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Stochastic Evolution Equations of Jump Type: Existence, Uniqueness and Large Deviation Principles

Michael Röckner · Tusheng Zhang

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Abstract This paper has two parts. In part I, existence and uniqueness results are established for solutions of stochastic evolution equations driven both by Brownian motion and by Poisson point processes. Exponential integrability of the solution are also proved. In part II, a large deviation principle is obtained for stochastic evolution equations driven by additive Lévy noise.

Key words evolution equations · Poisson measure · exponential integrability · large deviation principles

Mathematics Subject Classifications (2000) Primary 60H15 · Secondary 93E20 · 35R60

1 Introduction

Stochastic evolution equations and stochastic partial differential equations driven by Wiener processes have been studied by many people. There exists a great amount of literature on the subject, see, for example the monograph [22]. In contrast, there has not been very much study of stochastic partial differential equations driven by jump processes. However, it begun to gain attention recently. In [3] we obtained existence and uniqueness for solutions of stochastic reaction diffusion equations

M. Röckner
Department of Mathematics, University of Bielefeld, 33501 Bielefeld, Germany

M. Röckner
Departments of Mathematics and Statistics, Purdue University, West Lafayette, IN 47907, USA

T. Zhang (✉)
Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, England, UK
e-mail: tzhang@maths.man.ac.uk

driven by Poisson random measures. In [10], Malliavin calculus was applied to study the absolute continuity of the law of the solutions of stochastic reaction diffusion equations driven by Poisson random measures. In [19], a minimal solution was obtained for the stochastic heat equation driven by non-negative Lévy noise with coefficients of polynomial growth. In [20], a weak solution is established for the stochastic heat equation driven by stable noise with coefficients of polynomial growth.

In this paper, we consider the following evolution equation:

$$dY_t = -AY_t dt + b(Y_t)dt + \sigma(Y_t)dB_t + \int_X f(Y_{t-}, x)\tilde{N}(dt, dx), \tag{1.1}$$

$$Y_0 = h \in H, \tag{1.2}$$

in the framework of a Gelfand triple :

$$V \subset H \cong H^* \subset V^*, \tag{1.3}$$

where H, V are Hilbert spaces, A is the infinitesimal generator of a strongly continuous semigroup, b, σ, f are measurable mappings from H into H. The solutions are considered to be weak solutions (in the PDE sense) in the space V and not as mild solutions in H as is more common in the literature. The stochastic evolution equations of this type driven by Wiener processes were first studied by E.Pardoux in [21] and subsequently in [17]. For stochastic equations with general Hilbert space valued semimartingales replacing the Brownian motion we refer to [14, 15] and [13]. A large deviation principle for this type of stochastic evolution equations driven by Wiener process was obtained by P.Chow in [4].

The purpose of this paper is twofold. The first one is to establish the existence and uniqueness for solutions of Eq. 1.1. Our approach is similar to the one in [13–15, 21]. We, however, don't use Galerkin approximations. Instead, we get the solution via successive approximations. Secondly we will study the large deviation principle and exponential integrability of the solutions. We will prove that the solution is exponentially integrable both as a random variable in $D([0, 1] \to H)$ and in $L^2([0, 1] \to V)$. These estimates are of their own interest and also necessary for the study of large deviations. For the large deviation principle, we confine ourselves to the case of additive Lévy noise. The situation is quite different from the Gaussian case. If $X_t, t \geq 0$ is a Wiener process, the solution of the equation:

$$dY_t^n = -AY_t^n dt + \frac{1}{n} dX_t$$

is still Gaussian. The large deviations of Y^n follows from the well known large deviations of Gaussian processes. However, if $X_t, t \geq 0$, is a Lévy process, the solution Y^n is no longer a Lévy process. The additive noise case is already quite involved. Large deviations for stochastic evolution equations and stochastic partial differential equations have been investigated in many papers, see e.g. [6–8, 24]. To the best of our knowledge, this is the first to study large deviations for stochastic evolution equations driven by Lévy processes. Large deviations for Lévy processes on Banach spaces and large deviations for solutions of stochastic differential equations driven by Poisson measures were studied by de Acosta in [1, 2].

The rest of the paper is organized as follows. In Section 2 we present our framework. Existence and uniqueness are proved in Section 3. In Section 4, exponential integrability is established for the solutions. Section 5 is devoted to the large deviation principle.

2 Framework

Let V, H be two separable Hilbert spaces such that V is continuously, densely imbedded in H. Identifying H with its dual we have

$$V \subset H \cong H^* \subset V^*, \tag{2.1}$$

where V^* stands for the topological dual of V. Let A be a bounded linear operator from V to V^* satisfying the following coercivity hypothesis: There exist constants $\alpha > 0$ and $\lambda_0 \geq 0$ such that

$$2\langle Au, u\rangle + \lambda_0 |u|_H^2 \geq \alpha ||u||_V^2 \qquad \text{for all } u \in V. \tag{2.2}$$

$\langle Au, u\rangle = Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.

We remark that A is generally not bounded as an operator from H into H. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let $\{B_t, t \geq 0\}$ be a real-valued $\mathcal{F}_t$- Brownian motion. $\{W_t, t \geq 0\}$ will denote an H-valued $\mathcal{F}_t$- Brownian motion with covariance operator Q which could be degenerate. Let $(X, \mathcal{B}(X))$ be a measurable space and $\nu(dx)$ a σ-finite measure on it. Let $p = (p(t)), t \in D_p$ be a stationary $\mathcal{F}_t$-Poisson point process on X with characteristic measure ν. See [16] for the details on Poisson point processes. Denote by $N(dt, dx)$ the Poisson counting measure associated with p, i.e., $N(t, A) = \sum_{s\in D_p, s\leq t} I_A(p(s))$. Let $\tilde{N}(dt, dx) := N(dt, dx) - dt\nu(dx)$ be the compensated Poisson mesasure. Let b, σ be measurable mappings from H into H, and $f(y, x)$ a measurable mapping from $H \times X$ into H. For a separable Hilbert space L, we denote by $M^2([0, T], L)$ the Hilbert space of progressively measurable, square integrable, L-valued processes equipped with the inner product $< a, b >_M = E[\int_0^T < a_t, b_t >_L dt]$. Denote by $M^{\nu,2}([0, T] \times X, L)$ the collection of predictable mappings:

$$g(s, x, \omega) : [0, T] \times X \times \Omega \to L$$

such that $E[\int_0^T \int_X |g(s, x, \omega)|_L^2 ds\nu(dx)] < \infty$. Denote by $D([0, T], L)$ the space of all cadlag paths from $[0, T]$ into L. Consider the stochastic evolution equation:

$$dY_t = -AY_t dt + b(Y_t)dt + \sigma(Y_t)dB_t + \int_X f(Y_{t-}, x)\tilde{N}(dt, dx), \tag{2.3}$$

$$Y_0 = h \in H. \tag{2.4}$$

We introduce

(H.1) There exists a constant $C < \infty$ such that

$$|b(y)|_H^2 + |\sigma(y)|_H^2 + \int_X |f(y, x)|_H^2 \nu(dx) \leq C\left(1 + |y|_H^2\right), \tag{2.5}$$

for all $y \in H$.

(H.2) There exists a constant $C < \infty$ such that

$$\begin{aligned}&|b(y_1) - b(y_2)|_H^2 + |\sigma(y_1) - \sigma(y_2)|_H^2\\&\quad + \int_X |f(y_1, x) - f(y_2, x)|_H^2 \nu(dx)\\&\quad \le C|y_1 - y_2|_H^2,\end{aligned} \tag{2.6}$$

for all $y_1, y_2 \in H$.

We finish this section by two examples.

Example 2.1 Let $H = L^2(\mathbf{R}^d)$, and set

$$V = H_2^1(\mathbf{R}^d) = \left\{u \in L^2(R^d);\, \nabla u \in L^2(\mathbf{R}^d \to \mathbf{R}^d)\right\}.$$

Denote by $a(x) = (a_{ij}(x))$ a matrix-valued function on $\mathbf{R}^d$ satisfying the uniform ellipticity condition:

$$\frac{1}{c} I_d \le a(x) \le c I_d \qquad \text{for some constant } \ c \in (0, \infty).$$

Let $b(x)$ be a vector field on $\mathbf{R}^d$ with $b \in L^p(\mathbf{R}^d)$ for some $p > d$. Define

$$Au = -div(a(x)\nabla u(x)) + b(x) \cdot \nabla u(x).$$

Then Eq. 2.2 is fulfilled for (H, V, A).

Example 2.2 Stochastic evolution equations associated with fractional Laplacian:

$$dY_t = \Delta_\alpha Y_t dt + dL_t, \tag{2.7}$$

$$Y_0 = h \in H, \tag{2.8}$$

where Δ_α denotes the generator of the symmetric α-stable process in R^d, $0 < \alpha \le 2$. Δ_α is called the fractional Laplace operator. L_t stands for a Lévy process. It is well known that the Dirichlet form associated with Δ_α is given by

$$\mathcal{E}(u, v) = K(d, \alpha) \int\int_{R^d \times R^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dxdy,$$

$$D(\mathcal{E}) = \left\{u \in L^2\left(R^d\right): \quad \int\int_{R^d \times R^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} dxdy < \infty\right\},$$

where $K(d, \alpha) = \alpha 2^{\alpha-3} \pi^{-\frac{d+2}{2}} sin(\frac{\alpha\pi}{2}) \Gamma(\frac{d+\alpha}{2}) \Gamma(\frac{\alpha}{2})$. To study Eq. 2.7, we choose $H = L^2(\mathbf{R}^d)$, and $V = D(\mathcal{E})$ with the inner product $< u, v >= \mathcal{E}(u, v) + (u, v)_{L^2(R^d)}$.

Define

$$Au = -\Delta_\alpha.$$

Then Eq. 2.2 is fulfilled for (H, V, A). See [12] for details about the fractional Laplace operator.

3 Existence and Uniqueness

In [14, 15] and [13] it is explained how under certain conditions results on existence and uniqueness of solutions for equations with random measures as driving term can be derived from there results. In this section we prove existence and uniqueness for solutions of our Eq. 1.1, however, directly by a different method.

Proposition 3.1 *Let* $b \in M^2([0,T],H)$, $\sigma \in M^2([0,T],H)$ *and* $f \in M^{\nu,2}([0,T]\times X, H)$. *There exists a unique solution* $Y_t, t \geq 0$ *to the following equation:*

$$\begin{aligned} Y &\in M^2([0,T],V) \cap D([0,T],H), \\ dY_t &= -AY_t dt + b(t,\omega)dt + \sigma(t,\omega)dB_t \\ &\quad + \int_X f(t,x,\omega)\tilde{N}(dt,dx), \qquad (3.1) \\ Y_0 &= h \in H. \qquad (3.2) \end{aligned}$$

Proof We prove the existence in two steps.

Step 1. Assume $b \in M^2([0,T],V)$, $\sigma \in M^2([0,T],V)$ and $f \in M^{\nu,2}([0,T]\times X, V)$. Set

$$pU_t = \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t \int_X f(s,x)\tilde{N}(ds,dx).$$

It is easy to see that $U \in M^2([0,T],V)$. Consider the random equation:

$$\begin{aligned} dv_t &= (-Av_t - AU_t)dt, \qquad (3.3) \\ v_0 &= h. \end{aligned}$$

It is known from [18] that there exists a unique solution v to Eq. 3.3 such that $v \in M^2([0,T],V) \cap C([0,T],H)$. Set $Y_t := v_t + U_t$. Then $Y \in M^2([0,T],V) \cap D([0,T],H)$. Moreover, it solves Eq. 3.1.

Step 2. General case.
Choose $b_n \in M^2([0,T],V)$, $\sigma_n \in M^2([0,T],V)$ and $f_n \in M^{\nu,2}([0,T]\times X, V)$ such that $b_n \to b$, $\sigma_n \to \sigma$ in $M^2([0,T],H)$ and $f_n \to f$ in $M^{\nu,2}([0,T]\times X, H)$

as $n \to \infty$. Denote by Y_t^n the unique solution to Eq. 3.1 with b, σ, f replaced by b_n, σ_n, f_n. Such a Y^n exists by step 1. By Ito's formula, we have

$$
\begin{aligned}
|Y_t^n - Y_t^m|_H^2 = &-2\int_0^t < Y_s^n - Y_s^m, A(Y_s^n - Y_s^m) > ds \\
&+2\int_0^t < Y_s^n - Y_s^m, b_n(s) - b_m(s) > ds \\
&+2\int_0^t < Y_s^n - Y_s^m, \sigma_n(s) - \sigma_m(s) > dB_s \\
&+\int_0^t |\sigma_n(s) - \sigma_m(s)|_H^2 ds \\
&+\int_0^t \int_X \Big(| f_n(s,x) - f_m(s,x)|_H^2 + 2 < Y_{s-}^n - Y_{s-}^m, \\
&\qquad f_n(s,x) - f_m(s,x) >_H \Big) \tilde{N}(ds,dx) \\
&+\int_0^t \int_X | f_n(s,x) - f_m(s,x)|_H^2 ds\nu(dx).
\end{aligned}
\tag{3.4}
$$

In the following, C will denote a generic constant whose values might change from line to line. Set

$$
\begin{aligned}
M_t = \int_0^t \int_X \;\; &\Big(| f_n(s,x) - f_m(s,x)|_H^2 + 2 < Y_{s-}^n \\
&- Y_{s-}^m, f_n(s,x) - f_m(s,x) >_H \Big) \tilde{N}(ds,dx).
\end{aligned}
$$

Then,

$$
\begin{aligned}
[M,M]_t^{\frac{1}{2}} &= \Bigg\{ \sum_{s\in D_p, s\le t} \Big(| f_n(s,p(s)) - f_m(s,p(s))|_H^2 + 2 < Y_{s-}^n \\
&\qquad - Y_{s-}^m, f_n(s,p(s)) - f_m(s,p(s)) >_H \Big)^2 \Bigg\}^{\frac{1}{2}} \\
&\le C\Big(\sum_{s\in D_p, s\le t} | f_n(s,p(s)) - f_m(s,p(s))|_H^4 \Big)^{\frac{1}{2}} \\
&\quad + C\Big(\sum_{s\in D_p, s\le t} |Y_{s-}^n - Y_{s-}^m|_H^2 | f_n(s,p(s)) - f_m(s,p(s))|_H^2 \Big)^{\frac{1}{2}} \\
&\le C \sum_{s\in D_p, s\le t} | f_n(s,p(s)) - f_m(s,p(s))|_H^2 \\
&\quad + C \sup_{0\le s\le t} (|Y_{s-}^n - Y_{s-}^m|_H) \Big(\sum_{s\in D_p, s\le t} | f_n(s,p(s)) - f_m(s,p(s))|_H^2 \Big)^{\frac{1}{2}} \\
&\le C \sum_{s\in D_p, s\le t} | f_n(s,p(s)) - f_m(s,p(s))|_H^2 + \frac{1}{4} \sup_{0\le s\le t} \big(|Y_{s-}^n - Y_{s-}^m|_H^2 \big).
\end{aligned}
$$

By Burkholder's inequality,

$$E\left[\sup_{0\le s\le t}|M_s|\right]\le CE([M,M]_t^{\frac{1}{2}})$$
$$\le CE\Big[\sum_{s\in D_p,s\le t}|f_n(s,p(s))-f_m(s,p(s))|_H^2\Big]+\frac{1}{4}E\left[\sup_{0\le s\le t}|Y_{s-}^n-Y_{s-}^m|_H^2\right]$$
$$=CE\left[\int_0^t\int_X|f_n(s,x)-f_m(s,x)|_H^2 ds\nu(dx)\right]+\frac{1}{4}E\left[\sup_{0\le s\le t}|Y_s^n-Y_s^m|_H^2\right]. \tag{3.5}$$

It follows from Eqs. 3.4 and 2.2 that

$$E\left[\sup_{0\le s\le t}|Y_s^n-Y_s^m|_H^2\right]\le-\alpha E\left[\int_0^t||Y_s^n-Y_s^m||_V^2ds\right]$$
$$+(\lambda_0+C)E\left[\int_0^t|Y_s^n-Y_s^m|_H^2ds\right]$$
$$+CE\left[\int_0^t|b_n(s)-b_m(s)|_H^2ds\right]$$
$$+CE\left[\left(\int_0^t< Y_s^n-Y_s^m,\sigma_n(s)-\sigma_m(s)>_H^2ds\right)^{\frac{1}{2}}\right]$$
$$+CE\left[\int_0^t|\sigma_n(s)-\sigma_m(s)|_H^2ds\right]$$
$$+CE\left([M,M]_t^{\frac{1}{2}}\right)$$
$$+CE\left[\int_0^t\int_X|f_n(s,x)-f_m(s,x)|_H^2ds\nu(dx)\right].$$

Applying Eq. 3.5 we have

$$E\left[\sup_{0\le s\le t}|Y_s^n-Y_s^m|_H^2\right]$$
$$\le-\alpha E\left[\int_0^t||Y_s^n-Y_s^m||_V^2ds\right]+CE\left[\int_0^t|Y_s^n-Y_s^m|_H^2ds\right]$$
$$+\frac{1}{2}E\left[\sup_{0\le s\le t}(|Y_{s-}^n-Y_{s-}^m|_H^2\right]+CE\left[\int_0^t|b_n(s)-b_m(s)|_H^2ds\right]$$
$$+CE\left[\int_0^t|\sigma_n(s)-\sigma_m(s)|_H^2ds\right]+CE\left[\int_0^t\int_X|f_n(s,x)-f_m(s,x)|_H^2ds\nu(dx)\right]. \tag{3.6}$$

By Gronwall's inequality, this implies that

$$E\left[\sup_{0\le s\le t}|Y^n_s-Y^m_s|^2_H\right]\le Ce^{Ct}\ \Bigg\{E\left[\int_0^t|b_n(s)-b_m(s)|^2_H ds\right] + CE\left[\int_0^t|\sigma_n(s)-\sigma_m(s)|^2_H ds\right] + CE\left[\int_0^t\int_X|f_n(s,x)-f_m(s,x)|^2_H ds\nu(dx)\right]\Bigg\}. \tag{3.7}$$

Therefore,

$$\lim_{n,m\to\infty}E\left[\sup_{0\le s\le t}|Y^n_s-Y^m_s|^2_H\right]=0. \tag{3.8}$$

This further implies by Eq. 3.6 that $Y^n, n\ge 1$ is also a Cauchy sequence in $M^2([0, T], V)$. Let $Y_t, t\ge 0$ denote an element in $M^2([0, T], V)$ such that

$$\lim_{n\to\infty}E\left[\sup_{0\le s\le t}|Y^n_s-Y_s|^2_H\right]=0,$$

and

$$\lim_{n\to\infty}E\left[\int_0^t||Y^n_s-Y_s||^2_V ds\right]=0.$$

Letting $n\to\infty$, we see that $Y_t, t\ge 0$ is a solution to Eq. 3.1.

Uniqueness: If X_t, Y_t are two solutions to Eq. 3.1, then

$$\begin{cases}\frac{d(X_t-Y_t)}{dt}=-A(X_t-Y_t),\\ X_0-Y_0=0.\end{cases}$$

By the chain rule, we have

$$\begin{aligned}|X_t-Y_t|^2_H&=-2\int_0^t<X_s-Y_s, A(X_s-Y_s)>ds\\ &\le-\alpha\int_0^t||X_s-Y_s||^2_V ds+\lambda_0\int_0^t|X_s-Y_s|^2_H ds\end{aligned}$$

By Gronwall's inequality, we obtain that $Y_t=X_t$, which completes the proof. □

Theorem 3.2 *Assume* (*H.1*) *and* (*H.2*). *Then there exists a unique* H*-valued progressively measurable process* (Y_t) *such that*

(1) $Y\in M^2(0,T;V)\cap D(0,T;H)$ *for any* $T>0$,
(2) $Y_t=h-\int_0^t AY_s ds+\int_0^t b(Y_s)ds+\int_0^t\sigma(Y_s)dB_s+\int_0^{t+}\int_X f(Y_{s-},x)\tilde{N}(ds,dx)$ *a.s.*,
(3) $Y_0=h\in H$.

Proof
Existence of solution Let $Y_t^0 := h, t \geq 0$. For $n \geq 0$, define $Y^{n+1} \in M^2(0, T; V) \cap D(0, T; H)$ to be the unique solution to the following equation:

$$\begin{aligned} dY_t^{n+1} &= -AY_t^{n+1}dt + b(Y_t^n)dt + \sigma(Y_t^n)dB_t \\ &\quad + f(Y_{t-}^n, x)\tilde{N}(dt, dx), \end{aligned} \tag{3.9}$$

$$Y_0^n = h. \tag{3.10}$$

The solution Y^{n+1} of the above equation exists according to Proposition 3.1. We are going to show that $\{Y^n, n \geq 1\}$ forms a Cauchy sequence. Using Itô's formula, we find that

$$\begin{aligned} |Y_t^{n+1} - Y_t^n|_H^2 =& -2\int_0^t \langle Y_s^{n+1} - Y_s^n, A\big(Y_s^{n+1} - Y_s^n\big)\rangle ds \\ &+ 2\int_0^t < Y_s^{n+1} - Y_s^n, b\big(Y_s^n\big) - b\big(Y_s^{n-1}\big) > ds \\ &+ 2\int_0^t < Y_s^{n+1} - Y_s^n, \sigma\big(Y_s^n\big) - \sigma\big(Y_s^{n-1}\big) > dB_s \\ &+ \int_0^t |\sigma\big(Y_s^n\big) - \sigma\big(Y_s^{n-1}\big)|_H^2 ds \\ &+ \int_0^{t+}\int_X \big[|f\big(Y_{s-}^n, x\big) - f\big(Y_{s-}^{n-1}, x\big)|_H^2 + 2 < Y_s^{n+1} - Y_s^n, f\big(Y_{s-}^n, x\big) \\ &\qquad - f\big(Y_{s-}^{n-1}, x\big) >\big]\tilde{N}(ds, dx) \\ &+ \int_0^t \int_X |f\big(Y_s^n, x\big) - f\big(Y_s^{n-1}, x\big)|_H^2 ds\nu(dx). \end{aligned} \tag{3.11}$$

By a similar calculation as in Proposition 3.1, it follows from Eq. 3.11 that

$$\begin{aligned} &E\Big[\sup_{0\leq s\leq t} |Y_s^{n+1} - Y_s^n|_H^2\Big] \\ &\leq -\alpha E\Big[\int_0^t ||Y_s^{n+1} - Y_s^n||_V^2 ds\Big] + CE\Big[\int_0^t |Y_s^{n+1} - Y_s^n|_H^2 ds\Big] \\ &\quad + \frac{1}{2}E\Big[\sup_{0\leq s\leq t} |Y_s^{n+1} - Y_s^n|_H^2\Big] + CE\Big[\int_0^t |b\left(Y_s^n\right) - b\left(Y_s^{n-1}\right)|_H^2 ds\Big] \\ &\quad + CE\Big[\int_0^t |\sigma\left(Y_s^n\right) - \sigma\left(Y_s^{n-1}\right)|_H^2 ds\Big] \\ &\quad + CE\Big[\int_0^t \int_X |f\left(Y_s^n, x\right) - f\left(Y_s^{n-1}, x\right)|_H^2 ds\nu(dx)\Big]. \end{aligned} \tag{3.12}$$

By virtue of (H.1), this implies that

$$E\left[\sup_{0\le s\le t}|Y_s^{n+1}-Y_s^n|_H^2\right]\le CE\left[\int_0^t|Y_s^{n+1}-Y_s^n|_H^2ds\right] + CE\left[\int_0^t|Y_s^n-Y_s^{n-1}|_H^2ds\right]. \tag{3.13}$$

Define

$$g_t^n=E\left[\sup_{0\le s\le t}|Y_s^n-Y_s^{n-1}|_H^2\right],\quad G_t^n=\int_0^t g_s^n ds.$$

We have

$$g_t^{n+1}\le CG_t^{n+1}+CG_t^n. \tag{3.14}$$

Multiplying above inequality by e^{-Ct}, we get that

$$\frac{d\left(G_t^{n+1}e^{-Ct}\right)}{dt}\le Ce^{-Ct}G_t^n. \tag{3.15}$$

Therefore,

$$G_t^{n+1}\le Ce^{Ct}\int_0^t e^{-Cs}G_s^n ds\le Ce^{Ct}tG_t^n. \tag{3.16}$$

Combining Eqs. 3.14 and 3.16 we see that for a fixed $T>0$, and $t\le T$,

$$g_t^{n+1}\le C^2e^{Ct}tG_t^n+CG_t^n\le C_T\int_0^t g_s^n ds, \tag{3.17}$$

for some constant C_T. Iterating Eq. 3.17, we obtain that

$$E\left[\sup_{0\le s\le T}|Y_s^{n+1}-Y_s^n|_H^2\right]\le C\frac{(C_TT)^n}{n!}.$$

This implies that there exists $Y\in D([0,T],H)$ such that

$$\lim_{n\to\infty}E\left[\sup_{0\le s\le T}|Y_s^n-Y_s|_H^2\right]=0.$$

In view of Eq. 3.12 we see that Y^n also converges to Y in $M^2(0,T;V)$. Letting $n\to\infty$ in Eq. 3.9 it is seen that Y is a solution to equation (2) in the statement of the theorem.

Uniqueness Let X, Y be two solutions to (2) in $M^2(0, T; V) \cap D(0, T; H)$. By Ito's formula, we have

$$
\begin{aligned}
|Y_t - X_t|_H^2 = &-2\int_0^t \langle Y_s - X_s, A(Y_s - X_s)\rangle ds \\
&+ 2\int_0^t < Y_s - X_s, b(Y_s) - b(X_s) > ds \\
&+ 2\int_0^t < Y_s - X_s, \sigma(Y_s) - \sigma(X_s) > dB_s \\
&+ \int_0^t |\sigma(Y_s) - \sigma(X_s)|_H^2 ds \\
&+ \int_0^t \int_X \big[| f(Y_{s-}, x) - f(X_{s-}, x)|_H^2 + 2 < Y_s - X_s, f(Y_{s-}, x) \\
&\qquad - f(X_{s-}, x) >\big] \tilde{N}(ds, dx) \\
&+ \int_0^t \int_X | f(Y_s, x) - f(X_s, x)|_H^2 ds\nu(dx).
\end{aligned} \tag{3.18}
$$

By virtue of (H.2), it follows that

$$
\begin{aligned}
E\left[|Y_t - X_t|_H^2\right] \leq &-\alpha E\left[\int_0^t ||Y_s - X_s||_V^2 ds\right] + CE\left[\int_0^t |Y_s - X_s|_H^2 ds\right] \\
&+ \frac{1}{2}E\left[\sup_{0\leq s\leq t} |Y_s - X_s|_H^2\right] + CE\left[\int_0^t |b(Y_s) - b(X_s)|_H^2 ds\right] \\
&+ CE\left[\int_0^t |\sigma(Y_s) - \sigma(X_s)|_H^2 ds\right] \\
&+ CE\left[\int_0^t \int_X | f(Y_s, x) - f(X_s, x)|_H^2 ds\nu(dx)\right] \\
\leq &\, CE\left[\int_0^t |Y_s - X_s|_H^2 ds\right].
\end{aligned} \tag{3.19}
$$

Hence, $X_t = Y_t$.

Next we move to a more general equation which includes terms involving also Poisson measures. Let U be a set in $B(X)$ such that $\nu(X \setminus U) < \infty$. Let $g(y, x)$ be a measurable mapping from $H \times X$ into H. Introduce the following conditions:

(H.3) There exists a constant $C < \infty$ such that

$$
|b(y)|_H^2 + |\sigma(y)|_H^2 + \int_U |g(y, x)|_H^2 \nu(dx) \leq C\left(1 + |y|_H^2\right) \tag{3.20}
$$

for all $y \in H$.

(H.4) There exists a constant $C < \infty$ such that

$$|b(y_1) - b(y_2)|_H^2 + |\sigma(y_1) - \sigma(y_2)|_H^2$$
$$+ \int_U |g(y_1, x) - g(y_2, x)|_H^2 \nu(dx) \quad (3.21)$$
$$\leq C|y_1 - y_2|_H^2 \quad (3.22)$$

for all $y_1, y_2 \in H$.

Consider the stochastic evolution equation:

$$Y_t = h - \int_0^t AY_s ds + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s$$
$$+ \int_0^t \int_U g(Y_{t-}, x)\tilde{N}(dt, dx) + \int_0^t \int_{X\setminus U} g(Y_{t-}, x)N(dt, dx). \quad (3.23)$$

□

Theorem 3.3 *Assume* (*H.3*) *and* (*H.4*). *Then there exists a unique H-valued progressively measurable process* (Y_t) *such that*

(1) $Y \in M^2(0, T; V) \cap D(0, T; H)$ *for any* $T > 0$,
(2)

$$Y_t = h - \int_0^t AY_s ds + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s$$
$$+ \int_0^t \int_U g(Y_{t-}, x)\tilde{N}(dt, dx) + \int_0^t \int_{X\setminus U} g(Y_{t-}, x)N(dt, dx), \quad (3.24)$$

(3) $Y_0 = h \in H$.

Proof Having Theorem 3.2 in hand, this theorem can be proved in the same way as in the finite dimensional case (see [16, 22]). For completeness we sketch the proof. Let $\tau_1 < \tau_2 < \cdots$ be the enumeration of all elements in $D = \{s \in D_p; \quad p(s) \in X \setminus U\}$. It is clear that τ_n is an $(\mathcal{F}_t)$-stopping time and $\lim_{n\to\infty} \tau_n = \infty$. First we solve the equation on the time interval $[0, \tau_1]$. Consider the equation

$$X_t = h - \int_0^t AX_s ds + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s$$
$$+ \int_0^t \int_U g(X_{t-}, x)\tilde{N}(dt, dx). \quad (3.25)$$

Following the same proof as that of Theorem 3.2, it is seen that there exists a unique solution $X_t, t \geq 0$ to Eq. 3.25. Set

$$Y_t^1 = X_t, 0 \leq t < \tau_1, \quad = Y_{\tau_1 -} + g(Y_{\tau_1 -}, p(\tau_1)), t = \tau_1.$$

Clearly the process $\{Y_t^1\}_{t\in[0,\tau_1]}$ is the unique solution to Eq. 3.24. Now, set $\hat{B}_t = B_{t+\tau_1} - B_{\tau_1}$, $\hat{p}(s) = p(s + \tau_1)$. We can construct the process Y_t^2 on $[0, \hat{\tau}_1]$ with respect to the initial value $Y_0^2 = Y_{\tau_1}^1$, Brownian motion $\hat{B}$ and Poisson point process $\hat{p}$ in the same way as Y_t^1. Note that $\hat{\tau}_1$ defined with respect to $\hat{p}$ coincides with $\tau_2 - \tau_1$. Define

$$Y_t = Y_t^1, t \in [0, \tau_1], \quad = Y_{t-\tau_1}, t \in [\tau_1, \tau_2].$$

It is easy to see that $\{Y_t\}_{t\in[0,\tau_2]}$ is the unique solution to Eq. 3.24 in the interval $[0, \tau_2]$. Continuing this procedure successively, we get the unique solution Y to Eq. 3.24. □

4 Exponential Integrability

(H.5) There exists a measurable function $\bar{f}$ on X satisfying

$$\sup_{y\in H} |f(y, x)|_H \leq \bar{f}(x), \tag{4.1}$$

and

$$\int_X (\bar{f}(x))^2 exp(a\bar{f}(x))\nu(dx) < \infty, \quad \text{for all} \quad a > 0. \tag{4.2}$$

In this section, for simplicity we assume that $b = 0$, $\sigma = 0$ in Eq. 1.1. Again we denote the solution of Eq. 1.1 by Y_t.

Lemma 4.1 *For $g \in C_b^2(H)$, $M_t^g = exp(g(Y_t) - g(h) - \int_0^t h(Y_s)ds)$ is an $\mathcal{F}_t$-local martingale, where*

$$h(y) = < -Ay, g'(y) > + \int_X (exp[g(y + f(y, x)) - g(y)] - 1 - < g'(y), f(y, x) >)\nu(dx).$$

Proof Applying Itô's formula first to $exp(g(Y_t))$ and then to $exp(g(Y_t) - g(h))exp(-\int_0^t h(Y_s)ds)$ proves the lemma. □

Proposition 4.2 *Assume* Eq. 2.2 *with* $\lambda_0 = 0$ *and also* (*H.5*). *Then for* $r > 0$ *and any* $\lambda > 0$, *there exists a constant* C_λ *such that*

$$P\left(\sup_{0\leq t\leq 1} |Y_t|_H > r\right) \leq C_\lambda e^{-(1+\lambda r^2)^{\frac{1}{2}}}.$$

Proof For $\lambda > 0$, set $g(y) = (1 + \lambda|y|_H^2)^{\frac{1}{2}}$. Then

$$g'(y) = \lambda\left(1 + \lambda|y|_H^2\right)^{-\frac{1}{2}} y,$$
$$g''(y) = -\lambda^2\left(1 + \lambda|y|_H^2\right)^{-\frac{3}{2}} y \times y + \lambda\left(1 + \lambda|y|_H^2\right)^{-\frac{1}{2}} I_H.$$

where I_H stands for the identity operator. It is easy to see that

$$\sup_y |g''(y)| \leq \lambda, \qquad \sup_y |g'(y)| \leq \lambda^{\frac{1}{2}}.$$

Moreover,

$$< -Ay, g'(y) >= \lambda\left(1 + \lambda|y|_H^2\right)^{-\frac{1}{2}} < -Ay, y > \leq 0 \tag{4.3}$$

for $y \in V$. Write $G(y) = e^{g(y)}$. By Taylor's expansion, there exists θ between 0 and 1 such that

$$\begin{aligned} & exp\big[g(y+f(y,x)) - g(y)\big] - 1 - < g'(y), f(y,x) > \\ & \quad = e^{-g(y)}\big[G(y+f(y,x)) - G(y) - G(y) < g'(y), f(y,x) >\big] \\ & \quad = \frac{1}{2} e^{-g(y)} < G''(y+\theta f(y,x)), f(y,x) \times f(y,x) > . \end{aligned} \tag{4.4}$$

Note that

$$G''(y) = G(y) g'(y) \times g'(y) + G(y) g''(y).$$

It follows that

$$|G''(y)|_{L(H)} \le \lambda G(y), \qquad \text{for all} \quad y \in H. \tag{4.5}$$

By Eq. 4.4,

$$\begin{aligned} & |exp[g(y+f(y,x)) - g(y)] - 1 - < g'(y), f(y,x) > | \\ & \quad \le \lambda exp(g(y+\theta f(y,x)) - g(y)) | f(y,x)|_H^2 \\ & \quad = \lambda exp(< g'(y+\theta_1 f(y,x)), \theta f(y,x) >) | f(y,x)|_H^2 \\ & \quad \le \lambda exp\left(\lambda^{\frac{1}{2}} | f(y,x)|_H\right) | f(y,x)|_H^2 . \end{aligned} \tag{4.6}$$

Applying Lemma 4.1, with the above choice of g, $M_t^g = exp(g(Y_t) - g(y) - \int_0^t h(Y_s) ds)$ is an $\mathcal{F}_t$-local martingale, where

$$\begin{aligned} h(y) &= < -Ay, g'(y) > + \int_X (exp[g(y+f(y,x)) - g(y)] - 1 - < g'(y), f(y,x) >) \nu(dx) \\ &\le \int_X \lambda exp(\lambda^{\frac{1}{2}} | f(y,x)|_H) | f(y,x)|_H^2 \nu(dx) \\ &\le \int_X \lambda exp(\lambda^{\frac{1}{2}} | \bar{f}(x)|_H) | \bar{f}(x)|_H^2 \nu(dx) = M_\lambda < \infty. \end{aligned} \tag{4.7}$$

We have

$$\begin{aligned} P\left(\sup_{0\le t\le 1} |Y_t|_H > r\right) &= P\left(\sup_{0\le t\le 1} g(Y_t) \ge \left(1+\lambda r^2\right)^{\frac{1}{2}}\right) \\ &= P\left(\sup_{0\le t\le 1} \left(g(Y_t) - g(h) - \int_0^t h(Y_s) ds + g(h) + \int_0^t h(Y_s) ds\right)\right. \\ &\qquad \left. \ge \left(1+\lambda r^2\right)^{\frac{1}{2}}\right) \\ &\le P\left(\sup_{0\le t\le 1} \left(g(Y_t) - g(h) - \int_0^t h(Y_s) ds\right) + g(h) + M_\lambda \ge (1+\lambda r^2)^{\frac{1}{2}}\right) \\ &= P\left(\sup_{0\le t\le 1} \left(g(Y_t) - g(h) - \int_0^t h(Y_s) ds\right) \ge \left(1+\lambda r^2\right)^{\frac{1}{2}} - g(h) - M_\lambda\right) \\ &\le E\left[\sup_{0\le t\le 1} M_t^g\right] exp\left(-\left(1+\lambda r^2\right)^{\frac{1}{2}} + g(h) + M_\lambda\right). \end{aligned} \tag{4.8}$$

Since M_t^g is a non-negative local martingale (hence, a supermartingale), $E[\sup_{0\le t\le 1} M_t^g] \le 1$. Therefore the assertion follows with $C_\lambda = exp(g(h) + M_\lambda)$. □

Corollary 4.3 *Assume Eq.* 2.2 *with* $\lambda_0 = 0$ *and also* (*H.5*). *Then for any* $l > 0$,

$$E\left[exp\left(l \sup_{0\le t\le 1} |Y_t|_H\right)\right] < \infty.$$

Proposition 4.4 *Assume Eq.* 2.2 *with* $\lambda_0 = 0$ *and also* (*H.5*). *Then for any* $l > 0$,

$$E\left[exp\left(l||Y||_{L^2([0,1]\to V)}\right)\right] < \infty.$$

Proof Let $Z_\lambda = \int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}} ||Y_s||_V^2 ds$. We first prove that

$$P(Z_\lambda > r) \le exp\left(-\alpha\lambda r + M_\lambda + \left(1+\lambda|h|_H^2\right)^{\frac{1}{2}}\right), \tag{4.9}$$

where M_λ is the same constant as in Eq. 4.7. For $\lambda > 0$, define $g(y) = (1+\lambda|y|_H^2)^{\frac{1}{2}}$. In view of Eq. 2.2 we have

$$\begin{aligned} < -Ay, g'(y) > &= \lambda\left(1+\lambda|y|_H^2\right)^{-\frac{1}{2}} < -Ay, y > \\ &\le -\alpha\lambda\left(1+\lambda|y|_H^2\right)^{-\frac{1}{2}} ||y||_V^2. \end{aligned} \tag{4.10}$$

So the estimate in Eq. 4.7 can be strengthened as follows:

$$h(y) \le -\alpha\lambda\left(1+\lambda|y|_H^2\right)^{-\frac{1}{2}} ||y||_V^2 + M_\lambda. \tag{4.11}$$

Let $M_t^g, t \ge 0$ be defined as in the proof of Proposition 4.2. By Eq. 4.11, we have

$$\begin{aligned} P(Z_\lambda > r) &= P\left(\alpha\lambda\int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}} ||Y_s||_V^2 ds > \alpha\lambda r\right) \\ &\le P\left(g(Y_1) + \alpha\lambda\int_0^1 \left(1+\lambda|Y_s|_H^2\right)^{-\frac{1}{2}} ||Y_s||_V^2 ds > \alpha\lambda r\right) \\ &= P\left(g(Y_1) - g(h) - \int_0^1 h(Y_s)ds + g(h) + \int_0^1 h(Y_s)ds \right. \\ &\qquad \left. + \alpha\lambda\int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}} ||Y_s||_V^2 ds > \alpha\lambda r\right) \\ &\le P\left(g(Y_1) - g(h) - \int_0^1 h(Y_s)ds\right) + g(h) + M_\lambda > \alpha\lambda r\right) \\ &= P\left(g(Y_1) - g(h) - \int_0^1 h(Y_s ds) > \alpha\lambda r - g(h) - M_\lambda\right) \\ &\le E[M_1^g]exp(-\alpha\lambda r + g(h) + M_\lambda) \\ &\le exp(-\alpha\lambda r + g(h) + M_\lambda) \end{aligned} \tag{4.12}$$

which proves Eq. 4.9. It is easy to see from Eq. 4.9 that for any $l > 0$, one can choose $\lambda_l > 0$ large enough so that $E\left[exp(lZ_{\lambda_l})\right] < \infty$. Now for every $\lambda > 0$,

$$\begin{aligned}||Y||_{L^2([0,1]\to V)} &= \left(\int_0^1 ||Y_s||_V^2 ds\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 (1+\lambda|Y_s|_H^2)^{-\frac{1}{2}}||Y_s||_V^2 ds\right)^{\frac{1}{2}} \left(1+\lambda \sup_{0\leq s\leq 1}|Y_s|_H^2\right)^{\frac{1}{4}} \\ &\leq \frac{1}{2}Z_\lambda + \frac{1}{2}\left(1+\lambda \sup_{0\leq s\leq 1}|Y_s|_H^2\right)^{\frac{1}{2}}. \end{aligned} \tag{4.13}$$

By Hölder's inequality, for $l > 0$,

$$\begin{aligned}E\left[exp\left(l||Y||_{L^2([0,1]\to V)}\right)\right] &\leq E\left[exp\left(\frac{1}{2}lZ_\lambda\right) exp\left(\frac{1}{2}l\left(1+\lambda \sup_{0\leq s\leq 1}|Y_s|_H^2\right)^{\frac{1}{2}}\right)\right] \\ &\leq \left(E[exp(lZ_\lambda)]\right)^{\frac{1}{2}} \times \left(E\left[exp\left(l\left(1+\lambda \sup_{0\leq s\leq 1}|Y_s|_H^2\right)^{\frac{1}{2}}\right)\right]\right)^{\frac{1}{2}}. \end{aligned} \tag{4.14}$$

According to Eq. 4.9, we can choose λ such that $E[exp(lZ_\lambda)] < \infty$. On the other hand $E\left[exp\left(l\left(1+\lambda \sup_{0\leq s\leq 1}|Y_s|_H^2\right)^{\frac{1}{2}}\right)\right] < \infty$ for all $\lambda > 0$ according to Corollary 4.3. So we conclude that $E[exp(l||Y||_{L^2([0,1]\to V)})] < \infty$ proving the assertion. □

5 Large Deviations

In this section we consider the following Lévy process:

$$L_t = bt + W_t + \int_0^t \int_X f(x)\tilde{N}(ds, dx),$$

where W is the H-valued Brownian motion introduced in Section 2, b is a constant vector in H and f is a measurable mapping from X into H. The following Ornstein–Uhlenbeck type stochastic evolution equation was first studied in [5] and subsequently by many other authors (cf. e.g. [11]).

$$dY_t = -AY_t dt + dL_t, \tag{5.1}$$

$$Y_0 = h \in H \tag{5.2}$$

The following Theorem can be proved similarly as in Section 3. See also [5].

Theorem 5.1 *There exists a unique H-valued progressively measurable process (Y_t) such that*

(1) $Y \in M^2(0, T; V) \cap D(0, T; H)$ *for any* $T > 0$,
(2) $Y_t = h - \int_0^t AY_s ds + L_t$ *a.s.*,
(3) $Y_0 = h \in H$.

To get large deviations estimates, it is natural to impose the following exponential integrability. Assume throughout this section that

$$\int_X |f(x)|_H^2 exp(a|f(x)|_H)\nu(dx) < \infty, \quad \text{for all} \quad a > 0. \tag{5.3}$$

Consider the stochastic evolution equation:

$$Y_t^n = x - \int_0^t AY_s^n ds + bt + \frac{1}{n^{\frac{1}{2}}}W_t + \frac{1}{n}\int_0^t \int_X f(x)\tilde{N}_n(ds, dx), \tag{5.4}$$

where $\tilde{N}_n(ds, dx)$ denotes the compensated Poisson measure with intensity measure $n\nu$. The purpose is to establish a large deviation principle for the law μ_n of $Y_t^n, t \geq 0$ on $D([0, 1] \to H)$. To this end, we first do some preparations.

For $g \in D([0, 1] \to V)$, define $\phi(g) \in D([0, 1] \to H) \cap L^2([0, 1] \to V)$ as the solution to the following equation:

$$\phi_t(g) = x - \int_0^t A\phi_s(g)ds + g(t). \tag{5.5}$$

Lemma 5.2 *The mapping* ϕ *from* $D([0, 1] \to V)$ *into* $\phi(g) \in D([0, 1] \to H) \cap L^2([0, 1] \to V)$ *is continuous in the topology of uniform convergence.*

Proof Let $v_t(g) = \phi_t(g) - g(t)$. it is easy to see that $v(g)$ satisfies the equation:

$$v_t(g) = x - \int_0^t Av_s(g)ds - \int_0^t Ag(s)ds.$$

It suffices to show that the mapping

$$v(\cdot) : D([0, 1] \to V) \to D([0, 1] \to H) \cap L^2([0, 1] \to V)$$

is continuous. Taking $\beta < \alpha$, where α is the constant in Eq. 2.2, by the chain rule and Eq. 2.2,

$$\begin{aligned}
|v_t(g_n) - v_t(g)|_H^2 &= -2\int_0^t < A(v_s(g_n) - v_s(g)), v_s(g_n) - v_s(g) > ds \\
&\quad -2\int_0^t < A(g_n - g)(s), v_s(g_n) - v_s(g) > ds \\
&\leq -\alpha\int_0^t ||v_s(g_n) - v_s(g)||_V^2 ds + \lambda_0\int_0^t |v_s(g_n) - v_s(g)|_H^2 ds \\
&\quad +2\int_0^t ||v_s(g_n) - v_s(g)||_V ||A(g_n - g)(s)||_{V^*} ds \\
&\leq -\alpha\int_0^t ||v_s(g_n) - v_s(g)||_V^2 ds + \lambda_0\int_0^t |v_s(g_n) - v_s(g)|_H^2 ds \\
&\quad + \beta\int_0^t ||v_s(g_n) - v_s(g)||_V^2 ds + C_\beta\int_0^t ||A(g_n - g)(s)||_{V^*}^2 ds.
\end{aligned}$$

This gives that

$$|v_t(g_n) - v_t(g)|_H^2 + (\alpha - \beta)\int_0^t ||v_s(g_n) - v_s(g)||_V^2 ds$$
$$\leq \lambda_0 \int_0^t |v_s(g_n) - v_s(g)|_H^2 ds + C_\beta ||A|| \int_0^t ||g_n - g(s)||_V^2 ds$$

Applying Gronwall's inequality it is easily seen that the mapping $v(\cdot)$ is continuous, which completes the proof. □

For $l \in H$, define

$$F(l) = \int_X [exp(< f(x), l >) - 1- < f(x), l >]\nu(dx)+ < Ql, l > + < b, l > .$$

Set, for $z \in H$,

$$F^*(z) = \sup_{l\in H}[< z, l > -F(l)]. \tag{5.6}$$

Define a functional $I_0(\cdot)$ on $D([0, 1] \to H)$ as follows: if $g \in D([0, 1] \to H)$ and $g' \in L^1([0, 1] \to H)$, $I_0(g) = \int_0^1 F^*(g'(s))ds$; otherwise $I(g) = \infty$.

Lemma 5.3 *Let $a > 0$. Then $\mathcal{G} = \{|g'|; I_0(g) \leq a\}$ is uniformly integrable on the probability space* $([0, 1], \mathcal{B}, m)$*, where m denotes the Lebesgue measure.*

Proof Recall that $\mathcal{G}$ is uniformly integrable if and only if

(1) $\mathcal{G}$ is equi-absolutely continuous, i.e., for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $m(A) < \delta$ implies $\int_A |g'|_H m(ds) < \varepsilon$ for all $g \in \mathcal{G}$.
(2) $\sup_{g\in\mathcal{G}} \int_0^1 |g'|_H m(ds) < \infty$.

We will modify the proof of Theorem 3.1 in [2] to get (1) and (2). Let $a_i, b_i, i = 1, ...n$ be any given numbers such that $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq 1$. For any partition $\tau^i = \{t_0^i = a_i < t_1^i < \cdots < t_{i_m}^i = b_i\}$ of $[a_i, b_i]$ and any $\eta_k^i \in H$ with $|\eta_k^i|_H \leq 1$, define $\beta \in M([0, 1], H)$ by

$$\beta = \sum_{i=1}^n \sum_{k=0}^{i_m} \eta_k^i (\delta_{t_k^i} - \delta_{t_{k-1}^i}),$$

where $M([0, 1], H)$ denotes the space of $H-$ valued vector measures on $([0, 1], \mathcal{B})$. Let μ be the law of $\int_0^1 \int_X f(x)\tilde{N}(ds, dx)$ on H. Denote the characteristic functional of μ by $\hat{\mu}$. Then,

$$\int_0^1 log\hat{\mu}(\beta(s, 1])ds = \sum_{i=1}^n \sum_{k=0}^{i_m} log\hat{\mu}(\eta_k^i)(t_k^i - t_{k-1}^i).$$

Let $\rho > 0$. By the characterization of I_0 in [2], for $g \in \mathcal{G}$, we have

$$\rho \int_0^1 < g, d\beta > = \rho \sum_{i=1}^{n} \sum_{k=0}^{i_m} < g(t_k^i) - g(t_{k-1}^i), \eta_k^i >$$
$$\leq \int_0^1 log\hat{\mu}(\rho\beta(s, 1])ds + I_0(g)$$
$$\leq \sup_{i,k} |log\hat{\mu}(\rho\eta_k^i)| \sum_{i=1}^{n} \sum_{k=0}^{i_m} (t_k^i - t_{k-1}^i) + I_0(g)$$
$$\leq log\left(\int_H exp(\rho|x|_H)\mu(dx)\right) \sum_{i=1}^{n}(b_i - a_i) + a. \tag{5.7}$$

Taking sup in Eq. 5.7 over all possible $\eta_k^i \in H$ with $|\eta_k^i|_H \leq 1$ we get

$$\sum_{i=1}^{n} \sum_{k=0}^{i_m} |g(t_k^i) - g(t_{k-1}^i)|_H \leq \rho^{-1} log\left(\int_H exp(\rho|x|_H)\mu(dx)\right) \sum_{i=1}^{n}(b_i - a_i) + \rho^{-1}a. \tag{5.8}$$

Let $V(g)[a, b]$ denote the total variation of g over the interval $[a, b]$. Taking sup in Eq. 5.8 over all possible partitions we obtain

$$\sum_{i=1}^{n} V(g)[a_i, b_i] = \sum_{i=1}^{n} \int_{a_i}^{b_i} |g'(s)|_H ds = \int_{\cup_{i=1}^{n}(a_i,b_i)} |g'(s)|_H ds$$
$$\leq \rho^{-1} log\left(\int_H exp(\rho|x|_H)\mu(dx)\right) \sum_{i=1}^{n}(b_i - a_i) + \rho^{-1}a \tag{5.9}$$

For every $\varepsilon > 0$, choose first ρ_0 large enough such that $\rho_0^{-1}a \leq \frac{\varepsilon}{2}$. Set $\delta = \frac{1}{3}[\rho_0^{-1} log(\int_H exp(\rho|x|_H)\mu(dx))]^{-1}\varepsilon$. If $\cup_{i=1}^{n}(a_i, b_i) \subset [0, 1]$ with $m(\cup_{i=1}^{n}(a_i, b_i)) < \delta$, by Eq. 5.9 we have $\int_{\cup_{i=1}^{n}(a_i,b_i)} |g'(s)|_H ds < \varepsilon$. for all $g \in \mathcal{G}$. This implies (1). Take particularly $a_1 = 0, b_1 = 1$ in the above proof to see that (2) also holds. □

Let $T_t, t \geq 0$ denote the semigroup generated by $-A$. For $g \in L^1([0, 1] \to H)$, define the operator

$$Rg(t) = \int_0^t T_{t-s} g(s)ds, \quad t \geq 0,$$

which is the mild solution of the equation:

$$\phi(t) = -\int_0^t A\phi(s)ds + \int_0^t g(s)ds.$$

Proposition 5.4 *Assume that $T_t, t > 0$ are compact operators. If $\mathcal{G} \subset L^1([0, 1] \to H)$ is uniformly integrable, then $\mathcal{S} = R(\mathcal{G})$ is relatively compact in $C([0, 1] \to H)$.*

Proof The proof is a modification of the proof of Proposition 8.4 in [9]. According to the Ascoli–Arzela theorem we need to show

(1) For every $t \in [0, 1]$ the set $\{Rg(t); g \in \mathcal{G}\}$ is relatively compact in H;
(2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \le s \le t \le 1, t - s \le \delta$,

$$|Rg(t) - Rg(s)|_H \le \varepsilon \quad \text{for all} \quad g \in \mathcal{G}. \tag{5.10}$$

To prove (1), fix $t \in (0, 1]$ and define for $\varepsilon > 0$ $R^\varepsilon g(t) = \int_0^{t-\varepsilon} T_{t-s} g(s) ds$. Since

$$R^\varepsilon g(t) = T_\varepsilon \int_0^{t-\varepsilon} T_{t-\varepsilon-s} g(s) ds$$

and $T_\varepsilon, \varepsilon > 0$ is compact, $\{R^\varepsilon g(t), g \in \mathcal{G}\}$ is relatively compact in H for every $\varepsilon > 0$. On the other hand,

$$|R^\varepsilon g(t) - Rg(t)|_H \le M \int_{t-\varepsilon}^{t} |g(s)|_H ds, \tag{5.11}$$

where $M = sup_{t\in[0,1]} ||T_t||$. Since $\mathcal{G}$ is uniformly integrable, (4.52) implies that

$$lim_{\varepsilon \to 0} \sup_{g\in\mathcal{G}} |R^\varepsilon g(t) - Rg(t)|_H = 0$$

which furthermore implies that $\{Rg(t); g \in \mathcal{G}\}$ is also relatively compact. Let us now prove (2). For $0 \le t \le t + u \le 1$, we have

$$\begin{aligned} &|Rg(t+u) - Rg(t)|_H \\ &\le \int_0^t ||T_{t+u-s} - T_{t-s}|| |g(s)|_H ds + \int_t^{t+u} ||T_{t+u-s}|| |g(s)|_H ds \\ &:= I_g^u + II_g^u. \end{aligned}$$

By the uniform integrability of $\mathcal{G}$, it is clear that

$$\lim_{u\to 0} \sup_{g\in\mathcal{G}} II_g^u \le M \lim_{u\to 0} \sup_{g\in\mathcal{G}} \int_t^{t+u} |g(s)|_H ds = 0.$$

Since the semigroup $T_\cdot$ is compact, $||T_{t+u-s} - T_{t-s}|| \to 0$ for any $t - s > 0$ as $u \to 0$. By the dominated convergence theorem, we have that

$$\lim_{u\to 0} \int_0^t ||T_{t+u-s} - T_{t-s}|| ds = 0. \tag{5.12}$$

Now we prove

$$\lim_{u\to 0} \sup_{g\in\mathcal{G}} I_g^u = 0. \tag{5.13}$$

For given $\varepsilon > 0$, sine $\mathcal{G}$ is uniformly integrable, one can choose $\rho > 0$ such that $2M \int_{|g|>\rho} |g(s)|_H ds < \frac{\varepsilon}{2}$ for all $g \in \mathcal{G}$. For the fixed $\rho > 0$ above, there exists $\delta > 0$ such that $u \le \delta$ implies that

$$\rho \int_0^t ||T_{t+u-s} - T_{t-s}|| ds \le \frac{\varepsilon}{2}$$

for all $t \in [0, 1]$. Therefore if $u \leq \delta$, for all $g \in \mathcal{G}$, $t \in [0, 1]$,

$$\begin{aligned} I_g^u &= \int_{|g|>\rho} ||T_{t+u-s} - T_{t-s}|||g(s)|_H ds + \int_{|g|\leq\rho} ||T_{t+u-s} - T_{t-s}|||g(s)|_H ds \\ &\leq 2M \int_{|g|>\rho} |g(s)|_H ds + \rho \int_0^t ||T_{t+u-s} - T_{t-s}||ds \\ &\leq \varepsilon. \end{aligned} \tag{5.14}$$

This proves (2), hence the assertion. □

Theorem 5.5 *Assume that A has discrete spectrum with eigenvalues*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$$

and H-normalized eigenfunctions $\{e_i\}$. *Suppose that the Brownian motion* W_t *admits the following representation:*

$$W_t = \sum_{i=1}^{\infty} q_i \beta_i(t) e_i, \tag{5.15}$$

where $\beta_i(t), i \geq 1$ *are independent standard Brownian motions, and* $q_i, i \geq 1$ *are non-negative real numbers satisfying* $\sum_{i=1}^{\infty} q_i^2 < \infty$. *Then* $\{\mu_n, n \geq 1\}$ *satisfies a large deviation principle on* $D([0, 1] \to H)$ (*equipped with the topology of uniform convergence*) *with a rate functional I given for* $k \in D([0, 1] \to H)$ *by*

$$\begin{aligned} I(k) &= \inf\{I_0(g); g \in D([0, 1] \to H) \quad \textit{satisfying} \\ k(t) &= T_t x + \int_0^t T_{t-s} g'(s) ds\}. \end{aligned} \tag{5.16}$$

Proof Denote by $P_m : H \to H$ the projection operator defined by

$$P_m x = \sum_{i=1}^{m} < x, e_i > e_i.$$

As $\{e_i\} \subset V$, we have $Rang(P_m) \subset V$. For any integer $m \geq 1$, introduce a mapping $\phi^m(\cdot)$ from $D([0, 1] \to H)$ into $D([0, 1] \to H)$ as follows: for $g \in D([0, 1] \to H)$ define $\phi_t^m(g)$ as the solution of the following equation:

$$\phi_t^m(g) = x - \int_0^t A\phi_s^m(g) ds + P_m g(t). \tag{5.17}$$

By Lemma 5.2 the mapping $\phi^m(\cdot)$ is continuous. Let

$$L_t^n = bt + \frac{1}{n^{\frac{1}{2}}} W_t + \frac{1}{n} \int_0^t \int_X f(x) \tilde{N}_n(ds, dx).$$

Then it is easy to see that $Y_t^{n,m} := \phi_t^m(L_t^n)$ is the solution to the following equation:

$$Y_t^{n,m} = x - \int_0^t AY_s^{n,m} ds + b^m t + \frac{1}{n^{\frac{1}{2}}} W_t^m + \frac{1}{n} \int_0^t \int_X f^m(x) \tilde{N}_n(ds, dx), \tag{5.18}$$

where $f^m(x) = P_m f(x) = \sum_{i=1}^m < f(x), e_i > e_i$, $W_t^m = P_m W_t$ and $b^m = P_m b$. Let ν_n be the law of $L^n_\cdot$. It was proved in [2] that $\{\nu_n, n \geq 1\}$ satisfies a large deviation principle with rate function I_0. Applying the contraction principle, we see that $\{Y^{n,m}_\cdot\}$ satisfies a large deviation principle on $D([0,1] \to H)$ with a rate functional I_m given by, for $k \in D([0,1] \to H)$,

$$I_m(k) = \inf\{I_0(g); g \in D([0,1] \to H) \quad \text{satisfying}$$
$$k(t) = T_t x + \int_0^t T_{t-s} P_m g'(s) ds\}. \tag{5.19}$$

According to the generalized contraction principle Theorem 4.2 in [9], the theorem now follows from the following two lemmas. □

Lemma 5.6 *For any* $\delta > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} log P\left(\sup_{0\leq t\leq 1} |Y_t^{n,m} - Y_t^n|_H > \delta\right) = -\infty. \tag{5.20}$$

Proof Set $X_t^{n,m}(Y_t^{n,m} - Y_t)$. Then it can be seen that

$$X_t^{n,m} = -\int_0^t A X_s^{n,m} ds + \int_0^t \int_X (f^m(x) - f(x)) \tilde{N}_n(ds, dx)$$
$$+ n(b^m - b)t + n^{\frac{1}{2}}(W_t^m - W_t) \tag{5.21}$$

For $\lambda > 0$, set $g(y) = (1 + \lambda |y|_H^2)^{\frac{1}{2}}$. As in Section 4, we know that $M_t^g = exp(g(X_t^{n,m}) - g(0) - \int_0^t h(X_s^{n,m}) ds)$ is an $\mathcal{F}_t$-local martingale, where

$$h(y) = n \int_X (exp[g(y + f^m(x) - f(x)) - g(y)] - 1 - < g'(y), f^m(x) - f(x) >) \nu(dx)$$
$$- < Ay, g'(y) > + n < b^m - b, g'(y) >$$
$$+ n \sum_{i=m+1}^{\infty} q_i^2 < (g'(y) \otimes g'(y) + g''(y)) e_i, e_i > . \tag{5.22}$$

Furthermore (See Section 4), we have

$$h(y) \leq c_{\lambda,m} n, \tag{5.23}$$

where

$$c_{\lambda,m} = \lambda \int_X exp(\lambda^{\frac{1}{2}} | f^m(x) - f(x)|_H)(| f^m(x) - f(x)|_H)^2 \nu(dx)$$
$$+ \lambda^{\frac{1}{2}} |b^m - b|_H + 2\lambda \sum_{i=m+1}^{\infty} q_i^2. \tag{5.24}$$

We have

$$P\left(\sup_{0\le t\le 1}|X_t^{n,m}|_H > r\right) = P\left(\sup_{0\le t\le 1} g(X_t^{n,m}) \ge (1+\lambda r^2)^{\frac{1}{2}}\right)$$

$$= P\left(\sup_{0\le t\le 1}(g(X_t^{n,m}) - g(0) - \int_0^t h(X_s^{n,m})ds + 1 + \int_0^t h(X_s^{n,m})ds) \ge (1+\lambda r^2)^{\frac{1}{2}}\right)$$

$$\le P\left(\sup_{0\le t\le 1}(g(X_t^{n,m}) - g(0) - \int_0^t h(X_s^{n,m})ds) + 1 + c_{\lambda,m}n \ge (1+\lambda r^2)^{\frac{1}{2}}\right)$$

$$= P\left(\sup_{0\le t\le 1}(g(X_t^{n,m}) - g(0) - \int_0^t h(X_t^{n,m})ds) \ge (1+\lambda r^2)^{\frac{1}{2}} - 1 - c_{\lambda,m}n\right)$$

$$\le E\left[\sup_{0\le t\le 1} M_t^g\right] exp\left(-(1+\lambda r^2)^{\frac{1}{2}} + 1 + c_{\lambda,m}n\right). \tag{5.25}$$

This gives that

$$P\left(\sup_{0\le t\le 1}|Y_t^{n,m} - Y_t^n|_H > \delta\right) = P\left(\sup_{0\le t\le 1}|X_t^{n,m}|_H > n\delta\right)$$

$$\le exp(-(1+\lambda(n\delta)^2)^{\frac{1}{2}} + 1 + c_{\lambda,m}n).$$

Therefore,

$$\limsup_{n\to\infty}\frac{1}{n}log P\left(\sup_{0\le t\le 1}|Y_t^{n,m} - Y_t^n|_H > \delta\right)$$

$$\le \limsup_{n\to\infty}\frac{1}{n}[-(1+\lambda(n\delta)^2)^{\frac{1}{2}} + 1 + c_{\lambda,m}n]$$

$$\le -\lambda\delta + c_{\lambda,m}. \tag{5.26}$$

Note that by the dominated convergence theorem, for fixed λ, $\lim_{m\to\infty} c_{\lambda,m} = 0$. Taking $m\to\infty$ in Eq. 5.26 we get that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}log P\left(\sup_{0\le t\le 1}|Y_t^{n,m} - Y_t^n|_H > \delta\right) \le -\lambda\delta.$$

Let $\lambda\to\infty$ to get Eq. 5.20. □

For $g\in D([0,1]\to H)$, let $\phi_t(g), t\ge 0$, be defined as in Eq. 5.5.

Lemma 5.7 *For any $r > 0$,*

$$\lim_{m\to\infty}\sup_{\{f;I_0(f)\le r\}}\sup_{0\le t\le 1}|\phi_t^m(f) - \phi_t(f)| = 0. \tag{5.27}$$

Proof For $f \in D([0,1] \to H)$ with $I_0(f) < \infty$, we note that

$$\phi_t^m(f) = \int_0^t T_{t-s} P_m f'(s) ds = P_m \int_0^t T_{t-s} f'(s) ds = P_m \phi_t(f), \tag{5.28}$$

where we have used the fact that A has a discrete spectrum to exchange P_m and T_s. Since by Lemma 5.3 $L_r := \{f'; I_0(f) \le r\}$ is uniformly integrable on the probability space $([0,1], \mathcal{B}, m)$, it follows from Proposition 5.4 that $\mathcal{S} = \{\phi(f); I_0(f) \le r\}$ is relatively compact in $C([0,1] \to H)$. Therefore, for any $\varepsilon > 0$, there exist $f_1, f_2, ..., f_N \in \{f; I_0(f) \le r\}$ such that $\mathcal{S} \subset \cup_{k=1}^N B(\phi(f_k), \frac{\varepsilon}{3})$, where $B(\phi(f_k), \frac{\varepsilon}{3})$ stands for the ball centered at $\phi(f_k)$ with radius $\frac{\varepsilon}{3}$ in $C([0,1] \to H)$. Since $\lim_{m\to\infty} \sup_{0\le t\le 1} |\phi_t^m(f_k) - \phi_t(f_k)| = 0$ for every k, there exists $m_0 \ge 1$ such that

$$\sup_{0\le t\le 1} |\phi_t^m(f_k) - \phi_t(f_k)| \le \frac{\varepsilon}{3} \quad \text{for all} \quad k \le N, m \ge m_0. \tag{5.29}$$

Fix any f with $I_0(f) \le r$. Then there is $k \le N$ such that $\phi(f) \in B(\phi(f_k), \frac{\varepsilon}{3})$. Hence, if $m \ge m_0$,

$$\begin{aligned}\sup_{0\le t\le 1} |\phi_t^m(f) - \phi_t(f)| &\le \sup_{0\le t\le 1} |\phi_t^m(f) - P_m\phi_t(f_k)| + \sup_{0\le t\le 1} |\phi_t^m(f_k) - \phi_t(f_k)| \\ &\quad + \sup_{0\le t\le 1} |\phi_t(f_k) - \phi_t(f)| \\ &\le 2 \sup_{0\le t\le 1} |\phi_t(f_k) - \phi_t(f)| + \sup_{0\le t\le 1} |\phi_t^m(f_k) - \phi_t(f_k)| \\ &\le \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned} \tag{5.30}$$

which proves Eq. 5.27.

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