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Self-Similar Stable Processes Arising from High-Density Limits of Occupation Times of Particle Systems

Tomasz Bojdecki · Luis G. Gorostiza · Anna Talarczyk

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Abstract We extend results on time-rescaled occupation time fluctuation limits of the (d, α, β)-branching particle system $(0 < \alpha \leq 2, 0 < \beta \leq 1)$ with Poisson initial condition. The earlier results in the homogeneous case (i.e., with Lebesgue initial intensity measure) were obtained for dimensions $d > \alpha/\beta$ only, since the particle system becomes locally extinct if $d \leq \alpha/\beta$. In this paper we show that by introducing high density of the initial Poisson configuration, limits are obtained for all dimensions, and they coincide with the previous ones if $d > \alpha/\beta$. We also give high-density limits for the systems with finite intensity measures (without high density no limits exist in this case due to extinction); the results are different and harder to obtain due to the non-invariance of the measure for the particle motion. In both cases, i.e., Lebesgue and finite intensity measures, for low dimensions [$d < \alpha(1+\beta)/\beta$ and $d < \alpha(2+\beta)/(1+\beta)$, respectively] the limits are determined by non-Lévy self-similar stable processes. For the corresponding high dimensions the limits are qualitatively different: $\mathcal{S}'(\mathbb{R}^d)$-valued Lévy processes in the Lebesgue case, stable processes constant in time on $(0, \infty)$ in the finite measure case. For high dimensions, the laws of all limit processes are expressed in terms of Riesz potentials. If $\beta = 1$, the limits are Gaussian. Limits are also given for particle systems without branching, which yields

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T. Bojdecki · A. Talarczyk (✉)
Institute of Mathematics, University of Warsaw, ul. Banacha 2,
02-097 Warszawa, Poland
e-mail: annatal@mimuw.edu.pl

T. Bojdecki
e-mail: tobojd@mimuw.edu.pl

L. G. Gorostiza
Centro de Investigación y de Estudios Avanzados,
A.P. 14-740, Mexico 07000 D.F., Mexico
e-mail: lgorosti@math.cinvestav.mx

in particular weighted fractional Brownian motions in low dimensions. The results are obtained in the setup of weak convergence of $\mathcal{S}'(\mathbb{R}^d)$-valued processes.

Keywords Self-similar stable process · Long-range dependence · Branching particle system · Occupation time · Functional limit theorem · $\mathcal{S}'(\mathbb{R}^d)$-valued process

Mathematics Subject Classifications (2000) 60G52 · 60G18 · 60F17 · 60J80

1 Introduction

In order to explain the motivations for this paper, we refer briefly to previous results on occupation times of the (d, α, β)-branching particle system, which has been widely studied, and is described as follows. At time $t = 0$ particles are distributed in $\mathbb{R}^d$ according to a Poisson random measure, and then they evolve moving and branching independently of each other. The motion is given by the symmetric α-stable Lévy process, $0 < \alpha \leq 2$ (called standard α-stable process), the lifetime is exponentially distributed with parameter V, and the branching law has generating function

$$s + \frac{1}{(1+\beta)}(1-s)^{1+\beta}, \quad 0 < s < 1, \tag{1.1}$$

where $0 < \beta \leq 1$. This law is critical and belongs to the domain of attraction of a stable law with exponent $1 + \beta$. The case $\beta = 1$ corresponds to binary branching (0 or 2 particles). This is the simplest in a class of branching particle systems that yield essentially the same results. We also consider the system without branching ($V = 0$).

If the initial particle configuration is given by a homogeneous Poisson random measure, i.e., whose intensity is the Lebesgue measure λ, then the system without branching is in equilibrium, the branching system converges towards a non-trivial equilibrium state as time tends to infinity for $d > \alpha/\beta$, and it becomes locally extinct in probability for $d \leq \alpha/\beta$ [18].

Let $(N_t)_{t\geq 0}$ denote the empirical measure process of the system (with or without branching), i.e., $N_t(A)$ is the number of particles in the set $A \subset \mathbb{R}^d$ at time t. The rescaled occupation time fluctuation process with accelerated time is defined by

$$X_T(t) = \frac{1}{F_T}\int_0^{Tt}(N_s - \mathrm{E}N_s)ds, \quad t \geq 0, \tag{1.2}$$

where F_T is a suitable norming for convergence as $T \to \infty$. Note that if λ is the intensity of the initial Poisson configuration, then $\mathrm{E}N_t = \lambda$ for all t due to the invariance of λ for the standard α-stable process and the criticality of the branching (or no branching).

With homogeneous Poisson initial condition, functional limit theorems for the process X_T in the branching case were obtained in [5, 6] for $\beta = 1$, where the limit processes are Gaussian, and in [7, 8] for $\beta < 1$, with $(1+\beta)$-stable limit processes. The limits are dimension-dependent, their main qualitative properties being that for the intermediate dimensions, $\alpha/\beta < d < \alpha(1+\beta)/\beta$, the process has long-range dependence, while for the critical and high dimensions, $d = \alpha(1+\beta)/\beta$ and $d > \alpha(1+\beta)/\beta$, respectively, the processes have independent increments. For high dimensions the limits are $\mathcal{S}'(\mathbb{R}^d)$-valued $\big(\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions, the dual of $\mathcal{S}(\mathbb{R}^d)$, the space of smooth rapidly decreasing functions$\big)$, and their laws are expressed in terms of Riesz potentials. There is a functional ergodic theorem for $d = \alpha/\beta$ [25]. For intermediate dimensions the limit has the form $X = K\lambda\xi$, where K is a constant, and $(\xi_t)_{t\geq 0}$ is a real non-Lévy self-similar $(1+\beta)$-stable process, which for $\beta = 1$ is a *sub-fractional Brownian motion*, whose properties are described in [4].

Other papers related to occupation times of branching particle systems and superprocesses are [2, 12, 14–17, 19–23, 26]. Birkner and Zähle [2] consider occupation time limits for branching random walks on d-dimensional lattices.

The first motivation for this paper comes from the fact that in the homogeneous case with $\beta = 1$ and $d < \alpha$, the covariance of the process X_T has a non-trivial limit as $T \to \infty$, which corresponds to a process X of the same form as above, with a different Gaussian process instead of sub-fractional Brownian motion, but X is not the limit of X_T because, as recalled above, the particle system becomes locally extinct if $d < \alpha$. Therefore the question arises if it is possible to give a probabilistic meaning (related with the particle system) to the process X, by taking a different type of limit. Our objective is to show that this can be achieved by letting the density of the initial Poisson configuration tend to infinity in a suitable way as $T \to \infty$. We will prove a limit theorem for the process X_T for low dimensions, $d < \alpha(1+\beta)/\beta$ (which includes the old intermediate dimensions), and obtain results for the critical and high dimensions as well, by taking an initial Poisson configuration with intensity measure $H_T\lambda$, where $H_T \to \infty$ as $T \to \infty$ (and new normings F_T). It turns out that the limits coincide with the known ones in the cases where the latter exist, i.e., for $d > \alpha/\beta$, and they are new processes for $d \leq \alpha/\beta$, which are also of the form $X = K\lambda\xi$. For $\beta < 1$ and $d < \alpha/\beta$, ξ is an extension of a non-Lévy $(1+\beta)$-stable process obtained in [7] for intermediate dimensions (the process in [7] has the interesting property that it has two different long-range dependence regimes). For $\beta = 1$ and $d < \alpha$, ξ is a *negative sub-fractional Brownian motion*, which is a real centered Gaussian process with covariance

$$E\xi_s\xi_t = \frac{1}{2}[(s+t)^h + |s-t|^h] - s^h - t^h, \quad s, t \geq 0, \tag{1.3}$$

where $h = 3 - d/\alpha$. For $\beta = 1$ and $d = \alpha$, ξ is a centered Gaussian process with covariance

$$E\xi_s\xi_t = \frac{1}{2}\left[(s+t)^2 \log(s+t) + (s-t)^2 \log|s-t|\right] - s^2 \log s - t^2 \log t, \quad s, t \geq 0. \tag{1.4}$$

Some properties of these processes are studied in [10], independently of their origin in particle systems.

Thus, the high-density limits extend the ranges of the parameters of the branching particle system for convergence of X_T obtained in [5–8] without high density, so that all cases are now covered, including dimensions d below and at the extinction border α/β .

For completeness, we will also include high-density limits for the system without branching, but there are no novelties in the sense that the limits coincide with those for the homogeneous Poisson case without high density.

The second motivation is the question of what happens with the occupation times of the particle systems if the initial Poisson configuration has finite intensity measure. In this case the branching system becomes extinct a.s., while the non-branching system becomes locally extinct a.s. if $d > \alpha$, and if $d \leq \alpha$, then $(1/F_T)\int_0^T E\langle N_s, \varphi\rangle ds$ converges to a finite limit for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $(1/F_T)\int_0^{Tt} N_s ds$ has a non-trivial limit in law (see [9], the latter result is akin to the Darling–Kac occupation time theorem [13]). For these reasons it does not make sense to study asymptotic occupation time fluctuations. We will show that high density of the initial Poisson condition can be used to compensate extinction and obtain non-trivial limits for X_T. We will consider an initial Poisson configuration with intensity measure $H_T\mu$, where μ is a finite measure and $H_T \to \infty$ as $T \to \infty$. This yields results for the occupation time fluctuations of the branching and the non-branching systems, with new types of limits. These results are different, and significantly more difficult to obtain than the previous ones, because the Poisson intensity measure is not invariant for the standard α-stable process (if the intensity measure is μ, then $\mathrm{E}N_t = \mu\mathcal{T}_t$, where $\mathcal{T}_t$ is the semigroup of the standard α-stable process).

For the branching system with finite measure μ, the low, critical and high dimensions are $d < \alpha(2+\beta)/(1+\beta)$, $d = \alpha(2+\beta)/(1+\beta)$, and $d > \alpha(2+\beta)/(1+\beta)$, respectively. In the first two cases the limit processes are of the form $K\lambda\xi$. For low dimensions, ξ is a non-Lévy $(1+\beta)$-stable process, which is different from the one obtained in the homogeneous case. For the critical dimension, ξ is a process constant in time on $(0, \infty)$, given by a $(1+\beta)$-stable random variable. In these two cases the measure μ figures only through its total mass, which appears as a constant. For the high dimensions the limit is a process constant in time on $(0, \infty)$, given by an $\mathcal{S}'(\mathbb{R}^d)$-valued $(1+\beta)$-stable random variable whose law is expressed by means of a Riesz potential. In this case μ has a non-trivial effect on the spatial distribution of the limit process. So, in addition to the critical borders being different for Lebesgue and finite measures, the limit processes are qualitatively different for the two cases in the corresponding critical and high dimensions.

For the non-branching system, the low, critical and high dimensions for the high-density limits with finite measure are $d < \alpha$, $d = \alpha$ and $d > \alpha$, respectively. For $d < \alpha$ the limit has the form $K\lambda\rho$, where ρ is a special case of a *weighted fractional Brownian motion* studied in [10], i.e., centered Gaussian with covariance

$$E\rho_s\rho_t = \int_0^{s\wedge t} u^{-d/\alpha}[(t-u)^{1-d/\alpha} + (s-u)^{1-d/\alpha}]du, \quad s, t \geq 0. \tag{1.5}$$

For $d = \alpha$ and $d > \alpha$, the limits are constant in time on $(0, \infty)$, analogously to the branching case in the corresponding critical and high dimensions. They are Gaussian with covariances expressed by means of Riesz potentials.

The proofs in this paper are analogous to those in [5–9], but there are new complexities that require a more comprehensive approach. We will explain the general scheme at the beginning of the proofs, but we stress that its implementation in specific cases is not at all straightforward, and it becomes quite cumbersome technically in the case of finite measure. We will refer often to our previous papers (specially [7]) for some technical points, in order to shorten the length of this article, and the main parts of the proofs given here are devoted to arguments that involve something new. The general setting is weak convergence of $\mathcal{S}'(\mathbb{R}^d)$-valued processes, which covers the cases where the limit process is measure-valued and those where it is "truly" $\mathcal{S}'(\mathbb{R}^d)$-valued.

We will use the following notions of weak convergence of $\mathcal{S}'(\mathbb{R}^d)$-valued processes (recall that $\mathcal{S}(\mathbb{R}^d)$ denotes the space of smooth rapidly decreasing functions, $\mathcal{S}'(\mathbb{R}^d)$, the dual of $\mathcal{S}(\mathbb{R}^d)$, is the space of tempered distributions, and $\langle \cdot, \cdot \rangle$ stands for duality pairing):

$\Rightarrow_C$ is the convergence in law in $C([0,\tau], \mathcal{S}'(\mathbb{R}^d))$ for each $\tau > 0$;
$\Rightarrow_{C,\varepsilon}$ is the convergence in law in $C([\varepsilon,\tau], \mathcal{S}'(\mathbb{R}^d))$ for each $0 < \varepsilon < \tau$;
$\Rightarrow_f$ is the convergence of finite-dimensional distributions;
$\Rightarrow_i$ is the convergence in the integral sense, i.e., $X_T \Rightarrow_i X$ as $T \to \infty$ if, for any $\tau > 0$, the $\mathcal{S}'(\mathbb{R}^{d+1})$-random variables $\widetilde{X}_T$ converge in law to $\widetilde{X}$, where $\widetilde{X}$ (and, analogously, $\widetilde{X}_T$) is defined by

$$\langle \widetilde{X}, \Phi \rangle = \int_0^\tau \langle X(t), \Phi(\cdot, t) \rangle dt, \quad \Phi \in \mathcal{S}\left(\mathbb{R}^{d+1}\right). \tag{1.6}$$

We denote generic constants by $C, C_1, C_2, \ldots,$ with possible dependencies in parenthesis.

2 Results

Before stating the results we introduce two $(1+\beta)$-stable processes which appear in the theorems below ($0 < \beta \le 1$ is fixed).

Let M be the independently scattered $(1+\beta)$-stable measure on $\mathbb{R}^{d+1}$ with control measure λ_{d+1} (Lebesgue measure) and skewness intensity 1, i.e., for each $A \in \mathcal{B}(\mathbb{R}^{d+1})$ such that $0 < \lambda_{d+1}(A) < \infty$, $M(A)$ is a $(1+\beta)$-stable random variable with characteristic function

$$\exp\left\{-\lambda_{d+1}(A)|z|^{1+\beta}\left(1 - i(\operatorname{sgn} z)\tan\frac{\pi}{2}(1+\beta)\right)\right\}, \quad z \in \mathbb{R},$$

the values of M are independent on disjoint sets, and M is σ-additive a.s. (see [24], Definition 3.3.1).

Let $p_t(x)$ denote the transition density of the standard α-stable process in $\mathbb{R}^d$.

We define the following processes:

$$\xi_t = \int_{\mathbb{R}^{d+1}} \left(1\!\!1_{[0,t]}(r) \int_r^t p_{u-r}(x) du \right) M(drdx), \quad t \ge 0, \tag{2.1}$$

$$\zeta_t = \int_{\mathbb{R}^{d+1}} \left(1\!\!1_{[0,t]}(r) p_r^{1/(1+\beta)}(x) \int_r^t p_{u-r}(x) du \right) M(drdx), \quad t \ge 0, \tag{2.2}$$

where the integral with respect to M is understood in the sense of [24] (3.2–3.4).

Proposition 2.1 *The process ξ is well defined if $d < \alpha(1+\beta)/\beta$, and the process ζ is well defined if $d < \alpha(2+\beta)/(1+\beta)$.*

The process ξ is an extension of the one studied in [6].

We denote by $\mathcal{T}_t$ the semigroup of the standard α-stable process, i.e., $\mathcal{T}_t\varphi = p_t * \varphi$. For $d > \alpha$, we denote by G the potential operator

$$G\varphi(x) = \int_0^\infty \mathcal{T}_t\varphi(x)dt = C_{\alpha,d}\int_{\mathbb{R}^d} \frac{\varphi(y)}{|x-y|^{d-\alpha}}dy, \tag{2.3}$$

where

$$C_{\alpha,d} = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2}\Gamma(\frac{\alpha}{2})}. \tag{2.4}$$

We start with the high-density branching system described in the Introduction, where the intensity measure of the initial Poisson configuration is $H_T\lambda$.

Theorem 2.2 *Consider the (d, α, β)-branching particle system with branching mechanism (1.1) and initial intensity $H_T\lambda$, $H_T \to \infty$. Let X_T be defined by* Eq. 1.2.

(a) *Assume*

$$d < \frac{\alpha(1+\beta)}{\beta}. \tag{2.5}$$

Let H_T be such that

$$\lim_{T\to\infty} H_T^{-\beta} T^{1-d\beta/\alpha} = 0, \tag{2.6}$$

and

$$F_T^{1+\beta} = H_T T^{2+\beta-d\beta/\alpha}. \tag{2.7}$$

Then $X_T \Rightarrow_C K\lambda\xi$ as $T \to \infty$, where ξ is defined by Eq. 2.1 and

$$K = \left(-\frac{V}{1+\beta}\cos\frac{\pi}{2}(1+\beta)\right)^{1/(1+\beta)}. \tag{2.8}$$

(b) *Assume $d = \alpha(1+\beta)/\beta$ and $F_T^{1+\beta} = H_T T\log T$. Then $X_T \Rightarrow_i K_1\lambda\eta$ and $X_T \Rightarrow_f K_1\lambda\eta$ as $T \to \infty$, where η is a real $(1+\beta)$-stable process with stationary independent increments whose distribution is determined by*

$$E\exp\{iz\eta_t\} = \exp\left\{-t|z|^{1+\beta}\left(1 - i(\mathrm{sgn} z)\tan\frac{\pi}{2}(1+\beta)\right)\right\}, \quad z \in \mathbb{R}, \quad t \geq 0, \tag{2.9}$$

and

$$K_1 = \left(-V\int_{\mathbb{R}^d}\left(\int_0^1 p_r(x)dr\right)^\beta p_1(x)dx\cos\frac{\pi}{2}(1+\beta)\right)^{1/(1+\beta)}.$$

Moreover, if $\beta = 1$, the convergence holds in the sense $\Rightarrow_C$.

(c) *Assume* $d > \alpha(1+\beta)/\beta$ *and* $F_T^{1+\beta} = H_T T$.

(1) *If* $0 < \beta < 1$, *then* $X_T \Rightarrow_i X$ *and* $X_T \Rightarrow_f X$ *as* $T \to \infty$, *where* X *is an* $\mathcal{S}'(\mathbb{R}^d)$-*valued* $(1+\beta)$-*stable process with stationary independent increments whose distribution is determined by*

$$E\exp\{i\langle X(t),\varphi\rangle\}$$
$$= \exp\left\{-K^{1+\beta}t\int_{\mathbb{R}^d}|G\varphi(x)|^{1+\beta}\left(1-i(\operatorname{sgn}G\varphi(x))\tan\frac{\pi}{2}(1+\beta)\right)dx\right\},$$
$$\varphi\in\mathcal{S}(\mathbb{R}^d),\quad t\geq 0,$$

K is given by Eq. 2.8 *and G by Eq.* 2.3.

(2) *If* $\beta = 1$, *then* $X_T \Rightarrow_C W$ *as* $T \to \infty$, *where* W *is an* $\mathcal{S}'(\mathbb{R}^d)$-*valued Wiener process with covariance*

$$E(\langle W(s),\varphi_1\rangle\langle W(t),\varphi_2\rangle) = (s\wedge t)\int_{\mathbb{R}^d}\left[V(G\varphi_1(x))(G\varphi_2(x)) + 2\varphi_1(x)G\varphi_2(x)\right]dx,$$
$$\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^d),\quad s,t\geq 0.$$

Remark 2.3

(a) For $d > \alpha/\beta$, the limits in Theorem 2.2 are exactly the same as in the model without high density [5–8]. Thus, if the limits without high density exist, then increasing the initial density of particles does not change the results.

(b) Observe that assumption 2.6 is a restriction only if $d < \alpha/\beta$.

(c) If $d \leq \alpha/\beta$, then the limit processes are extensions of those studied before [5–8] in the sense that the ranges of the parameters are increased.
In [7] we discussed some basic properties of ξ defined by Eq. 2.1 for $\alpha/\beta < d < \alpha(1+\beta)/\beta$. It turns out that ξ has the same properties also for the full ranges of parameters. We collect them in the following proposition.

Proposition 2.4 *Assume Eq.* 2.5.

(a) ξ *is* $(1+\beta)$-*stable, totally skewed to the right if* $\beta < 1$.

(b) ξ *is self-similar with index* $b = (2+\beta-d\beta/\alpha)/(1+\beta)$, *i.e.,*

$$(\xi_{at_1},\ldots,\xi_{at_k}) \stackrel{d}{=} a^b(\xi_{t_1},\ldots,\xi_{t_k}),\quad a>0.$$

(c) ξ *has continuous paths.*

(d) ξ *has the long-range dependence property with dependence exponent*

$$\kappa = \begin{cases} \dfrac{d}{\alpha} & \text{if either } \alpha = 2, \text{ or } \alpha < 2 \text{ and } \beta > \dfrac{d}{d+\alpha}, \\[2ex] \dfrac{d}{\alpha}\left(1+\beta-\dfrac{d}{d+\alpha}\right) & \text{if } \qquad \alpha < 2 \text{ and } \beta \leq \dfrac{d}{d+\alpha}. \end{cases} \tag{2.10}$$

All these properties are obtained the same way as in [7]. Property (a) follows from the definition, (b) and (c) are consequences of Theorem 2.2, and (d) can be obtained exactly as in Theorem 2.7 in [7]. Recall that the *dependence exponent* of ξ is defined by

$$\kappa = \inf_{z_1 z_2 \in \mathbb{R}} \inf_{0 \le u < v < s < t} \sup\{\gamma > 0 : D_T(z_1, z_2; u, v, s, t) = o(T^{-\gamma}) \quad \text{as} \quad T \to \infty\}, \tag{2.11}$$

where

$$\begin{aligned} &D_T(z_1, z_2; u, v, s, t) \\ &\quad = |\log E e^{i(z_1(\xi_v - \xi_u) + z_2(\xi_{T+t} - \xi_{T+s})} - \log E e^{i z_1(\xi_v - \xi_u)} - \log E e^{i z_2(\xi_{T+t} - \xi_{T+s})}|, \end{aligned} \tag{2.12}$$

see Definition 2.5 in [7].

The process ξ can be described more explicitly in the case $\beta = 1$.

Proposition 2.5 *If $\beta = 1$ and $d < 2\alpha$, then ξ is a centered Gaussian process with covariance*

$$E\xi_s\xi_t = \begin{cases} \dfrac{p_1(0)}{(1-\frac{d}{\alpha})(2-\frac{d}{\alpha})(3-\frac{d}{\alpha})}\left(\dfrac{1}{2}[(s+t)^{3-d/\alpha} + |s-t|^{3-d/\alpha}] - s^{3-d/\alpha} - t^{3-d/\alpha}\right) & \text{if } d \ne \alpha, \quad (2.13) \\ \dfrac{p_1(0)}{2}\left(\dfrac{1}{2}[(s+t)^2\log(s+t) + (s-t)^2\log|s-t|] - s^2\log s - t^2\log t\right) & \text{if } d = \alpha, \quad (2.14) \end{cases}$$

$s, t \ge 0$.

The Gaussian process ξ with covariance (2.13) is (up to a multiplicative constant) a sub-fractional Brownian motion if $\alpha < d < 2\alpha$, and a negative sub-fractional Brownian motion if $d < \alpha$. These processes are studied in [4] and [10], respectively. The latter paper also contains a proof of the non-semimartingale property of the process with covariance (2.14).

Next we consider the system without branching. In this case it is known that if the initial intensity measure is λ, then the limit of X_T exists for all dimensions [5, 6]. The observation in Remark 2.3 (a) also applies here, i.e., introducing high density of the initial configuration does not have any effect on the results. For completeness we give the corresponding theorem.

Theorem 2.6 *Let X_T be defined by Eq. 1.2 for a system without branching with initial intensity $H_T\lambda$, $H_T \to \infty$.*

(a) *If $d < \alpha$ and $F_T = H_T^{1/2} T^{1-d/2\alpha}$, then $X_T \Rightarrow_C K\lambda\vartheta$ as $T \to \infty$, where ϑ is a fractional Brownian motion with Hurst parameter $1 - d/2\alpha$, i.e., a centered Gaussian process with covariance*

$$E\vartheta_s\vartheta_t = \frac{1}{2}(s^{2-d/\alpha} + t^{2-d/\alpha} - |s-t|^{2-d/\alpha}), \;\; s, t \ge 0,$$

and

$$K = \left(\frac{2\Gamma(d/\alpha)}{\pi\alpha(2-d/\alpha)(1-d/\alpha)}\right)^{1/2}.$$

(b) *If* $d=\alpha$ *and* $F_T=(H_T T\log T)^{1/2}$, *then* $X_T\Rightarrow_C K_1\lambda\vartheta^{(1)}$ *as* $T\to\infty$, *where* $\vartheta^{(1)}$ *is a standard Brownian motion and*

$$K_1=(2^{d-2}\pi^{d/2}d\Gamma(d/2))^{-1/2}.$$

(c) *If* $d>\alpha$ *and* $F_T=(H_T T)^{1/2}$, *then* $X_T\Rightarrow_C W^{(0)}$ *as* $T\to\infty$, *where* $W^{(0)}$ *is an* $\mathcal{S}'(\mathbb{R}^d)$*-valued Wiener process with covariance*

$$E(\langle W^{(0)}(s),\varphi_1\rangle\langle W^{(0)}(t),\varphi_2\rangle)=(s\wedge t)2\int_{\mathbb{R}^d}\varphi_1(x)G\varphi_2(x)dx,\ \ s,t\geq 0,$$

where G *is given by* (2.3).

An analysis of the proofs of Theorems 2.1 in [5] and [6] shows that the same argument can be employed in the present case, therefore we omit the proof of Theorem 2.6.

We now pass to the branching system with finite initial intensity measure.

Theorem 2.7 *Consider the* (d,α,β)*-branching particle system with initial Poisson intensity* $H_T\mu$, *where* μ *is a finite measure and* $H_T\to\infty$. *Let* X_T *be defined by Eq.* 1.2.

(a) *Assume*

$$d<\frac{\alpha(2+\beta)}{1+\beta}. \tag{2.15}$$

Let H_T *be such that*

$$\lim_{T\to\infty} H_T^{-\beta}T=0, \tag{2.16}$$

and

$$F_T^{1+\beta}=H_T T^{2+\beta-(d/\alpha)(1+\beta)}. \tag{2.17}$$

Then $X_T\Rightarrow_C K\lambda\zeta$ *as* $T\to\infty$, *where* ζ *is defined by Eq.* 2.2 *and*

$$K=\left(-\frac{V}{1+\beta}\mu\left(\mathbb{R}^d\right)\cos\frac{\pi}{2}(1+\beta)\right)^{1/(1+\beta)}.$$

(b) *Assume*

$$d=\frac{\alpha(2+\beta)}{(1+\beta)}, \tag{2.18}$$

let H_T *satisfy Eq.* 2.16, *and*

$$F_T^{1+\beta}=H_T\log T. \tag{2.19}$$

Then $X_T\Rightarrow_{C,\varepsilon} K_1\lambda\eta_1$ *as* $T\to\infty$, *where* η_1 *is a* $(1+\beta)$*-stable random variable, totally skewed to the right* (*see Eq.* 2.9), *and*

$$K_1=C_{\alpha,d}\left(-\frac{V}{1+\beta}\mu\left(\mathbb{R}^d\right)\int_{\mathbb{R}^d}\frac{p_1(y)}{|y|^{(d-\alpha)(1+\beta)}}dy\cos\frac{\pi}{2}(1+\beta)\right)^{1/(1+\beta)},$$

where $C_{\alpha,d}$ *is given by Eq.* 2.4.

(c) *Assume*

$$d > \frac{\alpha(2+\beta)}{(1+\beta)}, \tag{2.20}$$

let H_T *satisfy Eq. 2.16 and*

$$F_T^{1+\beta} = H_T. \tag{2.21}$$

(1) *If* $0 < \beta < 1$, *then* $X_T \Rightarrow_{C,\varepsilon} X$ *as* $T \to \infty$, *where* X *is an* $\mathcal{S}'(\mathbb{R}^d)$*-valued random variable with characteristic function*

$$Ee^{i\langle X,\varphi\rangle} = \exp\left\{-\frac{V}{1+\beta}\int_{\mathbb{R}^d}|G\varphi(x)|^{1+\beta}\right.$$
$$\left.\left[1 - i(\mathrm{sgn}G\varphi(x))\tan\frac{\pi}{2}(1+\beta)\right]G\mu(dx)\cos\frac{\pi}{2}(1+\beta)\right\}, \tag{2.22}$$

where G *is given by* (2.3).

(2) *If* $\beta = 1$, *then* $X_T \Rightarrow_{C,\varepsilon} X$ *as* $T \to \infty$, *where* X *is a centered* $\mathcal{S}'(\mathbb{R}^d)$*-valued Gaussian random variable with covariance*

$$E(\langle X,\varphi_1\rangle\langle X,\varphi\rangle) = 2\int_{\mathbb{R}^d}\left[\varphi_1(x)G\varphi_2(x) + \frac{V}{2}(G\varphi_1(x))(G\varphi_2(x))\right]G\mu(dx). \tag{2.23}$$

Remark 2.8

(a) In parts (a) and (b) of Theorem 2.7 the dependence of the limit processes on μ is quite weak; $\mu(\mathbb{R}^d)$ appears only in constants. On the other hand, for high dimensions [part (c)] μ has a non-trivial effect on the spatial structure of the limit.

(b) The limit processes in parts (a) of Theorems 2.2 and 2.7 are similar, while parts (b) and (c) of these theorems (the time structures of the limits) are substantially different. Note also that for $\beta < 1$ in the present case the convergence is stronger ($\Rightarrow_{C,\varepsilon}$ instead of $\Rightarrow_i$ and $\Rightarrow_f$). On the other hand, it is clear that one cannot expect to have convergence on the whole interval [0, 1], since the limit process is discontinuous at 0.

(c) For large dimensions [part (c)], analogously to the case of the Lebesgue measure, the limit for $\beta = 1$ is not obtained from Eq. 2.22 by putting $\beta = 1$. An additional term appears in the covariance, related to the system without branching, due to slower growth of F_T (see Eq. 2.26 below).

In the next proposition we collect properties of the process ζ in Theorem 2.7 (a).

Proposition 2.9 *Assume Eq. 2.15 and let* ζ *be defined by Eq. 2.2.*

(a) ζ *is* $(1+\beta)$*-stable, totally skewed to the right if* $\beta < 1$.
(b) ζ *is self-similar with index* $(2+\beta)/(1+\beta) - d/\alpha$.
(c) ζ *has continuous paths.*
(d) ζ *has long-range dependence exponent* d/α.

The long-range dependence exponent of ζ does not depend on β, whereas the process ξ has two long-range dependence regimes, one depending on β (cf. Eq. 2.10).

We remark that the covariance of the Gaussian process ζ in the case $\beta = 1$ does not have a simple form (in contrast with ξ, see Proposition 5.2).

Finally, we turn to the non-branching high-density system with finite initial intensity measure.

Theorem 2.10 *Let X_T be defined by Eq. 1.2 for a system without branching with initial Poisson intensity $H_T\mu$, where μ is a finite measure and $H_T \to \infty$.*

(a) *If $d < \alpha$ and*

$$F_T = H_T^{1/2} T^{1-d/\alpha}, \tag{2.24}$$

then $X_T \Rightarrow_C (2\mu(\mathbb{R}^d)/(1-d/\alpha))^{1/2} p_1(0)\lambda\rho$ as $T \to \infty$, where ρ is a centered Gaussian process with covariance

$$E\rho_s\rho_t = \int_0^{t\wedge s} u^{-d/\alpha}[(t-u)^{1-d/\alpha} + (s-u)^{1-d/\alpha}]du, \quad s,t \geq 0. \tag{2.25}$$

(b) *If $d = \alpha$ and $F_T = H_T^{1/2} \log T$, then $X_T \Rightarrow_{C,\varepsilon} \left(2\mu\left(\mathbb{R}^d\right)\right)^{1/2} p_1(0)\gamma$ as $T \to \infty$, where γ is a standard Gaussian random variable.*

(c) *If $d > \alpha$ and $F_T = H_T^{1/2}$, then $X_T \Rightarrow_{C,\varepsilon} X$ as $T \to \infty$, where X is a centered $\mathcal{S}'\left(\mathbb{R}^d\right)$-valued Gaussian random variable with covariance*

$$E(\langle X, \varphi_1\rangle\langle X, \varphi_2\rangle) = 2\int_{\mathbb{R}^d} \varphi_1(x)G\varphi_2(x)G\mu(dx), \tag{2.26}$$

with G given by (2.3).

Remark 2.11

(a) As in the branching case (Theorem 2.6), there is a substantial difference in the time structures of the limits for $d \geq \alpha$.

(b) The process ζ with covariance (2.25) belongs to a class of weighted fractional Brownian motions which is discussed in [10], in particular its long-range dependence is studied.

3 Proofs

Proof of Proposition 2.1 It is known that existence of the processes ξ and ζ defined by Eqs. 2.1 and 2.2 is equivalent to

$$\int_{\mathbb{R}^d}\int_0^t \left(\int_r^t p_{u-r}(x)dr\right)^{1+\beta} drdx < \infty, \quad t \geq 0, \tag{3.1}$$

and

$$\int_{\mathbb{R}^d}\int_0^t p_r(x)\left(\int_r^t p_{u-r}(x)du\right)^{1+\beta} drdx < \infty, \quad t \geq 0, \tag{3.2}$$

respectively (see [24]). On the other hand, from Lemma A.1 in [17] it follows that

$$\int_{\mathbb{R}^d}\left(\int_0^t p_u(x)du\right)^{1+\beta}dx<\infty,\quad t\geq 0\quad\text{if}\quad d<\frac{\alpha(1+\beta)}{\beta},\tag{3.3}$$

and

$$\int_{\mathbb{R}^d}\left(\int_0^t p_u(x)du\right)^{2+\beta}dx<\infty,\quad t\geq 0\quad\text{if}\quad d<\frac{\alpha(2+\beta)}{1+\beta}.\tag{3.4}$$

Equation 3.1 is an immediate consequence of Eq. 3.3, and Eq. 3.2 follows from the Hölder inequality and Eq. 3.4. □

3.1 General Scheme

We present a general scheme which will be employed in the convergence proofs. We consider a general (d,α,β)-branching system, initially Poisson with intensity measure ν_T. Without loss of generality we take the time interval $[0,1]$, i.e., $\tau=1$ (see the end of Section 1). Let X_T be defined by Eq. 1.2.

Analogously as in [7] (Theorem 2.2) and [8] (Theorem 2.1), we prove that

$$\lim_{T\to\infty}Ee^{-\langle\widetilde{X}_T,\Phi\rangle}=Ee^{-\langle\widetilde{X},\Phi\rangle},\tag{3.5}$$

where X is the corresponding limit process, $\Phi\in\mathcal{S}(\mathbb{R}^{d+1})$, $\Phi\geq 0$, and $\widetilde{X}_T$ and $\widetilde{X}$ are defined by Eq. 1.6. As explained in [7], due to the special form of the limit (either Gaussian or $(1+\beta)$-stable totally skewed to the right), Eq. 3.5 implies $X_T\Rightarrow_i X$. To prove convergence $\Rightarrow_C$ (or $\Rightarrow_{C,\varepsilon}$), according to the space-time approach [3], it suffices to show additionally that the family $\{\langle X_T,\varphi\rangle\}_{T\geq 0}$ is tight in $C([0,1],\mathbb{R})$ [or $C([\varepsilon,1],\mathbb{R})$].

For simplicity we consider Φ of the form

$$\Phi(x,t)=\varphi\otimes\psi(x,t)=\varphi(x)\psi(t),\quad \varphi\in\mathcal{S}(\mathbb{R}^d),\ \psi\in\mathcal{S}(\mathbb{R}),\quad \varphi,\psi\geq 0.$$

It will be clear from the proofs that for general Φ the argument is analogous.

Denote

$$\varphi_T=\frac{1}{F_T}\varphi,\quad \chi(t)=\int_t^1\psi(s)ds,\quad \chi_T(t)=\chi\left(\frac{t}{T}\right).\tag{3.6}$$

Let

$$v_T(x,t)=1-E\exp\left\{-\int_0^t\langle N_r^x,\varphi_T\rangle\chi_T(T-t+r)dr\right\},\ 0\leq t\leq T,\tag{3.7}$$

where N^x is the empirical process of the branching system started from a single particle at x. It is known that v_T satisfies the equation

$$v_T(x,t)=\int_0^t\mathcal{T}_{t-u}\left[\varphi_T\chi_T(T-u)(1-v_T(\cdot,u))-\frac{V}{1+\beta}v_T^{1+\beta}(\cdot,u)\right](x)du,\ 0\leq t\leq T,\tag{3.8}$$

(see [7], Eq. 3.3). From Eqs. 3.7 and 3.8 we obtain immediately

$$0\leq v_T\leq 1,\tag{3.9}$$

and

$$v_T(x,t) \leq \int_0^t \mathcal{T}_{t-u}\varphi_T(x)\chi_T(T-u)du. \tag{3.10}$$

By Eq. 1.2, the Poisson property, Eq. 3.7 and $E\langle N_t^x, \varphi\rangle = \mathcal{T}_t\varphi(x)$, we have

$$Ee^{-\langle \widetilde{X}_T, \varphi\otimes\psi\rangle} = \exp\left\{-\int_{\mathbb{R}^d} v_T(x,T)\nu_T(dx) + \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_u\varphi_T(x)\chi_T(u)du\nu_T(dx)\right\}. \tag{3.11}$$

Hence, by Eq. 3.8,

$$Ee^{-\langle \widetilde{X}_T, \varphi\otimes\psi\rangle} = \exp\left\{\frac{V}{1+\beta}I_1(T) + I_2(T) - \frac{V}{1+\beta}I_3(T)\right\}, \tag{3.12}$$

where

$$I_1(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left[\left(\int_0^s \mathcal{T}_{s-u}\varphi_T\chi_T(T-u)du\right)^{1+\beta}\right](x)ds\nu_T(dx), \tag{3.13}$$

$$I_2(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}(\varphi_T\chi_T(T-s)v_T(\cdot,s))(x)ds\nu_T(dx), \tag{3.14}$$

$$I_3(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left[\left(\int_0^s \mathcal{T}_{s-u}\varphi_T\chi_T(T-u)du\right)^{1+\beta} - v_T^{1+\beta}(\cdot,s)\right](x)ds\nu_T(dx). \tag{3.15}$$

In most of the cases (with the exception of large dimensions and $\beta = 1$, where I_2 has a nontrivial limit), we prove

$$\lim_{T\to\infty} e^{(V/(1+\beta))I_1(T)} = Ee^{-\langle \widetilde{X}, \varphi\otimes\psi\rangle}, \tag{3.16}$$

$$\lim_{T\to\infty} I_2(T) = 0, \tag{3.17}$$

and

$$\lim_{T\to\infty} I_3(T) = 0. \tag{3.18}$$

Note that if $\nu_T = H_T\lambda$, then formulas (3.13–3.15) have simpler forms due to invariance of λ for $\mathcal{T}_t$. If ν_T is finite (hence not invariant under $\mathcal{T}_t$), then the proofs are more involved.

To prove Eq. 3.17 we will use the inequality

$$I_2(T) \leq \frac{C}{F_T^2}\int_{\mathbb{R}^d}\int_0^T\int_0^T \mathcal{T}_s(\varphi\mathcal{T}_u\varphi)(x)duds\nu_T(dx), \tag{3.19}$$

which is an easy consequence of Eqs. 3.6 and 3.10.

To obtain Eq. 3.18 we apply the elementary inequality

$$(a+b)^{1+\beta} - a^{1+\beta} \leq b^{1+\beta} + (1+\beta)a^{(1+\beta)/2}b^{(1+\beta)/2}, \quad a,b\geq 0, 0<\beta\leq 1,$$

then by Eqs. 3.10 and 3.8 we obtain

$$0 \le I_3(T) \le \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\Big(\int_0^s \mathcal{T}_{s-u}(\varphi_T\chi_T(T-u)v_T)du + \int_0^s \mathcal{T}_{s-u}v_T^{1+\beta}(\cdot,u)du\Big)^{1+\beta}(x)\nu_T(dx)$$
$$+ (1+\beta)\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\Big[\Big(\int_0^s \mathcal{T}_{s-u}(\varphi_T\chi(T-u)v_T(\cdot,u)) + \int_0^s \mathcal{T}_{s-u}v_T^{1+\beta}(\cdot,u)du\Big)^{(1+\beta)/2}$$
$$\times\ v_T^{(1+\beta)/2}(\cdot,s)\Big](x)\nu_T(dx).$$

We apply the Schwarz inequality to the second term, then we use $(a+b)^{1+\beta} \le C(a^{1+\beta}+b^{1+\beta})$, $a, b \ge 0$, in both terms, and finally, by Eq. 3.10, we arrive at

$$0 \le I_3(T) \le C\Big(J_1(T) + J_2(T) + (J_1(T)+J_2(T))^{1/2}I_1(T)^{1/2}\Big), \tag{3.20}$$

where

$$J_1(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\Big[\Big(\int_0^s \mathcal{T}_{s-u}\Big(\varphi_T\int_0^u \mathcal{T}_{u-r}\varphi_T dr\Big)du\Big)^{1+\beta}\Big](x)ds\nu_T(dx)$$
$$\le \frac{1}{F_T^{2+2\beta}}\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_s\Big[\Big(\int_0^T \mathcal{T}_u\Big(\varphi\int_0^T \mathcal{T}_r\varphi dr\Big)du\Big)^{1+\beta}\Big](x)ds\nu_T(dx), \tag{3.21}$$

and

$$J_2(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\Big[\Big(\int_0^s \mathcal{T}_{s-u}\Big(\int_0^u \mathcal{T}_{u-r}\varphi_T dr\Big)^{1+\beta}du\Big)^{1+\beta}\Big](x)ds\nu_T(dx)$$
$$\le \frac{1}{F_T^{(1+\beta)(1+\beta)}}\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_s\Big[\Big(\int_0^T \mathcal{T}_u\Big(\int_0^T \mathcal{T}_r\varphi dr\Big)^{1+\beta}du\Big)^{1+\beta}\Big](x)ds\nu_T(dx). \tag{3.22}$$

Given Eq. 3.16, in order to prove Eq. 3.18 it suffices to show that

$$\lim_{T\to\infty} J_1(T) = 0 \tag{3.23}$$

and

$$\lim_{T\to\infty} J_2(T) = 0. \tag{3.24}$$

Note that our method of proof of $\Rightarrow_i$ convergence (based on Eqs. 3.8 and 3.11) gives also convergence of finite-dimensional distributions (see, e.g., the proof of Theorem 2.1 in [8]).

In the proofs of tightness of $\{\langle X_T,\varphi\rangle\}_{T\ge 2}$ we follow the idea of [7] (proof of Proposition 3.3). Fix $0\le t_1\le t_2\le 1$ (or $\varepsilon\le t_1\le t_2\le 1$ in the proofs of $\Rightarrow_{C,\varepsilon}$ convergence), and let $\psi\in\mathcal{S}(\mathbb{R}^d)$ be such that the corresponding χ (see Eq. 3.6) satisfies

$$0 \le \chi \le 1\!\!1_{[t_1,t_2]}. \tag{3.25}$$

We now repeat the argument of the previous part with φ replaced by $i\theta\varphi$, $\theta > 0$. Let $v_{\theta,T}$ be the analogue of Eq. 3.7. Using the inequality

$$|1-e^z| \le 2|z| \quad \text{if} \quad |e^z| \le 1, \quad z\in\mathbb{C}, \tag{3.26}$$

we have

$$|v_{\theta,T}(x,t)| \leq 2\theta \int_0^t \langle N_s^x, \varphi_T\rangle \chi_T(T-t+s)ds$$
$$= 2\theta \int_0^t \mathcal{T}_{t-s}\varphi_T(x)\chi_T(T-s)ds. \tag{3.27}$$

The function $v_{\theta,T}$ also satisfies Eq. 3.8 with $i\theta\varphi$ (we have not assumed $\psi \geq 0$, but it is not needed for Eq. 3.8 to hold). Hence by Eq. 3.11 we obtain

$$E\exp\{-i\theta\langle \widetilde{X}_T, \varphi\otimes\psi\rangle\} = \exp\{A_\theta(T) + B_\theta(T)\}, \tag{3.28}$$

where

$$A_\theta(T) = i\theta \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\Big(\varphi_T\chi_T(T-s)v_{\theta,T}(\cdot,s)\Big)(x)ds\nu_T(dx), \tag{3.29}$$

$$B_\theta(T) = \frac{V}{1+\beta}\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\Big(v_{\theta,T}^{1+\beta}(\cdot,s)\Big)(x)ds\nu_T(dx). \tag{3.30}$$

From Eqs. 3.28, again by Eq. 3.26, we have

$$0 \leq 1 - \mathrm{Re}\, E\exp\{-i\theta\langle \widetilde{X}_T, \varphi\otimes\psi\rangle\} \leq 2(|A_\theta(T)| + |B_\theta(T)|), \tag{3.31}$$

and this implies

$$P(|\langle \widetilde{X}_T, \varphi\otimes\psi\rangle| \geq \delta) \leq C\delta\int_0^{1/\delta}(|A_\theta(T)| + |B_\theta(T)|)d\theta, \quad \delta > 0, \tag{3.32}$$

(see e.g., [11], Proposition 8.29). The tightness will be proved if we show that

$$|A_\theta(T)| \leq C(\varphi,\sigma,h)\theta^2(t_2^h - t_1^h)^{1+\sigma}, \tag{3.33}$$

and

$$|B_\theta(T)| \leq C(\varphi,\sigma,h,V,\beta)\theta^{1+\beta}(t_2^h - t_1^h)^{1+\sigma}, \tag{3.34}$$

for some $\sigma, h > 0$. Indeed, Eqs. 3.32–3.34 imply, for $0 < \sigma < 1$,

$$P(|\langle \widetilde{X}_T, \varphi\otimes\psi\rangle| \geq \sigma) \leq \frac{C_1}{\delta^2}(t_2^h - t_1^h)^{1+\sigma}. \tag{3.35}$$

We take ψ approximating $\delta_{t_2} - \delta_{t_1}$, and we see that the left-hand side of Eq. 3.35 can be replaced by $P(|\langle X_T(t_2),\varphi\rangle - \langle X_T(t_1),\varphi\rangle| \geq \sigma)$. Hence tightness follows by a well-known criterion [1]. (In the case of $\Rightarrow_{C,\varepsilon}$ convergence we use additionally the fact that, as observed above, $\langle X_T(\varepsilon),\varphi\rangle$ converges in law).

In the proofs of Eqs. 3.33 and 3.34, we combine Eq. 3.27 with 3.29 or 3.30, respectively, obtaining

$$|A_\theta(T)| \leq 2\theta^2 A(T), \tag{3.36}$$

$$|B_\theta(T)| \leq \frac{2^{1+\beta}V}{1+\beta}\theta^{1+\beta}I_1(T), \tag{3.37}$$

where

$$A(T) = \frac{1}{F_T^2}\int_{\mathbb{R}^d}\int_0^T\int_0^s \mathcal{T}_{T-s}(\varphi\mathcal{T}_{s-u}\varphi)(x)\chi\left(1-\frac{s}{T}\right)\chi\left(1-\frac{u}{T}\right)duds\nu_T(dx), \quad (3.38)$$

and $I_1(T)$ is given by Eq. 3.13.

Hence we have reduced the proof of tightness to estimating $A(T)$ and $I_1(T)$ by $C(t_2^h - t_1^h)^{1+\sigma}$.

A similar scheme is applied in the cases without branching. We also have Eq. 3.11 where v_T satisfies Eq. 3.8 with $V = 0$. Then instead of Eq. 3.12 we have

$$Ee^{-\langle \widetilde{X}_T, \varphi\otimes\psi\rangle} = e^{\mathit{II}_1(T)-\mathit{II}_2(T)}, \quad (3.39)$$

where

$$\mathit{II}_1(T) = \int_{\mathbb{R}^d}\int_0^T\int_0^s \mathcal{T}_{T-s}(\varphi_T\mathcal{T}_{s-u}\varphi_T)(x)\chi_T(T-u)\chi_T(T-s)duds\nu_T(dx), \quad (3.40)$$

$$\mathit{II}_2(T) = \int_{\mathbb{R}^d}\int_0^T\int_0^s \mathcal{T}_{T-s}(\varphi_T\mathcal{T}_{s-u}\varphi_T v_T(\cdot,u))(x)\chi_T(T-u)\chi_T(T-s)duds\nu_T(dx), \quad (3.41)$$

and we show that

$$\lim_{T\to\infty} e^{\mathit{II}_1(T)} = Ee^{-\langle \widetilde{X}, \varphi\otimes\psi\rangle}, \quad (3.42)$$

and

$$\lim_{T\to\infty} \mathit{II}_2(T) = 0. \quad (3.43)$$

Also, the proof of tightness uses the same method as before with $B_\theta(T) = 0$ (see Eq. 3.30).

This general scheme is applied in all the proofs (with $\nu_T = H_T\lambda$ or $\nu_T = H_T\mu$, μ finite measure). However, as we have mentioned in the Introduction, its implementation in specific cases is not straightforward.

Proof of Theorem 2.2 We will prove only part (a) of this theorem, as the remaining parts can be obtained the same way as in [6] and [8]. Also, since the proof of (a) is similar to the proof of Theorem 2.2. in [7], we present only the main steps.

We follow the general scheme. Recall that in this case $\nu_T = H_T\lambda$. In order to show Eq. 3.16 it suffices to prove

$$\lim_{T\to\infty} I_1(T) = \int_{\mathbb{R}^d}\int_0^1\left(\int_{\mathbb{R}^d}\int_s^1 \varphi(y)\psi(r)\int_s^r p_{u-s}(x)dudrdy\right)^{1+\beta} dsdx, \quad (3.44)$$

and this can be done the same way as Eq. 3.21 in [7]. Note that H_T cancels out in $I_1(T)$ (see Eq. 3.13), and in the proof of Eq. 3.21 in [7], $\alpha/\beta < d$ was not used, only Eq. 3.1 was important.

Next, we prove Eq. 3.17. By Eq. 2.7, after obvious substitutions Eq. 3.19 has the form

$$I_2(T) \leq CH_T^{1-2/(1+\beta)} T^{2(d\beta/\alpha-1)/(1+\beta)} \int_{\mathbb{R}^d} \int_0^1 \varphi(x) \mathcal{T}_{Tu}\varphi(x) du dx$$
$$\leq C_1 T^{2(d\beta/\alpha-1)/(1+\beta)-1} \int_{\mathbb{R}^d} \frac{1-e^{-T|x|^\alpha}}{|x|^\alpha} |\widehat{\varphi}(x)|^2 dx, \tag{3.45}$$

where we have used $1-2/(1+\beta) \leq 0$, the Plancherel formula, and the fact that $\widehat{\mathcal{T}_u\varphi}(x) = e^{-u|x|^\alpha}\widehat{\varphi}(x)$ ($\widehat{\ }$ denotes Fourier transform, defined by $\widehat{\varphi}(z) = \int_{\mathbb{R}^d} e^{ix\cdot z}\varphi(x)dx$, $z \in \mathbb{R}^d$, where $\cdot$ is the scalar product in $\mathbb{R}^d$). Hence it is clear that Eq. 3.17 holds if $\alpha/\beta < d < \alpha(1+\beta)/\beta$ and if $d < \alpha/\beta$ (we use $(1-e^{-T|x|^\alpha})/(T|x|^\alpha) \leq C$).

For $d = \alpha/\beta$, we estimate the right-hand side of Eq. 3.45 by

$$C_1 T^{-1/2} \int_{\mathbb{R}^d} \left(\frac{1-e^{-T|x|^\alpha}}{T|x|^\alpha} \right)^{1/2} \frac{1}{|x|^{\alpha/2}} |\widehat{\varphi}(x)|^2 dx,$$

which tends to 0 as $T \to \infty$, since $\alpha \leq d$.

To prove Eq. 3.18 we show Eqs. 3.23 and 3.24. By Eq. 2.7, on the right-hand side of Eq. 3.21 H_T appears only as a factor H_T/H_T^2 (which is bounded), and the remaining term tends to 0 by the same argument as in [7] (see the proof of Eq. 3.33 therein, where only Eq. 3.1 was used). Hence we obtain Eq. 3.23.

So far we have not used the assumption 2.6; it will be needed in the proof of Eq. 3.24.

By Eq. 3.22, repeating the argument of [7] (see Eq. 3.35 therein and the estimates following it), we obtain

$$J_2(T) \leq CH_T^{-\beta} T^{1-d\beta/\alpha} \to 0,$$

by assumption Eq. 2.6. This completes the proof of Eq. 3.5 by 3.44 and 2.1.

In order to prove tightness, we show Eqs. 3.33 and 3.34 with $h = 1$ and

$$0 < \sigma < \left(1 + \beta - \frac{d\beta}{\alpha}\right) \wedge \beta. \tag{3.46}$$

Note that in Eq. 3.13 H_T cancels out, and then the proof of Eq. 3.34 follows the lines of the proof of Eq. 3.49 in [7]. The assumption $\sigma < \beta$ is needed in order to have $(1+\beta)/(1+\sigma) > 1$ (see Eq. 3.56 in [7]).

It remains to show Eq. 3.33. By Eq. 3.38 we have

$$A(T) = \frac{1}{(2\pi)^d} \frac{H_T}{H_T^{2/(1+\beta)}} \frac{T^2}{T^{2(2+\beta-d\beta/\alpha)/(1+\beta)}}$$
$$\int_0^1 \chi(s) \int_{\mathbb{R}^d} |\widehat{\varphi}(x)|^2 \int_s^1 e^{-T(u-s)|x|^\alpha} \chi(u) du dx ds$$
$$\leq \frac{1}{(2\pi)^d} T^{-2(1-d\beta/\alpha)/(1+\beta)} (t_2 - t_1)^{1+\sigma}$$
$$\int_{\mathbb{R}^d} |\widehat{\varphi}(x)|^2 \left(\frac{1}{1+T|x|^\alpha} \right)^{1-\sigma} dx, \tag{3.47}$$

where in the last estimate we used

$$\int_s^1 e^{-T(u-s)|x|^\alpha}\chi(u)du \le \left(\int_s^1 e^{-T(u-s)|x|^\alpha}dr\right)^{1-\sigma}\left(\int_s^1 \chi(u)du\right)^\sigma$$
$$\le \left(\frac{1}{1+T|x|^\alpha}\right)^{1-\sigma}(t_2-t_1)^\sigma \tag{3.48}$$

for any $0<\sigma\le 1$. Hence Eq. 3.33 follows immediately if $d\le\alpha/\beta$. For $d>\alpha/\beta$ (this case was also treated in [7]) we write $(1+T|x|^\alpha)^{-(1-\sigma)}\le T^{\sigma-1}|x|^{\alpha(\sigma-1)}$, we use $\alpha<d$ and Eq. 2.5, and we see that for σ satisfying Eq. 3.46 the estimate 3.33 holds since the term involving T tends to 0. □

Proof of Proposition 2.5 From Eq. 2.1, for $\beta=1$ we have

$$E\xi_s\xi_t = \int_0^{s\wedge t}\int_{\mathbb{R}^d}\int_r^s\int_r^t p_{u-r}(x)p_{u'-r}(x)du'dudxdr$$
$$= p_1(0)\int_0^{s\wedge t}\int_r^s\int_r^t (u+u'-2r)^{-d/\alpha}du'dudr, \tag{3.49}$$

by the Chapman–Kolmogorov identity and the self-similarity of the standard α-stable process. Hence Eqs. 2.13 and 2.15 follow by calculus. □

Proof of Theorem 2.7 Proof of part (a). According to the general scheme, we show Eq. 3.16, which amounts to proving

$$\lim_{T\to\infty} I_1(T) = \mu(\mathbb{R}^d)\int_0^1\int_{\mathbb{R}^d} p_s(y)\left(\int_s^1 p_{u-s}(y)\chi(u)du\right)^{1+\beta}dyds\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}. \tag{3.50}$$

In Eq. 3.13 with $\nu_T = H_T\mu$ we substitute $u'=(T-u)/T$, $s'=(T-s)/T$, we use the self-similarity of the α-stable density,

$$p_{st}(x) = t^{-d/\alpha}p_s(xt^{-1/\alpha}), \tag{3.51}$$

and by Eqs. 2.17 and 3.6 we obtain

$$I_1(T) = T^{-d/\alpha}\int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_s((x-y)T^{-1/\alpha})$$
$$\times\left(\int_s^1\int_{\mathbb{R}^d} p_{u-s}((y-z)T^{-1/\alpha})\varphi(z)\chi(u)dzdu\right)^{1+\beta} dyds\mu(dx)$$
$$= \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_s(xT^{-1/\alpha}-y)$$
$$\left(\int_s^1\int_{\mathbb{R}^d} p_{u-s}(y-z)\widetilde{\varphi}_T(z)\chi(u)dzdu\right)^{1+\beta} dyds\mu(dx), \tag{3.52}$$

where

$$\widetilde{\varphi}_T(z) = T^{d/\alpha}\varphi(zT^{1/\alpha}). \tag{3.53}$$

We denote

$$h_s(y) = \int_s^1 p_{u-s}(y)\chi(u)du, \tag{3.54}$$

and we write

$$I_1(T) = I_1'(T) + I_1''(T), \tag{3.55}$$

where

$$I_1'(T) = \int_{\mathbb{R}^d}\int_0^1 p_s * h_s^{1+\beta}(xT^{-1/\alpha})ds\mu(dx)\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}, \tag{3.56}$$

$$I_1''(T) = \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_s(xT^{-1/\alpha}-y)\Big[(h_s * \widetilde{\varphi}_T(y))^{1+\beta} - \left(h_s(y)\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}\Big]dydx\mu(dx). \tag{3.57}$$

By Eq. 3.4, it is not difficult to see that $I_1'(T)$ converges to the right-hand side of Eq. 3.50. Therefore, to obtain Eq. 3.50 it suffices to show that $\lim_{T\to\infty} I_1''(T) = 0$. Fix any δ satisfying

$$\frac{d}{\alpha} - \frac{1}{1+\beta} < \delta < 1, \tag{3.58}$$

(such δ exists by Eq. 2.15). We estimate Eq. 3.57 applying the Hölder inequality to the integrals with respect to the measure $dys^{-\delta}ds\mu(dx)$, obtaining

$$|I_1''(T)| \le \left(\int_{\mathbb{R}^d}\int_0^1 s^{-\delta}\int_{\mathbb{R}^d}\left(s^{\delta}p_s(xT^{-1/\alpha}-y)\right)^{2+\beta}dyds\mu(dx)\right)^{1/(2+\beta)} \times \left(\int_{\mathbb{R}^d}\int_0^1 s^{-\delta}\int_{\mathbb{R}^d}\left|(h_s * \widetilde{\varphi}_T(y))^{1+\beta} - \left(h_s(y)\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}\right|^{(2+\beta)/(1+\beta)} dyds\mu(dx)\right)^{(1+\beta)/(2+\beta)}. \tag{3.59}$$

The first factor does not depend on T and is finite by Eqs. 3.51, 3.58 and finiteness of μ.

By Eq. 3.4 and the form of $\widetilde{\varphi}_T$ (see Eq. 3.53) $(h_s * \widetilde{\varphi}_T)^{1+\beta}$ converges to $(h_s\int_{\mathbb{R}}\varphi(z)dz)^{1+\beta}$ in $L^{(2+\beta)/(1+\beta)}(\mathbb{R}^d)$ for any $s \in [0,1]$. Moreover, $h_s(y) \le \int_0^1 p_u(y)du$ (see Eq. 3.54), hence it is not hard to see that the dominated convergence theorem can be applied to show that the right-hand side of Eq. 3.59 tends to 0 as $T \to \infty$. So Eq. 3.50 is proved, and therefore so is Eq. 3.16.

To show Eq. 3.17 we make obvious substitutions in the right-hand side of Eq. 3.19 and use self-similarity, obtaining

$$I_2(T) \le C\frac{H_T T^{2-2d/\alpha}}{F_T^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}} f(xT^{1/\alpha}-y)f(y-z)\widetilde{\varphi}_T(y)\widetilde{\varphi}_T(z)dzdy\mu(dx), \tag{3.60}$$

where $\widetilde{\varphi}_T$ is given by Eq. 3.53, and

$$f(x) = \int_0^1 p_s(x)ds. \tag{3.61}$$

By the Hölder inequality applied to the integral on z, y, we have

$$I_2(T) \le C\frac{H_T T^{2-2d/\alpha}}{F_T^2}\mu(\mathbb{R}^d)||f||_{2+\beta}^2||\widetilde{\varphi}_T||^2_{(2+\beta)/(1+\beta)}.$$

Eqs. 3.4, 3.53 and 2.17 imply

$$I_2(T) \le C_1 T^{2(d/\alpha(2+\beta)-1/(1+\beta))} \to 0,$$

by Eq. 2.15.

To complete the proof of Eq. 3.5 we show Eqs. 3.23 and 3.24. From Eq. 3.21, by a similar argument as in Eq. 3.60 we obtain

$$J_1(T) \le \frac{H_T T^{1+2(1+\beta)-2(d/\alpha)(1+\beta)}}{F_T^{2(1+\beta)}}\int_{\mathbb{R}^d} f*(f*\widetilde{\varphi}_T(f*\widetilde{\varphi}_T))^{1+\beta}(xT^{-1/\alpha})\mu(dx),$$

with $f, \widetilde{\varphi}_T$ as above. Applying the Hölder and Young inequalities several times we obtain

$$||f*(f*\widetilde{\varphi}_T(f*\widetilde{\varphi}_T))^{1+\beta}||_\infty \le ||f||_{2+\beta}^{3+\beta}||\widetilde{\varphi}_T||_1^{1+\beta}||\widetilde{\varphi}_T||^{1+\beta}_{(2+\beta)/(1+\beta)}.$$

Hence, by Eqs. 2.17, 3.53, 3.61 and 3.4,

$$J_1(T) \le CT^{(d/\alpha)(1+\beta)/(2+\beta)-1} \to 0,$$

by Eq. 2.15.

Finally, by Eq. 3.22 and the usual argument we get

$$J_2(T) \le \frac{H_T T^{2+\beta+(1+\beta)(1+\beta)-(d/\alpha)(1+\beta)(1+\beta)}}{F_T^{(1+\beta)(1+\beta)}}\int_{\mathbb{R}^d} f*(f*(f*\widetilde{\varphi}_T)^{1+\beta})^{1+\beta}(xT^{-1/\alpha})\mu(dx).$$

In this case

$$||f*(f*(f*\widetilde{\varphi}_T)^{1+\beta})^{1+\beta}||_\infty \le ||f||_{2+\beta}^{2+\beta}||f||_{1+\beta}^{(1+\beta)(1+\beta)}||\widetilde{\varphi}_T||_1^{(1+\beta)(1+\beta)} \le C,$$

since $||\widetilde{\varphi}_T||_1 = ||\varphi||_1$. Hence, by Eqs. 2.16 and 2.17,

$$J_2(T) \le C\frac{T}{H_T^\beta} \to 0.$$

We now pass to the proof of tightness. To prove Eq. 3.33 we rewrite Eq. 3.38 as

$$A(T) = \frac{H_T T^2}{F_T^2}\int_0^1\int_s^1\int_{\mathbb{R}^d}\varphi(x)\mathcal{T}_{T(u-s)}\varphi(x)(\mu\mathcal{T}_{Ts})(dx)\chi(u)\chi(s)duds.$$

We use the following identity, which holds for any finite measure m,

$$\int_{\mathbb{R}^d}\varphi_1(x)\varphi_2(x)m(dx) = \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^{2d}}\widehat{\varphi}_1(x)\widehat{\varphi}_2(y)\overline{\widehat{m}(x+y)}dxdy,$$

obtaining

$$A(T)=\frac{H_T T^2}{(2\pi)^{2d} F_T^2}\int_0^1\int_s^1\int_{\mathbb{R}^{2d}}\widehat{\varphi}(x)e^{-T(u-s)|y|^\alpha}\widehat{\varphi}(y)e^{-Ts|x+y|^\alpha}\overline{\widehat{\mu}(x+y)}dxdy\chi(u)\chi(s)duds. \tag{3.62}$$

Fix h satisfying

$$\left(1-\frac{d}{\alpha}\right)^+ < h < \left(\frac{2+\beta}{1+\beta}-\frac{d}{\alpha}\right)\wedge 1. \tag{3.63}$$

The function $r \to r^{1-h}e^{-r}$ is bounded on $[0, \infty)$, hence we have from Eq. 3.62

$$\begin{aligned}A(T) &\le C\frac{H_T T^2}{F_T^2 T^{2(1-h)}}\int_0^1\int_s^1 (u-s)^{h-1}s^{h-1}\chi(u)\chi(s)duds\\ &\quad\int_{\mathbb{R}^{2d}}|\widehat{\varphi}(x)\widehat{\varphi}(y)||y|^{\alpha(h-1)}|x+y|^{\alpha(h-1)}dxdy\\ &\le C_1\frac{T^{-2(2+\beta)/(1+\beta)+2d/\alpha+2h}}{H_T^{(1-\beta)/(1+\beta)}}\int_{t_1}^{t_2}\int_s^{t_2}(u-s)^{h-1}s^{h-1}duds,\end{aligned}$$

by Eqs. 2.17, 3.25, and since $\alpha(1-h) < d$ by Eq. 3.63. The right-hand side of Eq. 3.63 implies that the term involving T is bounded, so it is easy to see that Eq. 3.33 is obtained with $\sigma = h$.

In order to prove Eq. 3.34 we use Eq. 3.37. By Eqs. 3.52 and 3.25 we have

$$I_1(T) \le \int_{\mathbb{R}^d}\left[R_1(xT^{-1/\alpha}) + R_2(xT^{-1/\alpha})\right]\mu(dx),$$

where

$$R_1(x) = \int_0^{t_1}\int_{\mathbb{R}^d} p_s(x-y)\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d} p_{u-s}(y-z)\widetilde{\varphi}_T(z)dzdy\right)^{1+\beta} dyds, \tag{3.64}$$

$$R_2(x) = \int_{t_1}^{t_2}\int_{\mathbb{R}^d} p_s(x-y)\left(\int_s^{t_2}\int_{\mathbb{R}^d} p_{u-s}(y-z)\widetilde{\varphi}_T(z)dzdy\right)^{1+\beta} dyds. \tag{3.65}$$

Since μ is finite, it is enough to show that

$$\sup_{x\in\mathbb{R}^d} R_j(x) \le C(t_2^h - t_1^h)^{1+\sigma}, \quad j = 1, 2 \tag{3.66}$$

for some positive h and σ.

Fix $\delta > 0$ satisfying Eq. 3.58 and

$$\delta\beta > \frac{(1+\beta)^2}{2+\beta}\frac{d}{\alpha} - 1. \tag{3.67}$$

Equation 3.67 holds for δ sufficiently close to 1 because from Eq. 2.15 it follows that

$$\frac{(1+\beta)^2}{2+\beta}\frac{d}{\alpha} - 1 < \beta.$$

For any fixed $s \in [0, t_1]$, by the Jensen inequality applied to the measure

$$\frac{(u-s)^{-\delta}}{\int_{t_1}^{t_2}(r-s)^{-\delta}dr}\mathbb{1}_{[t_1,t_2]}(u)du$$

(this trick is borrowed from [17]), we have

$$\begin{aligned}R_1(x) &\le \int_0^{t_1}\int_{\mathbb{R}^d} p_s(x-y)\left(\int_{t_1}^{t_2}(r-s)^{-\delta}dr\right)^{\beta}\\&\quad\times\int_{t_1}^{t_2}(u-s)^{-\delta}\left(\int_{\mathbb{R}^d}(u-s)^{\delta}p_{u-s}(y-z)\widetilde{\varphi}_T(z)dz\right)^{1+\beta}dudyds\\&\le C(t_2-t_1)^{(1-\delta)\beta}\int_{t_1}^{t_2}\int_0^{t_1}(u-s)^{\delta\beta}\\&\quad\int_{\mathbb{R}^d}p_s(x-y)(p_{u-s}*\widetilde{\varphi}_T(y))^{1+\beta}dydsdu.\end{aligned} \tag{3.68}$$

By the Hölder and Young inequalities,

$$\begin{aligned}\int_{\mathbb{R}^d}\dots dy &\le \left(\int_{\mathbb{R}^d}p_s^{2+\beta}(y)dy\right)^{1/(2+\beta)}\left(\int_{\mathbb{R}^d}p_{u-s}^{2+\beta}(y)dy\right)^{(1+\beta)/(2+\beta)}\left(\int_{\mathbb{R}^d}\widetilde{\varphi}_T(z)dz\right)^{1+\beta}\\&= Cs^{-(d/\alpha)(1+\beta)/(2+\beta)}(u-s)^{-(d/\alpha)(1+\beta)^2/(2+\beta)},\end{aligned} \tag{3.69}$$

where we have used Eqs. 3.51 and 3.53. Observe that by Eqs. 2.15 and 3.67,

$$1-\frac{d}{\alpha}\frac{1+\beta}{2+\beta}>0 \quad \text{and} \quad 1+\delta\beta-\frac{d}{\alpha}\frac{(1+\beta)^2}{2+\beta}>0.$$

Hence, combining Eq. 3.69 with 3.68, substituting $s' = s/u$ and estimating the integral on s' by the corresponding value of the beta function,

$$\begin{aligned}R_1(x) &\le C_1(t_2-t_1)^{(1-\delta)\beta}\int_{t_1}^{t_2}u^{1-(d/\alpha)(1+\beta)/(2+\beta)+\delta\beta-(d/\alpha)(1+\beta)^2/(2+\beta)}du\\&\le C_2(t_2^{h'}-t_1^{h'})^{1+(1-\delta)\beta},\end{aligned} \tag{3.70}$$

where

$$h' = \min\{1, 2-(d/\alpha)(1+\beta)/(2+\beta)+\delta\beta-(d/\alpha)(1+\beta)^2/(2+\beta)\}.$$

To estimate R_2 (see Eq. 3.65) we use the Hölder inequality as in Eq. 3.59, and then the Young inequality, obtaining

$$\begin{aligned}
R_2(x) &\le \left[\int_{t_1}^{t_2} s^{-\delta}\int_{\mathbb{R}^d}(s^\delta p_s(x-y))^{2+\beta}dyds\right]^{1/(2+\beta)}\\
&\quad\times\left[\int_{t_1}^{t_2} s^{-\delta}\int_{\mathbb{R}^d}\left(\int_0^{t_2-t_1}\int_{\mathbb{R}^d}p_u(y-z)\widetilde{\varphi}_T(z)dzdu\right)^{2+\beta}dyds\right]^{(1+\beta)/(2+\beta)}\\
&= C\left(\int_{t_1}^{t_2} s^{(\delta-d/\alpha)(1+\beta)}ds\right)^{1/(2+\beta)}(t_2^{1-\delta}-t_1^{1-\delta})^{(1+\beta)/(2+\beta)}\\
&\quad\times\left[\int_{\mathbb{R}^d}\left(\int_0^{t_2-t_1}p_u*\widetilde{\varphi}_T(y)du\right)^{2+\beta}dy\right]^{(1+\beta)/(2+\beta)}\\
&\le C_1(t_2^{h''}-t_1^{h''})Q^{(1+\beta)/(2+\beta)}, \qquad (3.71)
\end{aligned}$$

where

$$h''=\min\left\{1-\delta,1+\left(\delta-\frac{d}{\alpha}\right)(1+\beta)\right\}$$

(note that $h''>0$ by Eq. 3.58), and

$$Q=\int_{\mathbb{R}^d}\left(\int_0^{t_2-t_1}p_u(y)du\right)^{2+\beta}dy.$$

To estimate Q we substitute $u'=u/(t_2-t_1)$, we use self-similarity and Eq. 3.4, obtaining

$$Q=C(t_2-t_1)^{2+\beta-(d/\alpha)(1+\beta)},$$

the exponent being positive by Eq. 2.15. Combining this with Eq. 3.71 we have

$$R_2(x)\le C_2(t_2^{h''}-t_1^{h''})^{2+\beta-(d/\alpha)(1+\beta)^2/(2+\beta)}.$$

This and Eq. 3.70 imply 3.66 with

$$h=\min\{h',h''\}\quad\text{and}\quad\sigma=\min\left\{(1-\delta)\beta,1+\beta-\frac{d}{\alpha}\frac{(1+\beta)^2}{2+\beta}\right\}.$$

This proves Eq. 3.34 and completes the proof of part (a) of the theorem.

Proof of part (b) According to the general scheme, we prove Eq. 3.16, and it is easy to see that to this end it suffices to show that

$$\lim_{T\to\infty}I_1(T)=\mu(\mathbb{R}^d)C_{\alpha,d}^{1+\beta}\int_{\mathbb{R}^d}p_1(y)|y|^{-(d-\alpha)(1+\beta)}dy\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}(\chi(0))^{1+\beta}. \qquad (3.72)$$

Observe that Eq. 2.18 implies that $d>\alpha$, hence

$$\sup_{x\in\mathbb{R}^d}G\varphi(x)<\infty, \qquad (3.73)$$

where G is defined by Eq. 2.3. This fact implies that if one of the limits below exists, then so does the other, and

$$\lim_{T\to\infty} I_1(T) = \lim_{T\to\infty} I_1'(T), \tag{3.74}$$

where

$$I_1'(T) = \frac{1}{\log T}\int_{\mathbb{R}^d}\int_0^{T-1} \mathcal{T}_{T-s}\left(\int_0^s \mathcal{T}_{s-u}\varphi\chi\left(\frac{T-u}{T}\right)du\right)^{1+\beta}(x)ds\mu(dx)$$

(see Eqs. 3.13, 3.6 and 2.19). By obvious substitutions,

$$\begin{aligned} I_1'(T) &= \frac{1}{\log T}\int_{\mathbb{R}^d}\int_1^T\int_{\mathbb{R}^d} p_s(x-y)\left(\int_0^{T-s}\int_{\mathbb{R}^d} p_u(y-z)\varphi(z)\chi\left(\frac{u}{T}+\frac{s}{T}\right)dzdu\right)^{1+\beta} \\ &\quad dyds\mu(dx) \\ &= \frac{1}{\log T}\int_{\mathbb{R}^d}\int_1^T\int_{\mathbb{R}^d} p_1(xs^{-1/\alpha}-y) \\ &\quad \times\left(\int_0^{T-s}\int_{\mathbb{R}^d} s^{-d/\alpha}p_{u/s}(y-zs^{-1/\alpha})\varphi(z)\chi\left(\frac{u}{T}+\frac{s}{T}\right)dzdu\right)^{1+\beta} dyds\mu(dx), \end{aligned}$$

where we have used self-similarity and the substitution $y' = ys^{-1/\alpha}$. Next, we substitute $u' = u/s$, and using Eq. 2.18 we get

$$\begin{aligned} I_1'(T) &= \frac{1}{\log T}\int_{\mathbb{R}^d}\int_1^T\int_{\mathbb{R}^d} s^{-1}p_1(xs^{-1/\alpha}-y) \\ &\quad \times\left(\int_0^{T/s-1}\int_{\mathbb{R}^d} p_u(y-zs^{-1/\alpha})\varphi(z)\chi\left(\frac{us}{T}+\frac{s}{T}\right)dzdu\right)^{1+\beta} dyds\mu(dx). \end{aligned}$$

Now we make the substitution $s' = \log s/\log T$, which is the main trick in calculating the limit. We obtain

$$\begin{aligned} I_1'(T) &= \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d} p_1(xT^{-s/\alpha}-y) \\ &\quad \times\left(\int_0^{T^{1-s}-1}\int_{\mathbb{R}^d} p_u(y-zT^{-s/\alpha})\varphi(z)\chi((u+1)T^{s-1})dzdu\right)^{1+\beta} dyds\mu(dx). \end{aligned} \tag{3.75}$$

It is now seen that taking the limit as $T\to\infty$ we arrive at the right-hand side of Eq. 3.72. It remains to justify this procedure.

Denote

$$U_1(T,s,y) = \int_0^{T^{1-s}-1}\int_{\mathbb{R}^d} p_u(y-zT^{-s/\alpha})\varphi(z)\chi((u+1)T^{s-1})dzdu$$

and

$$U_2(T,x) = \int_0^1\int_{\mathbb{R}^d} p_1(xT^{-s/\alpha}-y)U_1^{1+\beta}(T,s,y)dyds.$$

We will need the following fact, which can be found, e.g., in [20] (Lemma 5.3)

$$\sup_{x\in\mathbb{R}^d}(1+|x|^{d-\alpha})|G\varphi(x)|<\infty,\quad \varphi\in\mathcal{S}(\mathbb{R}^d),\, d>\alpha. \tag{3.76}$$

We have

$$\begin{aligned} U_1(T,s,y) &\le C\int_{\mathbb{R}^d}\frac{1}{|y-zT^{-s/\alpha}|^{d-\alpha}}\varphi(z)dz \\ &= \frac{C_1}{|y|^{d-\alpha}}|yT^{s/\alpha}|^{d-\alpha}G\varphi(yT^{s/\alpha}) \\ &\le \frac{C_2}{|y|^{d-\alpha}}, \end{aligned}$$

by Eq. 3.76. On the other hand, using the well known estimate

$$p_1(x)\le\frac{C}{1+|x|^{d+\alpha}}, \tag{3.77}$$

we have

$$\int_0^1 p_1(xT^{-s/\alpha}-y)ds\le C_3\frac{1+|x|^{d+\alpha}}{1+|y|^{d+\alpha}},$$

hence it is not hard to see that $U_2(T,x)$ converges pointwise as $T\to\infty$, since $(d-\alpha)(1+\beta)<d$ by Eq. 2.18. Moreover, it is bounded in T, x, since

$$\int_{\mathbb{R}^d}p_1(xT^{-s/\alpha}-y)|y|^{-(d-\alpha)(1+\beta)}dy\le C.$$

This proves Eqs. 3.72 by 3.74 and 3.75 because μ is finite.

Next observe that Eq. 3.19 implies that

$$I_2(T)\le C\frac{H_T}{F_T^2}\int_{\mathbb{R}^d}G(\varphi G\varphi)(x)\mu(dx), \tag{3.78}$$

hence Eq. 3.17 follows by Eqs. 3.73 and 2.19. Using Eq. 3.73, by Eq. 3.21 we have

$$\begin{aligned} J_1(T) &\le \frac{C}{F_T^{1+\beta}}\frac{H_T}{F_T^{1+\beta}}\int_{\mathbb{R}^d}\int_0^T\mathcal{T}_{T-s}\left(\int_0^s\mathcal{T}_{s-u}\varphi du\right)^{1+\beta}(x)ds\mu(dx) \\ &\le \frac{C_1}{F_T^{1+\beta}}\to 0, \end{aligned}$$

since from the proof of Eq. 3.72 it follows that

$$\sup_{T\ge 2}\sup_{x\in\mathbb{R}^d}\frac{1}{\log T}\int_0^T\mathcal{T}_{T-s}\left(\int_0^s\mathcal{T}_{s-u}\varphi du\right)^{1+\beta}(x)ds<\infty. \tag{3.79}$$

To prove Eq. 3.24 it suffices to note that by Eqs. 3.22 and 3.79,

$$J_2(T) \le C\frac{H_T}{F_T^{(1+\beta)^2}}T(\log T)^{1+\beta} = C\frac{T}{H_T^{\beta}} \to 0,$$

by (2.16). This completes the proof of Eq. 3.5.

To show tightness we prove Eqs. 3.33 and 3.34 with $h = 1$ and σ satisfying Eq. 3.46 (such σ exists by Eq. 2.18). Recall that now we consider t_1, t_2 such that $0 < \varepsilon < t_1 < t_2 \le 1$, hence in Eq. 3.62 the integral on s is taken over $[\varepsilon, 1]$. In Eq. 3.62 we estimate $|\widehat{\varphi}(x)\widehat{\mu}(x+y)|$ by a constant and we integrate with respect to x, obtaining

$$A(T) \le C\frac{H_T T^{2-d/\alpha}}{F_T^2}\int_\varepsilon^1\int_s^1\int_{\mathbb{R}^d} s^{-d/\alpha}|\widehat{\varphi}(y)|e^{-T(u-s)|y|^\alpha}\chi(u)\chi(s)dyduds.$$

By Eqs. 3.48 and 2.19 we have

$$\begin{aligned}A(T) &\le C_1\varepsilon^{-d/\alpha}T^{1-d/\alpha+\sigma}(t_2-t_1)^\sigma\int_\varepsilon^1\chi(s)ds\int_{\mathbb{R}^d}|\widehat{\varphi}(y)||y|^{\alpha(\sigma-1)}dy\\ &\le C_2(\varepsilon)T^{1-d/\alpha+\sigma}(t_2-t_1)^{1+\sigma}.\end{aligned}$$

Hence Eq. 3.33 follows by Eqs. 3.36, 3.46 and 2.18.

Now we pass to the proof of Eq. 3.34. In this case the formula 3.52 has the form

$$I_1(T) = Q_1(T) + Q_2(T), \tag{3.80}$$

where

$$\begin{aligned}Q_1(T) = \frac{1}{\log T}&\int_{\mathbb{R}^d}\int_0^{\varepsilon/2}p_s(xT^{-1/\alpha}-y)\\ &\left(\int_s^1\int_{\mathbb{R}^d}p_{u-s}(y-z)\widetilde{\varphi}_T(z)\chi(u)dzdu\right)^{1+\beta}dyds\mu(dx),\end{aligned} \tag{3.81}$$

$$Q_2(T) = \frac{1}{\log T}\int_{\mathbb{R}^d}\int_{\varepsilon/2}^1\int_{\mathbb{R}^d}\ldots dyds\mu(dx). \tag{3.82}$$

In Eq. 3.81 we have $u - s > \varepsilon/2$, hence $p_{u-s}(y-z) \le C(\varepsilon/2)^{-d/\alpha}$. Therefore

$$Q_1(T) \le C_1(\varepsilon)\left(\int_{\mathbb{R}^d}\varphi(z)dz\right)^{1+\beta}\mu(\mathbb{R}^d)(t_2-t_1)^{1+\beta} \le C_2(\varepsilon)(t_2-t_1)^{1+\sigma}. \tag{3.83}$$

In Eq. 3.82 we estimate $p_s(xT^{-1/\alpha}-y)$ by $C(\varepsilon/2)^{-d/\alpha}$, hence

$$Q_2(T) \le C_3(\varepsilon)\int_0^1\int_{\mathbb{R}^d}\left(\int_s^1\int_{\mathbb{R}^d}p_{u-s}(y-z)\widetilde{\varphi}_T(z)dz\chi(u)du\right)^{1+\beta}dyds.$$

The last expression is identical with the estimate of $I\!I$ in [7], and it was shown there that it can be estimated by $C(t_2-t_1)^{1+\sigma}$, provided that $d < \alpha(1+\beta)/\beta$, which holds in our case by Eq. 2.18. This, together with Eqs. 3.83, 3.80 and 3.37, proves Eq. 3.34, so the proof of part (b) is complete.

Proof of part (c) First we show that under the assumption Eq. 2.20,

$$\sup_{x\in\mathbb{R}^d} G((G\varphi)^{1+\beta})(x) < \infty, \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{3.84}$$

We have

$$G(G\varphi)^{1+\beta}(x) = C_{\alpha,d}\int_{|x-y|<1}\frac{1}{|x-y|^{d-\alpha}}(G\varphi)^{1+\beta}(y)dy + C_{\alpha,d}g*(G\varphi)^{1+\beta}(x),$$

where $g(x) = 1\!\!1_{[1,\infty)}(|x|)|x|^{\alpha-d}$. The first term is bounded since $G\varphi$ is bounded. To show that the second term is bounded it suffices to find $p, q \geq 1$, $1/p + 1/q = 1$, such that $g \in L^p$ and $(G\varphi)^{1+\beta} \in L^q$. Fix q such that

$$\frac{d}{\alpha} > q > \max\left\{\frac{d}{(1+\beta)(d-\alpha)}, 1\right\}$$

(such q exists by Eq. 2.20). Then Eq. 3.76 implies that $G^{1+\beta}\varphi \in L^q$, and it is clear that $g \in L^p$ for the corresponding p.

We now study the convergence of I_1, I_2 and I_3 defined by Eqs. 3.13–3.15. We have

$$I_1(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_s\left(\int_0^{T-s}\mathcal{T}_u\varphi\chi\left(\frac{s}{T}+\frac{u}{T}\right)du\right)^{1+\beta}(x)ds\mu(dx). \tag{3.85}$$

It is not difficult to see that Eq. 3.84 implies that

$$\lim_{T\to\infty} I_1(T) = \int_{\mathbb{R}^d} G((G\varphi)^{1+\beta})(x)\mu(dx)(\chi(0))^{1+\beta}. \tag{3.86}$$

By Eq. 3.21, Eqs. 3.73 and 3.84 we have

$$J_1(T) \leq \frac{C}{H_T}\int_{\mathbb{R}^d} G((G\varphi)^{1+\beta})(x)\mu(dx) \to 0.$$

Similarly, using Eqs. 3.22 and 3.84,

$$J_2(T) \leq C\frac{T}{H_T^{\beta}} \to 0,$$

by Eq. 2.16. Hence we obtain Eq. 3.18, by Eqs. 3.20 and 3.86. Next, for $\beta < 1$, Eq. 3.78 implies 3.17. This together with Eq. 3.18, 3.86 and 3.12 yield Eq. 3.5 in the case $\beta < 1$ with X determined by Eq. 2.22. To obtain Eq. 3.5 in the case $\beta = 1$ it remains to show that

$$\lim_{T\to\infty} I_2(T) = \int_{\mathbb{R}^d} G(\varphi G\varphi)\mu(dx)(\chi(0))^2. \tag{3.87}$$

Using Eqs. 3.14 and 3.8 we write

$$I_2(T) = I_2'(T) - I_2''(T) - I_3'''(T),$$

where

$$I_2'(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left(\varphi\chi\left(\frac{T-s}{T}\right)\int_0^s \mathcal{T}_{s-u}\varphi\chi\left(\frac{T-s}{T}\right)du\right)(x)ds\mu(dx),$$

$$I_2''(T) = \int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left(\varphi\chi\left(\frac{T-s}{T}\right)\int_0^s \mathcal{T}_{s-u}\varphi\chi\left(\frac{T-u}{T}\right)v_T(\cdot,u)du\right)(x)ds\mu(dx),$$

$$I_2'''(T) = \frac{V}{2}H_T^{1/2}\int_{\mathbb{R}^d}\int_0^T \mathcal{T}_{T-s}\left(\varphi\chi\left(\frac{T-s}{T}\right)\int_0^s \mathcal{T}_{s-u}v_T^2(\cdot,u)du\right)(x)ds\mu(dx).$$

It is easy to see that $I_2'(T)$ converges to the right-hand side of Eq. 3.87. To show that $I_2''(T)$ and $I_2'''(T)$ converge to 0, we first apply Eq. 3.10, and then use Eqs. 3.73 and 3.84.

Finally, we pass to the proof of tightness. For $0 < \varepsilon \le t_1 < t_2$, by Eqs. 3.38 and 3.25 we have

$$\begin{aligned} A(T) &\le C\mu(\mathbb{R}^d) \sup_{x\in\mathbb{R}^d} \int_0^T \mathcal{T}_s\left(\varphi \int_s^T \mathcal{T}_{u-s}\varphi\chi\left(\frac{u}{T}\right)du\right)(x)\chi\left(\frac{s}{T}\right)ds \\ &\le C_1 \sup_{x\in\mathbb{R}^d} \int_{t_1T}^{t_2T}\int_{\mathbb{R}^d} s^{-d/\alpha} p_1((x-y)x^{-1/\alpha})\varphi(y)\int_0^{t_2T-s}\mathcal{T}_u\varphi(y)dudyds \\ &\le C_2\varepsilon^{-d/\alpha}T^{1-d/\alpha}(t_2-t_1)\int_{\mathbb{R}^d}\varphi(y)dy\sup_y\int_0^{(t_2-t_1)T}\mathcal{T}_u\varphi(y)du \\ &\le C_3(\varepsilon)T^{1-d/\alpha+\sigma}(t_2-t_1)^{1+\sigma}(\sup_y G\varphi(y))^{1-\sigma} \\ &\le C_4(\varepsilon)(t_2-t_1)^{1+\sigma}, \end{aligned}$$

for any

$$0 < \sigma < \left(\frac{d}{\alpha}-1\right)\wedge 1, \tag{3.88}$$

so we obtain Eq. 3.33.

To derive Eq. 3.34 we use Eqs. 3.37, 3.85 and 3.25, obtaining

$$I_1(T) \le \mu(\mathbb{R}^d)\sup_{x\in\mathbb{R}^d}(Z_1(T,x)+Z_2(T,x)+Z_3(T,x)), \tag{3.89}$$

where

$$Z_1(T,x) = \int_0^{t_1T/2}\mathcal{T}_s\left(\int_{t_1T-s}^{t_2T-s}\mathcal{T}_u\varphi du\right)^{1+\beta}(x)ds, \tag{3.90}$$

$$Z_2(T,x) = \int_{t_1T/2}^{t_1T}\mathcal{T}_s\left(\int_{t_1T-s}^{t_2T-s}\mathcal{T}_u\varphi du\right)^{1+\beta}(x)ds, \tag{3.91}$$

$$Z_3(T,x) = \int_{t_1T}^{t_2T}\mathcal{T}_s\left(\int_0^{t_2T-s}\mathcal{T}_u\varphi du\right)^{1+\beta}(x)ds. \tag{3.92}$$

By self-similarity we have

$$Z_1(T,x) \le C\int_0^{t_1T/2}\left(\int_{t_1T-s}^{t_2T-s}u^{-d/\alpha}du\right)^{1+\beta}ds.$$

As $u \ge t_1T - s \ge t_1T/2 \ge \varepsilon T/2$, we get

$$\begin{aligned} Z_1(T,x) &\le C_1\varepsilon^{1-(d/\alpha)(1+\beta)}T^{2+\beta-(d/\alpha)(1+\beta)}(t_2-t_1)^{1+\beta}, \\ &\le C_2(\varepsilon)(t_2-t_1)^{1+\beta}, \end{aligned} \tag{3.93}$$

by Eq. 2.20.

To estimate Z_2 we first use the bound $p_s(x-y) \leq C(T\varepsilon/2)^{-d/\alpha}$ for $s \geq t_1 T/2$. After obvious substitutions we have

$$Z_2(T,x) \leq C(\varepsilon)(Z_2'(T,x) + Z_2''(T,x)), \tag{3.94}$$

where

$$Z_2'(T,x) = T^{-d/\alpha} \int_0^1 \int_{\mathbb{R}^d} \left(\int_s^{(t_2-t_1)T+s} \mathcal{T}_u\varphi(y)du \right)^{1+\beta} dyds, \tag{3.95}$$

$$Z_2''(T,x) = T^{-d/\alpha} \left(\int_1^{t_1 T/2} \int_{\mathbb{R}^d} \dots dyds \right)^+. \tag{3.96}$$

For any $0 < \sigma \leq \beta$ we have

$$\left(\int_s^{(t_2-t_1)T+s} \mathcal{T}_u\varphi(y)du \right)^{\beta} \leq (G\varphi(y))^{\beta-\sigma} \left(\sup_{y\in\mathbb{R}^d} \varphi(y) \right)^{\sigma} ((t_2-t_1)T)^{\sigma}.$$
$$\leq C(t_2-t_1)^{\sigma} T^{\sigma}, \tag{3.97}$$

by Eq. 3.73. Applying this to Eq. 3.95 we obtain

$$Z_2'(T,x) \leq C_1 T^{-d/\alpha+1+\sigma}(t_2-t_1)^{1+\sigma} \leq C_1(t_2-t_1)^{1+\sigma}, \tag{3.98}$$

provided that

$$0 < \sigma < \left(\frac{d}{\alpha} - 1 \right) \wedge \beta. \tag{3.99}$$

In order to estimate Z_2'' we notice that for $d > \alpha$ and $0 < a < b$,

$$\int_a^b \mathcal{T}_u\varphi(y)du \leq C \int_a^b u^{-d/\alpha}du \leq \begin{cases} C(b-a)a^{-d/\alpha}, \\ C_1 a^{1-d/\alpha}. \end{cases}$$

Using these two bounds, instead of Eq. 3.97 we now have for $0 < \sigma \leq \beta$,

$$\left(\int_s^{(t_2-t_1)T+s} \mathcal{T}_u\varphi(y)du \right)^{\beta} \leq C_2 s^{(1-d/\alpha)(\beta-\sigma)}((t_2-t_1)Ts^{-d/\alpha})^{\sigma}.$$

Putting this into Eq. 3.96 we obtain for $t_1 T/2 > 1$,

$$\begin{aligned} Z_2''(T,x) &\leq C_3 T^{-d/\alpha+1+\sigma} \int_1^{t_1 T/2} s^{-\sigma+(1-d/\alpha)\beta} dx (t_2-t_1)^{1+\sigma} \\ &\leq C_3 T^{-d/\alpha+1+\sigma} \max(1, T^{1-\sigma+(1-d/\alpha)\beta} \log T)(t_2-t_1)^{1+\sigma} \\ &\leq C_4(t_2-t_1)^{1+\sigma}, \end{aligned} \tag{3.100}$$

provided that Eq. 3.99 holds, and we also use Eq. 2.20. Combining Eqs. 3.94, 3.98 and 3.100 we arrive at

$$Z_2(T,x) \leq C(t_2-t_1)^{1+\sigma} \tag{3.101}$$

for σ satisfying Eq. 3.99.

Finally, by the Hölder inequality and using the fact that $t_2T - s \leq (t_2 - t_1)T$, we have

$$\begin{aligned} Z_3(T,x) &\leq \left(\int_{t_1T}^{t_2T}\int_{\mathbb{R}^d} p_s(x-y)dyds\right)^{1/(2+\beta)} \\ &\quad \times\left[\int_{t_1T}^{t_2T}\int_{\mathbb{R}^d} p_s(x-y)\left(\int_0^{(t_2-t_1)T}\mathcal{T}_u\varphi(y)du\right)^{2+\beta}dyds\right]^{(1+\beta)/(2+\beta)} \\ &\leq C((t_2-t_1)T)^{1/(2+\beta)} \\ &\quad \times\left[\int_{t_1T}^{t_2T}\int_{\mathbb{R}^d} s^{-d/\alpha}(G\varphi(y))^{2+\beta-\sigma}((t_2-t_1)T)^{\sigma}dyds\right]^{(1+\beta)/(2+\beta)} \end{aligned} \tag{3.102}$$

for any $0 < \sigma \leq 2+\beta$, by an argument as in Eq. 3.97. Observe that by Eq. 3.76,

$$\int_{\mathbb{R}^d}(G\varphi(y))^{2+\beta-\sigma}dy < \infty$$

for σ sufficiently small, satisfying

$$\frac{d}{\alpha} > \frac{2+\beta-\sigma}{1+\beta-\sigma}. \tag{3.103}$$

Hence, by Eq. 3.102 we have

$$\begin{aligned} Z_3(T,x) &\leq C(\varepsilon)(t_2-t_1)^{1+\sigma(1+\beta)/(2+\beta)}T^{1+\sigma(1+\beta)/(2+\beta)-(d/\alpha)(1+\beta)/(2+\beta)} \\ &\leq C(\varepsilon)(t_2-t_1)^{1+\sigma(1+\beta)/(2+\beta)}, \end{aligned} \tag{3.104}$$

provided that

$$\sigma < \frac{d}{\alpha}\frac{1+\beta}{2+\beta} - 1. \tag{3.105}$$

Combining Eqs. 3.89, 3.93, 3.101 and 3.104, we conclude that Eq. 3.34 holds (with $\sigma(1+\beta)/(2+\beta)$ instead of σ) for any σ satisfying Eqs. 3.99, 3.103 and 3.105.

The proof of Theorem 2.7 is complete. □

Proof of Proposition 2.9 Only part (d) of the proposition needs to be proved. The argument is similar to that used in the proof of Theorem 2.7 in [7].

Observe that the finite-dimensional distributions of the process ζ defined by Eq. 2.2 are determined by

$$\begin{aligned} &E\exp\{i(z_1\xi_{t_1} + \ldots + z_k\xi_{t_k})\} \\ &= \exp\left\{-\int_{\mathbb{R}^{d+1}}\left[\left|\sum_{j=1}^k z_j p_r^{1/(1+\beta)}(x)\mathbb{1}_{[0,t_j]}(r)\int_r^{t_j}p_{u-r}(x)du\right|^{1+\beta}\right.\right. \\ &\quad \left.\left.\times\left(1 - i\,\mathrm{sgn}\left(\sum_{j=1}^k z_j p_r^{1/(1+\beta)}(x)\mathbb{1}_{[0,t_j]}(r)\int_r^{t_j}p_{u-r}(x)du\right)\tan\frac{\pi}{2}(1+\beta)\right)\right]drdx \right. \end{aligned} \tag{3.106}$$

(see Proposition 3.4.2 of [24]).

Denote

$$D_T^+ = D_T(1, z; u, v, s, t), \quad z > 0,$$
$$D_T^- = D_T(1, -z; u, v, s, t), \quad z > 0,$$

(see Eq. 2.12). It suffices to show that for fixed $0 \le u < v < s < t$ and $z > 0$,

$$D_T^+ \le CT^{-d/\alpha}, \quad D_T^- \le CT^{-d/\alpha}, \tag{3.107}$$

and for T sufficiently large,

$$D_T^+ \ge CT^{-d/\alpha} \tag{3.108}$$

(see Eq. 2.11).

It will be convenient to denote

$$f = f(x,r) = z\int_{s+T}^{t+T} p_{r'-r}(x)dr',$$
$$g_1 = g_1(x,r) = \int_u^v p_{r'-r}(x)dr',$$
$$g_2 = g_2(x,r) = \int_r^v p_{r'-r}(x)dr'.$$

It is not difficult to see that by Eq. 3.106,

$$D_T^+ = C\left[\int_0^u\int_{\mathbb{R}^d} p_r(x)((f+g_1)^{1+\beta} - f^{1+\beta} - g_1^{1+\beta})dxdr \right.$$
$$\left. + \int_u^v\int_{\mathbb{R}^d} p_r(x)((f+g_2)^{1+\beta} - f^{1+\beta} - g_2^{1+\beta})dxdr\right]. \tag{3.109}$$

By the elementary inequality

$$0 \le (a+b)^{1+\beta} - a^{1+\beta} - b^{1+\beta} \le (1+\beta)ab^{\beta}, \quad a, b \ge 0,\ 0 < \beta \le 1,$$

and the estimate

$$f(x,r) \le CT^{-d/\alpha},$$

we have

$$D_T^+ \le C_1\int_{\mathbb{R}^d}\left[\int_0^u p_r(x)\, fg_1^{\beta}dr + \int_u^v p_r(x)\, fg_2^{\beta}dr\right]dx$$
$$\le C_2T^{-d/\alpha}\int_{\mathbb{R}^d}\left(\int_0^v p_r(x)dr\right)^{1+\beta}dx$$
$$\le C_3T^{-d/\alpha}, \tag{3.110}$$

by Eq. 3.3. One can show that for D_T^- the estimate (3.110) also holds (see [7] for details). Hence Eq. 3.107 follows.

Next, by Eq. 3.109,

$$D_T^+ \geq C \int_u^{(u+v)/2} \int_{|x|\leq 1} p_r(x)((f+g_2)^{1+\beta} - f^{1+\beta} - g^{1+\beta})dxdr$$
$$\geq C_1 \int_u^{(u+v)/2} \int_{|x|\leq 1} ((f+g_2)^{1+\beta} - f^{1+\beta} - g^{1+\beta})dxdr,$$

and this is exactly the right-hand side of (4.18) in [7], and it was proved there that it is greater than $CT^{-d/\alpha}$ for large T. Thus Eq. 3.108 holds. □

Proof of Theorem 2.10 The theorem can be proved using the corresponding version of the general scheme (see Eq. 3.39) and the discussion following it). The arguments are similar to those carried out in the branching case and they are easier, therefore we omit the proof. We only indicate how to obtain the process ζ in part (a).

It is easy to see that $I\!I_1(T)$ defined by Eq. 3.40 can be written as

$$I\!I_1(T) = \int_{\mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d} \int_s^1 \int_{\mathbb{R}^d} s^{-d/\alpha} p_1((x-y)T^{-1/\alpha}s^{-1/\alpha})\varphi(y)\chi(s)(u-s)^{-d/\alpha}$$
$$\times p_1((y-z)T^{-1/\alpha}(u-s)^{-1/\alpha})\varphi(z)\chi(u)dzdudydsd\mu(dx)$$
$$\to p_1^2(0)\mu(\mathbb{R}^d)\left(\int_{\mathbb{R}^d}\varphi(y)dy\right)^2 \int_0^1\int_s^1 (u-s)^{-d/\alpha}s^{-d/\alpha}\chi(s)\chi(u)duds,$$

and this is exactly the logarithm of right-hand side of Eq. 3.42 with

$$X = \left(\frac{2p_1^2(0)\mu(\mathbb{R}^d)}{1-\frac{d}{\alpha}}\right)^{1/2}\lambda\rho,$$

and ρ is Gaussian with covariance (2.25). □

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