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Quasi-regular Dirichlet Forms and L^p-resolvents on Measurable Spaces

Lucian Beznea · Nicu Boboc · Michael Röckner

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Abstract We prove that for any semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on a measurable Lusin space E there exists a Lusin topology with the given σ-algebra as the Borel σ-algebra so that $(\mathcal{E}, D(\mathcal{E}))$ becomes quasi-regular. However one has to enlarge E by a zero set. More generally a corresponding result for arbitrary L^p-resolvents is proven.

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Introduction

Let E be a Lusin topological space (i.e., E is homeomorphic to a Borel subset of a compact metric space) with Borel σ-algebra $\mathcal{B}$. Let m be a σ-finite measure on $(E, \mathcal{B})$ and $L^p(E, m)$, $p \in [1, \infty]$, the corresponding (real) L^p-spaces. Let $(\mathcal{E}, D(\mathcal{E}))$ be a semi-Dirichlet form on $L^2(E, m)$ in the sense of [14]. Modifying the main result of [2, 14], in [13] an analytic characterization of all semi-Dirichlet forms on $L^2(E, m)$ which are associated with a nice Markov process (more precisely a so-called m-special standard process) was proved. Such semi-Dirichlet forms are called quasi-regular. An elaborate theory for such Dirichlet forms has been developed both for its analytic

L. Beznea (✉)
Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764,
RO-014700 Bucharest, Romania
e-mail: lucian.beznea@imar.ro

N. Boboc
Faculty of Mathematics and Informatics, University of Bucharest, str. Academiei 14,
RO-010014 Bucharest, Romania

M. Röckner
Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, D-33501 Bielefeld, Germany
e-mail: roeckner@mathematik.uni-bielefeld.de

and probabilistic components with numerous applications (cf. [14]). In particular, invariance properties under change of topology, more precisely, the invariance under quasi-isomorphism of the theory was discovered (cf. [1, 6] and the Appendix in [9]) and exploited subsequently (see, e.g., Chap. VI in [14]).

A fundamental question, however, remained open, namely whether it is enough to have a measurable structure only, in the following sense: Let $(E, \mathcal{B})$ be merely a Lusin measurable space (i.e., it is measurably isomorphic with $(F, \mathcal{B}(F))$, where F is some Lusin topological space equipped with Borel σ-algebra $\mathcal{B}(F)$) and $(\mathcal{E}, D(\mathcal{E}))$ a semi-Dirichlet form on $L^2(E, m)$ with m a σ-finite measure. Can we find a topology on E with Borel σ-algebra equal to the given $\mathcal{B}$ and making E a Lusin topological space such that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular with respect to this topology? As a consequence one could apply all results on quasi-regular Dirichlet forms only depending on the measurable structure (such as measure representations for potentials, spectral analysis, Beurling-Deny type representations etc.) for Dirichlet forms on arbitrary Lusin-measurable state spaces.

This question has been addressed in [7] as a question formulated by G. Mokobodzki in 1991. In [7] a necessary and sufficient condition on $(\mathcal{E}, D(\mathcal{E}))$ and $\mathcal{B}$ was formulated so that the answer to the above question is positive. This condition is, however, quite close to what is needed in the proof and, therefore, not very useful in applications (see the example in [7]). The main purpose of this paper is to show that it is always possible to find a Lusin topology on E making $(\mathcal{E}, D(\mathcal{E}))$ quasi-regular, however, one has to enlarge E by a set of m-zero measure (cf. Corollary 3.4 below). Our strategy of proof reveals that such an enlargement is probably necessary in general, though we cannot formally prove that.

For illustration (and following the kind advice of a very conscientious referee) we discuss an explicit example on the unit interval equipped with the Euclidean topology, namely the classical maximal Dirichlet form (i.e., with Neumann boundary conditions for its generator). It is well known that this Dirichlet form is (quasi-) regular on [0, 1], but it is not when considered in [0, 1) (cf. [17]). Preserving the Borel σ-algebra we, however, construct another topology on [0, 1) so that this Dirichlet form becomes quasi-regular (cf. the example after Corollary 3.4). On the other hand, equipped with this new topology, [0, 1) turns out to be isomorphic to [0, 1] with the usual Euclidean topology. Moreover, the latter is proved to be (essentially) necessary for a topology on [0, 1) to make the classical maximal Dirichlet form quasi-regular (cf. Proposition 3.5 below).

The organization of this paper is as follows. In Section 2 we first formulate and prove a corresponding result more generally for L^p-resolvents (cf. Theorem 2.2) and apply it subsequently to semi-Dirichlet forms in Section 3 (see Theorem 3.3 and Corollary 3.4). Our proof relies heavily on results in [3], in particular the characterization of resolvents of kernels which are associated to right processes. Therefore, in Section 1 we recall the most essential notions, and list all relevant results. In particular, we prove that the above characterization of resolvent kernels can be generalized to the non-transient case (see Theorem 1.3).

1. Preliminaries on Sub-Markovian Resolvents of Kernels

Below we follow the terminology of [3]. Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on a Lusin measurable space $(E, \mathcal{B})$. Recall that the resolvent

$\mathcal{U}$ is called *proper* provided there exists a strictly positive function $f \in bp\mathcal{B}$ such that $Uf \leq 1$, where $U = \sup_{\alpha>0} U_\alpha$ is the *initial kernel* of $\mathcal{U}$; $p\mathcal{B}$ (resp. $bp\mathcal{B}$ denotes the set of all positive numerical (resp. bounded positive) $\mathcal{B}$-measurable functions on E. If $\beta > 0$ then the family $\mathcal{U}_\beta = (U_{\beta+\alpha})_{\alpha>0}$ is also a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$, having U_β as (bounded) initial kernel. Recall also that a function $s \in p\mathcal{B}$ is termed $\mathcal{U}$*-supermedian* if $\alpha U_\alpha s \leq s$ for all $\alpha > 0$. A $\mathcal{U}$-supermedian function s is named $\mathcal{U}$*-excessive* if in addition $\sup_{\alpha>0} \alpha U_\alpha s = s$. We denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}$-measurable $\mathcal{U}$-excessive functions on E. If s is $\mathcal{U}$-supermedian then the function $\widehat{s}$ defined by $\widehat{s}(x) = \sup_{\alpha>0} \alpha U_\alpha s(x)$, $x \in E$, is $\mathcal{U}$-excessive and the set $M = \{x \in E|\ s(x) \neq \widehat{s}(x)\}$ is $\mathcal{U}$*-negligible*, i.e., $U_\alpha(1_M) = 0$ for one (and, therefore, for all) $\alpha > 0$. We denote by $D_\mathcal{U}$ the set of all *non-branch* points with respect to $\mathcal{U}$,

$$D_\mathcal{U} = \left\{x \in E|\ \inf(s,t)(x) = \widehat{\inf(s,t)}(x) \text{ for all } s, t \in \mathcal{E}(\mathcal{U}), \widehat{1}(x) = 1\right\}.$$

If $\mathcal{U}$ is proper then, since $\mathcal{B}$ is countably generated, we have $D_\mathcal{U} \in \mathcal{B}$ and the set $E \setminus D_\mathcal{U}$ is $\mathcal{U}$-negligible. Notice that in this case $D_\mathcal{U} = D_{\mathcal{U}_\beta}$ for all $\beta > 0$.

Let $(E', \mathcal{B}')$ be a second Lusin measurable space such that $E \subset E'$, $E \in \mathcal{B}'$ and $\mathcal{B} = \mathcal{B}'|_E$. For all $\alpha > 0$ define the kernel U'_α on $(E', \mathcal{B}')$ by

$$U'_\alpha f = 1_E U_\alpha(f|_E) + \frac{1}{1+\alpha} 1_{E'\setminus E} f \quad f \in p\mathcal{B}'.$$

Then the family $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on $(E', \mathcal{B}')$, called the *trivial extension of* $\mathcal{U}$ *to* E'. If $\beta > 0$ then a function $s \in p\mathcal{B}'$ will be $\mathcal{U}'_\beta$-excessive if and only if $s|_E$ is $\mathcal{U}_\beta$-excessive. Particularly we have $D_{\mathcal{U}'_\beta} = D_{\mathcal{U}_\beta} \cup (E' \setminus E)$ and: $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}$ if and only if $\sigma(\mathcal{E}(\mathcal{U}'_\beta)) = \mathcal{B}'$. If $\mathcal{U}$ is proper then $\mathcal{U}'$ is also proper.

Let $M \in \mathcal{B}$ be such that $U_\alpha(1_{E\setminus M}) = 0$ on M for one (and, therefore, for all) $\alpha > 0$. Then the family of kernels $\mathcal{U}|_M = (U_\alpha|_M)_{\alpha>0}$ on $(M, \mathcal{B}|_M)$ is a sub-Markovian resolvent of kernels, called the *restriction of* $\mathcal{U}$ *to* M; the kernel $U_\alpha|_M$ is defined by $U_\alpha|_M(g) = U_\alpha(\overline{g})|_M$ where $\overline{g} \in p\mathcal{B}$ and $\overline{g}|_M = g$.

Recall that a σ-finite measure ξ on $(E, \mathcal{B})$ is called $\mathcal{U}$*-excessive* if $\xi \circ \alpha U_\alpha \leq \xi$ for all $\alpha > 0$. We denote by $\mathsf{Exc}_\mathcal{U}$ the set of all $\mathcal{U}$-excessive measures. Further, let $L : \mathsf{Exc}_\mathcal{U} \times \mathcal{E}(\mathcal{U}) \longrightarrow \overline{\mathbb{R}}_+$ be the *energy functional* (associated with $\mathcal{U}$), $L(\xi, s) = \sup\{\mu(s)|\ \mu$ a σ-finite measure, $\mu \circ U \leq \xi\}$, for all $\xi \in \mathsf{Exc}_\mathcal{U}$ and $s \in \mathcal{E}(\mathcal{U})$. A $\mathcal{U}$-excessive measure of the form $\mu \circ U$ (where μ is a σ-finite measure) is called *potential*.

For the rest of this section we suppose that $D_{\mathcal{U}_\beta} = E$ and $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}$ for one (and, therefore, for all) $\beta > 0$.

1.1. The Transient Case

Suppose that $\mathcal{U}$ is proper. Notice that if $\mu \circ U = \nu \circ U \in \mathsf{Exc}_\mathcal{U}$ then $\mu = \nu$. Moreover, the set $\mathsf{Exc}_\mathcal{U}$ is an H-cone with respect to the usual order relation on the positive σ-finite measures; see, e.g., [11].

A $\mathcal{U}$-excessive measure ξ is called *purely excessive* (resp. *invariant*) if $\inf_\alpha \xi \circ \alpha U_\alpha = 0$ (resp. $\xi \circ \alpha U_\alpha = \xi$ for all $\alpha > 0$). Note that if $\xi \in \mathsf{Exc}_\mathcal{U}$ then the measure $\xi_o = \inf_\alpha \xi \circ \alpha U_\alpha$ is invariant and $\xi - \xi_o$ is purely excessive. Also, every potential is purely excessive.

The proof of the following lemma is given in the Appendix.

LEMMA 1.1. *If $\beta > 0$ then the following assertions hold.*

a) *Let $\xi \in \mathsf{Exc}_{\mathcal{U}}$. Then the measure $\xi' = \xi - \xi \circ \beta U_\beta$ is $\mathcal{U}_\beta$-excessive. If in addition ξ is purely excessive then $\xi = \xi' \circ (I + \beta U)$ and for every $\eta \in \mathsf{Exc}_{\mathcal{U}}$ with $\xi - \xi \circ \beta U_\beta \leq \eta - \eta \circ \beta U_\beta$ we have $\xi \leq \eta$.*
b) *If $\xi' \in \mathsf{Exc}_{\mathcal{U}_\beta}$ and the measure $\xi = \xi' \circ (I + \beta U)$ is σ-finite, then $\xi \in \mathsf{Exc}_{\mathcal{U}}$. Furthermore, it is purely excessive and $\xi' = \xi - \xi \circ \beta U_\beta$.*

We collect now some results on the semisaturation and saturation of E; cf. [3]. The set E is called *semisaturated* (resp. *saturated*) *with respect to* $\mathcal{U}$ provided that every $\mathcal{U}$-excessive measure dominated by a potential is also a potential (resp. every $\xi \in \mathsf{Exc}_{\mathcal{U}}$ with $L(\xi, 1) < \infty$ is a potential). If $\xi \in \mathsf{Exc}_{\mathcal{U}}$ then E is termed ξ*-semisaturated* if every $\mathcal{U}$-excessive measure dominated by a potential dominated by ξ is also a potential. The following assertions hold.

1) If E is saturated with respect to $\mathcal{U}$ then E is semisaturated with respect to $\mathcal{U}$.
2) The set E is semisaturated with respect $\mathcal{U}$ if and only if there exists a Lusin topology on E such that $\mathcal{B}$ is the σ-algebra of all Borel sets on E and there exists a right process with state space E, having $\mathcal{U}$ as associated resolvent.
3) There exist a second Lusin measurable space $(E_1, \mathcal{B}_1)$ such that $E \subset E_1$, $E \in \mathcal{B}_1$, $\mathcal{B} = \mathcal{B}_1|_E$, and a proper sub-Markovian resolvent of kernels $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ on $(E_1, \mathcal{B}_1)$ such that $D_{\mathcal{U}^1} = E_1$, $\sigma(\mathcal{E}(\mathcal{U}^1)) = \mathcal{B}_1$, $U^1_\alpha(1_{E_1\setminus E}) = 0$, E_1 is saturated with respect to $\mathcal{U}^1$ and $\mathcal{U}$ is the restriction of $\mathcal{U}^1$ to E. In particular, $\mathcal{U}^1$ is the resolvent of a right process with state space E_1 for a suitable Lusin topology on E_1. More precisely one can take E_1 as the set of all extreme points of the set $\{\xi \in \mathsf{Exc}_{\mathcal{U}} | L(\xi, 1) \leq 1\}$, endowed with the σ-algebra $\mathcal{B}_1$ generated by the functionals $\widetilde{s}$, $\widetilde{s}(\xi) = L(\xi, s)$ for all $\xi \in E_1$ and $s \in \mathcal{E}(\mathcal{U})$. The set E_1 is called the *saturation* of E.
4) Let $(E', \mathcal{B}')$ be a Lusin measurable space such that $E \subset E'$, $E \in \mathcal{B}'$, $\mathcal{B} = \mathcal{B}'|_E$, and there exists a proper sub-Markovian resolvent of kernels $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E', \mathcal{B}')$ with $D_{\mathcal{U}'} = E'$, $\sigma(\mathcal{E}(\mathcal{U}')) = \mathcal{B}'$, $U'_\alpha(1_{E'\setminus E}) = 0$, E' is saturated with respect to $\mathcal{U}'$ and $\mathcal{U}$ is the restriction of $\mathcal{U}'$ to E. Then the map $x \longmapsto \varepsilon_x \circ U'$ is a measurable isomorphism between $(E', \mathcal{B}')$ and the measurable space $(E_1, \mathcal{B}_1)$ defined in 3) above.
5) The set E is semisaturated (resp. ξ-semisaturated, where ξ is a fixed $\mathcal{U}$-excessive measure) if and only if $E_1 \setminus E$ is a polar (resp. ξ-polar) subset of E_1 (with respect to $\mathcal{U}^1$); recall that a set $M \in \mathcal{B}$ is *polar* (resp. ξ*-polar*) with respect to $\mathcal{U}$ if $\widehat{R^M 1} = 0$ (resp. $\widehat{R^M 1} = 0$ ξ-a.e.), where $R^M 1$ denotes the reduced function (with respect to $\mathcal{U}$) of 1 on M, $R^M 1 = \inf\{s \in \mathcal{E}(\mathcal{U}) | \, s \geq 1 \text{ on } M\}$.
6) If E is ξ-semisaturated then there exists a proper sub-Markovian resolvent of kernels $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ such that the set E is semisaturated with respect to $\mathcal{U}'$ and for all $f \in p\mathcal{B}$ and $\alpha > 0$ the set $[U_\alpha f \neq U'_\alpha f]$ is ξ-polar.
7) Let $A \in \mathcal{B}$ be such that $U_\alpha(1_{E\setminus A}) = 0$ on A and $\mathcal{U}'$ the trivial extension of $\mathcal{U}|_A$ to E. Then A is semisaturated with respect to $\mathcal{U}|_A$ if and only if E is semisaturated with respect to $\mathcal{U}'$.

PROPOSITION 1.2. *Let $\beta > 0$. Then E is semisaturated (resp. saturated) with respect to $\mathcal{U}$ if and only if it is semisaturated (resp. saturated) with respect to $\mathcal{U}_\beta$.*

Proof. Suppose that E is semisaturated with respect to $\mathcal{U}$ and let $\xi', \mu \circ U_\beta \in \mathsf{Exc}_{\mathcal{U}_\beta}$, $\xi' \leq \mu \circ U_\beta$. Clearly we may assume that μ is finite and thus ξ' is also a finite measure. By Lemma 1.1 it follows that the measure $\xi = \xi' \circ (I + \beta U)$ is $\mathcal{U}$-excessive and $\xi' = \xi - \xi \circ \beta U_\beta$. Since $\xi \leq \mu \circ U_\beta(I + \beta U) = \mu \circ U$ we deduce by hypothesis that there exists a σ-finite measure ν on $(E, \mathcal{B})$ such that $\xi = \nu \circ U$ and thus $\xi' = \nu \circ U(I - \beta U_\beta) = \nu \circ U_\beta$.

If E is saturated with respect to $\mathcal{U}$ and $\xi' \in \mathsf{Exc}_{\mathcal{U}_\beta}$ is such that $L_\beta(\xi', 1) < \infty$ (L_β denotes the energy functional associated with $\mathcal{U}_\beta$) then we claim that the measure $\xi = \xi' \circ (I + \beta U)$ is σ-finite. Indeed, let $(\mu_n)_n$ be a sequence of positive measures on $(E, \mathcal{B})$ such that $\mu_n \circ U_\beta \nearrow \xi'$. From $\mu_n(1) = L_\beta(\mu_n \circ U_\beta, 1) \leq L_\beta(\xi', 1)$ it follows that $\sup_n \mu_n(1) < \infty$. If $f_o \in bp\mathcal{B}$ is such that $Uf_o \leq 1$ then we get $\xi(f_o) = \xi' \circ (I + \beta U)(f_o) = \sup_n \mu_n \circ U_\beta(I + \beta U)(f_o) = \sup_n \mu_n(Uf_o) \leq \sup_n \mu_n(1) < \infty$. Hence the measure ξ is σ-finite and by Lemma 1.1 we obtain that ξ is $\mathcal{U}$-excessive and $\xi' = \xi \circ (I - \beta U_\beta)$. Since $L(\xi, 1) = \sup_n \mu_n(1) < \infty$ and E is saturated with respect to $\mathcal{U}$, it follows that there exists a σ-finite measure μ on $(E, \mathcal{B})$ such that $\xi = \mu \circ U$ and thus $\xi' = \mu \circ U_\beta$.

Assume now that E is semisaturated with respect to $\mathcal{U}_\beta$ and let $\xi, \mu \circ U \in \mathsf{Exc}_{\mathcal{U}}$, $\xi \leq \mu \circ U$. The measure ξ is purely excessive and we may suppose that μ is finite. Consequently the measure $\mu' = \mu \circ (I + \beta U)$ is σ-finite. Again by Lemma 1.1 it follows that the measure $\xi' = \xi \circ (I - \beta U_\beta)$ is $\mathcal{U}_\beta$-excessive. Since $\xi' \leq \mu \circ U = \mu' \circ U_\beta$, by hypothesis there exists a σ-finite measure ν on $(E, \mathcal{B})$ such that $\xi' = \nu \circ U_\beta$. As a consequence we get $\xi = \xi' \circ (I + \beta U) = \nu \circ U$.

Let us suppose now that E is saturated with respect to $\mathcal{U}_\beta$ and $\xi \in \mathsf{Exc}_{\mathcal{U}}$ is such that $L(\xi, 1) < \infty$. If E_1 is the saturation of E with respect to $\mathcal{U}$ then ξ is a potential on E_1 and thus it is purely excessive. Lemma 1.1 implies that the measure $\xi' = \xi - \xi \circ \beta U_\beta$ belongs to $\mathsf{Exc}_{\mathcal{U}_\beta}$ and $\xi = \xi' \circ (I + \beta U)$. We consider a sequence $(\mu_n)_n$ of positive σ-finite measures on $(E, \mathcal{B})$ such that $\mu_n \circ U_\beta \nearrow \xi'$. Consequently, we have $\mu_n \circ U \nearrow \xi$ and $L_\beta(\xi', 1) = \sup_n \mu_n(1) = L(\xi, 1) < \infty$. Therefore, there exists a σ-finite measure μ on $(E, \mathcal{B})$ such that $\xi' = \mu \circ U_\beta$ and so $\xi = \xi' \circ (I + \beta U) = \mu \circ U$. □

1.2. The Non-transient Case

Firstly recall some facts on Ray cones. Assume that the initial kernel U of $\mathcal{U}$ is bounded. A *Ray cone associated with* $\mathcal{U}$ is a convex cone $\mathcal{R}$ of bounded $\mathcal{U}$-excessive functions such that: $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$, $U((\mathcal{R} - \mathcal{R})_+) \subset \mathcal{R}$, $\sigma(\mathcal{R}) = \mathcal{B}$, $\mathcal{R}$ is min-stable, separable in the uniform norm and contains the positive constant functions.

We state here a slightly modified version of Proposition 1.5.1 in [3]: Let $\beta > 0$. Then there exists a Ray cone $\mathcal{R}_\beta$ associated with $\mathcal{U}_\beta$, such that $U_\alpha(\mathcal{R}_\beta) \subset \mathcal{R}_\beta$ for all $\alpha > 0$.

We claim that the above assertion 2) is true without assuming that $\mathcal{U}$ is proper. Namely the following result is a variant of assertion 2), in the case when the initial kernel U is not necessary a proper one; compare with [19].

THEOREM 1.3. *The set E is semisaturated with respect to $\mathcal{U}_\beta$ if and only if there exists a Lusin topology on E such that $\mathcal{B}$ is the σ-algebra of all Borel sets on E and there exists a right process with state space E, having $\mathcal{U}$ as associated resolvent.*

Proof. It is known that E is semisaturated with respect to $\mathcal{U}_\beta$ whenever $\mathcal{U}$ is the resolvent of a right process; see [11]. For the converse statement we shall adapt the proofs of Theorem 1.8.11 and Corollary 1.8.12 in [3].

First assume that E is saturated with respect to $\mathcal{U}_\beta$. Let $\mathcal{R}_\beta$ be a Ray cone associated with $\mathcal{U}_\beta$ such that $U_\alpha(\mathcal{R}_\beta) \subset \mathcal{R}_\beta$ for all $\alpha > 0$, and Y the (Ray) compactification of E with respect to $\mathcal{R}_\beta$. By Proposition 1.5.8 in [3] there exists a Ray resolvent $\widetilde{\mathcal{U}} = (\widetilde{U}_\alpha)_{\alpha>0}$ on Y such that $\widetilde{U}_\alpha(\widetilde{s}) = \widetilde{U_\alpha s}$ for all $s \in \mathcal{R}_\beta$ and $\alpha > 0$, where for each $s \in \mathcal{R}_\beta$ we have denoted by $\widetilde{s}$ the unique continuous extension of s to Y. Particularly $\widetilde{U}_\alpha(1_{Y\setminus E}) = 0$ on E for all $\alpha > 0$ and $\mathcal{U}$ is the restriction to E of $\widetilde{\mathcal{U}}$. Consequently (see, e.g., [18]) the restriction of $\widetilde{\mathcal{U}}$ to $D = D_{\widetilde{\mathcal{U}}}$ is the resolvent of a right process X with state space D, endowed with the Ray topology induced by $\mathcal{R}_\beta$ (i.e., the trace on D of the topology on Y). From Theorem 1.8.11 in [3] we have $E = \{x \in D |\ \widetilde{U}_\alpha(1_{D\setminus E})(x) = 0\}$. In addition E is a Borel subset of Y, $\widetilde{U}_\alpha(1_{D\setminus E}) = 0$ on E and it is a finely closed set with respect to $\widetilde{\mathcal{U}}_\beta$; the *fine topology* is the topology generated by $\mathcal{E}(\widetilde{\mathcal{U}}_\beta)$. As a consequence we may consider the restriction of X to E and $\mathcal{U}$ becomes the resolvent of this right process, since $\widetilde{\mathcal{U}}|_E = \mathcal{U}$.

If E is only semisaturated with respect to $\mathcal{U}_\beta$, then we consider the saturation E_1 of E with respect to $\mathcal{U}_\beta$ and let $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ be the resolvent of kernels on on $(E_1, \mathcal{B}_1)$ such that $D_{\mathcal{U}^1} = E_1$, $\sigma(\mathcal{E}(\mathcal{U}^1_\beta)) = \mathcal{B}_1$, $U^1_\alpha(1_{E_1\setminus E}) = 0$ and $\mathcal{U}^1|_E = \mathcal{U}$. By the first part of the proof there exists a right process X with state space E_1 (endowed with a Ray topology), having $\mathcal{U}^1$ as associated resolvent. By 5) we deduce that the set $E_1 \setminus E$ is polar (with respect to $\mathcal{U}^1_\beta$) and, therefore, the restriction of X to E is a right process with state space E and having $\mathcal{U}$ as associated resolvent, completing the proof. □

REMARK. By Proposition 1.2 it follows that the condition of semisaturation with respect to $\mathcal{U}_\beta$ in Theorem 1.3 does not depend on β.

Recall that a $\mathcal{U}$-excessive measure ξ is called *dissipative* (resp. *conservative*) provided that $\xi = \sup\{\mu \circ U |\ \mathsf{Exc}_{\mathcal{U}} \ni \mu \circ U \le \xi\}$ (resp. there is no non-zero potential $\mathcal{U}$-excessive measure dominated by ξ). The set of all dissipative (resp. conservative) $\mathcal{U}$-excessive measures is denoted by $\mathsf{Diss}_{\mathcal{U}}$ (resp. $\mathsf{Con}_{\mathcal{U}}$). As in [11] one can show that $\mathsf{Diss}_{\mathcal{U}}$ and $\mathsf{Con}_{\mathcal{U}}$ are solid convex subcones of $\mathsf{Exc}_{\mathcal{U}}$, $\mathsf{Diss}_{\mathcal{U}} \cap \mathsf{Con}_{\mathcal{U}} = \{0\}$ and every $\xi \in \mathsf{Exc}_{\mathcal{U}}$ has a unique decomposition of the form $\xi = \xi_d + \xi_c$, where $\xi_d \in \mathsf{Diss}_{\mathcal{U}}$ and $\xi_c \in \mathsf{Con}_{\mathcal{U}}$. Moreover, if $f \in p\mathcal{B}$ is strictly positive and $\xi(f) < \infty$ then $\xi_d = \xi|_{[Uf<\infty]}$ and $\xi_c = \xi|_{[Uf=\infty]}$; See also Proposition A1 in the Appendix.

The next result is an extension of assertion 6) to the non-transient case.

PROPOSITION 1.4. *Let $\xi \in \mathsf{Diss}_{\mathcal{U}}$ be such that E is ξ-semisaturated with respect to $\mathcal{U}$ (i.e., every $\mathcal{U}$-excessive measure dominated by a potential dominated by ξ is also a potential). Then there exists a proper sub-Markovian resolvent of kernels $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ such that E is semisaturated with respect to $\mathcal{U}'$ and the set $[U_\alpha f \neq U'_\alpha f]$ is ξ-polar with respect to $\mathcal{U}_\beta$ for all $f \in p\mathcal{B}$ and $\alpha > 0$. Moreover there exists a ξ-polar finely closed set $A \in \mathcal{B}$ such that $U(1_A) = 0$ on $E \setminus A$ and $\mathcal{U}'$ may be chosen as the trivial extension to E of the restriction of $\mathcal{U}$ to $E \setminus A$.*

Proof. Let $f \in p\mathcal{B}$ be strictly positive such that $\xi(f) < \infty$. The set $A = [Uf = \infty]$ is finely closed, $U(1_A) = 0$ on $E \setminus A$ and from $\xi \in \mathsf{Diss}_{\mathcal{U}}$ we get $\xi(A) = 0$. Therefore, the set A is ξ-polar with respect to $\mathcal{U}_\beta$. If $\mathcal{V}$ is the restriction of $\mathcal{U}$ to $E \setminus A$ then we deduce that $\mathcal{V}$ is a proper sub-Markovian resolvent of kernels on $(E \setminus A, \mathcal{B}|_{E\setminus A})$ such that $\sigma(\mathcal{E}(\mathcal{V})) = \mathcal{B}|_{E\setminus A}$ and $D_{\mathcal{V}} = E \setminus A$. Clearly the measure ξ belongs to $\mathsf{Exc}_{\mathcal{V}}$. We show that $E \setminus A$ is ξ-semisaturated with respect to $\mathcal{V}$. Indeed, let $\eta, \mu \circ V \in \mathsf{Exc}_{\mathcal{V}}$, with $\eta \le \mu \circ V \le \xi$, where μ is a σ-finite measure on $E \setminus A$. We deduce that $\eta, \mu \circ U \in \mathsf{Exc}_{\mathcal{U}}$ and $\eta \le \mu \circ U \le \xi$. Since E is ξ-semisaturated with respect to $\mathcal{U}$, there exists a σ-finite measure ν on E such that $\eta = \nu \circ U$. Since the set A is μ-polar and μ-negligible, it follows that it is also ν-negligible and consequently $\eta = \nu|_{E\setminus A} \circ V$. By 6) there exists a proper sub-Markovian resolvent of kernels $\mathcal{V}' = (V'_\alpha)_{\alpha>0}$ on $(E \setminus A, \mathcal{B}|_{E\setminus A})$ such that $E \setminus A$ is semisaturated with respect to $\mathcal{V}'$ and $V_\alpha f = V'_\alpha f$ on E_o for all $f \in p\mathcal{B}|_{E\setminus A}$ and $\alpha > 0$, where $E_o \in \mathcal{B}$ is such that $E_o \subset E \setminus A$, $E \setminus E_o$ is ξ-polar and $U(1_{E\setminus E_o}) = 0$ on E_o. From 7) we conclude that the trivial extension $\mathcal{U}'$ of $\mathcal{V}'|_{E_o}$ to E satisfies the required conditions. □

2. Right Processes Associated with L^p-resolvents

In the sequel μ will be a σ-finite measure on $(E, \mathcal{B})$.

Let $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ be a second sub-Markovian resolvent of kernels on $(E, \mathcal{B})$. We say that $\mathcal{U}$ and $\mathcal{U}'$ are *μ-equivalent* provided that $U_\alpha f = U'_\alpha f$ μ-a.e. for all $f \in p\mathcal{B}$ and $\alpha > 0$.

REMARK. There are examples of two sub-Markovian resolvents of kernels on the same space E, which are ξ-equivalent (where ξ is a σ-finite measure) and such that E is semisaturated with respect to only one of them. Indeed, let $\mathcal{U}^o$ be a sub-Markovian resolvent on a Lusin measurable space $(F, \mathcal{B}_o)$ such that F is not semisaturated with respect to $\mathcal{U}^o$. We denote by E the saturation of F with respect to $\mathcal{U}^o$ (i.e., $E = F_1$) and let $\mathcal{U}$ be the resolvent on E such that $\mathcal{U}|_F = \mathcal{U}^o$ and $E \setminus F$ is $\mathcal{U}$-negligible. Let further $\mathcal{U}'$ be the trivial extension of $\mathcal{U}^o$ to E. Then by 7) the set E is not semisaturated with respect to $\mathcal{U}'$. Clearly, since $U_\alpha(1_{E\setminus F}) = 0$, we deduce that $\mathcal{U}$ and $\mathcal{U}'$ are ξ-equivalent with respect to every $\xi \in \mathsf{Exc}_{\mathcal{U}}$.

LEMMA 2.1. *Let N be a bounded kernel on $(E, \mathcal{B})$ such that if $B \in \mathcal{B}$ and $\mu(B) = 0$ then $N(1_B) = 0$ μ-a.e. If $E_o \subset E$, $E_o \in \mathcal{B}$, is such that $\mu(E \setminus E_o) = 0$ then there exists $F \in \mathcal{B}$, $F \subset E_o$, such that $\mu(E \setminus F) = 0$ and $N(1_{E\setminus F}) = 0$ on F.*

Proof. Since $\mu(E \setminus E_o) = 0$ we get by hypothesis that $N(1_{E\setminus E_o}) = 0$ μ-a.e. Let $(E_n)_{n\ge1} \subset \mathcal{B}$ be the sequence defined inductively by $E_{n+1} = E_n \cap [N(1_{E\setminus E_n}) = 0]$ if $n \ge 0$. We have $\mu(E \setminus E_n) = 0$ for all n and let $F = \bigcap_n E_n$. Then $F \subset E_o$, $F \in \mathcal{B}$, $\mu(E \setminus F) = 0$ and if $x \in F$ then $N(1_{E\setminus E_n})(x) = 0$ for all n. Therefore, $N(1_{E\setminus F})(x) = N(1_{\bigcup_n E\setminus E_n})(x) = \sup_n N(1_{E\setminus E_n})(x) = 0$. □

REMARK. A procedure similar to Lemma 2.1 has been considered in [12] and [16].

THEOREM 2.2. *Let $p \in [1, +\infty]$ and $(V_\alpha)_{\alpha>0}$ be a sub-Markovian strongly continuous resolvent of contractions on $L^p(E, \mu)$, where $(E, \mathcal{B})$ is a Lusin measurable space and μ is a σ-finite measure on $(E, \mathcal{B})$. Then there exist a Lusin topological space E_1 with $E \subset E_1$, $E \in \mathcal{B}_1$ (the σ-algebra of all Borel subsets of E_1), $\mathcal{B} = \mathcal{B}_1|_E$, and a right process with state space E_1 such that its resolvent of kernels $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$, regarded on $L^p(E_1, \overline{\mu})$, coincides with $(V_\alpha)_{\alpha>0}$ and $U^1_\alpha(1_{E_1\setminus E}) = 0$, where $\overline{\mu}$ is the measure on $(E_1, \mathcal{B}_1)$ extending μ by zero on $E_1 \setminus E$.*

Proof. Let $(f_k)_k \subset bp\mathcal{B} \cap L^p(E,\mu)$ be a sequence separating the points of E. For every $\alpha > 0$ we consider a kernel $\overline{V}_\alpha$ on $(E,\mathcal{B})$ such that $\overline{V}_\alpha$ coincides with V_α as an operator on $L^p(E,\mu)$. By Proposition 1.4.13 in [3] there exists a sub-Markovian resolvent $\mathcal{W} = (W_\alpha)_{\alpha>0}$ on $(E,\mathcal{B})$ such that $W_\alpha f = \overline{V}_\alpha f$ μ-a.e. for all $f \in p\mathcal{B}$. Let us consider the set

$$E_o = \{x \in E | \lim_n \alpha_n W_{\alpha_n} f_k(x) = f_k(x) \text{ for all } k\}.$$

where $(\alpha_n)_n$ is a strictly increasing sequence of natural numbers such that $(\alpha_n W_{\alpha_n} f_k)_n$ converges μ-a.e. to f_k for all k. We have $E_o \in \mathcal{B}$ and $\mu(E \setminus E_o) = 0$. By Lemma 2.1 there exists $F \in \mathcal{B}$, $F \subset E_o$, such that $\mu(E \setminus F) = 0$ and $W_\alpha(1_{E\setminus F}) = 0$ on F for all $\alpha > 0$. Let $\beta > 0$ and F_o be the set of all non-branch points of F with respect to $\mathcal{W}_\beta|_F$. Then $F_o \in \mathcal{B}$, $\mathcal{W}_\beta|_{F_o}$ is a sub-Markovian resolvent of kernels on $(F_o, \mathcal{B}|_{F_o})$, $\mathcal{E}(\mathcal{W}_\beta|_{F_o})$ is min-stable, contains the positive constant functions and generates $\mathcal{B}|_{F_o}$. Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the trivial extension of $\mathcal{W}|_{F_o}$ to E. Then $\mathcal{U}$ is a sub-Markovian resolvent of kernels on $(E,\mathcal{B})$ such that $D_{\mathcal{U}_\beta} = E$, $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}$ and $(U_\alpha)_{\alpha>0}$ coincides with $(V_\alpha)_{\alpha>0}$ as a resolvent on $L^p(E,\mu)$. We consider now the set E_1, i.e. the saturation of E with respect $\mathcal{U}_\beta$ (see 3) in Section 1) and the resolvent of kernels $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ on $(E_1, \mathcal{B}_1)$ whose restriction to E is $\mathcal{U}$ and $U^1_\alpha(1_{E_1\setminus E}) = 0$. Since E_1 is saturated with respect to $\mathcal{U}^1_\beta$, we deduce from 1) and Theorem 1.3 that there exists a Lusin topology on E_1 such that $\mathcal{B}_1$ is the σ-algebra of all Borel sets on E_1 and $\mathcal{U}^1$ is the resolvent of a right process with state space E_1. Clearly $U^1_\alpha = V_\alpha$ for all $\alpha > 0$, regarded as an equality of operators on $L^p(E_1, \overline{\mu})$. □

REMARK 2.3. Under the assumptions of Theorem 2.2 we have proved that there exists a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E,\mathcal{B})$ such that for $\beta > 0$ we have $D_{\mathcal{U}_\beta} = E$, $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}$ and $U_\alpha = V_\alpha$ as operators on $L^p(E,\mu)$ for all $\alpha > 0$. Moreover the following assertions hold.

a) $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent of a right process with state space E if and only if E is semisaturated with respect to $\mathcal{U}_\beta$ (cf. Theorem 1.3).
b) If μ is $\mathcal{U}_\beta$-excessive and E is μ-semisaturated with respect to $\mathcal{U}_\beta$ (or if $\mu \in \mathsf{Diss}_\mathcal{U}$ and E is μ-semisaturated with respect to $\mathcal{U}$) then by 6), Proposition 1.4 and Theorem 1.3 there exist a Lusin topology on E and a right process with state space E such that its resolvent and $\mathcal{U}$ are μ-equivalent.

The following result is a consequence of Proposition 7.5.2 in [3], Theorem 2.2 and Remark 2.3.

COROLLARY 2.4. *Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on $(E,\mathcal{B})$ such that for $\beta > 0$ we have $D_{\mathcal{U}_\beta} = E$ and $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}$. If $\mu \in \mathsf{Exc}_\mathcal{U}$ then there exists a second sub-Markovian resolvent of kernels $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ on $(E,\mathcal{B})$ such that for $\beta > 0$ we have $D_{\mathcal{U}^*_\beta} = E$, $\sigma(\mathcal{E}(\mathcal{U}^*_\beta)) = \mathcal{B}$ and $\int_E fU_\alpha g d\mu = \int_E gU^*_\alpha f d\mu$ for all $f, g \in p\mathcal{B}$ and $\alpha > 0$.*

3. Tightness of Capacity and Quasi-regularity

In this section we shall give conditions on an L^p-resolvent to ensure tightness of the capacity induced by the reduction operator, the existence of quasi-continuous versions for the elements being in the domain of the generator and the standardness property of the associated right process.

Let $(V_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent on $L^p(E,\mu)$ as in Theorem 2.2 and $\beta > 0$. An element $u \in L^p_+(E,\mu)$ is called a *β-potential* provided that $\alpha V_{\beta+\alpha} u \le u$ for all $\alpha > 0$. We denote by $\mathcal{P}_\beta$ the set of all, β-potentials. It is known that (see, e.g., Proposition 3.1.10 in [3]) the ordered convex cone $\mathcal{P}_\beta$ is a cone of potentials in the sense of G. Mokobodzki, cf. [15] (see also [3]). Particularly if $u, u' \in \mathcal{P}_\beta$, $u \le u'$, then there exists $R_\beta(u-u') \in \mathcal{P}_\beta$, i.e. the *réduite* of $u-u'$, defined by $R_\beta(u-u') = \bigwedge\{v \in \mathcal{P}_\beta \mid v \ge u-u'\}$ (here $\bigwedge$ denotes the infimum in $\mathcal{P}_\beta$). An element $u \in \mathcal{P}_\beta$ is called *regular* if for every sequence $(u_n)_n \subset \mathcal{P}_\beta$ with $u_n \nearrow u$ we have $R_\beta(u-u_n) \searrow 0$.

REMARK 3.1.

a) If $f \in L^p(E,\mu)$ then $V_\beta f$ is regular. If $u \in \mathcal{P}_\beta$ then $V_\alpha u$ is regular for every $\alpha > 0$.
b) Let $u \in \mathcal{P}_\beta$. If there exists a sequence $(u_n)_n$ of regular elements from $\mathcal{P}_\beta$ with $u_n \nearrow u$ and $R_\beta(u-u_n) \searrow 0$ then by Proposition 3.2.3 in [3] it follows that u is regular. Consequently by *a*) we deduce that: u is regular if and only if $R_\beta(u - nV_n u) \searrow 0$.
c) Assume that $\boldsymbol{\mathcal{V}}_\beta = (V_{\beta+\alpha})_{\alpha>0}$ is the resolvent of a right process and let $u \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{V}}_\beta) \cap L^p(E,\mu)$, $u < \infty$. Then u is regular if and only if there exists a continuous additive functional whose potential equals u μ-a.e.

Let $f_o \in L^p(E,\mu)$ be strictly positive. We consider the following property of the resolvent $(V_\alpha)_{\alpha>0}$:

$$\textit{every } \beta-\textit{potential dominated by } V_\beta f_o \textit{ is regular.} \qquad (*)$$

REMARK. Since $V_\beta f_o > 0$, it follows from Propositions 2.4.6 and 2.4.7 in [3] that condition $(*)$ is equivalent with the following one: every β-potential dominated by a regular element from $\mathcal{P}_\beta$ is also regular.

PROPOSITION 3.2. *Condition* $(*)$ *does not depend on* β.

Proof. Let $\beta' > \beta > 0$ and assume that condition $(*)$ holds for β. If $(u_n)_n \subset \mathcal{P}_{\beta'}$, $u_n \nearrow u \in \mathcal{P}_{\beta'}$, $u \le V_{\beta'} f_o$, then the element $v = u + (\beta'-\beta)V_\beta u$ belongs to $\mathcal{P}_\beta$, $v \le V_\beta f_o$ and thus v is regular. Setting $v_n = u_n + (\beta'-\beta)V_\beta u$ we get $v_n \nearrow v$, $v_n = u_n + (\beta'-\beta)V_\beta u_n + (\beta'-\beta)V_\beta(u-u_n) \in \mathcal{P}_\beta$ and since $\mathcal{P}_\beta \subset \mathcal{P}_{\beta'}$ it follows that $R_{\beta'}(u-u_n) = R_{\beta'}(v-v_n) \le R_\beta(v-v_n) \searrow 0$.

Assume now that condition $(*)$ holds for β' and let $(u_n)_n \subset \mathcal{P}_\beta$, $u_n \nearrow u \in \mathcal{P}_\beta$. Then the element $v = u - (\beta'-\beta)V_{\beta'} u$ belongs to $\mathcal{P}_{\beta'}$. If $u \le V_\beta f_o$, since by Remark 3.1 $V_\beta f_o$ is a regular element of $\mathcal{P}_{\beta'}$, we deduce that v is regular in $\mathcal{P}_{\beta'}$. Let $(f_n)_n \subset L^p(E,\mu)$ be such that $V_{\beta'} f_n \nearrow v$. Then $R_{\beta'}(v - V_{\beta'} f_n) \searrow 0$ and $V_\beta f_n \nearrow u$. To show that u is regular, again by Remark 3.1 it suffices to prove that $R_\beta(u - V_\beta f_n) \searrow 0$. Notice that if $u', u'' \in \mathcal{P}_\beta$, $f = u' - u''$, then $R_\beta(f) \le (I + (\beta'-\beta)V_\beta)R_{\beta'}(f - (\beta'-\beta)V_{\beta'} f)$. We conclude that $R_\beta(u - V_\beta f_n) \le (I + (\beta'-\beta)V_\beta)R_{\beta'}(v - V_{\beta'} f_n) \searrow 0$. □

REMARK.

a) Let $\boldsymbol{\mathcal{U}}$ be the resolvent of kernels from Remark 2.3, $\boldsymbol{\mathcal{U}}^*$ be a second resolvent given by Corollary 2.4 and suppose that they are associated with two right processes with state space E. Then condition ($*$) is equivalent with the fact that 'the axiom of polarity' holds for $\mathcal{U}^*_\beta$, i.e. every semipolar set is μ-polar with respect to $\mathcal{U}^*_\beta$ (see Theorem 7.2.9 in [3]).

b) Recall (cf. [5], [13]) that a pair $(\mathcal{E}, D(\mathcal{E}))$ is called a *semi-Dirichlet form* on $L^2(E,\mu)$, provided that: $D(\mathcal{E})$ is a dense linear subspace of $L^2(E,\mu)$; $\mathcal{E} : D(\mathcal{E})\times D(\mathcal{E}) \longrightarrow \mathbb{R}$ is a bilinear form such that $\mathcal{E}(u,u) \geq 0$ for all $u \in D(\mathcal{E})$; $D(\mathcal{E})$ is a Hilbert space equipped with the inner product $\widetilde{\mathcal{E}}_1(u,v) = \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u)) + (u,v)_{L^2(E,\mu)}$; $(\mathcal{E}_1, D(\mathcal{E}))$ satisfies the *sector condition*, i.e., there exists a constant $K > 0$ such that $|\mathcal{E}_1(u,v)| \leq K\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}$ for all $u, v \in D(\mathcal{E})$; it has the following *unit contraction property*: for all $u \in D(\mathcal{E})$ we have $u^+ \wedge 1 \in D(\mathcal{E})$ and $\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$. If in addition $\mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0$, then $(\mathcal{E}, D(\mathcal{E}))$ is called a *Dirichlet form*.

Let $(V_\alpha)_{\alpha>0}$ be the resolvent of a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E,\mu)$, i.e., $V_\alpha(L^2(E,\mu)) \subset D(\mathcal{E})$ and $\mathcal{E}_\alpha(V_\alpha f, u) = (f,u)_{L^2(E,\mu)}$ for all $\alpha > 0$, $f \in L^2(E,\mu)$ and $u \in D(\mathcal{E})$, where $\mathcal{E}_\alpha = \mathcal{E} + \alpha(\,,\,)_{L^2(E,\mu)}$. Notice that the unit contraction property of $\mathcal{E}$ is equivalent with the property of $(V_\alpha)_{\alpha>0}$ to be sub-Markovian. It was shown in [3] (Theorem 7.5.19 and Corollary 7.7.8 applied to the (bounded) resolvent $(V_{\beta+\alpha})_{\alpha>0}$ associated with the semi-Dirichlet form $(\mathcal{E}_\beta, D(\mathcal{E}))$ which satisfies the sector condition; where $\beta > 0$ is fixed) that condition ($*$) holds and derived that a semi-Dirichlet form associated with a right process is quasi-regular; compare with [8, 13] and [14].

Assume further that in addition $f_o \in L^1(E,\mu)$, $f_o \leq 1$, $\lambda_o = f_o \cdot \mu$ and $m = \lambda_o \circ V_\beta$.

The next result is a consequence of Section 3.5 and Theorem 3.7.8 in [3] and Theorem 2.2; see also [4].

THEOREM 3.3. *Under the assumptions from Theorem 2.2 suppose that condition* ($*$) *holds. If* $(E_1, \mathcal{T})$ *is the Lusin topological space and* $\mathcal{U}^1$ *the resolvent of the right process with state space* E_1 *given by Theorem 2.2, then the following assertions hold.*

a) *There exists an increasing sequence* $(K_n)_n$ *of* $\mathcal{T}$*-compact subsets of* E_1 *such that*

$$\inf_n R_\beta^{E_1\setminus K_n} p_o = 0 \quad (m + \lambda_o)\text{-a.e.}$$

where $p_o = U^1_\beta \widetilde{f}_o$ ($\widetilde{f}_o \in p\mathcal{B}_1$, $\widetilde{f}_o|_E = f_o$) *and* $R^M_\beta p_o$ *denotes the reduced function (with respect to* $\mathcal{U}^1_\beta$*) of* p_o *on the set* M.

b) *Every* $\mathcal{U}^1_\beta$*-excessive function s is* $\mathcal{T}$*-quasi continuous, that is there exists an increasing sequence* $(K_n)_n$ *of* $\mathcal{T}$*-compact subsets of* E_1 *such that* $s|_{K_n}$ *is* $\mathcal{T}$*-continuous for all n and* $\inf_n R_\beta^{E_1\setminus K_n} p_o = 0$ $(m+\lambda_o)$*-a.e. Particularly, every element from* $V_\alpha(L^p(E,\mu))$ *(the domain of the generator of the resolvent* $(V_\alpha)_{\alpha>0}$*) possesses a* $\mathcal{T}$*-quasi continuous* μ*-version.*

c) *The right process having* $\mathcal{U}^1$ *as associated resolvent is* $(m+\lambda_o)$*-special standard.*

As a consequence of the previous theorem and the main result in [13] and [14] we obtain:

COROLLARY 3.4. *Let* $(\mathcal{E}, D(\mathcal{E}))$ *be a semi-Dirichlet form on* $L^2(E,\mu)$*, where* μ *is a* σ*-finite measure on the Lusin measurable space* $(E, \mathcal{B})$*. Then there exists a (larger) Lusin topological space* E_1 *such that* $E \subset E_1$*,* E *belongs to* $\mathcal{B}_1$ *(the* σ*-algebra of all Borel subsets of* E_1*),* $\mathcal{B} = \mathcal{B}_1|_E$*, and* $(\mathcal{E}, D(\mathcal{E}))$ *regarded as a semi-Dirichlet form on*

$L^2(E_1, \overline{\mu})$ is quasi-regular, where $\overline{\mu}$ is the measure on $(E_1, \mathcal{B}_1)$ extending μ by zero on $E_1 \setminus E$.

The next example completes the counterexample (*b*), page 194 in [17]; we thank the referee for suggesting us to present it related to our results.

EXAMPLE. Let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet form associated with reflecting Brownian motion on $[0, 1]$: $D(\mathcal{E}) = H^1(0, 1)$, $\mathcal{E}(u, v) = \frac{1}{2}\int_E u'v'dm$, m being the Lebesgue measure. It was shown in [17] that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is not quasi-regular, considered on $E = [0, 1)$ endowed with the canonical topology. However, regarded on the enlarged space $E_1 = [0, 1]$ (also endowed with with the canonical topology) the form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. We show that we can equip E with a second topology, preserving the Borel σ-algebra, such that $(\mathcal{E}, D(\mathcal{E}))$ becomes quasi-regular, without enlarging E. Indeed, let $M = \{1\} \cup \{1 - \frac{1}{n} | \, n \geq 2\}$ and consider the bijection $\varphi : E_1 \longrightarrow E$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \in E \setminus M, \\ \frac{1}{2} & \text{if } x = 1, \\ 1 - \frac{1}{n+1} & \text{if } x = \frac{1}{n}, n \geq 2. \end{cases}$$

Let $\mathcal{T}_1$ be the canonical topology on E_1 and $\mathcal{T} = \varphi(\mathcal{T}_1)$, the biggest topology on E making φ a continuous map. Since every element from $H^1(0, 1)$ has a $\mathcal{T}_1$-continuous m-version on E_1, it is easy to see that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form regarded on E endowed with the topology $\mathcal{T}$, and notice that every point of E is not $\mathcal{E}$-polar.

The next result is a converse of the last statement.

PROPOSITION 3.5. *Let $\mathcal{T}_o$ be a Lusin topology on $E = [0, 1)$ such that the Borel sets are the canonical ones and assume that the Dirichlet form associated with reflecting Brownian motion on* $[0, 1]$ *is quasi-regular, regarded on E equipped with $\mathcal{T}_o$ and every point of E is not $\mathcal{E}$-polar. Then the topological space $(E, \mathcal{T}_o)$ is compact and every element from $D(\mathcal{E})$ has a $\mathcal{T}_o$-continuous m-version. Moreover, there exists a bijection $\psi : E \longrightarrow E_1$ (where $E_1 = [0, 1]$ and it is endowed with the canonical topology $\mathcal{T}_1$) which is a topological homeomorphism between $(E, \mathcal{T}_o)$ and $(E_1, \mathcal{T}_1)$ and the set $\{x \in E | \, \psi(x) \neq x\}$ is m-negligible.*

Proof. We show firstly that every $\mathcal{E}$-exceptional subset of $(E, \mathcal{T}_o)$ is empty. Indeed, let $(F_n)_n$ be a $\mathcal{E}$-nest (of $\mathcal{T}_o$-closed subsets of E) and $u_n = 1 - R_\beta^{E \setminus F_n} 1$. Since the constant function 1 belongs $D(\mathcal{E})$, it follows that the sequence $(u_n)_n$ converges to 1 in $D(\mathcal{E})$. Assume that $E \setminus F_n \neq \emptyset$ for all n. The set $E \setminus F_n$ being not $\mathcal{E}$-polar, we get that $m(E \setminus F_n) > 0$ for all n. Lemma 5.1 in [17] leads now to the contradictory fact that there exists $x \in [0, 1]$ such that $\lim_{y \to x} 1(y) = 0$. Hence there exists n_0 such that $F_{n_0} = E$.

The form $(\mathcal{E}, D(\mathcal{E}))$ being quasi-regular on $(E, \mathcal{T}_o)$, it follows that $(E, \mathcal{T}_o)$ is a compact topological space and every $u \in D(\mathcal{E})$ has a m-version $\overline{u}$ which is $\mathcal{T}_o$-continuous on E. We shall denote by $\widetilde{u}$ the $\mathcal{T}_1$-continuous m-version of $u \in D(\mathcal{E})$ on E_1. Let D_o be a countable subset of $D(\mathcal{E})$ such that the family $\{\overline{u} | \, u \in D_o\}$ (resp.

$\{\widetilde{u}|\ u \in D_o\}$) separates the points of E (resp. of E_1) and generates the topology $\mathcal{T}_0$ (resp. $\mathcal{T}_1$). We consider the set

$$M_o = \{x \in E|\ \text{there exists } u \in D_o \text{ with } \overline{u}(x) \neq \widetilde{u}(x)\}.$$

Because $\overline{u} = \widetilde{u}$ m-a.e. for all $u \in D(\mathcal{E})$, we find that the set M_o is m-negligible and consequently $E \setminus M_o$ (resp. $E_1 \setminus M_o$) is a dense subset of $(E, \mathcal{T}_o)$ (resp. of $(E_1, \mathcal{T}_1)$). Therefore, for every $x \in E$ there exists a sequence $(x_n)_n \subset E \setminus M_o$ converging to x in $\mathcal{T}_o$. From $\overline{u}(x_n) = \widetilde{u}(x_n)$ for all n and $u \in D_o$, we deduce that the sequence $(x_n)_n$ converges in $\mathcal{T}_1$ to some point $\widetilde{x} \in E_1$ and $\overline{u}(x) = \widetilde{u}(\widetilde{x})$ for all $u \in D_o$. We may define a map $\psi : E \longrightarrow E_1$ by $\psi(x) = \widetilde{x}$ for all $x \in E$ and one can check that ψ is one-to-one, it is continuous and $\psi(x) = x$ for all $x \in E \setminus M_o$. □

Appendix

Proof of Lemma 1.1.

a) If $\alpha > 0$ then we have $\xi' \circ \alpha U_{\beta+\alpha} = \xi \circ \alpha U_{\beta+\alpha} - \xi \circ \beta\alpha U_\beta U_{\beta+\alpha} = \xi \circ (\alpha + \beta)U_{\beta+\alpha} - \xi \circ \beta U_\beta \leq \xi - \xi \circ \beta U_\beta = \xi'$. For $\alpha < \beta$ we have also $\xi' \circ (I + (\beta - \alpha)U_\alpha) = \xi \circ (I - \beta U_\beta + (\beta - \alpha)U_\alpha - (\beta - \alpha)\beta U_\alpha U_\beta) = \xi - \xi \circ \alpha U_\alpha$. If ξ is purely excessive then, letting $\alpha \longrightarrow 0$, we deduce that $\xi' \circ (I + \beta U) = \xi$. Let $\eta \in \mathsf{Exc}_{\mathcal{U}}$ be such that $\xi' \leq \eta - \eta \circ \beta U_\beta$. The measure $\eta_1 = \eta - \inf_\alpha \eta \circ \alpha U_\alpha$ is purely excessive and clearly $\eta - \eta \circ \beta U_\beta = \eta_1 - \eta_1 \circ \beta U_\beta$. Therefore, $\xi = \xi' \circ (I + \beta U) \leq (\eta_1 - \eta_1 \circ \beta U_\beta) \circ (I + \beta U) = \eta_1 \leq \eta$.

b) Assume that the measure $\xi = \xi' \circ (I + \beta U)$ is σ-finite and let $\alpha > 0$. Then $\xi \circ \alpha U_\alpha = \xi' \circ (I + \beta U)\alpha U_\alpha = \xi' \circ \alpha U_\alpha + \beta\xi' \circ (U - U_\alpha)$. Therefore, if $\alpha > \beta$ then $\xi \circ \alpha U_\alpha = \xi' \circ (\alpha - \beta)U_\alpha + \beta\xi' \circ U \leq \xi' + \beta\xi' \circ U = \xi$. If $\alpha \leq \beta$ then $\xi \circ \alpha U_\alpha \leq \beta\xi' \circ U \leq \xi' \circ (I + \beta U) = \xi$. Consequently the measure ξ is $\mathcal{U}$-excessive. From $\xi' \circ U_\alpha \leq \xi' \circ U$ we get $\xi \circ \alpha U_\alpha \leq \alpha\xi' \circ U + \beta\xi' \circ (U - U_\alpha)$. The measure $\xi' \circ U$ being σ-finite we conclude that $\inf_\alpha \xi \circ \alpha U_\alpha = 0$.

The following proposition is close to the results of R. K. Getoor from [10] and [11].

PROPOSITION A1. *If $\xi \in \mathsf{Exc}_{\mathcal{U}}$ then the following assertions are equivalent.*

1) *The measure ξ is dissipative.*
2) *If $f \in p\mathcal{B}$, $f > 0$ on E and $\xi(f) < \infty$ then $Uf < \infty$ ξ-a.e.*
3) *There exists $F \in \mathcal{B}$ such that $\xi(E \setminus F) = 0$, $U(1_{E\setminus F}) = 0$ on F and $\mathcal{U}|_F$ is proper.*
4) *There exists a finely continuous function $f \in bp\mathcal{B}$, $f > 0$ on E such that $Uf \leq 1$ ξ-a.e.*
5) *There exists $f \in bp\mathcal{B}$ such that $Uf > 0$ on E and $Uf \leq 1$ ξ-a.e.*
6) *There exists a sequence $(f_n)_n \subset p\mathcal{B}$ such that Uf_n is bounded ξ-a.e. for all n and $Uf_n \nearrow \infty$.*

Proof. The equivalence 1) $\Longleftrightarrow$ 2) follows from (2.11) in [11]. The implications 4) $\Longrightarrow$ 5) $\Longrightarrow$ 6) are clear. We have 3) $\Longrightarrow$ 1) since $\mathsf{Exc}_{\mathcal{U}} = \mathsf{Diss}_{\mathcal{U}}$ if $\mathcal{U}$ is proper.

2) $\Longrightarrow$ 3). Let $g \in bp\mathcal{B}$, $g > 0$ on E be such that $\xi(g) < \infty$. Then $Ug < \infty$ ξ-a.e. If we set $A_n = [Ug \leq n]$ then $(A_n)_n \subset \mathcal{B}$ is an increasing sequence, $\xi(E \setminus \cup_n A_n) = 0$ and $U(g1_{A_n}) \leq n$ for all n. The function $f = g(1_{A_\infty} + \sum_{n\geq 1} \frac{1}{n2^n} 1_{A_n})$ is strictly positive and

$Uf \leq 1$ on $[Ug < \infty]$, where $A_\infty = [Ug = \infty]$. Taking $E_o = [Uf \leq 1]$ and applying Lemma 2.1 we obtain the required set F.

3) $\Longrightarrow$ 4). Let $g \in bp\mathcal{B}$, $g > 0$ on E be such that $Ug \leq 1$ ξ-a.e. The function $f = U_1 g$ is bounded, finely continuous, strictly positive and we have ξ-a.e. $Uf = UU_1 g = Ug - U_1 g \leq Ug \leq 1$.

6) $\Longrightarrow$ 5). Let $(f_n)_n \subset p\mathcal{B}$ and $(\alpha_n)_n \subset \mathbb{R}_+^*$ such that $Uf_n \leq \alpha_n$ ξ-a.e. for all n and $Uf_n \nearrow \infty$. Consider the function $f = \sum_n \frac{1}{\alpha_n 2^n} f_n$. Clearly $f \in p\mathcal{B}$, $Uf \leq 1$ ξ-a.e. and $Uf > 0$ on E.

5) $\Longrightarrow$ 3). Let $g \in p\mathcal{B}$, $g \leq 1$, be such that $Ug > 0$ and $Ug \leq 1$ ξ-a.e., and let $F = [Ug < \infty]$. From $Ug = U_\alpha g + \alpha U_\alpha Ug$ we get that on F we have $U_\alpha(1_{E\setminus F}) = 0$ and, therefore, $U(1_{E\setminus F}) = 0$. The function $f = \alpha U_\alpha g \cdot 1_F + 1_{E\setminus F}$ belongs to $p\mathcal{B}$, $f \leq 1$ and $Uf \leq \alpha U_\alpha Ug + U(1_{E\setminus F}) \leq Ug < \infty$ on F. It remains to show that $f > 0$. If we assume that $f(x) = 0$ then $x \in F$ and $U_\alpha g(x) = 0$. Consequently, we get $\alpha U_\alpha Ug(x) = Ug(x)$ and thus $\beta U_\beta Ug(x) = Ug(x)$ for all $\beta > 0$, $U_\beta g(x) = 0$. This leads to the contradictory equality $Ug(x) = 0$. □

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