \beta_{n,m}$.
As in the proof of *part* (i), we assume that $B \subset\subset \Omega$. Let us consider an increasing smooth function (of class $\mathcal{C}^m$, say) $\varphi : \mathbb{R} \to [0, +\infty)$ such that $\varphi(t) = 0$, for $t \leq 0$ and $\varphi(t) = t - 1/2$, for $t \geq 1$. Let $a > 1$ and define the function $u_a : \Omega \to \mathbb{R}$ as

$$u_a(x) = \begin{cases} \varphi\left(\dfrac{\log \frac{1}{|x|}}{\log a}\right) & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Observe that $|\varphi'(t)| \leq 1$ for every $t \in \mathbb{R}$ and that the h-th order derivative $\varphi^{(h)}(t)$ is bounded for every $t \in \mathbb{R}$ and for every $h \in \mathbb{N}$. Furthermore, $u_a \in \mathcal{C}_0^m(\Omega)$.
Calculations show that, for every $h \in \mathbb{N}$,

$$\nabla^m u_a(x) = \begin{cases} \displaystyle\sum_{i=1}^{2h} c_i \varphi^{(i)}\left(\frac{\log \frac{1}{|x|}}{\log a}\right) \frac{1}{|x|^{2h} (\log a)^i} & \text{if } m = 2h \\ \displaystyle\sum_{i=1}^{2h+1} c_i \varphi^{(i)}\left(\frac{\log \frac{1}{|x|}}{\log a}\right) \frac{1}{|x|^{2h+1} (\log a)^i} & \text{if } m = 2h+1 \end{cases} \tag{25}$$

for $x \in B$, and $\nabla^m u_a(x) = 0$ otherwise. Here $c_1 = s(m) \dfrac{\beta_{n,m}}{n \omega_n^{m/n}}$, where $s(m) = (-1)^{m/2}$, for m even, and $s(m) = (-1)^{(m+2)/2}$, for m odd, and c_i, $i = 2, ..., m$, are constants depending only on n and m. Owing to Eq. 25, one has

$$|\nabla^m u_a|(x) \leq \frac{\beta_{n,m}}{n \omega_n^{m/n}} \frac{1}{|x|^m \log a} \left(1 + \frac{K}{\log a} + ... + \frac{K}{(\log a)^{m-1}}\right)$$

for some constant K depending on ϕ.
Fix any $\gamma > \beta_{n,m}$, choice $\epsilon > 0$ such that

$$\gamma > \beta_{n,m} (1 + \epsilon). \tag{26}$$

If a is sufficiently large, we have

$$|\nabla^m u_a|(x) \leq \frac{\beta_{n,m}}{n\, \omega_n^{m/n}} \frac{1}{|x|^m \, \log a} (1 + \epsilon).$$

Hence,

$$|||\nabla^m u_a|||_{L^{\frac{n}{m},\infty}(\Omega)} \leq \frac{\beta_{n,m}\,(1+\epsilon)}{n \log a}.$$

Equation 23 follows since

$$\begin{aligned}
&\int_\Omega \exp\left(\gamma \frac{|u_a(x)|}{|||\nabla^m u_a|||_{\frac{n}{m},\infty}}\right) dx \\
&\geq \int_B \exp\left(\gamma \frac{\varphi\left(\frac{\log(|x|^{-1})}{\log a}\right) n \log a}{\beta_{n,m}\,(1+\epsilon)}\right) dx \\
&\geq \int_{\{x\in B:|x|\leq 1/a\}} \exp\left(\gamma \frac{\left(\frac{\log(|x|^{-1})}{\log a} - \frac{1}{2}\right) n \log a}{\beta_{n,m}\,(1+\epsilon)}\right) dx \\
&= \omega_n \int_0^{\frac{1}{a}} \exp\left(\gamma \frac{n \log\left(r^{-1}\right)}{\beta_{n,m}\,(1+\epsilon)}\right) \exp\left(\gamma \frac{n \log a}{2\,\beta_{n,m}\,(1+\epsilon)}\right) r^{n-1}\, dr \\
&= \omega_n \exp\left(\gamma \frac{n \log a}{2\,\beta_{n,m}\,(1+\epsilon)}\right) \int_0^{\frac{1}{a}} r^{-\gamma \frac{n}{\beta_{n,m}\,(1+\epsilon)}+n-1}\, dr = +\infty,
\end{aligned}$$

by Eqs. 24 and 26. □

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