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Comparison of Green Functions and Harmonic Measures for Parabolic Operators

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Abstract. In this paper, we prove two-sided pointwise estimates for the Green function of a parabolic operator with singular first order term on a $C^{1,1}$-cylindrical domain Ω. Basing on these estimates, we establish the equivalence of the parabolic measure, the adjoint parabolic measure and the surface measure on the lateral boundary of Ω. These results are first studied by some authors for certain elliptic and less general parabolic operators.

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1. Introduction

To study the potential theory of a second-order elliptic or parabolic operator, one of the ways led up to understand the behavior of its Green function and harmonic measure. In the last years, this kind of problems have received much attention by several authors. In [18], Hueber and Sieveking proved the comparability of the Green function of an elliptic operator L with bounded Hölder continuous coefficients to the Laplacian Green function on a $C^{1,1}$-bounded domain. This result was also studied by Ancona [1] on a Lipschitz domain, and as a consequence he established the equivalence of the L-harmonic measure, the adjoint L-harmonic measure and the surface measure, which initially proved by Dahlberg [6] in the classical case. Some upper estimates for the Laplacian Green function on Liapunov–Dini domains were proved by Widman in [24] and later extended by Grüter and Widman in [14] to elliptic operators in divergence form with Dini continuous coefficients. Lower estimates were proved by Zhao in [28] and Hueber in [17] on $C^{1,1}$-bounded domains which provide a complete description of the boundary behavior of the Green function. In [5], Cranston and Zhao studied the operator $L = \frac{1}{2}\Delta + b \cdot \nabla$. Assuming that $|b|$ and $|b|^2$ are, respectively, in the Kato classes K_{n+1} and K_n (see definition in [5]), they proved by a probabilistic method the comparability of the L-Green function and the L-harmonic measure to those of Δ which enabled them to obtain some potential-theoretic results for L which are known to hold for Δ.

In the parabolic setting, it was shown that the fundamental solution satisfies upper and lower Gaussian bounds in different situations which were used to prove some interior and boundary estimates for weak solutions (boundness, Harnack inequalities, Hölder continuity, boundary Harnack principles, etc.). We refer the reader to [2, 3, 11, 12, 15] and [27] and the references therein. In [7] Davies proved, by using logarithmic Sobolev inequalities, upper estimates for the heat kernels of some elliptic operators in divergence form on some subdomains (see also [8], Theorem 4.6.9). In [19], Hui proved upper estimates for the Green function of the heat equation in a smooth cylindrical domain which played an important role to prove a Fatou theorem at the corner points. In the half-space, lower and upper estimates were established in [22] and used to study the boundary behavior of parabolic potentials. They were also used in [23] to prove boundary Harnack principles and the existence of the Martin kernel function. In [25], Wu studied the heat equation and he proved that if E is a null set for the surface measure on a general Lipschitz domain in $\mathbb{R}^{n+1}$, $n \geqslant 1$, then there is a decomposition of E in two subsets, E_1 with zero caloric measure, and E_2 with zero adjoint caloric measure. In this case the caloric measure and the surface measure may be mutually singular (see [20]). However, for the heat equation on a Lipschitz cylinder the equivalence of the caloric measure and the surface measure is established by Fabes and Salsa in [10]. In [15], Heurteaux extended this result to the parabolic operator $\partial/\partial t - \operatorname{div}(A(x,t)\nabla_x)$ on a Lip$(1, \frac{1}{2} + \varepsilon)$ domain. His idea is based on some boundary Harnack principles. Recently, Hofmann and Lewis [16] proved the same result for the operator $\partial/\partial t - \operatorname{div}(A(x,t)\nabla_x) + B(x,t) \cdot \nabla_x$ on a half-space provided that the coefficients satisfy some conditions defined by Carleson measures. They studied the operator as a pullback of the heat equation on certain time varying domains considered before by Levis and Murray and Hofmann and Levis (see [16] for all details). They also obtained the elliptic counterpart results.

Following the above mentioned works, our aim in this paper is to investigate the behavior of the Green function and the harmonic measure for a parabolic operator with first-order term in the so-called parabolic Kato class on a $C^{1,1}$-cylindrical domain. The parabolic Kato class is introduced in [26, 27], and is being proposed as a natural generalization of the Kato class in the elliptic case. Our ideas of proofs could be applied to other similar operators and our results imply their counterparts in the elliptic setting. More precisely, we will consider the parabolic operator

$$L = \frac{\partial}{\partial t} - \operatorname{div}(A(x,t)\nabla_x) + B(x,t) \cdot \nabla_x$$

on $\Omega = D \times]0, T[$, where D is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geqslant 1$ and $0 < T < \infty$. By a domain we mean an open connected set. The matrix $A(x,t) = (a_{ij}(x,t))_{1 \leqslant i,j \leqslant n}$ is assumed to be real, symmetric, and uniformly elliptic, i.e., $(1/\mu)\|\xi\|^2 \leqslant \langle A(x,t)\xi, \xi\rangle \leqslant \mu\|\xi\|^2$, for some $\mu \geqslant 1$, all $(x,t) \in \Omega$ and all $\xi \in \mathbb{R}^n$, with μ-Lipschitz coefficients with respect to the parabolic distance, i.e., $|a_{ij}(x,t) - a_{ij}(y,s)| \leqslant \mu(|x-y| \vee |t-s|^{1/2})$. The vector $B(x,t)$ is assumed to be in the parabolic Kato class, i.e., $B \in L^1_{\mathrm{loc}}(\Omega)$ and satisfies $\lim_{h\to 0} N^\alpha_h(B) = 0$, where

$$N^\alpha_h(B) \equiv \sup_{x,t} \int_{t-h}^{t} \int_D |B(y,s)| \frac{\exp(-\alpha \frac{|x-y|^2}{t-s})}{(t-s)^{(n+1)/2}} \,\mathrm{d}y\,\mathrm{d}s + \\ + \sup_{y,s} \int_{s}^{s+h} \int_D |B(x,t)| \frac{\exp(-\alpha \frac{|x-y|^2}{t-s})}{(t-s)^{(n+1)/2}} \,\mathrm{d}x\,\mathrm{d}t$$

for some constant $\alpha > 0$. Note that the parabolic Kato class depends on the parameter α. The choice of α will be fixed later. The existence and uniqueness of the L-Green function G for the initial Dirichlet problem on Ω are shown in [26] using an approximation argument and the standard theory in [3].

In Section 2, we are interested in the problem of bounding the L-Green function G on Ω. With the help of some known results of parabolic equations, we have proved lower and upper estimates for G which reveal its behavior especially near the boundary. The proof is done in two steps. The first step is concerned with the case $B \equiv 0$ and the second with the proof on its generality. As a consequence we deduce that G is comparable to the Green functions of parabolic operators of the form $\partial/\partial t - c\Delta_x$. In Section 3, we derive lower and upper estimates for the L-Poisson kernel on Ω which allow us to prove the equivalence of the L-parabolic measure, the adjoint L-parabolic measure and the surface measure on the lateral boundary of Ω. In Section 4, we apply the Green function estimates to prove the counterpart results for the elliptic operator $\mathrm{div}(A(x)\nabla_x) + B(x)\cdot\nabla_x$, with B only in the elliptic Kato class K_{n+1} on a bounded $C^{1,1}$-domain.

To prove our main results we need to give a few more notations and recall some known results. We call a bounded domain D in $\mathbb{R}^n$, $n \geqslant 2$, a $C^{1,1}$-domain if there exist positive constants c_0 and R_0 such that for every $z \in \partial D$ there exists a function $\psi_z : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $|\nabla\psi_z(x') - \nabla\psi_z(y')| \leqslant c_0|x'-y'|$ for all $x', y' \in \mathbb{R}^{n-1}$ and an orthonormal coordinate system CS_z such that if $y = (y', y_n)$ in the CS_z coordinate, then

$$B(z,R_0) \cap D = B(z,R_0) \cap \{y = (y', y_n) : y_n > \psi_z(y')\}$$

and

$$B(z,R_0) \cap \partial D = B(z,R_0) \cap \{y = (y', y_n) : y_n = \psi_z(y')\}.$$

We will call c_0 the $C^{1,1}$-constant of D and R_0 the localization radius of D.

It is well known that the bounded $C^{1,1}$ domain D satisfies the uniform interior and exterior ball condition: There exists $r_0 > 0$ depending only on D such that for any $z \in \partial D$, there exist two balls B^z_1 and B^z_2 of radius r_0 such that $B^z_1 \subset D$, $B^z_2 \subset \mathbb{R}^n \setminus \overline{D}$, and $\partial B^z_1 \cap \partial B^z_2 = \{z\}$ (see [4], p. 179, we may take $r_0 = \frac{R_0}{2} \wedge \frac{1}{c_0}$).

For $x \in D$, we denote by $d(x)$ the Euclidian distance from x to the boundary of D and $d(D)$ the diameter of D. For $r > 0$ small, we let $D_r = \{\xi \in D : d(\xi) > r\}$. Obviously for $r \in]0, r_0]$, D_r is an open connected set.

We will use Γ_0 and G_0 to denote, respectively, the fundamental solution and the Green function on Ω of the unperturbed operator $L_0 = \partial/\partial t - \operatorname{div}(A(x,t)\nabla_x)$. For Γ_0 and G_0 we recall the well known estimates:

(i) *Gaussian estimates:*

$$\frac{1}{k(t-s)^{n/2}} \exp\left(-\frac{|x-y|^2}{c(t-s)}\right) \leqslant \Gamma_0(x,t;y,s)$$
$$\leqslant \frac{k}{(t-s)^{n/2}} \exp\left(-c\frac{|x-y|^2}{t-s}\right),$$

for all $x, y \in \mathbb{R}^n$ and $s < t$, where $k = k(n,\mu) > 0$ and $c = c(n,\mu) > 0$ (see [2, 3, 11]).

(ii) *Estimate on the gradient of G_0:*

$$|\nabla_x G_0(x,t;y,s)| \leqslant \frac{k}{(t-s)^{(n+1)/2}} \exp\left(-c\frac{|x-y|^2}{t-s}\right),$$

for all $x, y \in D$ and $0 \leqslant s < t \leqslant T$, where $k = k(n,\mu,T,D) > 0$ and $c = c(n,\mu,D) > 0$.

The inequality (ii) follows easily from the upper estimate in (i) and, depending on whether or not $d(x)^2 \geqslant t-s$, from the interior or boundary estimates for solutions to parabolic equations given by Theorems 4.8 and 4.27 in [21]. It is the reason that we require the Hölder continuity of $A(x,t)$.

(iii) *Rescaling property:* For $0 < r \leqslant 1$, let $A_r(x,t) = A(rx, r^2t)$ which is μ-Lipschitz, $L_r = \partial/\partial t - \operatorname{div}(A_r(x,t)\nabla_x)$ and G_r the L_r-Green function on $\Omega_r = (r^{-1}D) \times]0, r^{-2}T[$. Then

$$G_r(x,t;y,s) = r^n G_0(rx, r^2t; ry, r^2s)$$

for all $x, y \in r^{-1}D$, $0 < s < t < r^{-2}T$.

(iv) *Reproducing property:* The L-Green function G on Ω satisfies:

$$G(x,t;y,s) = \int_D G(x,t;\xi,\tau)G(\xi,\tau;y,s)\,\mathrm{d}\xi,$$

for all $x, y \in D$ and $s < \tau < t$ ([3] and [26]).

Note that the constant c in the inequality (ii) does not depend on T because of the inequality on $D \times]0,1[$ and the reproducing property. The choice of the Kato class exponent α will be determined only by means of c.

For simplicity we will also use, when we need, for $a > 0$, the notation

$$\Gamma_a(x,t;y,s) = \frac{1}{(t-s)^{n/2}} \exp\left(-a\frac{|x-y|^2}{t-s}\right),$$

for all $x, y \in \mathbb{R}^n$ and $t > s$.

$(\frac{a}{\pi})^{n/2}\Gamma_a$ is the fundamental solution of the operator $\frac{\partial}{\partial t} - \frac{1}{4a}\Delta_x$ on $\mathbb{R}^n \times \mathbb{R}$ and so it satisfies the reproducing property.

2. Bounds for the Green Function G

The main result of this section is the following.

THEOREM 2.1. *Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ and G the L-Green function for the initial Dirichlet problem on $D \times]0, T[$, $0 < T < \infty$. Then, there exist positive constants k, c_1 and c_2 depending only on n, μ, T, r_0, $d(D)$ and on B in terms of the rate of convergence of $N_h^\alpha(B)$ to zero as $h \to 0$, such that*

$$\frac{1}{k}\gamma_{c_2}(x, t; y, s) \leqslant G(x, t; y, s) \leqslant k\gamma_{c_1}(x, t; y, s),$$

for all x, $y \in D$, $0 \leqslant s < t \leqslant T$, where

$$\gamma_a(x, t; y, s) = \min\left(1, \frac{d(x)}{\sqrt{t-s}}, \frac{d(y)}{\sqrt{t-s}}, \frac{d(x)d(y)}{t-s}\right)\frac{\exp(-a\frac{|x-y|^2}{t-s})}{(t-s)^{n/2}}.$$

Proof. By translation with respect to time, we may assume $s = 0$. We divide the proof into two steps.

Step 1: $B \equiv 0$, i.e., $L \equiv L_0$ and then $G \equiv G_0$.

We first prove the upper bound. Let x, $y \in D$ and $t \in]0, T[$ be fixed. Since $G_0(z, t; y, 0) = 0$ for $z \in \partial D$ then by the mean value inequality and (ii), we have

$$\begin{aligned} G_0(x, t; y, 0) &\leqslant d(x)|\nabla_x G_0(\bar{x}, t; y, 0)| \\ &\leqslant k\frac{d(x)}{t^{(n+1)/2}}\exp\left(-c\frac{|\bar{x}-y|^2}{t}\right), \end{aligned}$$

where $|\bar{x} - x| \leqslant d(x)$.

By using the inequality

$$|\bar{x}-y|^2 \geqslant \frac{1}{2}|x-y|^2 - |\bar{x}-x|^2 \geqslant \frac{1}{2}|x-y|^2 - d^2(x),$$

it follows that

$$G_0(x, t; y, 0) \leqslant k\frac{d(x)}{t^{(n+1)/2}}\exp\left(-c\left(\frac{|x-y|^2}{2t} - \frac{d^2(x)}{t}\right)\right),$$

which yields

$$G_0(x, t; y, 0) \leqslant k\frac{d(x)}{t^{(n+1)/2}}\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right), \tag{1}$$

for all x, $y \in D$, $t \in]0, T[$ with $0 < d(x)/\sqrt{t} \leqslant 1$.

We conclude the following cases:

Case 1: $0 < d(x)/\sqrt{t} \leqslant 1$ and $d(y)/\sqrt{t} \geqslant 1$.

The upper bound follows from (1).

Case 2: $0 < d(x)/\sqrt{t} \leqslant 1$ and $0 < d(y)/\sqrt{t} \leqslant 1$.

By the reproducing property (iv), the inequality (1), we have

$$\begin{aligned} G_0(x,t;y,0) &= \int_D G_0(x,t;\xi,t/2)G_0(\xi,t/2;y,0)\,\mathrm{d}\xi \\ &\leqslant k^2\frac{d(x)d(y)}{t}\int_D \Gamma_{c/2}(x,t;\xi,t/2)\Gamma_{c/2}(\xi,t/2;y,0)\,\mathrm{d}\xi \\ &\leqslant k'\frac{d(x)d(y)}{t}\Gamma_{c/2}(x,t;y,0) \\ &= k'\frac{d(x)d(y)}{t^{n/2+1}}\exp\left(-\frac{c}{2}\frac{|x-y|^2}{t}\right). \end{aligned}$$

Case 3: $d(x)/\sqrt{t} \geqslant 1$ and $d(y)/\sqrt{t} \geqslant 1$.

From the upper estimate in (i), we have

$$G_0(x,t;y,0) \leqslant \Gamma_0(x,t;y,0) \leqslant \frac{k}{t^{n/2}}\exp\left(-c\frac{|x-y|^2}{t}\right)$$

which completes the proof of the upper bound.

We next prove the lower bound.

We first note that by dividing $]0,T[$ into intervals of length r_0^2 and using the reproducing property, it suffices to prove the lower estimate for $t \in\,]0,r_0^2]$.

Case 1: $d(x)/\sqrt{t} \geqslant 1$ and $d(y)/\sqrt{t} \geqslant 1$.

Subcase 1: $|x-y|/\sqrt{t} \leqslant 1/2$.

Let $\tilde{G}_0$ be the L_0-Green function on $B(x,\sqrt{t}) \times\,]0,T[\, \subset \Omega$. From Lemma 5.1 in [11], there is a constant $k = k(n,\mu) > 0$ such that

$$G_0(x,t;y,0) \geqslant \tilde{G}_0(x,t;y,0) \geqslant \frac{1}{kt^{n/2}}.$$

Subcase 2: $|x-y|/\sqrt{t} > 1/2$.

Since D is a $C^{1,1}$ bounded domain then we can easily show that there exists $\lambda_0 \geqslant 1$ depending only on D in terms of the ratio $d(D)/r_0$ and a parameterized curve $l \subset D$ connecting x and y with length $|l| \leqslant \lambda_0|x-y|$ and $d(l,\partial D) \geqslant \sqrt{t}$. Hence by following the proof of Theorem 2.7 in [11] and using Subcase 1, we obtain

$$G_0(x,t;y,0) \geqslant \frac{1}{kt^{n/2}}\exp\left(-c'\frac{|x-y|^2}{t}\right),$$

where $k = k(n,\mu) > 0$ and $c' = c'(n,\mu,d(D)/r_0) > 0$.

Case 2: $0 < d(x)/\sqrt{t} \leqslant 1$ and $d(y)/\sqrt{t} \geqslant 1$.

A point x in $\mathbb{R}^n$ will be denoted by (x', x_n), where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Without loss of generality we may assume $(0,0) \in \partial D$ and let $\psi_{(0,0)}$ be the $C^{1,1}$-function which defines the boundary of D around $(0,0)$. By definition $\nabla_{x'}\psi_{(0,0)}$ is c_0-Lipschitz.

Let $V_{(0,0)} = \{x \in \mathbb{R}^n : |x'| < r_0, |x_n| < r_0\}$.

Since for $y \in D$ with $d(y) > r_0$, $G_0(\cdot,\cdot; y, 0)$ is a positive L_0-solution on $(D \cap V_{(0,0)})\times]0, T[$ vanishing on $(\partial D \cap V_{(0,0)})\times]0, T[$ then by Corollary 2.8 in [15], there exists a constant $k = k(n, \mu, c_0) > 0$ such that

$$G_0(x, r_0^2; y, 0) \geqslant \frac{1}{k}\frac{d(x)}{r_0} G_0((0, r_0/2), r_0^2/2; y, 0), \tag{2}$$

for $x \in D \cap V_{(0,0)}$ and $y \in D$ with $d(y) > r_0$.

From (2), Case 1 and the inequality

$$|(0, r_0/2) - y|^2 \leqslant 2|(0, r_0/2) - x|^2 + 2|x - y|^2 \leqslant 9r_0^2/2 + 2|x - y|^2,$$

it follows that

$$G_0(x, r_0^2; y, 0) \geqslant \frac{1}{k'}\frac{d(x)}{r_0^{n+1}} \exp\left(-2c'\frac{|x - y|^2}{r_0^2}\right), \tag{3}$$

for $x \in V_{(0,0)}$ and $y \in D$ with $d(y) > r_0$.

Clearly, by a compactness argument, we obtain the inequality (3) for all $x, y \in D$ with $0 < d(x) < r_0$ and $d(y) > r_0$.

Since $k' = k'(n, \mu, c_0)$ and $c' = c'(n, \mu, d(D)/r_0)$, then by using the rescaling property (iii) with $r = \sqrt{t}/r_0$ and taking into account that the $C^{1,1}$-constant of $r^{-1}D$ is equal to $rc_0 \leqslant c_0$, it follows that

$$G_0(x, t; y, 0) \geqslant \frac{1}{k'}\frac{d(x)}{t^{(n+1)/2}} \exp\left(-2c'\frac{|x - y|^2}{t}\right)$$

for all $x, y \in D$, $t \in]0, r_0^2]$ with $0 < d(x) < \sqrt{t}$ and $d(y) > \sqrt{t}$.

Case 3: $0 < d(x)/\sqrt{t} \leqslant 1$ and $0 < d(y)/\sqrt{t} \leqslant 1$.

We know that there exists exactly one point $y_0 \in \partial D$ with $y = y_0 + d(y)n_{y_0}$ where n_{y_0} is the unique inner normal at y_0 with $|n_{y_0}| = 1$. Put $z_0 = y_0 + \frac{3}{2}\sqrt{t}n_{y_0}$. We have $B(z_0, \sqrt{t}/2) \subset B_1^{y_0} \subset D$ and so $B(z_0, \sqrt{t}/2) \subset D_{\sqrt{t}/2}$. Hence from the reproducing property (iv), Case 2 and the inequality $|x-\xi|^2 \leqslant 2|x-y|^2+2|y-\xi|^2$, we have

$$\begin{aligned}
&G_0(x, t; y, 0)\\
&\quad\geqslant \int_{D_{\sqrt{t}/2}} G_0(x, t; \xi, t/2)G_0(\xi, t/2; y, 0)\,\mathrm{d}\xi\\
&\quad\geqslant \frac{1}{k^2}\frac{d(x)d(y)}{t^{n+1}} \int_{D_{\sqrt{t}/2}} \exp\left(-2c'\left(\frac{|x - \xi|^2 + |y - \xi|^2}{t}\right)\right) \mathrm{d}\xi\\
&\quad\geqslant \frac{1}{k^2}\frac{d(x)d(y)}{t^{n+1}} \exp\left(-4c'\frac{|x - y|^2}{t}\right) \int_{B(z_0,\sqrt{t}/2)} \exp\left(-6c'\frac{|y - \xi|^2}{t}\right) \mathrm{d}\xi\\
&\quad\geqslant \frac{1}{k^2}\frac{d(x)d(y)}{t^{n+1}} \exp\left(-4c'\frac{|x - y|^2}{t}\right) \exp(-24c') \int_{B(z_0,\sqrt{t}/2)} \mathrm{d}\xi\\
&\quad= \frac{1}{k'}\frac{d(x)d(y)}{t^{n/2+1}} \exp\left(-4c'\frac{|x - y|^2}{t}\right).
\end{aligned}$$

It follows that

$$G_0(x,t;y,0) \geqslant \frac{1}{k'}\frac{d(x)d(y)}{t^{n/2+1}}\exp\left(-4c'\frac{|x-y|^2}{t}\right),$$

for all $x, y \in D$ and $t \in]0, r_0^2]$, satisfying $0 < d(x)/\sqrt{t} \leqslant 1$ and $0 < d(y)/\sqrt{t} \leqslant 1$.

Step 2: General case: $B \in$ Kato class.

In order to prove the estimates in this case we need the following lemmas.

LEMMA 2.2 (see [27]). *For $b > a > 0$ and $\alpha = \min(b-a, a/2)$, there exists a constant $C_{a,b} > 0$ depending only on a and b such that for $0 < t \leqslant h$, we have*

$$(1)\quad \int_0^t\int_{\mathbb{R}^n} \frac{\exp(-a\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}}|B(z,\tau)|\frac{\exp(-b\frac{|z-y|^2}{\tau})}{\tau^{(n+1)/2}}\,\mathrm{d}z\,\mathrm{d}\tau \leqslant C_{a,b}N_h^\alpha(B)\frac{\exp(-a\frac{|x-y|^2}{t})}{t^{n/2}},$$

$$(2)\quad \int_0^t\int_{\mathbb{R}^n} \frac{\exp(-a\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{(n+1)/2}}|B(z,\tau)|\frac{\exp(-b\frac{|z-y|^2}{\tau})}{\tau^{(n+1)/2}}\,\mathrm{d}z\,\mathrm{d}\tau \leqslant C_{a,b}N_h^\alpha(B)\frac{\exp(-a\frac{|x-y|^2}{t})}{t^{(n+1)/2}}.$$

LEMMA 2.3. *There exists a constant $k > 0$ depending only on n, μ, T and D such that*

$$|\nabla_z G_0(z,\tau;y,0)| \leqslant k\min\left(1, \frac{d(y)}{\sqrt{\tau}}\right)\frac{\exp(-\frac{c}{2}\frac{|z-y|^2}{\tau})}{\tau^{(n+1)/2}},$$

for all $y, z \in D$ and $\tau \in]0, T[$.

Proof. From the reproducing property (iv), and the inequalities (i) and (ii), we have

$$\begin{aligned}|\nabla_z G_0(z,\tau;y,0)| &\leqslant \int_D |\nabla_z G_0(z,\tau;\xi,\tau/2)|G_0(\xi,\tau/2;y,0)\,\mathrm{d}\xi\\ &\leqslant 2k^2\int_D \frac{1}{\tau^{1/2}}\Gamma_c(z,\tau;\xi,\tau/2)\frac{d(y)}{\tau^{1/2}}\Gamma_{c/2}(\xi,\tau/2;y,0)\,\mathrm{d}\xi\\ &\leqslant k'\frac{d(y)}{\tau}\Gamma_{c/2}(z,\tau;y,0)\\ &= k'd(y)\frac{\exp(-\frac{c}{2}\frac{|z-y|^2}{\tau})}{\tau^{n/2+1}}.\end{aligned}$$

Combining the last inequality and (ii), we obtain the inequality given in the lemma. □

Now we are ready to prove the estimates of Theorem 2.1. By dividing $]0, T[$ into intervals of length h and using the reproducing property it suffices to prove the estimates for $t \in]0, h]$, where h is a small positive number. From [26] we know that G satisfies the integral equation:

$$\begin{aligned} &G(x, t; y, 0) \\ &\quad = G_0(x, t; y, 0) - \int_0^t \int_D G(x, t; z, \tau) B(z, \tau) \nabla_z G_0(z, \tau; y, 0)\, \mathrm{d}z\, \mathrm{d}\tau \\ &\quad \equiv G_0(x, t; y, 0) - G * (B \nabla G_0)(x, t; y, 0) \end{aligned}$$

for all $x, y \in D$ and $0 < t < T$.

By iteration we obtain

$$G(x, t; y, 0) = \sum_{m=0}^{+\infty} G_0 * (-B \nabla G_0)^{*m}(x, t; y, 0) \equiv \sum_{m=0}^{+\infty} J_m(x, t; y, 0). \tag{4}$$

We will show by recurrence that there exists a constant $k' > 0$ such that

$$|J_m(x, t; y, 0)| \leqslant k \big(k' N_h^{c/8}(B)\big)^m \gamma_{c/4}(x, t; y, 0), \tag{5}$$

for all $x, y \in D$ and $0 < t \leqslant h$.

In view of the formula $J_{m+1} = J_m * (-B \nabla G_0)$ with $J_0 = G_0 \leqslant k \gamma_{c/4}$ by Step 1, it is sufficient to prove that $\gamma_{c/4} * |B \nabla G_0| \leqslant k' N_h^{c/8}(B) \gamma_{c/4}$.

We have

$$\begin{aligned} &\gamma_{c/4} * |B \nabla G_0|(x, t; y, 0) \\ &\quad = \int_0^t \int_D \gamma_{c/4}(x, t; z, \tau) |B(z, \tau)| |\nabla_z G_0(z, \tau; y, 0)|\, \mathrm{d}z\, \mathrm{d}\tau \\ &\quad = \int_0^{t/2} \int_D \ldots\, \mathrm{d}z\, \mathrm{d}\tau + \int_{t/2}^t \int_D \ldots\, \mathrm{d}z\, \mathrm{d}\tau \\ &\quad \equiv I_1 + I_2. \end{aligned} \tag{6}$$

By Lemma 2.3, we have

$$\begin{aligned} &I_1(x, t; y, 0) \\ &\quad \leqslant k \int_0^{t/2} \int_D \min\bigg(1, \frac{d(x)}{\sqrt{t-\tau}}\bigg) \min\bigg(1, \frac{d(z)}{\sqrt{t-\tau}}\bigg) \frac{\exp(-\frac{c}{4} \frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}} \times \\ &\qquad \times |B(z, \tau)| \min\bigg(1, \frac{d(y)}{\sqrt{\tau}}\bigg) \frac{\exp(-\frac{c}{2} \frac{|z-y|^2}{\tau})}{\tau^{(n+1)/2}}\, \mathrm{d}z\, \mathrm{d}\tau. \end{aligned} \tag{7}$$

On the other hand by using the inequality $d(z) \leqslant d(y) + |z - y|$, we have

$$\min\bigg(1, \frac{d(z)}{\sqrt{t-\tau}}\bigg) \leqslant \min\bigg(1, \frac{d(z)}{d(y)} \frac{d(y)}{\sqrt{t-\tau}}\bigg)$$

$$\leqslant \min\left(1, \left(1 + \frac{|z-y|}{d(y)}\right)\frac{d(y)}{\sqrt{t-\tau}}\right)$$
$$\leqslant \left(1 + \frac{|z-y|}{d(y)}\right)\min\left(1, \frac{d(y)}{\sqrt{t-\tau}}\right),$$

which yields

$$\min\left(1, \frac{d(z)}{\sqrt{t-\tau}}\right)\min\left(1, \frac{d(y)}{\sqrt{\tau}}\right) \leqslant \left(1 + \frac{|z-y|}{\sqrt{\tau}}\right)\min\left(1, \frac{d(y)}{\sqrt{t-\tau}}\right). \tag{8}$$

Combining (7) and (8), using the inequality $(1+\theta)\mathrm{e}^{-\frac{c}{8}\theta^2} \leqslant 1 + (\frac{2}{c})^{1/2}$ and (1) of Lemma 2.2, we obtain

$$\begin{aligned}
I_1(x,t;y,0) &\leqslant k\int_0^{t/2}\int_D \min\left(1, \frac{d(x)}{\sqrt{t-\tau}}\right) \times \\
&\quad \times \min\left(1, \frac{d(y)}{\sqrt{t-\tau}}\right)\frac{\exp(-\frac{c}{4}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}} \times \\
&\quad \times |B(z,\tau)|\left(1 + \frac{|z-y|}{\sqrt{\tau}}\right)\frac{\exp(-\frac{c}{2}\frac{|z-y|^2}{\tau})}{\tau^{(n+1)/2}}\,\mathrm{d}z\,\mathrm{d}\tau \\
&\leqslant 2k\left(1 + \left(\frac{2}{c}\right)^{1/2}\right)\min\left(1, \frac{d(x)}{\sqrt{t}}\right)\min\left(1, \frac{d(y)}{\sqrt{t}}\right) \times \\
&\quad \times \int_0^{t/2}\int_D \frac{\exp(-\frac{c}{4}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}}|B(z,\tau)|\frac{\exp(-\frac{3c}{8}\frac{|z-y|^2}{\tau})}{\tau^{(n+1)/2}}\,\mathrm{d}z\,\mathrm{d}\tau \\
&\leqslant 2k\left(1 + \left(\frac{2}{c}\right)^{1/2}\right)CN_h^{c/8}(B)\gamma_{c/4}(x,t;y,0),
\end{aligned} \tag{9}$$

for $x, y \in D$ and $0 < t \leqslant h$.

Now we estimate I_2. From Lemma 2.3 and (2) of Lemma 2.2, we have

$$\begin{aligned}
I_2(x,t;y,0) &= \int_{t/2}^{t}\int_D \gamma_{c/4}(x,t;z,\tau)|B(z,\tau)||\nabla_z G_0(z,\tau;y,0)|\,\mathrm{d}z\,\mathrm{d}\tau \\
&\leqslant k\int_{t/2}^{t}\int_D \min\left(1, \frac{d(x)}{\sqrt{t-\tau}}\right)\frac{\exp(-\frac{c}{4}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}} \times \\
&\quad \times |B(z,\tau)|\min\left(1, \frac{d(y)}{\sqrt{\tau}}\right)\frac{\exp(-\frac{c}{2}\frac{|z-y|^2}{\tau})}{\tau^{(n+1)/2}}\,\mathrm{d}z\,\mathrm{d}\tau \\
&\leqslant 2k\min\left(1, \frac{d(x)}{\sqrt{t}}\right)\min\left(1, \frac{d(y)}{\sqrt{t}}\right)\sqrt{t} \times \\
&\quad \times \int_{t/2}^{t}\int_D \frac{\exp(-\frac{c}{4}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{(n+1)/2}}|B(z,\tau)|\frac{\exp(-\frac{c}{2}\frac{|z-y|^2}{\tau})}{\tau^{(n+1)/2}}\,\mathrm{d}z\,\mathrm{d}\tau \\
&\leqslant 2kCN_h^{c/8}(B)\gamma_{c/4}(x,t;y,0),
\end{aligned} \tag{10}$$

for $x, y \in D$ and $0 < t \leqslant h$.

Combining (6), (9) and (10), we then have

$$\gamma_{c/4} * |B\nabla G_0|(x,t;y,0) \leqslant kk' N_h^{c/8}(B)\gamma_{c/4}(x,t;y,0),$$

for $x, y \in D$ and $0 < t \leqslant h$.

By going back to (4) and (5) and choosing h sufficiently small so that $k' N_h^{c/8}(B) \leqslant 1/2$, we obtain

$$G(x,t;y,0) \leqslant 2k\gamma_{c/4}(x,t;y,0),$$

for all $x, y \in D$ and $0 < t \leqslant h$.

We next prove the lower bound. From (4), we have

$$G(x,t;y,0) - G_0(x,t;y,0) = \sum_{m=1}^{+\infty} J_m(x,t;y,0),$$

and then by (5), it follows that

$$|G(x,t;y,0) - G_0(x,t;y,0)| \leqslant kk' N_h^{c/8}(B)\gamma_{c/4}(x,t;y,0),$$

for $x, y \in D$ and $0 < t \leqslant h$.

By recalling that $G_0 \geqslant \frac{1}{k}\gamma_{c_2}$, we deduce

$$\begin{aligned} G(x,t;y,0) \ &\geqslant\ \min\left(1, \frac{d(x)}{\sqrt{t}}, \frac{d(y)}{\sqrt{t}}, \frac{d(x)d(y)}{t}\right)\frac{1}{t^{n/2}}\ \times \\ &\times \left[\frac{1}{k}\exp\left(-c_2\frac{|x-y|^2}{t}\right) - kk' N_h^{c/8}(B)\right] \end{aligned}$$

for all $x, y \in D$ and $0 < t \leqslant h$.

Then, for h so small that $k^2 k' \mathrm{e}^{c_2} N_h^{c/8}(B) \leqslant 1/2$, we obtain

$$G(x,t;y,0) \geqslant \frac{\mathrm{e}^{-c_2}}{2k}\min\left(1, \frac{d(x)}{\sqrt{t}}, \frac{d(y)}{\sqrt{t}}, \frac{d(x)d(y)}{t}\right)\frac{1}{t^{n/2}}, \tag{11}$$

for all $x, y \in D$ and $0 < t \leqslant h$ with $|x-y|^2/t \leqslant 1$.

Now, to prove the lower bound we first consider $x, y \in D_{\sqrt{t}}$. By following the proof of Theorem 2.7 in [11] and using (11), we obtain the existence of a constant $c' = c'(n, \mu, r_0, d(D)) > 0$ such that

$$G(x,t;y,0) \geqslant \frac{\mathrm{e}^{-c_2}}{2kt^{n/2}}\exp\left(-c'\frac{|x-y|^2}{t}\right), \tag{12}$$

for all $x, y \in D$ and $0 < t \leqslant h$.

For the general case, when x, y are arbitrary inside D, let $x_0, y_0 \in D_{\sqrt{t}}$ such that $|x - x_0| \leqslant \sqrt{t}$ and $|y - y_0| \leqslant \sqrt{t}$.

From (11) and (12), we have

$$\begin{aligned} G(x,t;y,0) &\geqslant \int_{B(x_0,\sqrt{t}/2)}\int_{B(y_0,\sqrt{t}/2)} G\Big(x,t;\xi_1,\frac{2t}{3}\Big)\times \\ &\quad \times G\Big(\xi_1,\frac{2t}{3};\xi_2,\frac{t}{3}\Big)G\Big(\xi_2,\frac{t}{3};y,0\Big)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &\geqslant \frac{1}{kt^{3n/2}}\min\Big(1,\frac{d(x)}{\sqrt{t}}\Big)\min\Big(1,\frac{d(y)}{\sqrt{t}}\Big)\times \\ &\quad \times \int_{B(x_0,\sqrt{t}/2)}\int_{B(y_0,\sqrt{t}/2)} \exp\Big(-c'\frac{|\xi_1-\xi_2|^2}{t}\Big)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \end{aligned}$$

Since

$$\begin{aligned} |\xi_1-\xi_2|^2 &\leqslant (|\xi_1-x|+|x-y|+|y-\xi_2|)^2 \\ &\leqslant (3\sqrt{t}+|x-y|)^2 \leqslant 18t+2|x-y|^2, \end{aligned}$$

then, we obtain

$$\begin{aligned} &G(x,t;y,0) \\ &\quad\geqslant \frac{1}{k}\Big(\frac{\sqrt{t}}{2}\Big)^{2n} w_n^2\min\Big(1,\frac{d(x)}{\sqrt{t}},\frac{d(y)}{\sqrt{t}},\frac{d(x)d(y)}{t}\Big)\frac{\exp(-2c'\frac{|x-y|^2}{t})}{t^{3n/2}} \\ &\quad= \frac{1}{k'}\gamma_{2c'}(x,t;y,0), \end{aligned}$$

for all $x, y \in D$ and $0 < t \leqslant h$, which ends the proof. □

The following is a simple consequence of Theorem 2.1.

COROLLARY 2.4. *There exists a constant $k' > 0$ depending only on n, μ, T, D and on B in terms of the rate of convergence of $N_h^{c/8}(B)$ to zero as $h \to 0$ such that*

$$\frac{1}{k'}G_{c_1/c_2} \leqslant G \leqslant k' G_{c_2/c_1},$$

where for $a > 0$, G_a denotes the Green function of $\frac{\partial}{\partial t} - a\Delta_x$ on Ω.

We also deduce the following estimate on the gradient.

COROLLARY 2.5. *There exists a constant $k' > 0$ depending only on n, μ, T, D and on B in terms of the rate of convergence of $N_h^{c/8}(B)$ to zero as $h \to 0$ such that*

$$|\nabla_x G(x,t;y,s)| \leqslant k\min\Big(1,\frac{d(y)}{\sqrt{t-s}}\Big)\frac{\exp(-\frac{c_1}{2}\frac{|x-y|^2}{t-s})}{(t-s)^{(n+1)/2}},$$

for all $x, y \in D$ and $0 < s < t < T$.

Proof. The estimate follows immediately from the equality

$$\nabla_x G(x,t;y,0) = \sum_{m=0}^{+\infty} \nabla_x G_0 * (-B\nabla G_0)^{*m}(x,t;y,0),$$

by using (ii), (2) of Lemma 2.2, the inequality $G \leqslant k\gamma_{c_1}$ and the argument given in Lemma 2.3. □

REMARKS 2.6. (1) From the proof we can see that the constants c_1 and c_2 which occur in Theorem 2.1 depend only on $n, \mu, r_0, d(D)$ and do not depend on T and B in any way. The independence on T is also clear from the estimates on $D \times]0,1[$ and the reproducing property.

(2) Since A is $C^{0,1}$ with respect to the space variables, we can also write L in the non-divergence form:

$$L = \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + (B+B')(x,t)\cdot\nabla_x,$$

where $B' = (B'_1,\dots,B'_n)$ with $B'_j = -\sum_{i=1}^n \frac{\partial a_{ij}}{\partial x_i} \in L^\infty(\Omega)$.

From this observation we see that the estimates on the Green function are valid for the operators in the non-divergence form as well.

(3) Although the adjoint operator $L^* = -\frac{\partial}{\partial t} - \mathrm{div}(A(x,t)\nabla_x) - \mathrm{div}(B(x,t).)$ has a different structure, by symmetry with respect to x and y, the estimates in Theorem 2.1 are also valid for the L^*-Green function $G^*(x,t;y,s) = G(y,s;x,t)$.

(4) When the domain D is only Lipschitz, Theorem 2.1 may fail to hold. This is clear from the following example. Let

$$D = \{x = (x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < x_1\}.$$

Fix $y \in D$ and let $V = D \cap B(0,r)$ with y does not belong to $\overline{B}(0,r)$. Consider the parabolic operators

$$L_1 = \frac{\partial}{\partial t} - \Delta_x, \qquad L_2 = \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2^2}\right),$$

and the functions

$$u_1(x) = x_1^3 x_2 - x_1 x_2^3, \qquad u_2(x) = x_1^2 x_2 - x_1 x_2^2.$$

The function u_i is a positive L_i-solution on $\Omega = D \times (0,\infty)$ for $i = 1, 2$. Let G_i denote the L_i-Green function on Ω for $i = 1, 2$. From the local comparison theorem (Theorem 1.6 in [12] or Theorem 1.4 in [15]), it follows that there exist two positive constants k_1 and k_2 such that, for $t > 0$ and r small be fixed, we have

$$\frac{1}{k_1} \leqslant \frac{u_1(x,t)}{G_1(x,t;y,0)} \leqslant k_1, \quad \text{for all } x \in V,$$

$$\frac{1}{k_2} \leqslant \frac{u_2(x,t)}{G_2(x,t;y,0)} \leqslant k_2, \quad \text{for all } x \in V.$$

Therefore

$$\frac{1}{k} \leqslant \frac{u_2(x,t)}{u_1(x,t)} \frac{G_1(x,t;y,0)}{G_2(x,t;y,0)} \leqslant k, \quad \text{for all } x \in V.$$

Suppose that the estimates of Theorem 2.1 are true, then

$$\frac{1}{k'} \exp\left(-c' \frac{|x-y|^2}{t}\right) \leqslant \frac{G_1(x,t;y,0)}{G_2(x,t;y,0)} \leqslant k' \exp\left(c' \frac{|x-y|^2}{t}\right), \quad \text{for all } x \in D.$$

The previous two-sided inequalities now imply that u_2/u_1 is bounded near zero, which is a contradiction.

(5) In general, the estimates of Theorem 2.1 are not global in time. This is clear from the following simple example. Consider $L = \partial/\partial t - \Delta_x - u \cdot \nabla_x$, where $u \in \mathbb{R}^n$ be fixed and $\Omega = B(0,1) \times (0,\infty)$. Denote by G the L-Green function on Ω. The L-fundamental solution is given by

$$\Gamma(x,t;y,0) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y+tu|^2}{4t}\right)$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$.

Suppose that there is a time global lower bound; then by recalling that $G \leqslant \Gamma$, it follows that

$$\frac{1}{k} \min\left(1, \frac{d(x)}{\sqrt{t}}\right) \min\left(1, \frac{d(y)}{\sqrt{t}}\right) \frac{\exp(-c_2 \frac{|x-y|^2}{t} s)}{t^{n/2}} \leqslant \Gamma(x,t;y,0)$$

for all $t > 0$ and $x, y \in B(0,1)$.

If we choose u, x, y such that $x = y \neq 0$ and $|u| = 1$, then we find

$$\min\left(1, \frac{d^2(x)}{t}\right) \leqslant k \exp\left(-\frac{t}{4}\right)$$

for all $t > 0$, which is a contradiction.

3. Estimates of the Poisson Kernel and Parabolic Measures

Let L be the parabolic operator introduced in Section 1 on the cylinder Ω. We will prove the equivalence of the L-parabolic measure, the L^*-parabolic measure and the surface measure on the lateral boundary $\partial D \times]0, T[$ of Ω. In particular, we present a simple proof based on the Green function estimates (Theorem 2.1). We first introduce the definition of the L-parabolic measure. From [9] and a limiting argument given in [26], for any $\varphi \in C(\partial_p\Omega)$, there exists a unique solution $u = H_\varphi^\Omega$ of the Dirichlet problem $Lu = 0$ on Ω and $u_{/\partial_p\Omega} = \varphi$. For all $M \in \Omega$, the map $\varphi \to H_\varphi^\Omega(M)$ is a linear positive continuous functional on $C(\partial_p\Omega)$ and so by the Riesz representation theorem there exists a unique Borel measure μ_M on $\partial_p\Omega$ such that

$$H_\varphi^\Omega(M) = \int_{\partial_p\Omega} \varphi(\xi)\,\mathrm{d}\mu_M(\xi).$$

μ_M will be called the L-parabolic measure at M. The L^*-parabolic measure μ_M^* at M is defined in a similar way. To establish our main result, we first prove a two-sided estimate for the L-Martin–Poisson kernel P. The existence and uniqueness of the L-Martin–Poisson kernel on Ω hold by using the Green function bounds (Theorem 2.1), the Harnack inequality (Theorem 1.1 in [26]) and by closely following the arguments in [23] (for all details we refer the reader to [23]). We have the following.

THEOREM 3.1. *The L-Poisson kernel P on Ω satisfies the following estimates: there exists a constant $k > 0$ depending only on n, μ, D, T and on B in terms of the rate of convergence of $N_h^{c/8}(B)$ to zero as $h \to 0$, such that*

$$\frac{1}{k}\min\left(1, \frac{d(x)}{\sqrt{t-s}}\right)\frac{\exp(-c_2\frac{|x-Q|^2}{t-s})}{(t-s)^{(n+1)/2}} \leqslant P(x,t;Q,s) \leqslant k\min\left(1, \frac{d(x)}{\sqrt{t-s}}\right)\frac{\exp(-c_1\frac{|x-Q|^2}{t-s})}{(t-s)^{(n+1)/2}},$$

for all $x \in D$, $Q \in \partial D$, $0 \leqslant s < t \leqslant T$.

Proof. Since Ω is of $C^{1,1}$-boundary then by the divergence theorem we have, for all $x \in D$, $Q \in \partial D$ and $0 \leqslant s < t \leqslant T$,

$$P(x,t;Q,s) = \frac{\partial G}{\partial N_{(Q,s)}}(x,t;Q,s),$$

where $N_{(Q,s)} = A(Q,s)n_Q$ with n_Q is the unit inner normal to ∂D at Q.

We write

$$N_{(Q,s)} = T_{(Q,s)} + a(Q,s)n_Q,$$

where $T_{(Q,s)}$ is a tangential vector field to ∂D at Q. Therefore

$$\frac{\partial G}{\partial N_{(Q,s)}}(x,t;Q,s) = \frac{\partial G}{\partial T_{(Q,s)}}(x,t;Q,s) + a(Q,s)\frac{\partial G}{\partial n_Q}(x,t;Q,s).$$

Since $G(x,t;\cdot,s) = 0$ on ∂D, then

$$\frac{\partial G}{\partial T_{(Q,s)}}(x,t;Q,s) \equiv \frac{\partial}{\partial T_{(Q,s)}}(G(x,t;\cdot,s))(Q) = 0,$$

and so

$$\begin{aligned} P(x,t;Q,s) &= a(Q,s)\frac{\partial G}{\partial n_Q}(x,t;Q,s) \\ &\equiv a(Q,s)\lim_{r\to 0^+}\frac{G(x,t;Q+rn_Q,s)}{r}. \end{aligned} \tag{13}$$

For $r > 0$ small we have $d(Q+rn_Q) = r$ and so from the estimates in Theorem 2.1, it follows that,

$$\begin{aligned} &\frac{1}{k}\min\left(1,\frac{d(x)}{\sqrt{t-s}}\right)\frac{\exp(-c_2\frac{|x-Q-rn_Q|^2}{t-s})}{(t-s)^{(n+1)/2}} \\ &\quad\leqslant \frac{G(x,t;Q+rn_Q,s)}{r} \\ &\quad\leqslant k\min\left(1,\frac{d(x)}{\sqrt{t-s}}\right)\frac{\exp(-c_1\frac{|x-Q-rn_Q|^2}{t-s})}{(t-s)^{(n+1)/2}}. \end{aligned} \tag{14}$$

By noting that $1/\mu \leqslant a(Q,s) = \langle N_{(Q,s)}, n_Q\rangle \leqslant \mu$, combining (13) and (14) and letting r to zero, we get the estimates stated in Theorem 3.1. □

For $(Q,s) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, let $T_r(Q,s)$ be the cylinder

$$T_r(Q,s) = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x-Q| < r, |t-s| < r^2\right\}.$$

Let σ be the surface measure on the lateral boundary $\partial D \times]0,T[$ of Ω. We deduce the following result.

COROLLARY 3.2. *Let* $(Q_0,s_0) \in \partial D \times]0,T[$ *such that* $M_0 = (Q_0,s_0) + (r_0 n_{Q_0}, 2r_0^2) \in \Omega$, $M_0^* = (Q_0,s_0) + (r_0 n_{Q_0}, -2r_0^2) \in \Omega$ *and set* $F = (\partial D \times]0,T[) \cap T_{r_0}(Q_0,s_0)$. *Then there exists a constant* $k > 0$ *depending only on* n, μ, r_0, $d(D)$, T *and on* B *in terms of the rate of convergence of* $N_h^{c/8}(B)$ *to zero as* $h \to 0$, *such that*

$$\frac{1}{k}\sigma_{/F} \leqslant \mu_{M_0/F} \leqslant k\sigma_{/F} \quad \textit{and} \quad \frac{1}{k}\sigma_{/F} \leqslant \mu^*_{M_0^*/F} \leqslant k\sigma_{/F}.$$

Proof. For any nonnegative continuous function $f : \partial D \times]0,T[\to \mathbb{R}$, the potential

$$u(x,t) = \int_0^t\int_{\partial D} P(x,t;Q,s)f(Q,s)\,\mathrm{d}\sigma(Q,s)$$

represents the L-solution of the Dirichlet problem with boundary data f on $\partial D\times]0,T[$ and zero on $\overline{D} \times \{0\}$.

Set $M_0 = (x_0, t_0) \equiv (Q_0, s_0) + (r_0 n_{Q_0}, 2r_0^2)$. By the estimates on the Poisson kernel in Theorem 3.1, for any Borel subset $A \subset F$, we then have

$$\begin{aligned}\mu_{M_0}(A) &= \iint_A P(x_0, t_0; Q, s)\,\mathrm{d}\sigma(Q, s)\\ &\leqslant k \iint_A \frac{\exp(-c_1\frac{|x_0-Q|^2}{t_0-s})}{(t_0-s)^{(n+1)/2}}\,\mathrm{d}\sigma(Q, s)\\ &\leqslant \frac{k}{r_0^{n+1}}\sigma(A).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mu_{M_0}(A) &= \iint_A P(x_0, t_0; Q, s)\,\mathrm{d}\sigma(Q, s)\\ &\geqslant \frac{1}{k}\iint_A \min\left(1, \frac{d(x_0)}{\sqrt{t_0-s}}\right)\frac{\exp(-c_2\frac{|x_0-Q|^2}{t_0-s})}{(t_0-s)^{(n+1)/2}}\,\mathrm{d}\sigma(Q, s).\end{aligned}$$

Since by the assumptions

$$\frac{|x_0-Q|^2}{t_0-s} \leqslant \frac{(|x_0-Q_0|+|Q_0-Q|)^2}{t_0-s} \leqslant \frac{(2r_0)^2}{r_0^2} = 4$$

and

$$\frac{d(x_0)}{\sqrt{t_0-s}} \geqslant \frac{r_0}{\sqrt{2r_0^2}} = \frac{1}{\sqrt{2}},$$

then it follows that

$$\mu_{M_0}(A) \geqslant \frac{1}{k(2r_0)^{n+1}}\frac{\mathrm{e}^{-4c_2}}{\sqrt{2}}\sigma(A).$$

In a similar way we prove that

$$\frac{1}{k}\sigma_{/F} \leqslant \mu^*_{M_0^*/F} \leqslant k\sigma_{/F}. \qquad \square$$

REMARK 3.3. Since the operator L could be written in the non-divergence form as is stated in Remark 2.6(2), then the previous results are also valid for such operators. The concept of parabolic measures in the non-divergence form case is introduced by Garofalo in [13] using an elementary barrier technique.

4. Applications to the Elliptic Operators

In this section we apply the estimates in Theorem 2.1 to obtain the counterpart results for the elliptic operator $\mathcal{L} = \operatorname{div}(A(x)\nabla_x) + B(x)\cdot\nabla_x$ on the $C^{1,1}$-bounded domain D in $\mathbb{R}^n$, $n \geqslant 3$ with matrix $A(x) = (a_{ij}(x))_{1\leqslant i,j\leqslant n}$ uniformly elliptic and μ-Lipschitz continuous, and vector $B = B(x)$ in the elliptic Kato class, i.e., $B \in L^1_{\mathrm{loc}}(D)$ and satisfies

$$\lim_{r\to 0} \sup_{x\in D} \int_{D\cap(|x-y|<r)} \frac{|B(y)|}{|x-y|^{n-1}}\,\mathrm{d}y = 0.$$

We point out that $\|B\| \equiv \sup_{x\in D} \int_D \frac{|B(y)|}{|x-y|^{n-1}}\,\mathrm{d}y < \infty$. Let g denotes the $\mathcal{L}$-Green function on D. We have the following estimates.

THEOREM 4.1. *Let B in the elliptic Kato class with $\|B\| \leqslant C_0$, for some suitable constant C_0. Then, there exists a constant $k > 0$ depending only on n, μ, r_0, $d(D)$ and $\|B\|$ such that*

$$\frac{1}{k|x-y|^{n-2}}\varphi(x,y) \leqslant g(x,y) \leqslant \frac{k}{|x-y|^{n-2}}\varphi(x,y),$$

for all $x, y \in D$, where $\varphi(x,y) = \min(1, \frac{d(x)}{|x-y|}, \frac{d(y)}{|x-y|}, \frac{d(x)d(y)}{|x-y|^2})$.

Proof. Let $L = \partial/\partial t - \mathrm{div}(A(x)\nabla_x) - B(x)\cdot\nabla_x$ and G its Green function on $D\times(0,\infty)$. Then $g(x,y) = \int_0^\infty G(x,t;y,0)\,\mathrm{d}t$.

By integrating with respect to time the lower bound in Theorem 2.1, we obtain

$$\frac{1}{k}\int_0^1 \min\left(1, \frac{d(x)}{\sqrt{t}}, \frac{d(y)}{\sqrt{t}}, \frac{d(x)d(y)}{t}\right)\frac{\exp(-c_2\frac{|x-y|^2}{t})}{t^{n/2}}\,\mathrm{d}t \leqslant g(x,y).$$

Making the change of variable $r = |x-y|^2/t$, it follows that

$$\frac{1}{k|x-y|^{n-2}}\int_{|x-y|^2}^\infty \min\left(1, \frac{r^{1/2}d(x)}{|x-y|}, \frac{r^{1/2}d(y)}{|x-y|}, r\frac{d(x)d(y)}{|x-y|^2}\right) r^{n/2-2}\mathrm{e}^{-c_2 r}\,\mathrm{d}r \\ \leqslant g(x,y).$$

This implies

$$\frac{1}{k|x-y|^{n-2}}\varphi(x,y)\int_{d(D)^2}^\infty r^{n/2-2}\mathrm{e}^{-c_2 r}\,\mathrm{d}r \leqslant g(x,y),$$

and so

$$\frac{1}{k'|x-y|^{n-2}}\varphi(x,y) \leqslant g(x,y)$$

for all $x, y \in D$, where $k' = k'(n,\mu,r_0,d(D)) > 0$, which proves the lower bound.

On the other hand from the reproducing property (iv), the upper bound in Theorem 2.1 and the global upper bound in Corollary 1.1 of [27], we have for all $t > 1$,

$$\begin{aligned}
G(x,t;y,0) &= \int_D G(x,t;\xi,t-1)G(\xi,t-1;y,0)\,\mathrm{d}\xi\\
&= \int_D G(x,1;\xi,0)G(\xi,t-1;y,0)\,\mathrm{d}\xi\\
&\leqslant \int_D k_1 d(x)\Gamma_{c_1}(x,1;\xi,0)k\Gamma_c(\xi,t-1;y,0)\,\mathrm{d}\xi\\
&= k_1 k d(x)\int_D \Gamma_{c_1}(x,t;\xi,t-1)\Gamma_c(\xi,t-1;y,0)\,\mathrm{d}\xi\\
&\leqslant k' d(x)\Gamma_{c_1}(x,t;y,0),
\end{aligned}$$

where k' is independent of t.

Using again the reproducing property, we also obtain

$$G(x,t;y,0)\leqslant k' d(x)d(y)\frac{\exp(-c_1\frac{|x-y|^2}{t})}{t^{n/2}}.$$

Then, we have, for all $t>1$,

$$G(x,t;y,0)\leqslant k'\min(1,d(x),d(y),d(x)d(y))\frac{\exp(-c_1\frac{|x-y|^2}{t})}{t^{n/2}}. \tag{15}$$

From (15) and the upper bound in Theorem 2.1, we have

$$\begin{aligned}
&g(x,y)\\
&\leqslant k\int_0^1 \min\left(1,\frac{d(x)}{\sqrt{t}},\frac{d(y)}{\sqrt{t}},\frac{d(x)d(y)}{t}\right)\frac{\exp(-c_1\frac{|x-y|^2}{t})}{t^{n/2}}\,\mathrm{d}t+\\
&\quad + k'\min(1,d(x),d(y),d(x)d(y))\int_1^\infty \frac{\exp(-c_1\frac{|x-y|^2}{t})}{t^{n/2}}\,\mathrm{d}t\\
&= \frac{k}{|x-y|^{n-2}}\int_{|x-y|^2}^\infty \min\left(1,\frac{r^{1/2}d(x)}{|x-y|},\frac{r^{1/2}d(y)}{|x-y|},\frac{rd(x)d(y)}{|x-y|^2}\right)r^{n/2-2}\mathrm{e}^{-c_1 r}\,\mathrm{d}r+\\
&\quad + k'\min(1,d(x),d(y),d(x)d(y))\int_1^\infty \frac{\exp(-c_1\frac{|x-y|^2}{t})}{t^{n/2}}\,\mathrm{d}t\\
&\leqslant k\int_0^\infty \mathrm{e}^{-\frac{c_1}{2}r}\,\mathrm{d}r\frac{1}{|x-y|^{n-2}}\varphi(x,y)+\\
&\quad + k'\min(1,d(x),d(y),d(x)d(y))\frac{1}{|x-y|^{n-2}}\\
&\leqslant \frac{k''}{|x-y|^{n-2}}\varphi(x,y),
\end{aligned}$$

which proves the upper bound. □

REMARK 4.2. The same upper estimate was first proved by Widman in [24] for the Laplacian Green function on Liapunov–Dini domains and later extended by Grüter and Widman in [14] to elliptic operator in divergence form with Dini continuous coefficients. The lower estimate was first proved by Zhao in [28] for the Laplacian Green function on bounded $C^{1,1}$-domains and extended by Hueber in [17] to elliptic operators with bounded Hölder continuous coefficients.

Let p denotes the $\mathcal{L}$-Poisson kernel on D. From Theorem 4.1 we derive the following estimates for p.

COROLLARY 4.3. *Let B in the elliptic Kato class with $\|B\| \leqslant C_0$, for some suitable constant C_0. Then, there exists a constant $k > 0$ depending only on n, μ, r_0, $d(D)$ and $\|B\|$ such that*

$$\frac{1}{k}\frac{d(x)}{|x-Q|^n} \leqslant p(x,Q) \leqslant k\frac{d(x)}{|x-Q|^n},$$

for all $x \in D$ and $Q \in \partial D$.

Proof. As in the proof of Theorem 3.1, the estimates hold by using that

$$p(x,Q) = a(Q)\frac{\partial g}{\partial n_Q}(x,Q) \equiv a(Q)\lim_{r\to 0^+}\frac{g(x,Q+rn_Q)}{r},$$

where $1/\mu \leqslant a(Q) = \langle A(Q)n_Q, n_Q\rangle \leqslant \mu$ and the estimates in Theorem 4.1. □

For $x \in D$ let m_x (resp. m_x^*) denotes the $\mathcal{L}$ (resp. $\mathcal{L}^*$)-harmonic measure at x on ∂D and σ the surface measure on ∂D. We have the following.

COROLLARY 4.4. *Let B in the elliptic Kato class with $\|B\| \leqslant C_0$, for some suitable constant C_0. Then, the measures m_x, m_x^* and σ are equivalent on ∂D.*

Proof. For any Borel subset $A \subset \partial D$, we have

$$m_x(A) = \int_A p(x,Q)\,\mathrm{d}\sigma(Q)$$

and so by Corollary 4.3, it follows that

$$\frac{1}{k}\frac{d(x)}{d(D)^n}\sigma(A) \leqslant m_x(A) \leqslant \frac{k}{d(x)^{n-1}}\sigma(A).$$

The same inequalities hold for m_x^*. □

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