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Space Semi-Discretisations for a Stochastic Wave Equation*

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Abstract. We study an approximation scheme for a nonlinear stochastic wave equation in one-dimensional space, driven by a spacetime white noise. The sequence of approximations is obtained by discretisation of the Laplacian operator. We prove L^p-convergence to the solution of the equation and determine the rate of convergence. As a corollary, almost sure convergence, uniformly in time and space, is also obtained. Finally, the speed of convergence is tested numerically.

Mathematics Subject Classifications (2000): 60H35, 60H15.

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1. Introduction

Nowadays, stochastic partial differential equations (spde's) are accepted as being a very suitable framework to understand complex phenomena. An aspect of the development of the theory consists in seeking methods of finding solutions numerically. Some of them are inspired on those used in the deterministic context. Let us mention for instance, finite differences [9, 10, 15, 16, 29], finite elements [7], splitting up methods [1, 2, 12, 18], Galerkin approximations [8] and time discretisation [17, 25]. Others are more genuine stochastic, based on the Wiener chaos decomposition [20, 21] or on truncations of the Fourier expansion of the noise [26, 27]. We refer the reader to [11] for a survey of some of these methods, together with a more extensive list of references.

Lattice approximation schemes for parabolic spde's in one spatial dimension, developed in [9, 10], have been the starting point of several further investigations. In [24], lattice schemes for parabolic spde's in any spatial dimension are considered and the influence of the particular covariance density of the noise given by Riesz kernels is studied. A class of parabolic evolution equations on Banach spaces with monotone operators are analysed in [14]. In [13], a finite difference approximation scheme for an elliptic spde in dimension $d = 1, 2, 3$ is

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studied. The results show how much the behaviour of this kind of approximation depends on the differential operator driving the spde and are one of the very few attempts of looking beyond the parabolic case. Let us also mention [5] for some results on numerical approximations for elliptic equations.

In this paper we consider strong approximations for a stochastic wave equation in spatial dimension one by a sequence obtained substituting the derivatives in space by finite differences. This is a first step towards the analysis of lattice approximations for hyperbolic spde's. In fact, to our best knowledge there are very few results on numerical approximation for the stochastic wave equation [22].

We consider the nonlinear stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}(t,x),$$

$t > 0$, $x \in (0,1)$, with initial conditions

$$u(0,x) = u_0,\ \frac{\partial u}{\partial t}(0,x) = v_0,\ x \in (0,1),$$

and Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0,\ t > 0.$$

We assume that u_0 and v_0 are functions defined on $[0,1]$, u_0 vanishes at $x = 0$ and $x = 1$, and that W is the Brownian sheet on $\mathbb{R}_+ \times [0,1]$; that is, $\{W(t,x), (t,x) \in \mathbb{R}_+ \times [0,1]\}$ is a Gaussian stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with mean zero and covariance function

$$E(W(t,x)W(s,y)) = (s \wedge t)(x \wedge y).$$

Following a classical approach to spde's, we attach a rigorous meaning to the formal problem described above by means of the evolution formulation, as follows:

$$\begin{aligned} u(t,x) = {} & \int_0^1 G(t,x,y)v_0(y)\mathrm{d}y + \frac{\partial}{\partial t}\left(\int_0^1 G(t,x,y)u_0(y)\mathrm{d}y\right) \\ & + \int_0^t \int_0^1 G(t-s,x,y)\sigma(s,y,u(s,y))W(\mathrm{d}s,\mathrm{d}y) \qquad (1) \\ & + \int_0^t \int_0^1 G(t-s,x,y)f(s,y,u(s,y))\mathrm{d}s\mathrm{d}y, \end{aligned}$$

$t \geqslant 0$, $x \in (0,1)$, where G is the Green function of the wave equation with homogeneous Dirichlet boundary conditions.

For any $n \geqslant 1$, we fix the spatial grid $x_k = k/n$, $k = 1, \cdots, n-1$, and consider the system of stochastic differential equations obtained by substituting the Laplacian by its discretisation and freezing the evolution equation (1) at the points of the grid (see (8)). This provides an implicit finite dimensional scheme. By linear interpolation, we obtain a sequence of evolution equations which is proved to converge in any $L^p(\Omega)$, uniformly in t, x, to the solution of Equation (1) with a given rate of convergence (see Theorem 1).

In comparison with parabolic examples, the rate of convergence differs substantially from the Hölder continuity order of the sample paths of the solution. Indeed, assuming for simplicity that the initial conditions vanishes, sample paths are jointly Hölder continuous in (t, x) of order $\alpha < \frac{1}{2}$, while the rate of convergence is of order $\rho < \frac{1}{3}$. In fact this is the order of the quadratic mean error in the approximation of the Green function by the approximation derived from the discretised scheme. We have checked with a numerical analysis that one cannot expect better results.

The paper is organized as follows. In the second section, we study the Hölder continuity of the sample paths of Equation (1). Section three is devoted to prove the main result on the approximation scheme. Finally, in an Appendix we test numerically the optimality of the rate of convergence proved in section three.

2. Some Properties of the Solution

In this section we prove some properties of the solution of Equation (1). In particular, we analyse sufficient conditions on the initial data ensuring joint Hölder continuity, in time and in space, of the sample paths of the solution.

We fix a finite time horizon T and assume that the coefficients f, σ are real-valued functions defined on $[0, T] \times [0, 1] \times \mathbb{R}$, satisfying the following conditions:

(L)

$$\sup_{t \in [0,T]} \big(|f(t,x,z) - f(t,y,v)| + |\sigma(t,x,z) - \sigma(t,y,v)|\big) \leqslant C(|x-y| + |z-v|),$$

(LG)

$$\sup_{(t,x) \in [0,T] \times [0,1]} \big(|f(t,x,z)| + |\sigma(t,x,z)|\big) \leqslant C(1 + |z|),$$

for every $x, y \in [0, 1]$ and $z, v \in \mathbb{R}$.

Throughout the paper we shall use the expansion of the Green function

$$G(t,x,y) = \sum_{j=1}^{\infty} \frac{\sin(j\pi t)}{j\pi} \varphi_j(x)\varphi_j(y), \tag{2}$$

where $\varphi_j(x) = \sqrt{2}\sin(j\pi x), j \geqslant 1$, is a complete orthonormal system of $L^2([0,1])$ (see for instance [6], p. 94). Alternatively, for $(t,x) \in [0,T] \times [0,1]$,

$$G(t,x,y) = \sum_{k=-\infty}^{+\infty} \left(\mathbf{1}_{\{0 \leqslant y \leq \leqslant 1, |y-x-2k| \leqslant t\}} - \mathbf{1}_{\{0 \leqslant y \leqslant 1, |y+x-2k| \leqslant t\}} \right) \tag{3}$$

(see [3]).

Assume that u_0, v_0 belong to $L^2([0,1])$. By the classical approach to the deterministic wave equation on $[0,1]$ with Dirichlet boundary conditions, we know that

$$\frac{\partial}{\partial t}\left(\int_0^1 G(t,x,y)u_0(y)dy \right) = \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle \cos(j\pi t)\varphi_j(x),$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $L^2([0,1])$ (see [19], p. 44).

Let $\mathcal{F}_t$, $t \in [0,T]$, be the σ-field generated by the random variables $W(s,x)$, $s \in [0,t]$, $x \in [0,1]$. Assume that the stochastic process $u = \{u(t,x), (t,x) \in [0,T] \times [0,1]\}$ in equation (1) is $\mathcal{F}_t$-adapted and satisfies $\sup_{(t,x) \in [0,T] \times \mathbb{R}} E\big(|u(t,x)|^2\big) < \infty$, then all terms in the right-hand side of equation (1) are well defined, when choosing as stochastic integral the extension of Itô's integral with respect to martingale measures developed by Walsh in [30].

By the standard technique based on Picard's iterations scheme, it is not difficult to prove the existence and uniqueness of a measurable, $\mathcal{F}_t$-adapted stochastic process $\{u(t,x), (t,x) \in [0,T] \times [0,1]\}$ such that

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} E\big(|u(t,x)|^2\big) < \infty$$

and satisfying Equation (1). Existence only requires the condition (LG), while uniqueness needs (L). We refer the reader to [4] (see also [3] and [23]) for a similar result on different types of equations that can be easily adapted to equation (1).

For a function g: $[0,1] \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, we define

$$\|g\|_{\alpha,2} := \left(\sum_{j=1}^{\infty} (1+j^2)^{\alpha} |\langle g, \varphi_j \rangle|^2 \right)^{\frac{1}{2}}$$

and denote by $H^{\alpha,2}([0,1])$ the set of functions g: $[0,1] \to \mathbb{R}$ such that $\|g\|_{\alpha,2} < \infty$. Notice that $H^{\alpha,2}([0,1])$ is a subspace of the fractional Sobolev space of fractional differential order α and integrability order $p = 2$ (see [28]).

The next result gives additional information on the existence and uniqueness of solution of equation (1).

PROPOSITION 1. *Assume that* $v_0 \in H^{\beta,2}([0,1])$ *for some* $\beta > -\frac{1}{2}$ *and* $u_0 \in H^{\alpha,2}([0,1])$ *for some* $\alpha > \frac{1}{2}$*; suppose also that the coefficients* σ *and* f *satisfy condition (LG). Then, for every* $p \geqslant 1$,

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|u(t,x)|^p) < +\infty.$$

Proof. Consider the decomposition

$$E(|u(t,x)|^p) \leqslant C \sum_{k=1}^{4} J_k(t,x),$$

with

$$J_1(t,x) = \left| \int_0^1 G(t,x,y) v_0(y) \mathrm{d}y \right|^p,$$

$$J_2(t,x) = \left| \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle \cos(j\pi t) \varphi_j(x) \right|^p,$$

$$J_3(t,x) = E\left(\left| \int_0^t \int_0^1 G(t-s,x,y) \sigma(s,y,u(s,y)) W(\mathrm{d}s, \mathrm{d}y) \right|^p \right),$$

$$J_4(t,x) = E\left(\left| \int_0^t \int_0^1 G(t-s,x,y) f(s,y,u(s,y)) \mathrm{d}s\mathrm{d}y \right|^p \right).$$

Since

$$\int_0^1 G(t,x,y) v_0(y) dy = \sum_{j=1}^{\infty} \frac{\sin(j\pi t)}{j\pi} \langle v_0, \varphi_j \rangle \varphi_j(x), \tag{4}$$

Cauchy–Schwarz inequality and the assumptions on v_0 yield

$$\begin{aligned} \sup_{(t,x)\in[0,T]\times[0,1]} J_1(t,x) &\leqslant \left(\sum_{j=1}^{\infty} j^{2\beta} |\langle v_0, \varphi_j \rangle|^2 \right)^{\frac{p}{2}} \\ &\quad \times \sup_{(t,x)\in[0,T]\times[0,1]} \left(\sum_{j=1}^{\infty} \frac{\sin^2(j\pi t)}{j^2\pi^2} |\varphi_j(x)|^2 j^{-2\beta} \right)^{\frac{p}{2}} \\ &\leqslant C \|v_0\|_{\beta,2}^p \left(\sum_{j=1}^{\infty} j^{-2(1+\beta)} \right)^{\frac{p}{2}} \leqslant C. \end{aligned}$$

Using similar arguments, we obtain that

$$\sup_{(t,x)\in[0,T]\times[0,1]} J_2(t,x) \leqslant C \|u_0\|_{\alpha,2}^p \left(\sum_{j=1}^{\infty} j^{-2\alpha} \right)^{\frac{p}{2}} \leqslant C.$$

Owing to the expression (3),

$$\sup_{(t,x)\in[0,T]\times[0,1]} \int_0^1 |G(t,x,y)|^2 \mathrm{d}y < +\infty. \tag{5}$$

In fact, $\sup_{(t,x,y)\in[0,T]\times[0,1]^2} |G(t,x,y)| < \infty$. Hence, the measure on $[0,T]\times[0,1]$ defined by $\mu_{t,x}(\mathrm{d}s,\mathrm{d}y) = |G(t-s,x,y)|^2 \mathrm{d}s\mathrm{d}y$ is finite, uniformly with respect to $(t,x)\in[0,T]\times[0,1]$.

Applying Burkholder's inequality and then Hölder's inequality with respect to $\mu_{t,x}(\mathrm{d}s,\mathrm{d}y)$ yield

$$\begin{aligned}
&\sup_{x\in[0,1]} J_3(t,x)\\
&\quad \leqslant C \sup_{x\in[0,1]} E\left(\left|\int_0^t \int_0^1 |G(t-s,x,y)|^2 |\sigma(s,y,u(s,y))|^2 \mathrm{d}s\mathrm{d}y\right|^{\frac{p}{2}}\right)\\
&\quad \leqslant C \sup_{x\in[0,1]} \left(\int_0^t \int_0^1 |G(t-s,x,y)|^2 E(|\sigma(s,y,u(s,y))|^p) \mathrm{d}s\mathrm{d}y\right)\\
&\quad \leqslant C\left(1 + \int_0^t \sup_{x\in[0,1]} E(|u(s,x)|^p)\mathrm{d}s\right),
\end{aligned}$$

where we have used the bound (5) and condition (LG) on σ.

For the term $J_4(t,x)$, we apply Hölder's inequality with respect to $\mu_{t,x}(\mathrm{d}s,\mathrm{d}y)$ and the property (LG) on f. We obtain

$$\sup_{x\in[0,1]} J_4(t,x) \leqslant C\left(1 + \int_0^t \sup_{x\in[0,1]} E(|u(s,x)|^p)\mathrm{d}s\right).$$

Bringing together the above estimates, we obtain

$$\sup_{x\in[0,1]} E(|u(t,x)|^p) \leqslant C\left(1 + \int_0^t \sup_{x\in[0,1]} E(|u(s,x)|^p)\mathrm{d}s\right),$$

with a constant C independent of t. We conclude applying Gronwall's lemma. □

We next study the Hölder property of the trajectories of the solution of equation (1).

PROPOSITION 2. *We assume that* $v_0 \in H^{\beta,2}([0,1])$, *for some* $\beta > -\frac{1}{2}$, $u_0 \in H^{\alpha,2}([0,1])$, *for some* $\alpha > \frac{1}{2}$, *and that the coefficients* σ *and* f *satisfy conditions*

(LG) *and* (L). *Then, for all* $p \geqslant 1$ *there exists a positive constant* C, *depending on* α, β, *such that*

$$E(|u(s,x)-u(t,y)|^{2p}) \leqslant C(|t-s|^{p(1+2\beta)} + |x-y|^{p(1+2\beta)} + |t-s|^{p(2\alpha-1)+|x-y|^{p(2\alpha-1)}} + |t-s|^{p} + |x-y|^{p}), \tag{6}$$

for every $s,t \in [0,T]$ *and* $x,y \in [0,1]$.

Consequently, the process u has a.s. Hölder-continuous sample paths of order δ, *for all* $\delta \in (0,\delta_0)$, *where* $\delta_0 = (\frac{1}{2}+\beta) \wedge (\alpha - \frac{1}{2}) \wedge \frac{1}{2}$.

Proof. Assume that $s \leqslant t$ and $y \leqslant x$. We set

$$H(t,x) = \int_0^t \int_0^1 G(t-s,x,z)\sigma(s,z,u(s,z))W(\mathrm{d}s,\mathrm{d}z),$$

$$F(t,x) = \int_0^t \int_0^1 G(t-s,x,z)f(s,z,u(s,z))\mathrm{d}s\mathrm{d}z.$$

Thus we have the decomposition

$$E(|u(s,x)-u(t,y)|^{2p}) \leqslant C\sum_{k=1}^{4} J_k(s,t,x,y),$$

where

$$J_1(s,t,x,y) = \left|\int_0^1 (G(s,x,z)-G(t,y,z))v_0(z)\mathrm{d}z\right|^{2p},$$

$$J_2(s,t,x,y) = \left|\sum_{j=1}^{\infty}\langle u_0,\varphi_j\rangle(\cos(j\pi s)\varphi_j(x)-\cos(j\pi t)\varphi_j(y))\right|^{2p},$$

$$J_3(s,t,x,y) = E(|H(s,x)-H(t,y)|^{2p}),$$

$$J_4(s,t,x,y) = E(|F(s,x)-F(t,y)|^{2p}).$$

The identity (4) and Cauchy–Schwarz inequality yield

$$J_1(s,t,x,y) = \left|\sum_{j=1}^{\infty}\left(\frac{\sin(j\pi s)}{j\pi}\varphi_j(x) - \frac{\sin(j\pi t)}{j\pi}\varphi_j(y)\right)\langle v_0,\varphi_j\rangle\right|^{2p}$$

$$\leqslant \|v_0\|_{\beta,2}^{p}\left(\sum_{j=1}^{\infty}\left(\frac{\sin(j\pi s)}{j\pi}\varphi_j(x) - \frac{\sin(j\pi t)}{j\pi}\varphi_j(y)\right)^2 j^{-2\beta}\right)^{p}.$$

Hence $J_1(s,t,x,y) \leqslant C(A_1(s,t,x,y) + A_2(s,t,x,y))$, where

$$A_1(s,t,x,y) = \left(\sum_{j=1}^{\infty}\left(\frac{\sin(j\pi s)}{j\pi} - \frac{\sin(j\pi t)}{j\pi}\right)^2 |\varphi_j(x)|^2 j^{-2\beta}\right)^p,$$

$$A_2(s,t,x,y) = \left(\sum_{j=1}^{\infty}(\varphi_j(x) - \varphi_j(y))^2 \left|\frac{\sin(j\pi t)}{j\pi}\right|^2 j^{-2\beta}\right)^p.$$

The mean value theorem yields

$$A_1(s,t,x,y) \leqslant C\left(\sum_{j=1}^{\infty} j^{-2(1+\beta)}\left(1 \wedge j^2(t-s)^2\right)\right)^p.$$

If $\beta > \frac{1}{2}$, we clearly have $A_1(s,t,x,y) \leqslant C(t-s)^{2p}$, for some positive constant C depending on β. Assume now that $\beta \in (-\frac{1}{2}, \frac{1}{2}]$. Obviously,

$$\left(\sum_{j=1}^{\infty} j^{-2(1+\beta)}\left(1 \wedge j^2(t-s)^2\right)\right)^p \leqslant C\left((t-s)^2 \sum_{j=1}^{N} j^{-2\beta} + \sum_{j=N+1}^{\infty} j^{-2(1+\beta)}\right)^p,$$

where $N = [1/(t-s)]$ and $[\cdot]$ stands for the integer value. Since

$$\sum_{j=1}^{N} j^{-2\beta} \leqslant CN^{-2\beta+1}, \quad \sum_{j=N+1}^{\infty} j^{-2(1+\beta)} \leqslant C(N+1)^{-1-2\beta},$$

and $1 + 2\,\beta \leqslant 2$, if $\beta \in (-\frac{1}{2}, \frac{1}{2}]$, we obtain

$$A_1(s,t,x,y) \leqslant C(t-s)^{p(1+2\beta)}.$$

Analogously, $A_2(s,t,x,y) \leqslant C(x-y)^{p(1+2\beta)}$. Thus,

$$J_1(s,t,x,y) \leqslant C\big((t-s)^{p(1+2\beta)} + (x-y)^{p(1+2\beta)}\big).$$

Let us now deal with the term J_2. By Cauchy–Schwarz inequality,

$$J_2(s,t,x,y) \leqslant C\|u_0\|_{\alpha,2}^p \left(\sum_{j=1}^{\infty} j^{-2\alpha}(\cos(j\pi s)\varphi_j(x) - \cos(j\pi t)\varphi_j(y))^2\right)^p.$$

Therefore $J_2(s,t,x,y) \leqslant C(B_1(s,t,x,y) + B_2(s,t,x,y))$, with

$$B_1(s,t,x,y) = \left(\sum_{j=1}^{\infty} j^{-2\alpha}(\cos(j\pi s) - \cos(j\pi t))^2\right)^p,$$

$$B_2(s,t,x,y) = \left(\sum_{j=1}^{\infty} j^{-2\alpha}(\varphi_j(x) - \varphi_j(y))^2\right)^p.$$

The same arguments used in the analysis of the terms $A_1(s,t,x,y)$ and $A_2(s,t,x,y)$ yield

$$J_2(s,t,x,y) \leqslant C\big((x-y)^{p(2\alpha-1)} + (t-s)^{p(2\alpha-1)}\big).$$

Let us now study the stochastic integral term by considering the decomposition

$$J_3(s,t,x,y) \leqslant C(D_1(s,t,x) + D_2(t,x,y)),$$

with

$$D_1(s,t,x) = E(|H(s,x) - H(t,x)|^{2p}),$$

$$D_2(t,x,y) = E(|H(t,x) - H(t,y)|^{2p}).$$

Set $h(r,z) = \sigma(r,z,u(r,z))$, $r \in [0,T]$, $z \in \mathbb{R}$. Observe that the assumption (LG) and Proposition 1 yield

$$\sup_{(r,z)\in[0,T]\times[0,1]} E(|h(r,z)|^q) < C, \tag{7}$$

for any $q \in [2,\infty)$.

Clearly, $D_1(s,t,x) \leqslant C(D_{11}(s,t,x) + D_{12}(s,t,x))$, with

$$D_{11}(s,t,x) = E\left(\left|\int_0^s \int_0^1 [G(t-r,x,z) - G(s-r,x,z)]h(r,z)W(\mathrm{d}r,\mathrm{d}z)\right|^{2p}\right),$$

$$D_{12}(s,t,x) = E\left(\left|\int_s^t \int_0^1 G(t-r,x,z)h(r,z)W(\mathrm{d}r,\mathrm{d}z)\right|^{2p}\right).$$

We apply first Burkholder's inequality and then Hölder's inequality with respect to the finite measure on $[0,T]\times[0,1]$ defined by $|G(t-r,x,y) - G(s-r,x,y)|^2 \mathrm{d}r\mathrm{d}y$. By virtue of Equation (7) we obtain

$$\begin{aligned}
&D_{11}(s,t,x)\\
&\quad\leqslant CE\left(\left|\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 |h(r,z)|^2 \mathrm{d}z\mathrm{d}r\right|^p\right)\\
&\quad\leqslant C\left(\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 \mathrm{d}z\mathrm{d}r\right)^{p-1}\\
&\qquad\times\left(\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 E(|h(r,z)|^{2p})\mathrm{d}z\mathrm{d}r\right)^p\\
&\quad\leqslant C\left(\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 \mathrm{d}z\mathrm{d}r\right)^p.
\end{aligned}$$

Replace the Green function G by its expansion given in Equation (2). Since the family $(\varphi_j, j \geqslant 1)$ is orthonormal in $L^2([0,1])$, we obtain

$$D_{11}(s,t,x) \leqslant C\Big(\int_0^s \sum_{j=1}^{\infty} \tfrac{1}{j^2} |\sin(j\pi(t-r)) - \sin(j\pi(s-r))|^2 dr\Big)^p$$

$$\leqslant C\Big(\sum_{j=1}^{\infty} j^{-2}\Big(1 \wedge j^2(t-s)^2\Big)\Big)^p.$$

Therefore, $\sup_{x\in[0,1]} D_{11}(s,t,x) \leqslant C(t-s)^p$.

We can obtain an upper bound for $D_{12}(s,t,x)$ by similar arguments, yielding

$$\sup_{x\in[0,1]} D_{12}(s,t,x) \leqslant C\left(\int_s^t \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \sin^2(j\pi(t-r))\right) dr\right)^p \leqslant C(t-s)^p.$$

Thus, $\sup_{x\in[0,1]} D_1(s,t,x) \leqslant C(t-s)^p$. Similarly, one can check that $\sup_{t\in[0,T]} D_2(t,x,y) \leqslant C(x-y)^p$.

Summarising, we have proved

$$J_3(s,t,x,y) \leqslant C\big((t-s)^p + (x-y)^p\big).$$

With the same type of arguments, but less effort one can check that

$$J_4(s,t,x,y) \leqslant C\big((t-s)^p + (x-y)^p\big).$$

We leave the details to the reader. This finishes the proof of the upper bound (6).

The last statement of the proposition follows from Kolmogorov's continuity criterion. □

3. Strong Approximations

This section is devoted to the proof of the main result of the paper. We start with the description of the discretisation of the stochastic wave equation and end up with a result on the rate of the convergence in $L^p(\Omega)$ of the approximations. As a by-product we also obtain almost sure convergence.

3.1. DISCRETISATION OF THE ONE-DIMENSIONAL STOCHASTIC WAVE EQUATION

Throughout this section we shall assume that u_0, v_0 belong to $L^2([0,1])$.

One can express the stochastic boundary value problem we are studying in this paper by means of a system of two first-order spde's, as follows:

$$\frac{\partial u}{\partial t}(t,x) = v(t,x),$$
$$\frac{\partial v}{\partial t}(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x)) \frac{\partial^2}{\partial t \partial x} W(t,x),$$

$t > 0$, $x \in (0,1)$, with initial conditions

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in (0,1),$$

and Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t > 0.$$

For any integer $n \geqslant 1$, set $x_k = k/n$, $k = 1, \ldots, n-1$. Consider the system of stochastic differential equations

$$\begin{aligned} du^n(t,x_k) &= v^n(t,x_k)\mathrm{d}t \\ dv^n(t,x_k) &= n^2(u^n(t,x_{k+1}) - 2u^n(t,x_k) + u^n(t,x_{k-1}))\mathrm{d}t \\ &\quad + f(t,x_k,u^n(t,x_k))\mathrm{d}t \\ &\quad + n\sigma(t,x_k,u^n(t,x_k))d(W(t,x_{k+1}) - W(t,x_k)), \end{aligned} \tag{8}$$

with initial conditions

$$u^n(0,x_k) = u_0(x_k),\ v^n(0,x_k) = v_0(x_k),$$

where

$$u_0(x_k) = \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle \varphi_j(x_k), \quad v_0(x_k) = \sum_{j=1}^{\infty} \langle v_0, \varphi_j \rangle \varphi_j(x_k),$$

$k = 1, \ldots, n-1$.

Conditions (LG) and (L) on the coefficients σ and f guarantee the existence and uniqueness of solution to the above system of equations.

We would like to write the stochastic system (8) in an evolution-like form. Let us introduce a simplified notation by setting

$$\begin{aligned} u_k^n(t) &= u^n(t,x_k), \\ v_k^n(t) &= v^n(t,x_k), \\ W_k^n(t) &= \sqrt{n}(W(t,x_{k+1}) - W(t,x_k)), \end{aligned}$$

$k = 1, \ldots, n-1$. Notice that $W^n(t) = \left(W_1^n(t), \ldots, W_{n-1}^n(t)\right)$ is a $(n-1)$-dimensional standard Brownian motion.

Then Equation (8) is equivalent to

$$\begin{aligned} \mathrm{d}u_k^n(t) &= v_k^n(t)\mathrm{d}t \\ \mathrm{d}v_k^n(t) &= n^2 \sum\nolimits_{i=1}^{n-1} \mathrm{d}_{ki} u_i^n(t)\mathrm{d}t + f(t,x_k,u_k^n(t))\mathrm{d}t \\ &\quad + \sqrt{n}\sigma(t,x_k,u_k^n(t))\mathrm{d}W_k^n(t), \end{aligned} \tag{9}$$

with $u_k^n(0) = u_0(x_k)$, $v_k^n(0) = v_0(x_k)$, $k = 1, \ldots, n-1$, where $d_{kk} = -2$, $d_{ki} = 1$ if $|k-i| = 1$ and $d_{ki} = 0$ if $|k-i| > 1$.

In the sequel we denote by D the square $(n-1)$-dimensional matrix whose entries are d_{ik}.

The system (9) can be written as the $\mathbb{R}^{2(n-1)}$-valued stochastic differential equation

$$d\underline{w}^n(t) = A^n \underline{w}^n(t) + F(\underline{w}^n(t)) + \Sigma(\underline{w}^n(t))d\underline{W}^n(t),$$

$\underline{w}^n(0) = (\underline{u}^n(0), \underline{v}^n(0))^*$, with the following notations: $\underline{u}^n(t) = \big(u_k^n(t), k = 1, \ldots, n-1\big)$, $\underline{v}^n(t) = \big(v_k^n(t), k = 1, \ldots, n-1\big)$, $\underline{w}^n(t) = \big(\underline{u}^n(t), \underline{v}^n(t)\big)^*$, where the superscript * means the transpose of the vector,

$$F(\underline{w}^n(t)) = (\overbrace{0\ldots0}^{n-1}, f(t, x_1, u_1^n(t)), \ldots, f(t, x_{n-1}, u_{n-1}^n(t)))^*,$$

and $\underline{W}^n = (Z, W^n)^*$, with Z a $(n-1)$-dimensional Brownian motion independent of W^n. Finally

$$A^n = \begin{pmatrix} 0 & I_{n-1} \\ n^2 D & 0 \end{pmatrix} \quad \text{and} \quad \Sigma(\underline{w}^n(t)) = \sqrt{n}\begin{pmatrix} 0 & 0 \\ 0 & B_\sigma \end{pmatrix},$$

where I_{n-1} denotes the identity matrix in $\mathbb{R}^{n-1}$ and B_σ is the diagonal matrix of size $n-1$ with diagonal elements $\sigma(t, x_k, u_k^n(t))$, for $k = 1, \ldots, n-1$.

Itô's formula yields

$$\underline{w}^n(t) = \mathrm{e}^{tA^n}\underline{w}^n(0) + \int_0^t \mathrm{e}^{(t-s)A^n} F(\underline{w}^n(s))\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A^n}\Sigma(\underline{w}^n(s))\mathrm{d}\underline{W}^n(s). \tag{10}$$

As mentioned in [9], the $n-1$-dimensional vectors

$$e_j = \left(\sqrt{\frac{2}{n}}\sin\left(j\frac{k}{n}\pi\right),\ k = 1, \ldots, n-1\right), \tag{11}$$

$j = 1, \ldots, n-1$ are an orthonormal basis of $\mathbb{R}^{n-1}$. In addition, they are eigenvectors of n^2D with eigenvalues

$$\lambda_j^n = -4n^2 \sin^2\left(\frac{j}{2n}\pi\right) = -j^2\pi^2 c_j^n,$$

respectively, where

$$c_j^n = \frac{\sin^2\left(\frac{j\pi}{2n}\right)}{\left(\frac{j\pi}{2n}\right)^2}.$$

It is easy to check that $4/\pi^2 \leqslant c_j^n \leqslant 1$.

With these ingredients, it is possible to compute the exponential matrix e^{rA^n}, $r \geqslant 0$. Indeed, by the definition of the matrix A^n, we have

$$(A^n)^{2k} = \begin{pmatrix} n^{2k}D^k & 0 \\ 0 & n^{2k}D^k \end{pmatrix}, \quad (A^n)^{2k+1} = \begin{pmatrix} 0 & n^{2k}D^k \\ n^{2(k+1)}D^{k+1} & 0 \end{pmatrix},$$

$k \in \mathbb{N} \cup \{0\}$, which implies that

$$e^{tA^n} = \begin{pmatrix} E_1(t,n) & E_2(t,n) \\ E_3(t,n) & E_1(t,n) \end{pmatrix},$$

where

$$E_1(t,n) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} n^{2k} D^k, \quad E_2(t,n) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} n^{2k} D^k,$$

$$E_3(t,n) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} n^{2k} D^{k+1}.$$

With these expressions, we can write equation (10) coordinatewise and in particular, the components of the vector $\underline{u}^n(t)$, as follows:

$$\begin{aligned} u_k^n(t) &= \frac{1}{n} \sum_{l=1}^{n-1} \sum_{j=1}^{n-1} \cos\left(j\pi t \sqrt{c_j^n}\right) \varphi_j(x_k)\varphi_j(x_l)u_0(x_l) \\ &+ \frac{1}{n} \sum_{l=1}^{n-1} \sum_{j=1}^{n-1} \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j(x_k)\varphi_j(x_l)v_0(x_l) \\ &+ \int_0^t \frac{1}{n} \sum_{l=1}^{n-1} \sum_{j=1}^{n-1} \frac{\sin\left(j\pi(t-s)\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j(x_k)\varphi_j(x_l) f(s, x_l, u_l^n(s))\mathrm{d}s \\ &+ \int_0^t \frac{1}{n} \sum_{l=1}^{n-1} \sum_{j=1}^{n-1} \frac{\sin\left(j\pi(t-s)\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j(x_k)\varphi_j(x_l)\sigma(s, x_l, u_l^n(s))\mathrm{d}W_l^n(s), \end{aligned}$$

where $\varphi_j(x) = \sqrt{2}\,\sin(j\pi x)$ and $k = 1, \ldots, n-1$.

Set

$$G^n(t,x,y) = \sum_{j=1}^{n-1} \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j^n(x)\varphi_j(\kappa_n(y)),$$

where $\kappa_n(y) = [ny]/n$, $\varphi_j^n(x) = \varphi_j(x_l)$ for $x = x_l$ and

$$\varphi_j^n(x) = \varphi_j(x_l) + (nx - l)\big(\varphi_j(x_{l+1}) - \varphi_j(x_l)\big)$$

for $x \in (x_l, x_{l+1})$.

We extend the definition of $u_k^n(t) = u^n(t, x_k)$ to any $x \in [0, 1]$ by linear interpolation, by setting

$$u^n(t,x) = u^n(t,x_k) + (nx - k)(u^n(t,x_{k+1}) - u^n(t,x_k)),$$

if $x \in [x_k, x_{k+1})$.

The sequence of processes $u^n = \{u^n(t,x), (t,x) \in [0,T] \times (0,1)\}$, $n \geqslant 1$, is the approximation scheme of the solution of the stochastic wave equation we are considering in this paper. Notice that u^n satisfies the evolution equation

$$\begin{aligned} u^n(t,x) = & \int_0^1 G^n(t,x,y) v_0(\kappa_n(y))\mathrm{d}y \\ & + \frac{\partial}{\partial t}\left(\int_0^1 G^n(t,x,y) u_0(\kappa_n(y))\mathrm{d}y\right) \\ & + \int_0^t \int_0^1 G^n(t-s,x,y) f(s,\kappa_n(y),u^n(s,\kappa_n(y)))\mathrm{d}s\mathrm{d}y \\ & + \int_0^t \int_0^1 G^n(t-s,x,y)\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))W(\mathrm{d}s,\mathrm{d}y), \end{aligned} \tag{12}$$

$t \in (0,T]$ and $x \in (0,1)$.

3.2. THE RATE OF CONVERGENCE IN L^p

This section is devoted to the proof of the main result of this paper, as follows.

THEOREM 1. *Suppose that* $u_0 \in H^{\alpha,2}([0,1])$, *with* $\alpha > \frac{3}{2}$, $v_0 \in H^{\beta,2}([0,1])$, *with* $\beta > \frac{1}{2}$. *We also assume that the coefficients* σ *and* f *satisfy conditions* (LG) *and* (L).

There exists a positive constant C *depending on* α, β *such that, for any* $n \geqslant 1$,

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|u^n(t,x) - u(t,x)|^{2p}) \leqslant \frac{C}{n^{2p\rho}}, \tag{13}$$

for all $\rho \in (0, \rho_0)$, *with* $\rho_0 = \frac{1}{3} \wedge (\alpha - \frac{3}{2}) \wedge (\beta - \frac{1}{2})$.

Moreover, $u^n(t,x)$ *converges to* $u(t,x)$ *almost surely, as* n *tends to infinity, uniformly with respect to* $(t,x) \in [0,T] \times [0,1]$.

Remark. Remember that, for any $\gamma > \frac{1}{2}$, $H^{\gamma,2}([0,1])$ is embedded in the space of δ-Hölder continuous functions on $(0,1)$, for any $\delta \in (0, \gamma - \frac{1}{2})$. Actually, one could state an analogue to Theorem 1 assuming Hölder continuity of the initial conditions.

We prepare the proof of this theorem with some preliminary results.
In the next lemma we will use the inequality

$$\int_0^1 |h(y) - h(\kappa_n(y))|^2 \mathrm{d}y \leqslant \frac{C}{n^2} \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}y} h(y) \right|^2 \mathrm{d}y, \tag{14}$$

valid for every function h in $\mathcal{C}^1([0,1])$ and any $n \geqslant 1$.

LEMMA 1. *For every $\delta \in (0, \frac{2}{3})$, there exists a positive constant C, depending on δ, such that*

$$\sup_{(t,x)\in[0,T]\times[0,1]} \int_0^1 |G(t,x,y) - G^n(t,x,y)|^2 \mathrm{d}y \leqslant \frac{C}{n^\delta},$$

for every $n \geqslant 1$.

Proof. Set

$$G_n(t,x,y) = \sum_{j=1}^{n-1} \frac{\sin(j\pi t)}{j\pi} \varphi_j(x)\varphi_j(y).$$

We consider the upper bound

$$\int_0^1 |G(t,x,y) - G^n(t,x,y)|^2 dy \leqslant C \sum_{k=1}^4 I_k^n(t,x),$$

where

$$I_1^n(t,x) = \int_0^1 |G(t,x,y) - G_n(t,x,y)|^2 \mathrm{d}y,$$

$$I_2^n(t,x) = \int_0^1 |G_n(t,x,y) - G_n(t,x,\kappa_n(y))|^2 \mathrm{d}y,$$

$$I_3^n(t,x) = \int_0^1 \left| \sum_{j=1}^{n-1} \left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \right) \varphi_j(x)\varphi_j(\kappa_n(y)) \right|^2 \mathrm{d}y,$$

$$I_4^n(t,x) = \int_0^1 \left| \sum_{j=1}^{n-1} \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} (\varphi_j(x) - \varphi_j^n(x))\varphi_j(\kappa_n(y)) \right|^2 \mathrm{d}y.$$

Owing to equation (2) and to the orthonormality of the family $(\varphi_j, j \geqslant 1)$,

$$I_1^n(t,x) \leqslant C \sum_{j=n}^{\infty} \frac{1}{j^2} \leqslant \frac{C}{n}.$$

Notice that $(\psi_j, j \geqslant 1)$ defined by $\psi_j(y) = \cos(j\pi y)$, is an orthogonal system in $L^2([0,1])$. Thus, by Equation (14),

$$I_2^n(t,x) \leqslant \frac{C}{n^2} \int_0^1 \left| \frac{\partial}{\partial y} G_n(t,x,y) \right|^2 \mathrm{d}y \leqslant \frac{C}{n^2} \sum_{j=1}^{n-1} \cos^2(j\pi t) |\varphi_j(x)|^2 \leqslant \frac{C}{n},$$

for all $n \geqslant 1$.

For $h, g \in L^2([0,1])$, set $\langle h, g \rangle^n = \int_0^1 h(\kappa_n(y)) g(\kappa_n(y)) \mathrm{d}y$. By its very definition, for any $j, l \geqslant 1$,

$$\langle \varphi_j, \varphi_l \rangle^n = \langle e_j, e_l \rangle_{n-1} = \delta_{j,l}, \tag{15}$$

where $\langle \cdot, \cdot \rangle_{n-1}$ denotes the Euclidean inner product in $\mathbb{R}^{n-1}$, $(e_j, j = 1 \ldots, n-1)$ is the basis of $\mathbb{R}^{n-1}$ defined in equation (11) and $\delta_{j,l}$ is the Kronecker symbol. Consequently,

$$I_3^n(t,x) = \sum_{j=1}^{n-1} \left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \right)^2 \varphi_j^2(x)$$

and

$$I_4^n(t,x) = \sum_{j=1}^{n-1} \left(\frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \right)^2 (\varphi_j(x) - \varphi_j^n(x))^2.$$

Set $I_3^n(t,x) \leqslant C(A_n + B_n)$, where

$$A_n = \sum_{j=1}^{n-1} \frac{1}{j^2} \left(1 - \frac{1}{\sqrt{c_j^n}} \right)^2 \sin^2(j\pi t),$$

$$B_n = \sum_{j=1}^{n-1} \frac{1}{j^2 c_j^n} \left(\sin(j\pi t) - \sin\left(j\pi t\sqrt{c_j^n}\right) \right)^2.$$

A Taylor expansion of the function $\sin x$ at $x = 0$ yields

$$1 - \sqrt{c_j^n} \leq C \frac{j^2}{n^2}, \tag{16}$$

for any $j = 1, \ldots, n-1$, and $n \geqslant 1$. Then, since c_j^n is bounded below by $4/\pi^2$, we obtain

$$A_n \leqslant C \sum_{j=1}^{n-1} \frac{1}{j^2} \left(1 - \sqrt{c_j^n} \right)^2 \leqslant \frac{C}{n}. \tag{17}$$

Let $\gamma \in (0, \frac{1}{6})$; the mean value theorem and Equation (16) yield

$$B_n \leqslant C \sum_{j=1}^{n-1} \frac{1}{j^2 c_j^n} \left(\sin(j\pi t) - \sin\left(j\pi t \sqrt{c_j^n} \right) \right)^{2\gamma}$$

$$\leqslant C \sum_{j=1}^{n-1} \frac{1}{j^{2(1-\gamma)}} \left(1 - \sqrt{c_j^n} \right)^{2\gamma}$$

$$\leqslant C \frac{1}{n^{4\gamma}} \sum_{j=1}^{\infty} \frac{1}{j^{2(1-3\gamma)}} \leq \frac{C}{n^{4\gamma}},$$

for some positive constant C depending on γ. Thus, for every $\delta \in (0, \frac{2}{3})$

$$B \leqslant \frac{C(\delta)}{n^\delta}. \tag{18}$$

Putting together Equations (17) and (18) we obtain that

$$I_3^n(t,x) \leqslant \frac{C}{n^\delta},$$

for all $\delta \in (0, \frac{2}{3})$.

Observe that

$$|\varphi_j(x) - \varphi_j^n(x)|^2 \leqslant C \frac{j^2}{n^2}. \tag{19}$$

Hence

$$I_4^n(t,x) \leqslant C \sum_{j=1}^{n-1} \frac{1}{j^2} |\varphi_j(x) - \varphi_j^n(x)|^2 \leqslant \frac{C}{n}.$$

The proof of the lemma is complete. □

PROPOSITION 3. *Assume that* $v_0 \in L^2([0,1])$, $u_0 \in H^{\alpha,2}([0,1])$, *for some* $\alpha > \frac{1}{2}$, *and that the coefficients* f *and* σ *satisfy condition* (LG). *Then for every* $p \geqslant 1$

$$\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times [0,1]} E\left(|u^n(t,x)|^{2p} \right) < +\infty.$$

Proof. By virtue of Equation (12), we have

$$E\left(|u^n(t,x)|^{2p} \right) \leqslant C \sum_{k=1}^{4} A_k(n,t,x),$$

with

$$A_1(n,t,x) = \left| \int_0^1 G^n(t,x,y) v_0(\kappa_n(y)) \mathrm{d}y \right|^{2p},$$

$$A_2(n,t,x) = \left| \sum_{j=1}^{n-1} \langle u_0, \varphi_j \rangle^n \cos\left(j\pi t \sqrt{c_j^n} \right) \varphi_j^n(x) \right|^{2p},$$

$$A_3(n,t,x) = E\left(\left| \int_0^t \int_0^1 G^n(t-s,x,y) \sigma(s,\kappa_n(y), u^n(s,\kappa_n(y))) W(\mathrm{d}s,\mathrm{d}y) \right|^{2p} \right),$$

$$A_4(n,t,x) = E\left(\left| \int_0^t \int_0^1 G^n(t-s,x,y) f(s,\kappa_n(y), u^n(s,\kappa_n(y))) \mathrm{d}s\mathrm{d}y \right|^{2p} \right).$$

We can easily prove that

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times[0,1]} \int_0^1 |G^n(t,x,y)|^2 \mathrm{d}y < \infty. \tag{20}$$

Let $h \in H^{\alpha,2}([0,1])$, with $\alpha > \frac{1}{2}$, then

$$\langle h, \varphi_j \rangle^n = \langle h, \varphi_j \rangle, \tag{21}$$

for every $n \geqslant 1$ and $j = 1, \ldots, n-1$. Indeed, Fubini's theorem and Equation (15) yield

$$\begin{aligned} \langle h, \varphi_j \rangle^n &= \int_0^1 \sum_{l=1}^{\infty} \langle h, \varphi_l \rangle \varphi_l(\kappa_n(y)) \varphi_j(\kappa_n(y)) \mathrm{d}y \\ &= \sum_{l=1}^{\infty} \langle h, \varphi_l \rangle \int_0^1 \varphi_l(\kappa_n(y)) \varphi_j(\kappa_n(y)) \mathrm{d}y \\ &= \langle h, \varphi_j \rangle. \end{aligned}$$

By Cauchy–Schwarz inequality,

$$A_1(n,t,x) \leqslant \left(\int_0^1 |G^n(t,x,y)|^2 \mathrm{d}y \right)^p \left(\int_0^1 |v_0(\kappa_n(y))|^2 \mathrm{d}y \right)^p.$$

Since $\sup_n \int_0^1 |v_0(\kappa_n(y))|^2 \mathrm{d}y$ is finite, using Equation (20) we obtain

$$\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times [0,1]} A_1(n,t,x) \leqslant C.$$

The identity (21) and Cauchy–Schwarz inequality yield

$$A_2(n,t,x) \leqslant C \left(\sum_{j=1}^{\infty} |\langle u_0, \varphi_j \rangle|^2 j^{2\alpha} \right)^p \left(\sum_{j=1}^{\infty} j^{-2\alpha} \right)^p \leqslant C \|u_0\|_{\alpha,2}^{2p}.$$

Applying first Burkholder's inequality and then Hölder's inequality with respect to the finite (uniformly with respect to n, t and x) measure $|G^n(t-s,x,y)|^2 \mathrm{d}s\mathrm{d}y$ on $[0,T] \times [0,1]$ yield

$$\begin{aligned} A_3 &\leqslant CE\left(\left| \int_0^t \int_0^1 |G^n(t-s,x,y)|^2 |\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y))|^2 \mathrm{d}s\mathrm{d}y \right|^p \right) \\ &\leqslant C\left(1 + \int_0^t \int_0^1 |G^n(t-s,x,y)|^2 E\left(|u^n(s,\kappa_n(y))|^{2p}\right) \mathrm{d}s\mathrm{d}y\right) \\ &\leqslant C\left(1 + \int_0^t \sup_{x \in [0,1]} E\left(|u^n(s,x)|^{2p}\right) \mathrm{d}s\right). \end{aligned}$$

Similarly,

$$A_4 \leqslant C\left(1 + \int_0^t \sup_{x \in [0,1]} E\left(|u^n(s,x)|^{2p}\right) \mathrm{d}s\right).$$

Therefore

$$\sup_{x \in [0,1]} E\left(|u^n(t,x)|^{2p}\right) \leqslant C + C \int_0^t \sup_{x \in [0,1]} E\left(|u^n(s,x)|^{2p}\right) \mathrm{d}s,$$

with a constant C independent of n.

We apply Gronwall's lemma to conclude the proof. □

With analogous arguments as those used in the study of the terms $J_3(s,t,x,y)$ and $J_4(s,t,x,y)$ in the proof of Proposition 2 we obtain the following.

LEMMA 2. *Let* $\{h(t,x), (t,x) \in [0,T] \times [0,1]\}$ *be an* $\mathcal{F}_t$*-adapted stochastic process such that for any* $p \geqslant 1$

$$\sup_{(t,x) \in [0,T] \times [0,1]} E\left(|h(t,x)|^{2p}\right) < \infty.$$

The stochastic processes defined by

$$H^n(t,x) = \int_0^t \int_0^1 G^n(t-s,x,z)h(s,z)W(\mathrm{d}s,\mathrm{d}z),$$

$$F^n(t,x) = \int_0^t \int_0^1 G^n(t-s,x,z)h(s,z)\mathrm{d}z\,\mathrm{d}s,$$

$(t,x) \in [0,T] \times [0,1]$*, are well defined. Moreover, there exists a positive constant* C *such that*

$$E\Big(|H^n(s,x) - H^n(t,y)|^{2p}\Big) \leqslant C(|t-s|^p + |x-y|^p),$$
$$E\Big(|F^n(s,x) - F^n(t,y)|^{2p}\Big) \leqslant C(|t-s|^p + |x-y|^p),$$

for all $s,t \in [0,T]$, $x,y \in [0,1]$, $n \geqslant 1$.

Therefore, almost all the sample paths of both, H^n *and* F^n*, are jointly Hölder continuous in time and in space, of any order* $\delta \in (0, \frac{1}{2})$.

In order to shorten the notation, we set

$$\nu(t,x) = \int_0^1 G(t,x,y)v_0(y)\mathrm{d}y, \tag{22}$$

$$\nu^n(t,x) = \int_0^1 G^n(t,x,y)v_0(\kappa_n(y))\mathrm{d}y,$$

$$\mu(t,x) = \sum_{j=1}^{\infty} \langle u_0, \varphi_j\rangle \cos(j\pi t)\varphi_j(x),$$

$$\mu^n(t,x) = \sum_{j=1}^{n-1} \langle u_0, \varphi_j\rangle^n \cos\Big(j\pi t\sqrt{c_j^n}\Big)\varphi_j^n(x),$$

$$w(t,x) = u(t,x) - \nu(t,x) - \mu(t,x),$$

$$w^n(t,x) = u^n(t,x) - \nu^n(t,x) - \mu^n(t,x). \tag{23}$$

From the proof of Proposition 2 and the above Lemma 2, we clearly have

$$\begin{aligned} &\sup_n \Big(E(|w^n(s,x) - w^n(t,y)|^{2p}) + E(|w(s,x) - w(t,y)|^{2p})\Big) \\ &\quad \leqslant C(|t-s|^p + |x-y|^p), \end{aligned} \tag{24}$$

for every $p \geqslant 1$, $s,t \in [0,T]$, $x,y \in [0,1]$.

In particular, these estimates imply that, if the initial conditions v_0, u_0 vanish, the trajectories of the stochastic processes u^n and u are a.s. jointly Hölder continuous in time and in space, of any order $\delta \in (0, \frac{1}{2})$.

PROPOSITION 4. *Assume that v_0 belongs to $H^{\beta,2}([0,1])$, for some $\beta > \frac{1}{2}$. There exists a positive constant C depending on β, such that*

$$\sup_{(t,x)\in[0,T]\times[0,1]} |\nu^n(t,x) - \nu(t,x)| \leqslant \frac{C}{n^{\epsilon}},$$

for each $n \geqslant 1$ and every $\epsilon \in (0, \epsilon_0)$, with $\epsilon_0 = \frac{1}{3} \wedge (\beta - \frac{1}{2})$.

Proof. Owing to Cauchy–Schwarz inequality and Equation (20) we have

$$|\nu^n(t,x) - \nu(t,x)| \leqslant C(N_1(n,t,x) + N_2(n)),$$

where

$$N_1(n,t,x) = \left(\int_0^1 |G^n(t,x,y) - G(t,x,y)|^2 \mathrm{d}y\right)^{\frac{1}{2}},$$

$$N_2(n) = \left(\int_0^1 |v_0(\kappa_n(y)) - v_0(y)|^2 \mathrm{d}y\right)^{\frac{1}{2}}.$$

From Lemma 1, it follows that $\sup_{(t,x)\in[0,T]\times[0,1]} N_1(n,t,x) \leqslant C/n^{\gamma}$, for every $\gamma \in (0, \frac{1}{3})$.

Moreover,

$$\begin{aligned} N_2(n) &= \left(\int_0^1 |\sum_{j=1}^{\infty} \langle v_0, \varphi_j\rangle (\varphi_j(\kappa_n(y)) - \varphi_j(y))|^2 dy\right)^{\frac{1}{2}} \\ &\leqslant \|v_0\|_{\beta,2} \left(\sum_{j=1}^{\infty} j^{-2\beta} \big(1 \wedge \frac{j^2}{n^2}\big)\right)^{\frac{1}{2}} \end{aligned} \tag{25}$$

Hence, for $\beta \in (\frac{1}{2}, \frac{3}{2}]$, (25) is bounded by $C/n^{\beta - \frac{1}{2}}$. If $\beta > \frac{3}{2}$, since the series $\sum_{j=1}^{\infty} j^{2(1-\beta)}$ is convergent, we can estimate Equation (25) by C/n. Consequently,

$$N_2(n) \leqslant \frac{C}{n^{\beta - \frac{1}{2}}}.$$

The proof is complete. □

PROPOSITION 5. *Assume that* $u_0 \in H^{\alpha,2}([0,1])$, *with* $\alpha > \frac{3}{2}$. *There exists a positive constant C depending on* α *such that*

$$\sup_{(t,x)\in[0,T]\times[0,1]} |\mu^n(t,x) - \mu(t,x)| \leqslant \frac{C}{n^\tau},$$

for each $n \geqslant 1$ *and every* $\tau \in (0, \tilde{\tau}_0)$, with $\tilde{\tau}_0 = (\alpha - \frac{3}{2}) \wedge 1$.

Proof. By Equation (21) we have that

$$|\mu^n(t,x) - \mu(t,x)| \leqslant C \sum_{k=1}^{3} I_k(t,x,n),$$

with

$$I_1(t,x,n) = \left| \sum_{j=n}^{\infty} \langle u_0, \varphi_j \rangle \cos(j\pi t) \varphi_j(x) \right|,$$

$$I_2(t,x,n) = \left| \sum_{j=1}^{n-1} \langle u_0, \varphi_j \rangle \Big(\cos(j\pi t) - \cos\Big(j\pi t \sqrt{c_j^n} \Big) \Big) \varphi_j(x) \right|,$$

$$I_3(t,x,n) = \left| \sum_{j=1}^{n-1} \langle u_0, \varphi_j \rangle \cos\Big(j\pi t \sqrt{c_j^n} \Big) (\varphi_j(x) - \varphi_j^n(x)) \right|.$$

Cauchy–Schwarz inequality yields

$$\begin{aligned} I_1(t,x,n) &\leqslant C \left(\sum_{j=n}^{\infty} |\langle u_0, \varphi_j \rangle|^2 j^{2\alpha} \right)^{\frac{1}{2}} \left(\sum_{j=n}^{\infty} j^{-2\alpha} \right)^{\frac{1}{2}} \\ &\leqslant \frac{C}{n^{\alpha - \frac{1}{2}}}. \end{aligned}$$

Cauchy–Schwarz inequality and Equation (16) yield

$$\begin{aligned} I_2(t,x,n) &\leqslant C\|u_0\|_{\alpha,2} \left(\textstyle\sum_{j=1}^{n-1} j^{-2\alpha} \left| \cos(j\pi t) - \cos\Big(j\pi t \sqrt{c_j^n} \Big) \right|^2 \right)^{\frac{1}{2}} \\ &\leqslant C \left(\textstyle\sum_{j=1}^{n-1} j^{-2(\alpha-1)} \Big(1 - \sqrt{c_j^n} \Big)^2 \right)^{\frac{1}{2}} \qquad (26) \\ &\leqslant C \Big(\tfrac{1}{n^4} \textstyle\sum_{j=1}^{n-1} j^{2(3-\alpha)} \Big)^{\frac{1}{2}}. \end{aligned}$$

For α, the last term in Equation (26) is bounded by $C/n^{\alpha-\frac{3}{2}}$. For $\alpha > \frac{7}{2}$, the series $\sum_{j=1}^{\infty} j^{2(3-\alpha)}$ converges and therefore the last term in Equation (26) is bounded by C/n^2. Hence, since $\alpha > \frac{3}{2}$, for any $n \geqslant 1$,

$$I_2(t,x,n) \leqslant \frac{C}{n^{\tau}},$$

for every $\tau \in (0, \tilde{\tau}_0)$, with $\tilde{\tau}_0 = (\alpha - \frac{3}{2}) \wedge 2$.

Similarly, by virtue of Equation (19),

$$I_3(t,x,n) \leqslant \frac{C}{n}.$$

Thus the proof of the statement is complete. □

PROPOSITION. 6 *Suppose that u_0 belongs to $H^{\alpha,2}([0,1])$, with $\alpha > \frac{3}{2}$, $v_0 \in H^{\beta,2}([0,1])$, with $\beta > \frac{1}{2}$, and that the coefficients σ and f satisfy conditions* (LG) *and* (L).

There exists a positive constant C, depending on α, β, such that

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|w^n(t,x) - w(t,x)|^{2p}) \leqslant \frac{C}{n^{2p\rho}},$$

for each $n \geqslant 1$ and any $\rho \in (0, \rho_0)$, with $\rho_0 = \frac{1}{3} \wedge (\alpha - \frac{3}{2}) \wedge (\beta - \frac{1}{2})$.

Proof. By definition of w^n and w,

$$E(|w^n(t,x) - w(t,x)|^{2p}) \leqslant C(A_1^n(t,x) + A_2^n(t,x)),$$

with

$$A_1^n(t,x) = E\bigg(\bigg|\int_0^t \int_0^1 G^n(t-s,x,y)\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))W(\mathrm{d}s,\mathrm{d}y)$$
$$- \int_0^t \int_0^1 G(t-s,x,y)\sigma(s,y,u(s,y))W(\mathrm{d}s,\mathrm{d}y)\bigg|^{2p}\bigg),$$

$$A_2^n(t,x) = E\bigg(\bigg|\int_0^t \int_0^1 G^n(t-s,x,y)f(s,\kappa_n(y),u^n(s,\kappa_n(y)))\mathrm{d}s\mathrm{d}y$$
$$- \int_0^t \int_0^1 G(t-s,x,y)f(s,y,u(s,y))\mathrm{d}s\mathrm{d}y\bigg|^{2p}\bigg).$$

Burkholder's inequality and Hölder's inequality with respect to the measures on $[0,T]\times[0,1]$ given by $|G^n(t-s,x,y)-G(t-s,x,y)|^2 \mathrm{d}s\mathrm{d}y$ and $|G(t-s,x,y)|^2$ $\mathrm{d}s\mathrm{d}y$, respectively, yield

$$A_1^n(t,x) \leqslant C(B_1^n(t,x)+B_2^n(t,x)),$$

where

$$\begin{aligned} B_1^n(t,x) = &\left(\int_0^t\int_0^1 |G^n(t-s,x,y)-G(t-s,x,y)|^2 \mathrm{d}s\mathrm{d}y\right)^{p-1} \\ &\times\left(\int_0^t\int_0^1 |G^n(t-s,x,y)-G(t-s,x,y)|^2\right. \\ &\left.\times E(|\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))|^{2p})\mathrm{d}s\mathrm{d}y\right), \end{aligned}$$

$$\begin{aligned} B_2^n(t,x) = &\int_0^t\int_0^1 |G(t-s,x,y)|^2 \\ &\times E(|\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))-\sigma(s,y,u(s,y))|^{2p})\mathrm{d}s\mathrm{d}y. \end{aligned}$$

The assumption (LG), Proposition 3 and Lemma 1 yield, for any $n\geqslant 1$,

$$B_1^n(t,x) \leqslant \left(\int_0^t\int_0^1 |G^n(t,x,y)-G(t,x,y)|^2 dyds\right)^p \leqslant \frac{C}{n^{\delta p}},$$

for all $\delta\in(0,\frac{2}{3})$.

From the Lipschitz condition (L) on σ and Equation (5) we have

$$\begin{aligned} B_2^n(t,x) \leqslant C\Bigg(&\frac{1}{n^{2p}}+\int_0^t \Big(\sup_{z\in[0,1]} E(|u^n(s,z)-u(s,z)|^{2p} \\ &+\sup_{z\in[0,1]} E(|u(s,\kappa_n(z))-u(s,z)|^{2p})\Big)ds\Bigg). \end{aligned}$$

Owing to Equations (22), (23) and the upper bounds provided by Propositions 2, 4 and 5, we obtain

$$B_2^n(t,x) \leqslant C\left(\frac{1}{n^{2p\rho}}+\int_0^t \sup_{z\in[0,1]} E(|w^n(s,z)-w(s,z)|^{2p})\mathrm{d}s\right),$$

for any $\rho\in(0,\rho_0)$, with $\rho_0=\frac{1}{3}\wedge(\alpha-\frac{3}{2})\wedge(\beta-\frac{1}{2})$.

Taking into account the upper bound obtained for the term $B_1^n(t,x)$, it follows that

$$\sup_{z\in[0,1]} E(|w^n(t,z)-w(t,z)|^{2p}) \leqslant C\left(\frac{1}{n^{2p\rho}}+\int_0^t \sup_{z\in[0,1]} E(|w^n(s,z)-w(s,z)|^{2p})ds\right),$$

for all $t\in[0,T]$, every $n\geqslant 1$ and the same range of ρ described before.

With Gronwall's lemma we complete the proof. □

We are now ready to end up with the proof of the main theorem.

Proof of Theorem 1. The first statement (see (13)) is a consequence of the previous Propositions 4, 5 and 6.

Owing to Propositions 4 and 5, in order to complete the proof we only need to check the almost sure convergence $w^n(t,x)\to w(t,x)$, uniformly in $(t,x)\in[0,T]\times[0,1]$.

Notice that, for any $p\in[2,\infty)$,

$$\sup_{(t,x)\in[0,T]\times[0,1]} |w^n(t,x)-w(t,x)|^{2p} \leqslant C(J_1+J_2+J_3),$$

with

$$J_1 := \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} |w^n(t_k^n,x_l^n)-w(t_k^n,x_l^n)|^{2p},$$

$$J_2 := \sup_k \sup_l \sup_{|t-t_k^n|\leqslant 1/n} \sup_{|x-x_l^n|\leqslant 1/n} |w^n(t_k^n,x_l^n)-w^n(t,x)|^{2p},$$

$$J_3 := \sup_k \sup_l \sup_{|t-t_k^n|\leqslant 1/n} \sup_{|x-x_l^n|\leqslant 1/n} |w(t_k^n,x_l^n)-w(t,x)|^{2p},$$

where $t_k^n := k\frac{T}{n}$, $x_l^n := \frac{l}{n}$, $k,l=0,1,\dots,n-1$. By Proposition 6,

$$E(J_1)\leqslant \frac{C}{n^{2(p\rho-1)}},$$

for all $\rho\in(0,\rho_0)$ and $n\geqslant 1$.

The joint Hölder continuity of the sample paths of the processes w^n and w (see (24)) yields

$$E(J_2+J_3)\leqslant \frac{C}{n^{2p\delta}},$$

for every $\delta\in(0,\frac{1}{2})$, $n\geqslant 1$. Consequently,

$$E\left(\sup_{(t,x)\in[0,T]\times[0,1]} |w^n(t,x)-w(t,x)|^{2p}\right) \leqslant C\frac{C}{n^{2(p\rho-1)}},$$

for all $\rho \in (0, \rho_0)$. Hence

$$P\left(\sup_{(t,x)\in[0,T]\times[0,1]} |w^n(t,x) - w(t,x)|^{2p} \geqslant \frac{1}{n^2}\right) \leqslant \frac{C}{n^{2p\rho-4}},$$

for all $n \geqslant 1$.

Let $p > 5/2\rho$, Borel–Cantelli's Lemma yields, with probability one,

$$\sup_{(t,x)\in[0,T]\times[0,1]} |w^n(t,x) - w(t,x)| < \left(\frac{1}{n}\right)^{\frac{1}{p}},$$

for n sufficiently large. The proof is now complete. □

Appendix

This section is devoted to a numerical analysis of the speed of convergence of the sequence of approximations u^n, $n \geqslant 1$, given in Theorem 1. We study the optimality of the restriction given by the value $\frac{1}{3}$ in the rate of convergence.

According to the proof of Propositions 6 and 4, the above-mentioned restriction comes from the uniform bound of the $L^2([0,1])$ norm of the difference $G^n(t,x,\cdot) - G(t,x,\cdot)$ stated in Lemma 1 and more precisely, from the upper bound proved for the term

$$I_3^n(t,x) = \sum_{j=1}^{n-1}\left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}}\right)^2 \varphi_j^2(x),$$

$(t,x) \in [0,T] \times [0,1]$.

With a computer program (written in C language) we compute $I_3^n(t,x)$ for different values of t, x and n. First, we check that the term $\varphi_j^2(x) = 2\sin^2(j\pi x)$ has no significant influence in the behaviour of $I_3^n(t,x)$, for fixed $t \in [0,T]$ and large n. Thus, it seems reasonable to focus our attention on

$$I_3^n(t) = \sum_{j=1}^{n-1}\left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}}\right)^2.$$

For a fixed t, we compute $I_3^n(t)$ for many different natural values of n in some range $[n_0, n_1]$. Then, we consider the function f_t defined by $f_t(z) = I_3^z(t)$, $z \in \mathbb{N}$ $p[n_0, n_1]$, and use the least square optimization method to fit f_t to a function of the form $g_t(z) = e^b/z^a$, for some $a > 0$ and $b \in \mathbb{R}$. In the following table, the values of a, b and the range of variation on n for some values of t are displayed.
We observe that the values of a in the above table are slightly less than $\frac{2}{3}$. It is

worthy mentioning that for values of t in the neighbourhood of integer numbers we need to compute $I_3^n(t)$ for larger values of n in order to obtain a suitable convergence.

In Figures 1 and 2, we simultaneously plot the functions $I_3^n(t)$ and $g_t(z)$, for $t = 1.002$ and $t = \sqrt{2}$, respectively. The coefficients a, b of the function g_t and

t	Range of n	a	b
1	$2 \leq n \leq 10000$	0.662039	−2.54703
1.002	$2 \leq n \leq 50000$	0.665005	−2.10701
$\sqrt{2}$	$2000 \leq n \leq 15000$	0.659738	−2.00993
8.7	$2 \leq n \leq 10000$	0.65826	−1.40663

the range of variation of n – and therefore of z – are specified in the above table. Notice the almost perfect matching for $t = \sqrt{2}$.

We conclude that the bound

$$I_3^n(t,x) \leqslant \frac{C}{n^\delta},$$

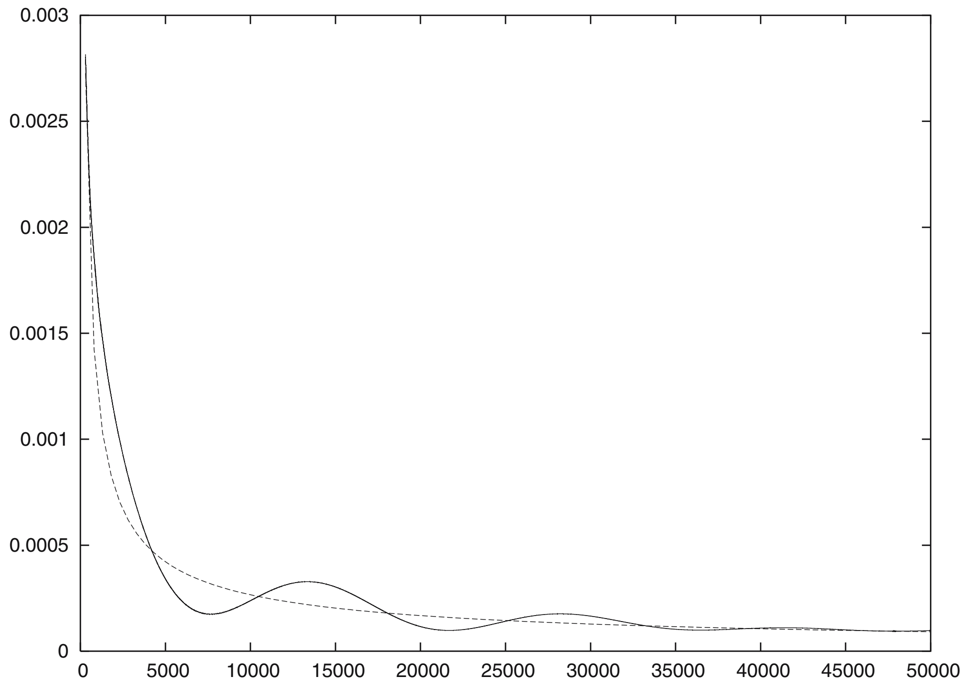

Figure 1. The dotted line corresponds to g_t and the continuous one to $I_3^n(t)$, for t=1.002.

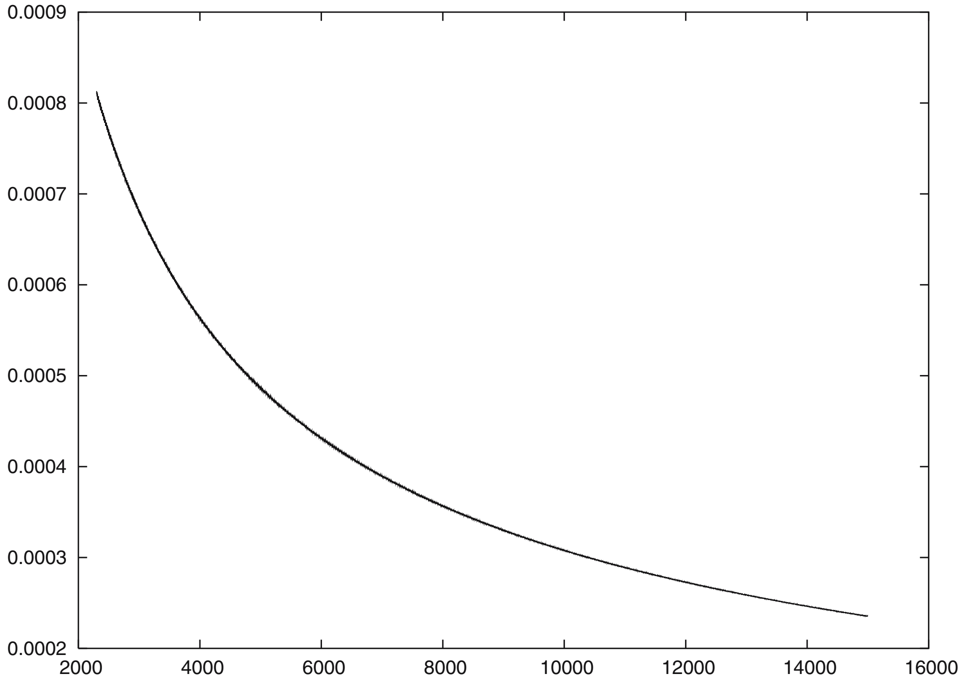

Figure 2. Graphs of g_t and $I_3^n(t)$ for $t = \sqrt{2}$.

$\delta \in (0, \frac{2}{3})$, is optimal. Therefore, the restriction in Theorem 1 given by the value $\frac{1}{3}$ is intrinsic to the model and it is not due to the method of the proof.

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