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Uniform Estimates of the Fundamental Solution for a Family of Hypoelliptic Operators

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Abstract In this paper we are concerned with a family of elliptic operators represented as sum of square vector fields: $L_\epsilon = \sum_{i=1}^m X_i^2 + \epsilon\Delta$, in $\mathbb{R}^n$ where Δ is the Laplace operator, $m < n$, and the limit operator $L = \sum_{i=1}^m X_i^2$ is hypoelliptic. It is well known that L_ϵ admits a fundamental solution Γ_ϵ. Here we establish some a priori estimates uniform in ϵ of it, using a modification of the lifting technique of Rothschild and Stein. As a consequence we deduce some a priori estimates uniform in ϵ, for solutions of the approximated equation $L_\epsilon u = f$. These estimates can be used in particular while studying regularity of viscosity solutions of nonlinear equations represented in terms of vector fields.

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1. Introduction

Let $X_1, \ldots, X_m$ be smooth real vector fields on an open set $\Omega \subset \mathbb{R}^n$ satisfying the Hörmander condition for the hypoellipticity

$$rank\ Lie(X_1, \ldots, X_m)(x) = n, \quad \forall x \in \Omega. \tag{1}$$

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It is well known that the operator

$$L = \sum_{i=1}^{m} X_i^2 \tag{2}$$

is hypoelliptic, and estimates of the its fundamental solution Γ are very well known (see [10, 18, 20], see also [15]). However, in many applications it is necessary to study elliptic regularization of this type of operators. For every fixed point x_0 there exist $\nu \geq n$ and there exist vector fields $X_{m+1}, \ldots, X_\nu$ (for example the complete list of commutators up to a fixed step s) such that

$$X_1, \ldots, X_m, X_{m+1}, \ldots, X_\nu \tag{3}$$

span the tangent space at x for every $x \in \Omega$. Then the operator

$$L_\epsilon = \sum_{i=1}^{m} X_i^2 + \epsilon^2 \sum_{i=m+1}^{\nu} X_i^2 \tag{4}$$

is uniformly elliptic in Ω. This approximation can be used to study interior regularity of viscosity solutions of nonlinear problems, when the vector fields $(X_i)_{i=1,\ldots,m}$ depend on the solution: $X_i = X_i(u, \nabla u)$. We refer to [2, 6, 22] and [7] for nonlinear differential equation of this type, arising in complex analysis or mathematics finance. A simple example could be

$$L\,u = \partial_x^2 u + (\partial_y + u\partial_z)^2 u = f, \quad u = u_0 \text{ on } \partial\Omega. \tag{5}$$

This problem cannot be studied directly, but, under very general assumptions on the open set Ω and the boundary datum u_0, the approximating problem:

$$L_\epsilon u = f, \quad u = u_0 \text{ on } \partial\Omega, \tag{6}$$

has a C^∞ solution u_ϵ with gradient bounded uniformly with respect to ϵ. In order to prove the existence of a classical solution of (5) it is natural to establish interior estimates uniform in ϵ for the C^∞ solutions of (6) and then let ϵ goes to 0. As a first step in this direction we consider operators with C^∞ coefficients, and establish uniform estimates for the fundamental solutions.

As it is well known, the fundamental solutions of any operator represented as sum of square of vector fields $X_1, \ldots, X_m, \epsilon X_{m+1}, \ldots, \epsilon X_\nu$ can be estimated in terms of the measure of the spheres of the control distance d_ϵ (see (16) for the definition). Precisely the fundamental solution of the operator L_ϵ can be locally estimated in terms of d_ϵ as

$$|\Gamma_\epsilon(x, y)| \leq C_\epsilon \frac{d_\epsilon^2(x, y)}{|B_\epsilon(x, d_\epsilon(x, y))|},$$

for every (x, y) in a neighborhood of 0, and for a suitable constant C_ϵ (see [18, 19]). Here $B_\epsilon(x, r)$ denotes the sphere of the metric d_ϵ and $|\ |$ its Lebesgue measure.

The dependence of C_ϵ on the variable $\epsilon > 0$ is completely unknown, at our knowledge. However it is known that the control distance d_ϵ associated to L_ϵ tends to the control distance d of L as ϵ tends to 0. This has been proved in [16]. We also refer to [1, 12], where the relation between the metric d_ϵ and d is investigated. This fact suggests to look for estimates of the fundamental solution of the operator L_ϵ uniform in ϵ. In fact we are able to prove that

THEOREM 1.1. *For every compact set $K \subset \Omega$ and for every $p \in \mathbb{N}$ there exist two positive constants C, C_p independent of ϵ such that*

$$|\epsilon^{j_\#} X_{i_1} \cdots X_{i_p} \Gamma_\epsilon(x, y)| \le C_p \frac{d_\epsilon^{2-p}(x, y)}{|B_\epsilon(x, d_\epsilon(x, y))|}, \quad i_1, \dots, i_p \in \{1, \dots, \nu\}, \tag{7}$$

for every $x, y \in K$ with $x \neq y$, where $j_\#$ denotes the number of indices $i \in \{m+1, \dots, \nu\}$ and $B_\epsilon(x, r)$ denotes the ball with center x and radius r of the distance d_ϵ. If $p = 0$ we mean that no derivative are applied on Γ_ϵ.

Besides, for every $0 < a < b$

$$\int_{a \le d_\epsilon(x,y) \le b} |\epsilon^{j_\#} X_{i_1} X_{i_2} \Gamma_\epsilon(x, y)| dy \le C(b - a,) \quad i_1, i_2 \in \{1, \dots, \nu\}. \tag{8}$$

□

The main idea of the proof is a new lifting method. These family of instruments has been first introduced by Rothschild and Stein [19], and subsequently improved by [5, 10, 11, 14]. Adding suitable variables and vector fields, the operator L_ϵ is lifted to a new operator $\tilde{L}_\epsilon$, sum of squares of a family of stratified and nilpotent vector fields. We introduce a new point of view, in order to obtain a lifting independent of the variable ϵ. In presence of a family of vector fields $(X_i, \dots, X_m, \epsilon X_{m+1}, \dots, \epsilon X_\nu)$, larger than the dimension of the space, the classical approach of [18] is to select for every point x, every r and every ϵ the correct sub-family defining the ball of center x and radius r in the metric d_ϵ. We use here a simpler method. For example ϵX_{m+1} and X_{m+1} are linearly dependent, but play different role, since ϵX_{m+1}, has to be considered a first derivative, while X_{m+1} is the commutator, so that it has a higher step. In order to eliminate this ambiguity introduce a new vector $\tilde{X}_{m+1}$ depending on completely new variables. Hence the vector fields $\epsilon X_{m+1} + \tilde{X}_{m+1}$ and X_{m+1} are linearly independent. If the choice of the added vector is more accurate, we can show that the metric induced by the old vectors is the projection on the initial space of the lifted metric. The idea of the lifting seem to be completely new, even thought, from a technical point of view, it is partially inspired to the one introduced in [19]. Let us give a sketch of the proof in the simplest case of the operator

$$L_\epsilon u = \partial_1^2 u + (\partial_2 + x_1\partial_3)^2 u + \epsilon^2 \partial_3^2 u \quad \text{on} \quad \mathbb{R}^3. \tag{9}$$

In this case the vector field $\epsilon\partial_3$ acts as a first derivative, while the direction ∂_3 can be obtained as a commutator and acts as a second derivative. Consequently we lift the operator L_ϵ to a new operator

$$\tilde{L}_\epsilon u = \partial_1^2 u + (\partial_2 + x_1\partial_3)^2 u + (\partial_4 + \epsilon\partial_3)^2 u \quad \text{on} \quad \mathbb{R}^4. \tag{10}$$

It is intuitively clear that, for ϵ small, the lifted vector field $\partial_4 + \epsilon\partial_3$ identifies a direction completely different from ∂_3. Besides the fundamental solution of this operator has the same behavior of the fundamental solution of

$$\tilde{L} u = \partial_1^2 u + (\partial_2 + x_1\partial_3)^2 u + \partial_4^2 u \quad \text{on} \quad \mathbb{R}^4. \tag{11}$$

This operator is independent of ϵ, but its fundamental solution coincides, up to a change of variables with the fundamental solution of $\tilde{L}_\epsilon$. Hence also the estimates of the fundamental solution of this last operator are uniform in ϵ. In turn this fundamental solution allows to construct a parametrix of the fundamental solution of L_ϵ simply via a projection on $x_4 = 0$. The method of parametrix we apply generalizes the technique already applied by [15, 18, 20], see also [4], and for homogeneous vector fields, [3].

Theorem 1.1 provides uniform estimates of fundamental solution of an operator, in terms of its control distance. It allows to deduce from regularity results known in the elliptic case, similar results, for the sub-elliptic situation. In general this approach allows to work with smooth solutions of an elliptic problem $L_\epsilon u_\epsilon = f$ in order to obtain uniform estimates for the limit equation.

Let $\Omega_0 \subset \Omega$, and $W^{p,q}_{\epsilon,X}(\Omega_0)$ be the set of functions $f \in L^q(\Omega_0)$ such that

$$\epsilon^{j_\#} X_{i_1} \cdots X_{i_p} f \in L^q(\Omega_0), \quad i_1, \dots, i_p \in \{1, \dots, n\},$$

with natural norm

$$||f||_{W^{p,q}_{\epsilon,X}(\Omega_0)} = \sum_{i_1,\dots,i_p \in \{1,\dots,n\}} \epsilon^{j_\#} ||X_{i_1} \dots X_{i_p} f||_{L^q(\Omega_0)},$$

where $j_\#$ denotes the number of indices $i \in \{m+1, \dots, n\}$. We have:

COROLLARY 1.1. *Assume that* $u \in L^q_{loc}(\Omega)$ *is a solution of*

$$L_\epsilon u = f \text{ in } \Omega,$$

with $f \in W^{p,q}_{\epsilon,X}(\Omega)$ *and let* $K_1 \subset\subset K_2 \subset\subset \Omega$. *Then there exists a constant C independent of* ϵ *such that*

$$||u||_{W^{p+2,q}_{\epsilon,X}(K_1)} \le C||f||_{W^{p,q}_{\epsilon,X}(K_2)},$$

for every $p \ge 1$.

Analogously, using this result, we can get interior hölder continuous estimates with optimal exponent, which improve a previous result of Krylov [17], where a slightly more general operator $L_\epsilon u = f$ is considered, and holder continuous estimates are provided, but the optimal exponent is far from been reached.

2. Preliminaries and Known Results

In this section we recall the properties of an Hörmander type operator already proved by [9, 19, 20]. Indeed we lift the operator in (4) to a new operator of this type.

Hence we consider now an arbitrary Hörmander type operator

$$L = \sum_{i=1}^{\nu} X_i \quad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{12}$$

where $(X_i)_{i=1,\dots,\nu}$ satisfy the rank condition (1) at every point. We say that a commutator has degree s and denote

$$deg(X) = s \quad \text{if} \quad X = Ad(X_{i_1}, \dots, X_{i_s}),$$

with $i_1, \ldots, i_s \in \{1, \ldots, \nu\}$. Thus, for a fixed point x_0 there is a number s such that the set of all commutators of degree smaller than s span the whole tangent space at every point in a neighborhood of x_0. Then we complete $X_1, \ldots, X_\nu$ with the collection

$$X_{\nu+1}, \ldots, X_N \tag{13}$$

of all the commutators of degree less of equal to s.

Different equivalent definitions of the control distance have been provided for example in [18]. The more natural is defined in terms of the vectors $X_1, \ldots, X_N$ alone. Let $C_1(r)$ denote the class of smooth curves $\phi : [0, 1] \to \Omega$ such that

$$\phi'(t) = \sum_{j=1}^{N} u_j(t) X_j(\phi(t)), \tag{14}$$

where u_j are continuous functions such that $|u_j| \leq r$. The control distance is defined

$$d_c(x, y) = inf\{r > 0 : \exists \phi \in C_1(r) : \phi(0) = 0, \phi(1) = y\}. \tag{15}$$

In (14) the coefficients u_j are continuous functions. In order to replace them with constants, an equivalent definition can given in terms of all the vectors $(X_i)_{i=1,\ldots,N}$. Since the family $(X_i)_{i=1,\ldots,N}$ has more that n elements, a basis of the space can be searched for within the subfamilies with the same cardinality of the space. For each n-tuple $I = (i_1, \ldots, i_n)$, $i_1, \ldots, i_n \in \{1, \ldots, N\}$ the set $C_2(r, I)$ denotes the class of smooth curves $\phi : [0, 1] \to \Omega$ such that

$$\phi'(t) = \sum_{j=1}^{n} u_j X_{i_j}(\phi(t)) \tag{16}$$

with constants u_j such that $|u_j| \leq r^{deg(X_{i_j})}$. If $C_2(r) = \bigcup_I C_2(r, I)$ a new distance is defined

$$d(x, y) = inf\{r > 0 : \exists \phi \in C_1(r) : \phi(0) = 0, \phi(1) = y\}. \tag{17}$$

REMARK 2.1. These two distances are locally equivalent, in the sense that for every $K \subset\subset \Omega$ there exist positive constants C_0, C_1 only dependent on the step of the Lie algebra such that

$$C_0\, d_c(x, y) \leq d(x, y) \leq C_1\, d_c(x, y),$$

for any $x, y \in K$. Since we are interested in establishing estimates we can use either the first or the second definition.

If $(X_i)_{i=1,\ldots,\nu}$ are free up to step s, then $n = \nu$ the $n-$tuple I in the previous definition is unique, and the distance is defined in terms of the exponential map. Indeed, for every fixed point x_0 in $\mathbb{R}^n$ there exist a neighborhood V of x_0 and for every $x \in V$ a neighborhood U_x of x in the Lie algebra, such that for every $x \in V$ the exponential mapping

$$u \longmapsto y = \exp\left(\sum_{i=1}^{n} u_i X_i\right)(x) \tag{18}$$

is defined in U_x. The definition of distance simply reduces to

$$d(x, y) = \sum_{i=1}^{n} |u_i|^{\frac{1}{deg(X_i)}}, \quad x, y \in W, \tag{19}$$

where u_i are defined in (18). Suitable restricting V and choosing $W \subset\subset V$ we can assume that for every $x \in W$ the map in (18) is defined on the same $U \subset U_x$ and it is a diffeomorphism from U onto the image. Its inverse mapping denoted $\Theta_x(u)$ satisfies $U \subseteq \Theta_x(V)$ for every $x \in W$. Finally

$$\Theta : W \times W \to \mathbb{R}^n,$$

defined by

$$\Theta(x, y) = \Theta_x(y) \quad \text{on} \quad W \times W \tag{20}$$

is C^∞. For a fixed x, the function Θ_x introduces a change of variable called {canonical}. The space is not homogeneous, but, due to the fact that the vector fields $(X_i)_{i=1,\dots,\nu}$ are free up to step s and together with their commutators of order s span the tangent space at every point, the following number is constant and it is called local homogeneous dimension of the space

$$Q = \sum_{i=1}^{n} deg(X_i). \tag{21}$$

In this case, for any $K \subset\subset \mathbb{R}^n$ there exists $R > 0$ such that for any $x \in K$ and $0 < r < R$ the measure of the ball is

$$C_0 r^Q \leq |B(x, r)| \leq C_1 r^Q, \tag{22}$$

with suitable positive constants C_0, C_1 depending only on K.

Assume now that the family

$$X_1 \dots, X_m, X_{m+1}, \dots, X_n$$

has the property that the variables defining the vectors $X_1 \dots, X_m$, and the one defining $X_{m+1}, \cdots X_n$, are completely different. Also assume that the two different families are free up to the same step s and span the tangent space at every point, together with their commutators of step up to s. Also in this case we can define the local dimension of the space as in (21), and the measure of the ball is estimated as in (22).

Let us go back to the properties of a general operator L. Note that, if the family of Lie algebra generated by $(X_i)_{i=1,\dots,\nu}$ is not free, it is always possible to lift it to a Lie algebra free up to a step s, via the lifting procedure introduced by Rothschild and Stein. Precisely

THEOREM 2.1. *(Theorem 4 in [19]) Let $(X_i)_{i=1,\dots,\nu}$ be C^∞ vector fields, which, together with their commutators up to step s, span the tangent space at a point x_0. Then we can find new variables $\hat{x}$ and vector fields defined in a neighborhood of x_0*

$$\tilde{X}_i(x, \hat{x}) = X_i(x) + Z_i, \quad Z_i = \sum_{j=1}^{l} a_i^j(x, \hat{x}) \frac{\partial}{\partial \hat{x}_j} \quad \forall i = 1, \dots, \nu$$

such that the system $(\tilde{X}_i)_{i=1,\dots,\nu}$ is free up to order s at x_0 and span $R^{\tilde{N}}$, where $\tilde{N} = n + l$.

In order to simplify notations we give the following definition

DEFINITION 2.1. We say that k is an regular kernel of type λ with respect to the vectors $X_1, \dots, X_\nu$, and the distance d in an open set W and we denote $k \in F_\lambda(X, d, W)$ if for every $p \in \mathbb{N}$ there exists a positive constant C_p such that, one has for every $x, y \in W$ with $x \neq y$

$$| X_{i_1} \cdots X_{i_p} k(x, y)| \leq C_p \frac{d^{\lambda-p}(x, y)}{|B(x, d(x, y))|}, \quad i_1, \dots, i_p \in \{1, \dots, \nu\}. \tag{23}$$

If $\lambda = 0$ we also require that there exists positive constant C_* such that for any $0 < a < b$

$$\int_{a \leq d(x,y) \leq b} k(x, y)|detJ\Theta_x(y)|dy \leq C_*(b - a). \tag{24}$$

3. The Lifting Procedure

In this section we introduce our lifting method, for the vector fields defining the operator L_ϵ in (4). By simplicity we call

$$X_{\epsilon,i} = X_i,\ i = 1, \dots, m, \quad X_{\epsilon,i} = \epsilon X_i,\ i = m + 1, \dots, \nu.$$

and we lift them to a new family $(\tilde{X}_{\epsilon,i})_{i=1,\dots,\nu}$ which satisfies the following assumptions:

for every $i = 1, \dots, \nu$ there exists a vector field $\tilde{Z}_i$ depending on variables different from the ones defining $X_{\epsilon,i}$ such that

$$\tilde{X}_{\epsilon,i} = X_{\epsilon,i} + \tilde{Z}_i, \tag{25}$$

$$\text{the family } (\tilde{X}_{\epsilon,i})_{i=1,\dots,\tilde{N}} \text{ is a basis of the tangent space at every } x. \tag{26}$$

3.1. Vector Fields Free Up to a Fixed Step

We start with describing the lifting procedure under the additional assumption that the vector fields $X_1, \dots, X_m$ defined in (2) are defined on R^n, generate a Lie algebra free up to a step s, and together with their commutators of order at most s span the tangent space at every point. This ensures that the list

$$X_{m+1}, \dots, X_n \tag{27}$$

of the commutators up to step s complete the collection $X_1, \dots, X_m$ to a basis of R^n. Hence the number ν in (3) simply coincides with n.

In turn the list $(X_{\epsilon,i})_{i=1,\dots,n}$ can be completed to a set of generators of the space, with the complete list of commutators up to step s

$$X_{\epsilon,n+1}, \dots, X_{\epsilon,N}.$$

Clearly these vectors are not linearly independent. We start by lifting the vector fields $(X_{\epsilon,i})_{i=m+1,\dots,n}$ to new vector fields linearly independent of the commutators of $(X_{\epsilon,i})_{i=1,\dots,m}$. More precisely, since the vector fields $(\epsilon X_i)_{i=m+1,\dots,n}$ have step s, we call μ the dimension of the free Lie algebra $g(s, n-m)$, with $n-m$ generators and step s and define $n-m$ new vector fields free and nilpotent of step s

$$\tilde{X}_{m+1}, \ldots, \tilde{X}_n, \tag{28}$$

in term of μ completely new variables. The initial family $(X_{\epsilon,i})_{i=1,\ldots,n}$ can be lifted as follows

$$\tilde{X}_{\epsilon,i} = X_i, \;\; i = 1, \ldots, m, \;\; \tilde{X}_{\epsilon,i} = \epsilon X_i + \tilde{X}_i, \;\; i = m+1, \ldots, n. \tag{29}$$

The vector fields $(\tilde{X}_{\epsilon,i})_{i=1,\ldots,n}$ generate a Lie algebra with n generators, direct sum of a free algebra of step s and an algebra free up to step s (and not necessarily free). In the lifted space $\mathbb{R}^{\tilde{N}}$ we can complete the family $(\tilde{X}_{\epsilon,i})_{i=1,\ldots,n}$ to a basis of the Lie algebra, with the complete list of all the non-zero commutators, up to step s denoted $(\tilde{X}_{\epsilon,i})_{i=1,\ldots,\tilde{N}}$. This list is ordered in such a way that the first N vectors are the lifted of the list $(X_{\epsilon,i})_{i=1,\ldots,N}$.

The correspondent sublaplacian is

$$\tilde{L}_\epsilon = \sum_{i=1}^{n} \tilde{X}_{\epsilon,i}^2 = \sum_{i=1}^{m} X_i^2 + \sum_{i=m+1}^{n} (\epsilon X_i + \tilde{X}_i)^2, \tag{30}$$

in $\mathbb{R}^{\tilde{N}} = \mathbb{R}^n \times \mathbb{R}^\mu$ We need a third operator $\tilde{L}$, with the same structure of L_ϵ, but independent of ϵ:

$$\tilde{L} = \sum_{i=1}^{m} X_i^2 + \sum_{i=m+1}^{n} \tilde{X}_i^2.$$

We also denote $\tilde{X}_i = X_i$ for all $i = 1, \ldots, m$, in such a way that $\tilde{L}$ can be represented as $\sum_{i=1}^n \tilde{X}_i^2$. And we complete the family $(\tilde{X}_i)_{i=1,\ldots,n}$ to a basis of the space

$$\tilde{X}_1, \ldots, \tilde{X}_n, \tilde{X}_{n+1}, \ldots, \tilde{X}_{\tilde{N}} \tag{31}$$

with the list of all commutators ordered as the list $(\tilde{X}_{\epsilon,i})_{i=n+1,\ldots,\tilde{N}}$.

3.2. The General Case

We conclude here the description of the lifting procedure, removing the assumptions on the Lie algebra generated by $X_1, \ldots, X_m$ formulated in the previous section. We will start from an operator L_ϵ, and perform a first lifting, it order to reduce to the case described in the previous section. In order to keep trace of the dependence on ϵ we first apply the lifting of Rothschild and Stein only on the initial family of vector fields $(X_i)_{i=1,\ldots,m}$. The vector fields $(X_i)_{i=1,\ldots,m}$ are lifted to $(X_i + Z_i)_{i=1,\ldots,m}$, free at step s in a neighborhood U of x_0, according to Theorem 2.1. The commutators will be accordingly lifted. Hence, also the vector fields ϵX_i are lifted to $\epsilon(X_i + Z_i)$ for every $i = m+1, \ldots, \nu$. Consequently we have a first lifting of the operator L_ϵ is lifted to the operator

$$\sum_{i=1}^{m} (X_i + Z_i)^2 + \sum_{i=m+1}^{\nu} (\epsilon X_i + \epsilon Z_i)^2.$$

Note that the family $(X_i + Z_i)_{i=1,\dots,m}$ defining the first part of the operator is free up to step s. The family $(\epsilon X_i + \epsilon Z_i)_{i=m+1,\dots,\nu}$ is a subfamily of the collection of all commutators. Hence we can apply the same procedure as before, denoting $(\tilde{X}_i)_{i=m+1,\dots,\nu}$ a family of vector fields generating a Lie algebra nilpotent and free of step s. And defining the vectors

$$\tilde{X}_{\epsilon,i} = X_i + Z_i, \quad \text{for all} \quad i = 1, \dots, m,$$

$$\tilde{X}_{\epsilon,i} = \epsilon X_i + \epsilon Z_i + \tilde{X}_i, \quad \text{for all} \quad i = m+1, \dots, \nu.$$

We can finally define

$$\tilde{L}_\epsilon = \sum_{i=1}^{\nu} \tilde{X}_{\epsilon,i}^2, \quad \tilde{L} = \sum_{i=1}^{\nu} \tilde{X}_i^2 \tag{32}$$

where $\tilde{X}_i = X_i + Z_i$ for all $i = 1, \dots, m$. Again we can complete the family $(\tilde{X}_{\epsilon,i})_{i=1,\dots,N}$ to a basis of the space with the complete list of commutators of step s denoted $(\tilde{X}_{\epsilon,i})_{i=1,\dots,\tilde{N}}$. The family $(\tilde{X}_i)_{i=1,\dots,N}$ will be similarly completed to the list of its commutators $(\tilde{X}_i)_{i=1,\dots,\tilde{N}}$.

4. Estimates of the Metric

In this section we provide the properties of the metric d_ϵ associated to the vector fields $(X_{\epsilon,i})_{i=1,\dots,n}$, the metric $\tilde{d}_\epsilon$ associated to the vector fields $(\tilde{X}_{\epsilon,i})_{i=1,\dots,\tilde{N}}$ and the metric $\tilde{d}$ associated to the vector fields $(\tilde{X}_i)_{i=1,\dots,\tilde{N}}$ according to definition (19).

4.1. The Metric $\tilde{d}_\epsilon$

The generic point of the lifted space $\mathbb{R}^{\tilde{N}}$ is denoted by $\tilde{x}$ and the homogeneous dimension is denoted $\tilde{Q}$ (see definition (21)). The function defining canonical coordinates with respect to the basis $\tilde{X}_1, \dots, \tilde{X}_{\tilde{N}}$ according to (31) is called $\Theta_{\tilde{x}}(\tilde{y}) = \tilde{u}$.

Let ψ_ϵ be a Lie algebra isomorphism defined as $\tilde{L}_\epsilon$ and $\tilde{L}$, as

$$\psi_\epsilon(\tilde{X}_i) = \tilde{X}_{\epsilon,i}, \quad i = 1, \dots, n.$$

Clearly ψ_ϵ can be extended on the whole algebra via the bracket, so that

$$\tilde{X}_{\epsilon,i} = \psi_\epsilon(\tilde{X}_i) = \tilde{X}_i + \sum_{j=1}^{\tilde{N}} a_{ij} \tilde{X}_j, \quad \forall i = 1, \dots, \tilde{N}, \tag{33}$$

where

$$|a_{ij}| \le \epsilon \quad \forall i, j \in \{1, \dots, \tilde{N}\}. \tag{34}$$

Since ψ_ϵ is linear and lower diagonal, with 1 on the diagonal, it has jacobian determinant 1. The function ψ_ϵ induces, via the mapping Θ, a change of variables on $\mathbb{R}^{\tilde{N}}$

$$\Phi_\epsilon : \mathbb{R}^{\tilde{N}} \to \mathbb{R}^{\tilde{N}}, \quad \Phi_\epsilon = Exp \circ \psi_\epsilon \circ \Theta_0. \tag{35}$$

Since the function Exp and Θ_0 are local diffeomorphisms, with determinant 1 in 0, and independent of ϵ, also the jacobian determinant of Φ_ϵ is independent of ϵ in a neighborhood of 0. Besides the distance can be computed as

$$\tilde{d}_\epsilon(\tilde{x}, \tilde{y}) = \tilde{d}(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

In particular, this provides a simple estimate of the ball of the metric $\tilde{d}_\epsilon$: There exist positive constants C_0 and C_1 independent of ϵ such that

$$C_0 r^{\tilde{Q}} \leq |\tilde{B}_\epsilon(\tilde{x}, r)| \leq C_1 r^{\tilde{Q}}.$$

4.2. The Metric d_ϵ and its Lifting

The generic point of the lifted space $\mathbb{R}^{\tilde{N}}$ is denoted by $\tilde{x} = (x, \hat{x})$, where $x \in \mathbb{R}^n$ is the initial variable and $\hat{x} \in R^{\tilde{N}-n}$ the added ones.

Let us study the relation between the metrics induced by the family of vector fields $(X_{\epsilon,i})_{i=1,\dots,\nu}$, and their lifted counterpart. The procedure we use is a generalization of the one introduced by [18], who study the relationship between the metric d_ϵ and $\tilde{d}_\epsilon$ under the assumptions (25), (26) on the defining vector fields. We follow the same ideas, but we also study the dependence on the variable ϵ. We denote $B_\epsilon(x, r)$ and $\tilde{B}_\epsilon(\tilde{x}, r)$ the balls in the two metrics d_ϵ and $\tilde{d}_\epsilon$. Precisely, if π_i denote the projections

$$\pi_1 : \mathbb{R}^{\tilde{N}} \longrightarrow \mathbb{R}^n, \quad \pi_1(x, \widehat{x}) = x, \quad \pi_2 : \mathbb{R}^{\tilde{N}} \longrightarrow \mathbb{R}^n, \quad \pi_1(x, \widehat{x}) = \widehat{x},$$

then

LEMMA 4.1 (Lemma 3.1 in [18]). *The projection π_1 has the following properties:*

$$\pi_1 : \tilde{B}_\epsilon((x, 0), r) \longrightarrow B_\epsilon(x, r),$$

and the map is onto. Besides

$$\pi_1\Big(\exp\Big(\sum_i u_i \tilde{X}_{\epsilon,i} \Big) \Big) = \exp\Big(\sum_i u_i X_{\epsilon,i} \Big).$$

Furthermore the following result is satisfied:

LEMMA 4.2 (Theorem 7 in [18]). *Let $K \subset\subset \mathbb{R}^n$. There exist constants $0 < \eta_2 < \eta_1 < 1$, and constants C_0, C_1 such that $\forall x \in K$, $\forall r > 0$, $\forall \epsilon > 0$ there exists a n-tuple I with the following properties*

$$|\lambda_I(\epsilon, x)|\, r^{deg(X_I)} \geq \eta_2 \max_J |\lambda_J(\epsilon, x)|\, r^{deg(X_J)}$$

where $\lambda_I(\epsilon, x) = det(X_{\epsilon,i} : i \in I)$ and $deg(X_I) = \sum_{i \in I} deg(X_i)$. Besides, if we define

$$\Phi_{\hat{u}}(u) = exp\Big(\sum_{i \in I} u_i X_{\epsilon,i} + \sum_{i \notin I} \hat{u}_i X_{\epsilon,i} \Big)(x), \tag{36}$$

the following is true:

- *if* $|\hat{u}_j| \le \eta_2 r^{deg(\tilde{X}_{\epsilon,j})}$ *the function* $\Phi_{\hat{u}}$ *is one to one on the ball* $B_\epsilon(x,r)$*;*
- *if* $|\hat{u}_j| \le \eta_2 r^{deg(\tilde{X}_{\epsilon,j})}$ *the function* $\Phi_{\hat{u}}$ *is non-singular, and if* $|J\Phi_{\hat{u}}|$ *denotes the jacobian, then on* $B_\epsilon(x,r)$

$$\frac{1}{4}|\lambda_I(\epsilon,x)| \le |J\Phi_{\hat{u}}| \le 4|\lambda_I(\epsilon,x)|;$$

- *the measure of the sphere can be estimated as*

$$\frac{r^{deg(X_I)}}{|\lambda_I(\epsilon,x)|} \le |B_\epsilon(x,r)| \le \frac{1}{|\lambda_I(\epsilon,x)|}\left(\frac{r}{\eta_1}\right)^{deg(X_I)}.$$

Using these results we prove the following lemma, which is the main result of this section.

LEMMA 4.3. *For every compact set* $K \subset \mathbb{R}^n$ *there exist positive constants* C_1, C_2 *independent of* ϵ *such that if* $\chi_{\tilde{B}_\epsilon((x,0),r)}$ *is the characteristic function of the ball* $\tilde{B}_\epsilon((x,0),r)$*, then for every* $x \in K$ *and* $r > 0$

$$\int \chi_{\tilde{B}_\epsilon((x,0),r)}(y,\hat{y})d\hat{y} \le C_2 \frac{r^{\tilde{Q}}}{|B_\epsilon(x,r)|}.$$

Note that the integration is performed only in the added variables $\hat{y}$*.*

Proof. We follow here the proof of Lemma 3.2 in [18], checking the independence on ϵ of all the constants. Now

$$\int \chi_{\tilde{B}_\epsilon((x,0),r)}(y,\tilde{y})d\hat{y} = \left|\left\{\hat{y} : (y,\hat{y}) \in \tilde{B}_\epsilon((x,0),r)\right\}\right|.$$

By definition of $\tilde{B}_\epsilon((x,0),r)$, if $(y,\hat{y}) \in \tilde{B}_\epsilon((x,0),r)$ then $y \in \pi_1(B_\epsilon((x,0),r))$. Let us choose I as in Lemma 4.2, and, for y fixed let us denote

$$\Sigma_y = \left\{\tilde{u} : y = \pi_1\Big(exp\Big(\sum_{i\in I} u_i \tilde{X}_{\epsilon,i} + \sum_{i\notin I} \hat{u}_i \tilde{X}_{\epsilon,i}\Big)(x,0), |\tilde{u}_i| \le \eta_2 r^{deg(\tilde{X}_{\epsilon,i})}\Big)\right\}.$$

By Lemma 4.2, the mapping $\Phi_{\hat{u}}$ defined in (36) is one to one, then $\forall y \in B_\epsilon(x,r)$, $\forall \hat{u}$, such that $|\hat{u}_j| \le \eta_2 r^{deg(\tilde{X}_{\epsilon,j})}$ for every $j = n+1,\ldots,\tilde{N}$, there exists unique $u = \theta(\hat{u})$, such that $|u_j| \le \eta_1 r^{deg(X_{\epsilon,j})}$ for every $j = 1,\ldots,n$, and

$$y = \Phi_{\hat{u}}(\theta(\hat{u})).$$

If we denote

$$\tilde{\Phi}(u,\hat{u}) = exp\Big(\sum_{i\in I} u_i \tilde{X}_{\epsilon,i} + \sum_{i\notin I} \hat{u}_i \tilde{X}_{\epsilon,i}\Big)(x,0),$$

then the preceding equality means that

$$y = \Phi_{\hat{u}}(\theta(\hat{u})) = \pi_1(\tilde{\Phi}(u,\hat{u})).$$

If we differentiate this equation with respect to $\hat{u}$, and use the fact, stated in Lemma 4.2, that the jacobian of $\Phi_{\hat{u}}$ is bounded from above and from below by $|\lambda_I(\epsilon, x)|$, we deduce that

$$|J\theta(\hat{u})| \leq |\lambda_I^{-1}(\epsilon, x)|.$$

Hence the map $g : \hat{u} \mapsto \hat{y} = \pi_2(\tilde{\Phi}(u, \hat{u}))$ is a diffeomorphism, with jacobian bounded by $|\lambda_I^{-1}(\epsilon, x)|$. Then

$$\Big| \int \chi_{\ \tilde{B}_\epsilon((x,0),r)} d\hat{y} \Big| \leq \Big| \int_{g\left(\left\{\hat{u}_\epsilon : |\hat{u}_{\epsilon,j}| \leq \eta_2 r^{deg(\tilde{X}_{\epsilon,j})},\ \forall j=n+1,\dots,N\right\}\right)} \Big|$$

$$= C \frac{r^{deg(X_I)}}{|\lambda_I^{-1}(\epsilon, x)|} \quad \leq C \frac{r^{\tilde{Q}}}{|B_\epsilon(x, r)|}.$$

□

Also note that we have the following local inclusions, which ensure that local estimates uniform in ϵ with respect to d_ϵ are local estimates uniform in ϵ with respect to the distance d:

LEMMA 4.4. *For every compact set $K \subset \mathbb{R}^n$ there exists a positive constant C independent of ϵ such that for every $x \in K$, for every $r > 0$*

$$B_\epsilon(x, r) \subset B(x, C(r + (\epsilon r)^{\frac{1}{s}}))$$

and

$$B(x, r) \subset B_\epsilon(x, C(r + (\epsilon r)^{\frac{1}{s}})),$$

where s is the step of Lie algebra.

Proof. Let us consider a point $y \in B_\epsilon(x, r)$. Then there exists $\hat{y} \in \mathbb{R}^{\tilde{N}-n}$ such that

$$(y, \hat{y}) = exp\Big(\sum_{i=1}^{\tilde{N}} \tilde{u}_{\epsilon,i} \tilde{X}_{\epsilon,i}\Big)(x, 0) \text{ and } |\tilde{u}_{\epsilon,i}|^{\frac{1}{deg(\tilde{X}_{\epsilon,i})}} < r. \tag{37}$$

By (33) we can write

$$(y, \hat{y}) = exp\Big(\sum_{i=1}^{\tilde{N}} \tilde{u}_{\epsilon,i}\big(\tilde{X}_i + \sum_{j=1}^{\tilde{N}} a_{ij}\tilde{X}_j\big)\Big)(x, 0) = exp\Big(\sum_{j=1}^{\tilde{N}}\sum_{i=1}^{\tilde{N}} \tilde{u}_{\epsilon,i}\big(\delta_{ij} + a_{ij}\big)\tilde{X}_j\Big)(x, 0).$$

Now

$$\Big|\tilde{u}_{\epsilon,i}\big(\delta_{ij} + a_{ij}\big)\Big|^{\frac{1}{deg(\tilde{X}_j)}} \leq$$

by (34) and the fact that $deg(\tilde{X}_j) \leq s$

$$\leq C|\tilde{u}_{\epsilon,j}|^{\frac{1}{deg(\tilde{X}_j)}} + |\epsilon \tilde{u}_{\epsilon,i}|^{\frac{1}{s}} \leq C\big(r + (\epsilon r)^{\frac{1}{s}}\big),$$

by (37). It follows that $y \in B(x, C\big(r + (\epsilon r)^{\frac{1}{s}}\big))$.

The proof of the second inclusion is similar. □

5. Uniform Estimates of the Fundamental Solution

In this section we conclude the uniform estimates of the fundamental solution of the operator L_ϵ. We first use the estimates of the fundamental solution of $\tilde{L}$ to obtain estimates of the fundamental solution of $\tilde{L}_\epsilon$ independent of ϵ. Then, with a generalization of the projection-result of Sánchez-Calle, we deduce the uniform estimates of the fundamental solution of L_ϵ from the uniform estimates of $\tilde{L}_\epsilon$.

5.1. Estimates of the Fundamental Solution of $\tilde{L}_\epsilon$

Using the function Φ_ϵ defined in (35) the fundamental solution of the operator $\tilde{L}_\epsilon$ can be represented in terms of the fundamental solution $\tilde{\Gamma}$ of the operator $\tilde{L}$

$$\tilde{\Gamma}_\epsilon(\tilde{x}, \tilde{y}) = \tilde{\Gamma}(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

Besides

$$\tilde{X}_{\epsilon,i}\tilde{\Gamma}_\epsilon(\tilde{x}, \tilde{y}) = (\tilde{X}_i\tilde{\Gamma})(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{y})).$$

Hence $\tilde{\Gamma}_\epsilon$ satisfies the estimates (7) and (8), with the same constants as $\tilde{L}$. Precisely, if $p \in \mathbb{N}$ then there exist positive constant C_p and C such that for every $i_1 \ldots, i_p \in \{1, \ldots, n\}$, one has

$$|\tilde{X}_{\epsilon,i_1}, \ldots, \tilde{X}_{\epsilon,i_p}\tilde{\Gamma}_\epsilon(\tilde{x}, \tilde{y})| \le C_p(\tilde{d}(\Phi_\epsilon(\tilde{x}), \Phi_\epsilon(\tilde{x}))^{2-Q-p} = C_p(\tilde{d}_\epsilon(\tilde{x}, \tilde{y}))^{2-Q-p}. \quad (38)$$

For every $0 < a < b$

$$\int_{a \le \tilde{d}_\epsilon(\tilde{x},\tilde{y}) \le b} |\tilde{X}_{i_1}\tilde{X}_{i_2}\tilde{\Gamma}_\epsilon(x, y)| d\tilde{y} \le C(b-a) \quad i_1, i_2 \in \{1, \ldots, n\}.$$

Also note that for every $\tilde{x}, \tilde{y}, \tilde{z}$ such that $\tilde{d}_\epsilon(\tilde{x}, \tilde{y}) \ge \tilde{d}_\epsilon(\tilde{y}, \tilde{z})$

$$|\tilde{\Gamma}_\epsilon(\tilde{x}, \tilde{y}) - \tilde{\Gamma}_\epsilon(\tilde{x}, \tilde{z})| \le \bar{C}\,\tilde{d}_\epsilon(\tilde{y}, \tilde{z})\Big(\tilde{d}_\epsilon^{-\tilde{Q}+1}(\tilde{x}, \tilde{y}) + \tilde{d}_\epsilon^{-\tilde{Q}+1}(\tilde{x}, \tilde{z})\Big).$$

The constants C_p, C and $\bar{C}$ are independent of ϵ.

5.2. Estimates of the Fundamental Solution of L_ϵ

The parametrix of the fundamental solution of L_ϵ is defined as the restriction on $\mathbb{R}^n$ of the fundamental solution $\tilde{\Gamma}_\epsilon$. Following [19], we define a restriction operator R mapping kernels in $\tilde{\Omega}$ to kernels on Ω as follows:

$$Rf(y) = \int f(y, \hat{y})\hat{a}(\hat{y})d\hat{y} \quad (39)$$

where $\hat{a}$ is a fixed function of class $C^\infty(\mathbb{R}^{\tilde{N}-n})$ with compact support, say in the sphere of radius η, and integral equal to 1.

With this definition we see that there is a natural relation between kernels of type $F_\lambda(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$, and their restriction on $\mathbb{R}^n$, which belongs to $F_\lambda(X_\epsilon, d_\epsilon, W)$, according to Definition 2.1. Indeed

PROPOSITION 5.1. *If $\tilde{k}(\tilde{x}, \tilde{y})$ is a kernel of class $F_\lambda(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$, then $k(x, y) = R\tilde{k}(\tilde{x}, \tilde{y})$ is a kernel of class $F_\lambda(X_\epsilon, d_\epsilon, W)$, which satisfies inequalities* (23), (24), *with the same constants as k.*

Proof. We develop the integral in diadic spheres:

$$\int k((x,0),(y,\hat{y}))\hat{a}(\hat{y})d\hat{y} \le \sum_j \int_{\eta 2^{-j}\le \tilde{d}_\epsilon((x,0),(y,\hat{y}))\le \eta 2^{-j+1}} \tilde{k}((x,0),(y,\hat{y}))\hat{a}(\hat{y})d\hat{y} \le$$

$$\le C\sum_j \int_{\eta 2^{-j}\le \tilde{d}_\epsilon((x,0),(y,\hat{y}))\le \eta 2^{-j+1}} \frac{\tilde{d}_\epsilon^\lambda((x,0),(y,\hat{y})))}{|\tilde{B}_\epsilon((x,0),\tilde{d}_\epsilon((x,0),(y,\hat{y})))|} d\hat{y} \le$$

since in each term of the sum $d_\epsilon(x,y) \le \tilde{d}_\epsilon((x,0),(y,\hat{y})) \le \eta 2^{-j+1}$.

$$\le C \sum_{d_\epsilon(x,y)\le \eta 2^{-j+1}} \frac{(\eta 2^{-j})^\lambda}{|\tilde{B}_\epsilon((x,0),\eta 2^{-j+1})|} \int \chi_{\tilde{B}_\epsilon((x,0),\eta 2^{-j+1})} \le$$

$$\le \sum_{d_\epsilon(x,y)\le \eta 2^{-j+1}} \frac{(\eta 2^{-j})^\lambda}{|B_\epsilon(x,\eta 2^{-j+1})|} \le$$

$$\le \frac{1}{|B_\epsilon(x,d_\epsilon(x,y))|} \sum_{d_\epsilon(x,y)\le \eta 2^{-j+1}} (\eta 2^{-j})^\lambda \le \frac{d_\epsilon^\lambda(x,y)}{|B_\epsilon(x,d_\epsilon(x,y))|}.$$

□

REMARK 5.1. In the sequel we will need to evaluate the derivatives of the restriction mapping R. Since the lifted vector fields are of the form:

$$\tilde{X}_{\epsilon,i} = X_{\epsilon,i} + \sum_j \lambda_{i,j}(x,\hat{x})\frac{\partial}{\partial \hat{x}_j},$$

differentiating the expression of R we get

$$X_{\epsilon,i}R(f)(y) = R(\tilde{X}_{\epsilon,i}\tilde{f})(y) + \sum_j \int \frac{\partial}{\partial \hat{y}_j}\Big(\lambda_{i,j}(y,\hat{y})\hat{a}(\hat{y})\Big)\tilde{f}(y,\hat{y})d\hat{y}. \tag{40}$$

More generally, for every $p \in \mathbb{N}$

$$X_{\epsilon,i_1}\cdots X_{\epsilon,i_p}R = R\tilde{X}_{\epsilon,i_1}\cdots \tilde{X}_{\epsilon,i_p} + R\tilde{M}_{\epsilon,p-1},$$

where the rest $\tilde{M}_{\epsilon,p-1}$ is the reduction of a suitable operator of order $p-1$:

$$\tilde{M}_{\epsilon,p-1} = \sum_{(\alpha_1\cdots\alpha_l),\ l\le p-1} \tilde{X}_{\epsilon,\alpha_1}\cdots \tilde{X}_{\epsilon,\alpha_j}$$

in particular the coefficients of $\tilde{M}_{\epsilon,p-1}$ are independent of ϵ.

THEOREM 5.1. *Let $a \in C_0^\infty(W)$ and let $p \in \mathbb{N}$. There exist kernels $K_{\epsilon,p} \in F_2(X_\epsilon, d_\epsilon, W)$ and $H_{\epsilon,p} \in F_p(X_\epsilon, d_\epsilon, W)$, satisfying (23), (24) with constants independent of ϵ such that for every $x, y \in W$*

$$L_\epsilon^y(K_{\epsilon,p}(x,y)) = a(x)\delta_y(x) + H_{\epsilon,p}(x,y) \tag{41}$$

with δ_y the Dirac distribution at y, and where L_ϵ^y means that the differentiation is in the y-variable. Analogously, there exist kernels $K'_{\epsilon,p} \in F_2(X_\epsilon, d_\epsilon, W)$ and *$H'_{\epsilon,p} \in F_p(X_\epsilon, d_\epsilon, W)$, satisfying (23), (24) with constants independent of ϵ such that*

$$L_\epsilon^x(K'_{\epsilon,p}(x,y)) = a(x)\delta_y(x) + H'_{\epsilon,p}(x,y) \tag{42}$$

for every $x, y \in W$ where L_ϵ^x means that the differentiation is in the x-variable.

Proof. The proof is similar to Theorem 2 in [20]. We set

$$\tilde{K}_{\epsilon,0}(\tilde{x},\tilde{y}) = \tilde{\Gamma}_\epsilon(\tilde{x},\tilde{y})a(y),$$

and

$$K_{\epsilon,0}(x,y) = R_y(\tilde{K}_{\epsilon,0}((x,0),\ \cdot\ ))(y) = \int \tilde{K}_{\epsilon,0}((x,0),(y,\hat{y}))\hat{a}(\hat{y})d\hat{y},$$

where the function and the operator R are defined in (39). Here the symbol R_y indicates that the reduction is made with respect to the variable y, so that the variable x acts as a parameter. Besides $K_{\epsilon,0}$ satisfies (23), (24) with constants independent of ϵ. We have

$$\tilde{L}_\epsilon^y(\tilde{K}_{\epsilon,0})(\tilde{x},\tilde{y}) = a(x)\delta_{\tilde{y}} + \tilde{P}_{\epsilon,1}(\tilde{x},\tilde{y}),$$

with $\tilde{P}_{\epsilon,1}(\tilde{x},\tilde{y}) \in F_1(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$. Then by the preceding remark there exists a suitable operator $\tilde{M}_{\epsilon,1}$ of first order such that

$$\begin{aligned} L_\epsilon^y(K_{\epsilon,0})(x,y) &= L_\epsilon^y(R_y(\tilde{K}_{\epsilon,0}((x,0),\cdot)))(y) \\ &= R_y(\tilde{L}_\epsilon^y(\tilde{K}_{\epsilon,0}((x,0),\cdot)))(y) + R_y(\tilde{M}_{\epsilon,1}(\tilde{K}_{\epsilon,0}((x,0),\cdot)))(y) \\ &= R_y(a(x)\delta_{\tilde{y}}) + R_y((\tilde{P}_{\epsilon,1} + \tilde{M}_{\epsilon,1}(\tilde{K}_{\epsilon,0}))((x,0),\cdot)))(y) \\ &= a(x)\delta_y(x) + H_{\epsilon,1}(x,y) \end{aligned}$$

where we have called $H_{\epsilon,1}(x,y) = R_y(\tilde{H}_{\epsilon,1}((x,0),\cdot))(y)$, $\tilde{H}_{\epsilon,1} = (\tilde{P}_{\epsilon,1} + \tilde{M}_{\epsilon,1}(\tilde{K}_{\epsilon,0}))$. Besides $\tilde{H}_{\epsilon,1} \in F_1(\tilde{X}_\epsilon, \tilde{d}_\epsilon, U)$ and $H_{\epsilon,1} \in F_1(X_\epsilon, d_\epsilon, W)$, with the constants in (23), (24) independent of ϵ. By construction $K_{\epsilon,0}$ and $H_{\epsilon,1}$ are represented in terms of the function a or its derivatives, so that for every $x, \hat{x}, \hat{y}$

$$\text{support}\Big(\tilde{K}_{\epsilon,0}((x,\hat{x}),(\ \cdot\ ,\hat{y})\Big) \subset \text{support}(a),$$

$$\text{support}\Big(\tilde{H}_{\epsilon,1}((x,\hat{x}),(\ \cdot\ ,\hat{y})\Big) \subset \text{support}(a).$$

Define

$$K_{\epsilon,1} = K_{\epsilon,0} - R_y(\tilde{H}_{\epsilon,1} * \tilde{\Gamma}_\epsilon).$$

Then

$$L^y_\epsilon(K_{\epsilon,1}) = L^y_\epsilon(K_{\epsilon,0}) - L^y_\epsilon R_y(\tilde H_{\epsilon,1} * \tilde\Gamma_\epsilon)$$

$$= a(x)\delta_y(x) + H_{\epsilon,1}(x,y) - R_y\Big(\tilde L^y_\epsilon(\tilde H_{\epsilon,1} * \tilde\Gamma_\epsilon)\Big) - R_y\Big(\tilde M^y_\epsilon(\tilde H_{\epsilon,1} * \tilde\Gamma_\epsilon)\Big)$$

$$= a(x)\delta_y(x) + H_{\epsilon,2}(x,y)$$

with $H_{\epsilon,2} = R_y(\tilde M^y_\epsilon(\tilde H_{\epsilon,1} * \tilde\Gamma_\epsilon)) \in F_2(X_\epsilon, d_\epsilon, W)$, with constants in (23), (24) independent of ϵ. Iterating this procedure we get the assertion.

The proof of the second assertion is similar, starting with

$$\tilde K^{'}_{\epsilon,0}(\tilde x,\tilde y) = \tilde\Gamma_\epsilon(\tilde x,\tilde y) \quad K^{'}_{\epsilon,0}(x,y) = \int \hat a(\hat x)\tilde K^{'}_{\epsilon,0}((x,\hat x),(y,0))d\hat x.$$

□

Proof of Theorem 1.1. Note that for every $f \in C^\infty_0(W)$ if $\hat a$ is a function of class $C^\infty_0(\mathbb{R}^{\tilde N - n})$ such that $\int \hat a^2(\hat x)d\hat x = 1$ then

$$f(x) = \int f(x)\hat a^2(\hat x)d\hat x$$

$$= \int \hat a(\hat x)\int\int \tilde\Gamma_\epsilon((x,\hat x),(y,\hat y))\tilde L_\epsilon(f(y)\hat a(\hat y))dy\,d\hat y\,d\hat x$$

$$= \int\int\int \tilde\Gamma_\epsilon((x,\hat x),(y,\hat y))\tilde L_\epsilon \hat a(\hat y)\hat a(\hat x)d\hat x\,d\hat y\,f(y)dy$$

$$-\int\int\int \tilde X_{\epsilon,i}\Big(\tilde\Gamma_\epsilon((x,\hat x),(y,\hat y))\tilde X_{\epsilon,i}\hat a(\hat y)\Big)\hat a(\hat x)d\hat x\,d\hat y\,f(y)dy$$

$$+\int\int\int \tilde\Gamma_\epsilon((x,\hat x),(y,\hat y))\hat a(\hat y)\hat a(\hat x)d\hat x\,d\hat y\,L_\epsilon f(y)dy$$

In particular we can apply this representation formula to $K'_{\epsilon,p} - a(x)\Gamma_\epsilon$. Since

$$L^x_\epsilon(K'_{\epsilon,p} - a\Gamma_\epsilon)(x,z) = H^{'}_{\epsilon,p}(x,z),$$

and $H^{'}_{\epsilon,p}(x,z)$ is arbitrary regular, then the desired estimate for Γ_ϵ follows. On the other hand the kernel arising in the preceding formula can be differentiated using Remark 5.1, and we get the estimates also for all derivatives of Γ_ϵ. □

Proof of Corollary 1.1. Let us start with a new estimate for Γ_ϵ. Let us fix $x, y, z \in \Omega$, with $d_\epsilon(x,y) \geq 4d_\epsilon(y,z)$. By the definition of distance (15) there exists a multiindex I such that $deg(X_i) = 1$ for every $i \in I$ and continuous functions $(u_i)_{i\in I}$ such that

$$z = \gamma(1), \quad \gamma'(t) = \sum_{i\in I} u_{\epsilon,i}X_{\epsilon,i}(\gamma), \quad \gamma(0) = 0.$$

Hence by the mean value theorem

$$|\Gamma_\epsilon(x, y) - \Gamma_\epsilon(x, z)| \leq \sum_{i\in I} |u_{\epsilon,i}||X_{\epsilon,j}\Gamma_\epsilon(x, s)| \leq C\, d_\epsilon(y, z)\frac{|d_\epsilon(x, s)|}{|B_\epsilon(x, d_\epsilon(x, s))|},$$

for a suitable point $s=\gamma(\tau), \tau \in [0, 1]$. On the other hand $d_\epsilon(x, s) \geq d_\epsilon(x, y) - d_\epsilon(y, s) \geq d_\epsilon(x, y) - d_\epsilon(y, z) \geq \frac{3}{4} d_\epsilon(x, y)$ and $d_\epsilon(x, s) \leq d_\epsilon(x, y) + d_\epsilon(y, s) \leq d_\epsilon(x, y) + d_\epsilon(y, z) \leq \frac{5}{4} d_\epsilon(x, y)$. Inserting in the previous expression, and using the doubling property of the spheres, we have

$$|\Gamma_\epsilon(x, y) - \Gamma_\epsilon(x, z)| \leq C\, d_\epsilon(y, z)\frac{|d_\epsilon(x, y)|}{|B_\epsilon(x, d_\epsilon(x, y))|},$$

with C positive constant independent of ϵ.

The properties (7) and (8) in Theorem 1.1 together with this last inequality allow to apply Theorem 6, page 290 in [19], and deduce that

$$||u||_{W^{k+2,q}_{\epsilon,X}(B_\epsilon(x,2r))} \leq C||f||_{W^{k,q}_{\epsilon,X}(B_\epsilon(x,4r))},$$

with a constant C only depending on the constant in the previously recalled assertions, hence independent of ϵ. Finally we can use Lemma 4.4 to deduce that, if ϵ is sufficiently small

$$\begin{aligned} \|u\|_{W^{k+2,q}_{\epsilon,X}(B(x,r))} &\leq C||u||_{W^{k+2,q}_{\epsilon,X}(B_\epsilon(x,2r))} \\ &\leq C||f||_{W^{k,q}_{\epsilon,X}(B_\epsilon(x,4r))} \leq C||f||_{W^{k,q}_{\epsilon,X}(B(x,8r))}. \end{aligned}$$

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