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Positivity

Positive solutions for nonlocal differential equations with concave and convex coefficients

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Abstract
In this paper, we study the positive solutions for nonlocal differential equations with concave and convex coefficients:

$$-A\left(\int_0^1 (u^p(s)+u^q(s))ds\right)u''(t)=f(t,u(t)),\quad t\in(0,1),$$

where $0<p<1\leq q$. Using the fixed point index theory and fixed point theorems on cones, existence and multiplicity results are obtained, when the nonlinear term $f(t,x)$ is continuous, has a singularity at $x=0$, changes sign, respectively.

Keywords Nonlocal differential equation · Nonlocal boundary condition · Concave and convex coefficients · Positive solution · Fixed point

Mathematics Subject Classification 34B10 · 34B18

1 Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions for the following nonlocal differential equation

$$-A\left(\int_0^1 (u^p(s)+u^q(s))ds\right)u''(t)=f(t,u(t)),\quad t\in(0,1), \tag{1.1}$$

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where $0 < p < 1 \leq q$, the function $A : \mathbb{R} \to \mathbb{R}$ is continuous. Note the differential Eq. (1.1) is nonlocal due to the coefficient function $A\left(\int_0^1 (u^p(s) + u^q(s))ds\right)$ involving an integral, and u^p is a concave function and u^q is a convex function.

In order to better study the positive solutions for nonlocal differential Eq. (1.1), we convert Eq. (1.1) coupled with different nonlocal boundary conditions into the following integral equation

$$u(t) = \gamma(t)H(\varphi(u)) + \int_0^1 \left(A\left(\int_0^1 (u^p(r) + u^q(r))dr\right)\right)^{-1} G(t,s)f(s,u(s))ds, \quad t \in [0,1], \tag{1.2}$$

where functions γ, H and φ are involved in boundary data, and then investigate the existence of positive solutions to the Eq. (1.2).

Nonlocal differential equations as (1.1) arise in various areas of physics and applied mathematics (see [1–3, 7, 9] and the references therein). The nonlocal differential equations and nonlocal boundary conditions are intensively studied by many scholars; for example, see [4–6, 8, 10, 12–22, 24–33].

Recently, Goodrich [14] studied the following nonlocal differential equation Dirichlet boundary value problems:

$$\begin{cases} -A\left(\int_0^1 |u(s)|^q ds\right) u''(t) = \lambda f(t,u(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

where $q \geq 1,\ \lambda > 0,\ A : [0,+\infty) \to \mathbb{R}$ is continuous, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. Under weak condition on the coefficient function A, the author established the existence of at least one positive solution by using a nonstandard order cone and fixed point index theory. In fact, (1.3) is a special case of the nonlocal elliptic PDE

$$-A\left(\int_\Omega |u|^q d\mathbf{s}\right) \Delta u(\mathbf{x}) = \lambda g(u(\mathbf{x})), \quad \mathbf{x} \in \Omega,$$

subject to $u(\mathbf{x}) \equiv 0$, for $\mathbf{x} \in \partial\Omega$, where Ω is an annular region. There are also many variants on (1.3). For example, the case u in coefficient function A of (1.2) is replaced by u' is a one-dimensional Kirchhoff-type problem (see [4, 13, 29]).

In [18], Goodrich discussed the following nonlocal differential equation:

$$-A\left(\int_0^1 (g \circ u)(s)ds\right) u''(t) = \lambda f(t,u(t)), \quad t \in (0,1), \tag{1.4}$$

where $A : [0,+\infty) \to \mathbb{R}$ is continuous, $g : [0,+\infty) \to [0,+\infty)$ is continuous concave strictly increasing, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. In contrast to [14], g is concave in the coefficient function. A model case of (1.4) is the nonlocal differential equation

$$-A\left(\int_0^1 |u(s)|^p ds\right) u''(t) = \lambda f(t, u(t)), \quad t \in (0, 1), \quad 0 < p < 1.$$

In [20], Goodrich considered the existence result of positive solution for the following perturbed Hammerstein integral equation:

$$u(t) = \gamma(t)H(\varphi(u)) + \lambda \int_0^1 \left(A\left(\int_0^1 |u(\xi)|^q d\xi\right)\right)^{-1} G(t, s) f(s, u(s))ds, \quad t \in (0, 1), \tag{1.5}$$

where $q \geq 1$, $A : [0, +\infty) \to \mathbb{R}$, $G : [0, 1] \times [0, 1] \to [0, +\infty)$, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$, $H : [0, +\infty) \to [0, +\infty)$, and $\gamma : [0, 1] \to [0, 1]$ are continuous, and $\varphi(u) = \int_0^1 u(s)d\alpha(s)$, where α is of bounded variation and monotone increasing on $[0, 1]$. Solutions of (1.5) can be associated to solutions of a boundary value problem, which possess two nonlocal elements. The nonlocality is embodied in differential equation itself and boundary condition.

More recently, Goodrich [17] studied the following nonlocal convolution-type differential equations

$$-A((a * (g \circ u))(1))u''(t) = \lambda f(t, u(t)), \quad t \in (0, 1), \tag{1.6}$$

where g is a continuous function, and there exist constants $c_1, c_2 \in (0, \infty)$ and $c_3 \in [0, \infty)$ such that

$$c_1 u^p \leq g(u) \leq c_2(c_3 + u^q), \quad u \geq 0, \ 1 \leq p \leq q < +\infty.$$

The author established the existence of positive solution by using fixed point index theory. If we choose $a(x) \equiv 1$, $g(u) = u^p + u^q$ and $\lambda = 1$, then convolution-type Eq. (1.6) reduces to the nonlocal differential Eq. (1.1). We note that Goodrich only consider the case $1 \leq p < q$.

Greatly inspired by above works, in this paper, we study the positive solutions for nonlocal differential Eq. (1.1) with concave and convex coefficients. Firstly, we establish the existence and multiplicity results of Eq. (1.2) when the nonlinear term f is continuous. Secondly, we obtain the existence of positive solutions of Eq. (1.2) when $f(t, x)$ is singular at $x = 0$. Finally, we discuss the existence result when f changes sign.

2 Multiple positive solutions for Eq. (1.2)

In this section, by using the fixed point index theory in cones, we give the existence and multiplicity results of positive solutions for integral Eq. (1.2), where $0 < p < 1 \leq q$, the functions $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ and $H : [0, +\infty) \to [0, +\infty)$ are continuous. $\varphi(u) = \int_0^1 u(s)d\alpha(s)$, where $\alpha : [0, 1] \to \mathbb{R}$ is of bounded variation and monotonically increasing on [0, 1].

For example, when $\gamma(t) = 1 - t$, and

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}$$

a solution of the integral Eq. (1.2) is equivalent to a solution of the boundary value problem of differential equation

$$\begin{cases} -A\left(\int_0^1 (u^p(s) + u^q(s))ds\right) u''(t) = f(t, u(t)), & t \in (0,1), \\ u(0) = H(\varphi(u)), \quad u(1) = 0. \end{cases}$$

Set $E = C([0, 1])$. Then E is a Banach space with the norm $\|u\| = \sup\limits_{t\in[0,1]} |u(t)|$. We define the cone

$$P = \left\{ u \in E : u(t) \ge 0,\ t \in [0,1],\ \min_{t\in[c,d]} u(t) \geqslant \eta_0 \|u\| \right\},$$

where $0 \le c < d \le 1$, and the constant η_0 will be given in (H_1). In addition, for $\rho > 0$, the sets $\widehat{V}_\rho$ and Ω_ρ are given by

$$\widehat{V}_\rho = \left\{ u \in P : \int_0^1 (u^p(s) + u^q(s))ds < \rho \right\}$$

and

$$\Omega_\rho = \{u \in P : \|u\| < \rho\}.$$

For $H : [0, +\infty) \to [0, +\infty)$ a continuous function and given numbers $0 \le a < b < +\infty$, we denote

$$H^m_{[a,b]} = \min_{y\in[a,b]} H(y), \quad H^M_{[a,b]} = \max_{y\in[a,b]} H(y).$$

For $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ a continuous function and given numbers $0 \le a_1 < b_1 \le 1$ and $0 \le a_2 < b_2 < +\infty$, then by $f^m_{[a_1,b_1]\times[a_2,b_2]}$ and $f^M_{[a_1,b_1]\times[a_2,b_2]}$ we will denote, respectively, the numbers

$$f^m_{[a_1,b_1]\times[a_2,b_2]} = \min_{(t,x)\in[a_1,b_1]\times[a_2,b_2]} f(t,x),$$
$$f^M_{[a_1,b_1]\times[a_2,b_2]} = \max_{(t,x)\in[a_1,b_1]\times[a_2,b_2]} f(t,x).$$

Denote by $\mathbf{1}$ the function $\mathbf{1} : \mathbb{R} \to \{1\}$. Similarly, the notation $\mathbf{0}$ will denote the function that is identically zero. Define the operator $T : E \to E$ by

$$(Tu)(t) = \gamma(t)H(\varphi(u)) + \int_0^1 \left(A\left(\int_0^1 (u^p(r) + u^q(r))dr\right)\right)^{-1} G(t,s)f(s,u(s))ds,\ t \in [0,1].$$

A fixed point of T is a solution of nonlocal differential Eq. (1.1) equipped with some boundary data.

The following assumptions are used in this section.

(H_1) $G : [0,1] \times [0,1] \to [0,+\infty)$ is continuous, and there exist a set $[c,d] \subseteq [0,1]$ and a constant $\eta_0 := \eta_0(c,d) \in (0,1]$ such that

$$\min_{t\in[c,d]} G(t,s) \geqslant \eta_0 \mathcal{G}(s), \quad s \in [0,1],$$

where $\mathcal{G}(s) = \max_{t\in[0,1]} G(t,s)$;

(H_2) $\gamma : [0,1] \to [0,1]$ is continuous, and $\min_{t\in[c,d]} \gamma(t) \geq \eta_0 \|\gamma\|$, where c, d and η_0 are the same as we defined in (H_1);

(H_3) $A : [0,+\infty) \to \mathbb{R}$ is continuous, and there exist $0 \leq \rho_1 \leq \rho_2$ such that $A(t) > 0$ for $t \in [\rho_1, \rho_2]$.

Denote

$$Q_1 = \min_{t\in[\rho_1,\rho_2]} A(t), \quad Q_2 = \max_{t\in[\rho_1,\rho_2]} A(t).$$

Remark 2.1 From the definitions of $\widehat{V}_\rho$ and Ω_ρ, it is clear that $\widehat{V}_\rho$ and Ω_ρ are relatively open sets in P.

Lemma 2.2 *For each fixed $\rho > 0$, it holds that*

$$\overline{\Omega_{M_\rho}} \subseteq \overline{\widehat{V}_\rho} \subseteq \overline{\Omega_{N_\rho}},$$

where $M_\rho \in \left(0, \rho^{\frac{1}{q}}\right)$ is the unique positive solution of $x^p + x^q = \rho$ and $N_\rho \in \left(0, \frac{1}{\eta_0}(\frac{\rho}{d-c})^{\frac{1}{q}}\right)$ is the unique positive solution of $(\eta_0 x)^p + (\eta_0 x)^q = \frac{\rho}{d-c}$.

Proof For any $u \in \overline{\Omega_{M_\rho}}$, we have $\|u\| \leq M_\rho$. Thus,

$$\int_0^1 (u^p(t) + u^q(t))dt \leq \|u\|^p + \|u\|^q \leq M_\rho^p + M_\rho^q = \rho.$$

Then we obtain $u \in \overline{\widehat{V}_\rho}$, i.e., $\overline{\Omega_{M_\rho}} \subseteq \overline{\widehat{V}_\rho}$.

Next, we prove $\overline{\widehat{V}_\rho} \subseteq \overline{\Omega_{N_\rho}}$. In fact, for any $u \in \overline{\widehat{V}_\rho}$, we have

$$\rho \geq \int_0^1 (u^p(t) + u^q(t))dt \geq \int_c^d ((\eta_0\|u\|)^p + (\eta_0\|u\|)^q)dt$$
$$= (d-c)((\eta_0\|u\|)^p + (\eta_0\|u\|)^q).$$

Then

$$\frac{\rho}{d-c} \geq (\eta_0\|u\|)^p + (\eta_0\|u\|)^q.$$

Let

$$h(x) = (\eta_0 x)^p + (\eta_0 x)^q - \frac{\rho}{d-c}.$$

It is easy to see that $h(x)$ is a strictly monotone increasing function on $[0, +\infty)$. Thus, we obtain $\|u\| \leq N_\rho$. Therefore, $u \in \overline{\Omega_{N_\rho}}$, i.e., $\overline{\widehat{V}_\rho} \subseteq \overline{\Omega_{N_\rho}}$. This completes the proof. □

By using standard arguments, we obtain the following result.

Lemma 2.3 *Assume that* (H_1)-(H_3) *hold. Then* $T : \overline{\widehat{V}_{\rho_2}} \backslash \widehat{V}_{\rho_1} \to P$ *is completely continuous.*

Lemma 2.4 [23] *Let* U *be a bounded open set, and with* K *a cone in a real Banach space* X. *Suppose both that* $U_K = U \cap K \supseteq \{0\}$ *and that* $\overline{U_K} \neq K$. *Assume that* $T : \overline{U_K} \to K$ *is completely continuous such that* $x \neq Tx$, *for any* $x \in \partial U_K$. *Then the fixed point index* $i_K(T, U_K)$ *has the following properties.*

(1) *If there exists* $e \in K \setminus \{0\}$ *such that* $x \neq Tx + \lambda e$ *for each* $x \in \partial U_K$ *and* $\lambda > 0$, *then* $i_K(T, U_K) = 0$.
(2) *If* $\mu x \neq Tx$ *for* $x \in \partial U_K$ *and* $\mu \geq 1$, *then* $i_K(T, U_K) = 1$.
(3) *Let* U_K^1 *be a open set in* X *with* $\overline{U_K^1} \subseteq U_K$. *If* $i_K(T, U_K) = 0$ *and* $i_K(T, U_K^1) = 1$, *then* T *has a fixed point in* $U_K \setminus \overline{U_K^1}$. *The same result holds if* $i_K(T, U_K) = 1$ *and* $i_K(T, U_K^1) = 0$.

Theorem 2.5 *Assume that* $(H_1) - (H_3)$ *hold. If*

$$(H_4)\ \int_0^1 \left[\left(\frac{1}{A(\rho_1)} f^m_{[c,d]\times[\eta_0 M_{\rho_1}, N_{\rho_1}]} \int_c^d G(t,s)ds\right)^p + \left(\frac{1}{A(\rho_1)} f^m_{[c,d]\times[\eta_0 M_{\rho_1}, N_{\rho_1}]} \int_c^d G(t,s)ds\right)^q\right] dt > \rho_1;$$

$$(H_5) \int_0^1 \left[\left(H^M_{[0,N_{\rho_2}\varphi(\mathbf{1})]} + \frac{1}{A(\rho_2)} f^M_{[0,1]\times[0,N\rho_2]} \int_0^1 G(t,s)ds \right)^p + \left(H^M_{[0,N_{\rho_2}\varphi(\mathbf{1})]} + \frac{1}{A(\rho_2)} f^M_{[0,1]\times[0,N\rho_2]} \int_0^1 G(t,s)ds \right)^q \right] dt < \rho_2,$$

then Eq. (1.2) *has at least one positive solution u with $M_{\rho_1} \le \|u\| \le N_{\rho_2}$.*

Proof Clearly, $\mathbf{0} \in \widehat{V}_{\rho_1} \subseteq \widehat{V}_{\rho_2}$ and $c\mathbf{1} \in \widehat{V}_{\rho_1}$ for small constant $c > 0$, so $\widehat{V}_{\rho_1}\backslash\{\mathbf{0}\} \neq \emptyset$. From Lemma 2.3 and the extension theorem of a completely continuous operator, there exists $\widetilde{T} : \widehat{V}_{\rho_2} \to P$, which is still completely continuous. Without loss of the generality, we still write it as T.

Now, we claim that $u \neq Tu + \mu\mathbf{1}$ for any $\mu > 0$ and $u \in \partial\widehat{V}_{\rho_1}$. Suppose that T has no fixed points on $\partial\widehat{V}_{\rho_1}$ (otherwise, the proof is finished). Assume by contradiction that there exist $u \in \partial\widehat{V}_{\rho_1}$ and $\mu > 0$ such that $u = Tu + \mu\mathbf{1}$. It follows from Lemma 2.2 that

$$M_{\rho_1} \le \|u\| \le N_{\rho_1}, \quad \forall\, u \in \partial\widehat{V}_{\rho_1}.$$

Therefore, for $s \in [c, d]$, we have

$$\eta_0 M_{\rho_1} \le \eta_0\|u\| \le u(s) \le \|u\| \le N_{\rho_1}.$$

Then for $u \in \partial\widehat{V}_{\rho_1}$,

$$(Tu)(t) \ge \frac{1}{A(\rho_1)} f^m_{[c,d]\times[\eta_0 M_{\rho_1}, N_{\rho_1}]} \int_c^d G(t,s)ds, \quad t \in [0,1].$$

It follows that

$$\begin{aligned}
\rho_1 &= \int_0^1 (u^p + u^q)(t)dt = \int_0^1 ((Tu + \mu\mathbf{1})^p + (Tu + \mu\mathbf{1})^q)(t)dt \\
&\ge \int_0^1 ((Tu)^p + (Tu)^q)(t)dt \ge \int_0^1 \left[\left(\frac{1}{A(\rho_1)} f^m_{[c,d]\times[\eta_0 M_{\rho_1}, N_{\rho_1}]} \int_c^d G(t,s)ds \right)^p \right. \\
&\quad \left. + \left(\frac{1}{A(\rho_1)} f^m_{[c,d]\times[\eta_0 M_{\rho_1}, N_{\rho_1}]} \int_c^d G(t,s)ds \right)^q \right] dt,
\end{aligned}$$

which is a contradiction to (H_4). So we get

$$i_K(T, \widehat{V}_{\rho_1}) = 0.$$

We next show that $\mu u \neq Tu$ for any $u \in \partial\widehat{V}_{\rho_2}$ and $\mu \ge 1$. If otherwise, there exist $u \in \partial\widehat{V}_{\rho_2}$ and $\mu \ge 1$ such that $\mu u = Tu$. For $u \in \partial\widehat{V}_{\rho_2}$, it is easy to check that $M_{\rho_2} \leq \|u\| \leq N_{\rho_2}$ and $0 \leq u(t) \leq \|u\| \leq N_{\rho_2}$ for $t \in [0, 1]$, and $0 \leq \varphi(u) \leq \|u\|\varphi(\mathbf{1}) \leq N_{\rho_2}\varphi(\mathbf{1})$. It follows that

$$(Tu)(t) \leq H^M_{[0, N_{\rho_2}\varphi(\mathbf{1})]} + \frac{1}{A(\rho_2)} f^M_{[0,1]\times[0,N_{\rho_2}]} \int_0^1 G(t, s)ds, \quad t \in [0, 1].$$

Then we deduce that

$$\begin{aligned}
\rho_2 &= \int_0^1 (u^p + u^q)(t)dt \leq \int_0^1 ((\mu u)^p + (\mu u)^q)(t)dt \\
&= \int_0^1 ((Tu)^p + (Tu)^q)(t)dt \\
&\leq \int_0^1 \left[\left(H^M_{[0, N_{\rho_2}\varphi(\mathbf{1})]} + \frac{1}{A(\rho_2)} f^M_{[0,1]\times[0,N_{\rho_2}]} \int_0^1 G(t, s)ds \right)^p \right. \\
&\quad \left. + \left(H^M_{[0, N_{\rho_2}\varphi(\mathbf{1})]} + \frac{1}{A(\rho_2)} f^M_{[0,1]\times[0,N_{\rho_2}]} \int_0^1 G(t, s)ds \right)^q \right] dt,
\end{aligned}$$

which contradicts (H_5). So we obtain

$$i_K(T, \widehat{V}_{\rho_2}) = 1.$$

It follows from Lemma 2.4 that T has a fixed point $u \in \widehat{V}_{\rho_2} \backslash \overline{\widehat{V}_{\rho_1}}$ with $M_{\rho_1} \leq \|u\| \leq N_{\rho_2}$, and u is a positive solution of the Eq. (1.2). □

Corollary 2.6 *If we reverse ρ_1 and ρ_2 in Theorem 2.5, then $i_K(T, \widehat{V}_{\rho_1}) = 1$, $i_K(T, \widehat{V}_{\rho_2}) = 0$. We can get the same result as Theorem 2.5.*

Next, we prove the following multiplicity results.

Theorem 2.7 *Assume that (H_1)–(H_4) hold. If*

(H_6) $N_{\rho_1} < M_{\rho_2}$;

(H_7) $$\int_0^1 \left[\left(\frac{1}{A(\rho_2)} f^m_{[c,d]\times[\eta_0 M_{\rho_2}, N_{\rho_2}]} \int_c^d G(t, s)ds \right)^p + \left(\frac{1}{A(\rho_2)} f^m_{[c,d]\times[\eta_0 M_{\rho_2}, N_{\rho_2}]} \int_c^d G(t, s)ds \right)^q \right] dt > \rho_2;$$

(H_8) $$H^M_{[0, N_{\rho_1}\varphi(\mathbf{1})]} + \frac{1}{Q_1} f^M_{[0,1]\times[0,N_{\rho_1}]} \int_0^1 \mathcal{G}(s)ds < N_{\rho_1},$$

then Eq. (1.2) has at least two positive solutions.

Proof Firstly, we prove that T has a fixed point which is either on $\partial\widehat{V}_{\rho_1}$ or in $\Omega_{N_{\rho_1}} \backslash \overline{\widehat{V}_{\rho_1}}$. If $x \neq Tx$ for $x \in \partial\widehat{V}_{\rho_1}$, by Theorem 2.5, we obtain

$$i_K(T, \widehat{V}_{\rho_1}) = 0.$$

We next prove that $\|Tu\| < \|u\|$ for any $u \in \partial\Omega_{N_{\rho_1}}$. From Lemma 2.2, it follows that

$$\partial\Omega_{N_{\rho_1}} \subseteq \overline{\Omega_{N_{\rho_1}}} \subseteq \overline{\Omega_{M_{\rho_2}}} \subseteq \overline{\widehat{V}_{\rho_2}}.$$

Then

$$\int_0^1 (u^p(r) + u^q(r))dr \leq \rho_2.$$

Owing to

$$\int_0^1 (u^p(r) + u^q(r))dr \geq \int_c^d (\eta_0\|u\|)^p + (\eta_0\|u\|)^q dr = \rho_1,$$

we have

$$Q_1 \leq A\left(\int_0^1 (u^p(r) + u^q(r))dr\right) \leq Q_2.$$

Since $0 \leq \varphi(u) \leq \|u\|\varphi(\mathbf{1}) = N_{\rho_1}\varphi(\mathbf{1})$, we have

$$\|Tu\| \leq H^M_{[0,N_{\rho_1}\varphi(\mathbf{1})]} + \frac{1}{Q_1} f^M_{[0,1]\times[0,N_{\rho_1}]} \int_0^1 \mathcal{G}(s)ds < N_{\rho_1} = \|u\|.$$

Therefore, it is obvious that $Tu \neq u$ for $u \in \partial\Omega_{N_{\rho_1}}$, and $\mu u \neq Tu$ for $u \in \partial\Omega_{N_{\rho_1}}$ and $\mu \geq 1$. By Lemma 2.4,

$$i_K(T, \Omega_{N_{\rho_1}}) = 1.$$

Since $\widehat{V}_{\rho_1} \subseteq \Omega_{N_{\rho_1}}$, by Lemma 2.4, T has a fixed point in $\Omega_{N_{\rho_1}}\backslash\overline{\widehat{V}_{\rho_1}}$. So T has a fixed point which is either on $\partial\widehat{V}_{\rho_1}$ or in $\Omega_{N_{\rho_1}}\backslash\overline{\widehat{V}_{\rho_1}}$.

On the other hand, we prove T has a fixed point which is either on $\partial\widehat{V}_{\rho_2}$ or in $\widehat{V}_{\rho_2}\backslash\overline{\Omega_{N_{\rho_1}}}$. If $x \neq Tx,\ x \in \partial\widehat{V}_{\rho_2}$, by Theorem 2.5, we conclude that

$$i_K(T, \widehat{V}_{\rho_2}) = 0.$$

It follows from Lemma 2.4 that T has a fixed point in $\widehat{V}_{\rho_2}\backslash\overline{\Omega_{N_{\rho_1}}}$. So T has a fixed point which is either on $\partial\widehat{V}_{\rho_2}$ or in $\widehat{V}_{\rho_2}\backslash\overline{\Omega_{N_{\rho_1}}}$.

Therefore, T has at least two fixed points u_1 and u_2 with $0 < M_{\rho_1} \leq \|u_1\| \leq N_{\rho_1} < M_{\rho_2} < \|u_2\| \leq N_{\rho_2}$, i.e., Eq. (1.2) has at least two positive solutions. □

Theorem 2.8 *Assume that* (H_1)-(H_3), (H_5) *and* (H_6) *hold. If*

$$(H_9)\ \int_0^1\left[\left(H^M_{[0,N_{\rho_1}\varphi(\mathbf{1})]}+\frac{1}{A(\rho_1)}f^M_{[0,1]\times[0,N_{\rho_1}]}\int_0^1 G(t,s)ds\right)^p + \left(H^M_{[0,N_{\rho_1}\varphi(\mathbf{1})]}+\frac{1}{A(\rho_1)}f^M_{[0,1]\times[0,N_{\rho_1}]}\int_0^1 G(t,s)ds\right)^q\right]dt<\rho_1;$$

$$(H_{10})\ \frac{\eta_0}{Q_2}f^m_{[c,d]\times[\eta_0N_{\rho_1},N_{\rho_1}]}\int_c^d \mathcal{G}(s)ds>N_{\rho_1},$$

then Eq. (1.2) *has at least two positive solutions.*

Proof Similar to the proof of Theorem 2.5, we obtain that $i_K(T,\widehat{V}_{\rho_1})=1$ and $i_K(T,\widehat{V}_{\rho_2})=1$. So it suffices to show that $i_K(T,\Omega_{N_{\rho_1}})=0$. Assume that there exist $u\in\partial\Omega_{N_{\rho_1}}$ and $\mu>0$ such that $u=Tu+\mu\mathbf{1}$. From Theorem 2.7, we have $Q_1\le A\left(\int_0^1(u^p(r)+u^q(r))dr\right)\le Q_2$. Then we have

$$N_{\rho_1}=\|u\|=\|Tu+\mu\mathbf{1}\|\ge(Tu)(c)\ge\frac{\eta_0}{Q_2}f^m_{[c,d]\times[\eta_0N_{\rho_1},N_{\rho_1}]}\int_c^d\mathcal{G}(s)ds,$$

which contradicts the assumption. Then for all $u\in\partial\Omega_{N_{\rho_1}}$ and $\mu>0$, we have $u\ne Tu+\mu\mathbf{1}$. It follows from Lemma 2.4 that $i_K(T,\Omega_{N_{\rho_1}})=0$. Then Eq. (1.2) has at least two positive solutions. □

By arguments similar to Theorems 2.5, 2.7 and 2.8, we have the following results.

Theorem 2.9 *Assume that* (H_1)–(H_3), (H_6), (H_7), (H_9) *and* (H_{10}) *hold. If*

$$(H_{11})\ H^M_{[0,M_{\rho_2}\varphi(\mathbf{1})]}+\frac{1}{Q_1}f^M_{[0,1]\times[0,M_{\rho_2}]}\int_0^1\mathcal{G}(s)ds<M_{\rho_2},$$

then Eq. (1.2) *has at least three positive solutions.*

Theorem 2.10 *Assume that* (H_1)-(H_6) *and* (H_8) *hold. If*

$$(H_{12})\ \frac{\eta_0}{Q_2}f^m_{[c,d]\times[\eta_0M_{\rho_2},M_{\rho_2}]}\int_c^d\mathcal{G}(s)ds>M_{\rho_2},$$

then Eq. (1.2) *has at least three positive solutions.*

Example 2.11 Let $p=\frac{1}{2},\ q=2,\ A(t)=\sin t,\ \varphi(u)=\frac{1}{2}u\left(\frac{1}{3}\right)+\frac{1}{50}u\left(\frac{1}{10}\right),\ \gamma(t)=1-t,\ H(t)=\frac{9}{100}\sqrt{t},\ f(t,x)=tx$. We consider the following nonlocal problem

$$\begin{cases}-\sin\left(\int_0^1(u^{\frac{1}{2}}(s)+u^2(s))ds\right)u''(t)=tu(t), & t\in(0,1),\\ u(0)=\frac{9}{100}\sqrt{\frac{1}{2}u\left(\frac{1}{3}\right)+\frac{1}{50}u\left(\frac{1}{10}\right)}, & u(1)=0.\end{cases}\tag{2.1}$$

Then

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1,\end{cases}$$

and $\mathcal{G}(s)=\max_{t\in[0,1]} G(t,s)=s(1-s)$. Choose $c=\frac{1}{4},\ d=\frac{3}{4}$. Then $\eta_0=\min\{c,1-d\}=\frac{1}{4}$. Obviously, (H_1) and (H_2) hold. Take $\rho_1=0.002,\ \rho_2=\frac{\pi}{2}$. Then $A(t)=\sin t>0$ on $[0.002,\frac{\pi}{2}]$, which implies (H_3) holds. It follows from Lemma 2.2 that $M_{\rho_1}\approx 4\times 10^{-6},\ N_{\rho_1}\approx 6.4\times 10^{-5}$, $M_{\rho_2}\approx 0.817,\ N_{\rho_2}\approx 5.598$. Direct computation demonstrates that

$$\begin{aligned}
&\int_0^1\left[\left(\frac{1}{A(\rho_1)}f^m_{[c,d]\times[\eta_0 M_{\rho_1},N_{\rho_1}]}\int_c^d G(t,s)ds\right)^p\right.\\
&\quad\left.+\left(\frac{1}{A(\rho_1)}f^m_{[c,d]\times[\eta_0 M_{\rho_1},N_{\rho_1}]}\int_c^d G(t,s)ds\right)^q\right]dt\\
&\approx\int_0^1\left[\left(\frac{1}{\sin(0.002)}\times\frac{1}{4}\times\frac{1}{4}\times 4\times 10^{-6}\times\int_{\frac{1}{4}}^{\frac{3}{4}}G(t,s)ds\right)^{\frac{1}{2}}\right.\\
&\quad\left.+\left(\frac{1}{\sin(0.002)}\times\frac{1}{4}\times\frac{1}{4}\times 4\times 10^{-6}\times\int_{\frac{1}{4}}^{\frac{3}{4}}G(t,s)ds\right)^{2}\right]dt\\
&\approx 0.0023>\rho_1,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left[\left(H^M_{[0,N_{\rho_2}\varphi(\mathbf{1})]}+\frac{1}{A(\rho_2)}f^M_{[0,1]\times[0,N\rho_2]}\int_0^1 G(t,s)ds\right)^p\right.\\
&\quad\left.+\left(H^M_{[0,N_{\rho_2}\varphi(\mathbf{1})]}+\frac{1}{A(\rho_2)}f^M_{[0,1]\times[0,N\rho_2]}\int_0^1 G(t,s)ds\right)^q\right]dt\\
&\approx\int_0^1\left[\left(\frac{9}{100}\times\sqrt{5.598\times\frac{13}{25}}+1\times 5.598\times\int_0^1 G(t,s)ds\right)^{\frac{1}{2}}\right.\\
&\quad\left.+\left(\frac{9}{100}\times\sqrt{5.598\times\frac{13}{25}}+1\times 5.598\times\int_0^1 G(t,s)ds\right)^{2}\right]dt\\
&\approx 1.202<\rho_2.
\end{aligned}$$

So assumptions (H_4) and (H_5) hold. By Theorem 2.5 we conclude that problem (2.1) has at least one positive solution u with $4\times 10^{-6}\le\|u\|\le 5.598$.

Remark 2.12 Note in Example 2.11 that the nonlocal coefficient function $z \mapsto \sin z$ is sign changing on $\mathbb{R}$. This is in considerable contrast to most of the the existing literature. Our main tool is topological fixed point theory in a nonstandard order cone due to Goodrich (for example, [14,17,18,20]).

Example 2.13 Let $p,\ q,\ A,\ \varphi,\ \gamma,\ c$ and d are same as those in Example 2.11. Take $\rho_1 = 0.01$ and $\rho_2 = \frac{\pi}{2}$. Then $(H_1) - (H_3)$ hold. From Lemma 2.2, we know that $M_{\rho_1} \approx 0.0001,\ N_{\rho_1} \approx 0.0016, M_{\rho_2} \approx 0.817$ and $N_{\rho_2} \approx 5.598$. Obviously, (H_6) holds. Let $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ be defined by

$$f(t,x) = 10^{-8}t + \begin{cases} \frac{1}{1250}x + \frac{3}{50000}, & 0 \le x < 0.1, \\ 140x - \frac{699993}{50000}, & x \ge 0.1. \end{cases}$$

Choose $H(t) = \frac{9}{1000}\sqrt{t}$. By calculation, we obtain

$$\begin{aligned}
&\int_0^1 \left[\left(\frac{1}{A(\rho_1)} f^m_{[c,d]\times[\eta_0 M_{\rho_1}, N_{\rho_1}]} \int_c^d G(t,s)ds \right)^p \right. \\
&\quad \left. + \left(\frac{1}{A(\rho_1)} f^m_{[c,d]\times[\eta_0 M_{\rho_1}, N_{\rho_1}]} \int_c^d G(t,s)ds \right)^q \right] dt \\
&\approx \int_0^1 \left[\left(\frac{1}{\sin(0.01)} \times \left(\frac{10^{-8}}{4} + \frac{0.0001}{5000} + \frac{3}{50000} \right) \times \int_{\frac14}^{\frac34} G(t,s)ds \right)^{\frac12} \right. \\
&\quad \left. + \left(\frac{1}{\sin(0.01)} \times \left(\frac{10^{-8}}{4} + \frac{0.0001}{5000} + \frac{3}{50000} \right) \times \int_{\frac14}^{\frac34} G(t,s)ds \right)^2 \right] dt \\
&\approx 0.016 > \rho_1, \\
&\int_0^1 \left[\left(\frac{1}{A(\rho_2)} f^m_{[c,d]\times[\eta_0 M_{\rho_2}, N_{\rho_2}]} \int_c^d G(t,s)ds \right)^p \right. \\
&\quad \left. + \left(\frac{1}{A(\rho_2)} f^m_{[c,d]\times[\eta_0 M_{\rho_2}, N_{\rho_2}]} \int_c^d G(t,s)ds \right)^q \right] dt \\
&\approx \int_0^1 \left[\left(\frac{1}{\sin\frac{\pi}{2}} \times \left(\frac{10^{-8}}{4} + 140 \times 0.204 - \frac{699993}{50000} \right) \times \int_{\frac14}^{\frac34} G(t,s)ds \right)^{\frac12} \right. \\
&\quad \left. + \left(\frac{1}{\sin\frac{\pi}{2}} \times \left(\frac{10^{-8}}{4} + 140 \times 0.204 - \frac{699993}{50000} \right) \times \int_{\frac14}^{\frac34} G(t,s)ds \right)^2 \right] dt \\
&\approx 1.642 > \rho_2, \\
&H^M_{[0, N_{\rho_1}\varphi(\mathbf{1})]} + \frac{1}{Q_1} f^M_{[0,1]\times[0,N_{\rho_1}]} \int_0^1 \mathcal{G}(s)ds
\end{aligned}$$

$$\approx \frac{9}{1000}\sqrt{0.0016 \times \frac{13}{25}} + \frac{1}{\sin(0.01)} \times \left(10^{-8} + \frac{1}{1250} \times 0.0016 + \frac{3}{50000}\right)$$
$$\times \int_0^1 s(1-s)ds \approx 0.0013 < 0.0016 \approx N_{\rho_1}.$$

Therefore, (H_4), (H_7) and (H_8) hold. By Theorem 2.7 we conclude that problem (2.1) has at least two positive solutions.

3 The case $f(t, x)$ is singular at $x = 0$

In this section, by applying the Guo–Krasnoselskii fixed point theorem on cones, we obtain the existence of positive solutions for Eq. (1.1) with nonlocal boundary conditions when the nonlinear term $f(t, x)$ has singularity at $x = 0$. It is worth mentioning that the restrictions on functions A and H are different from those in Sect. 2.

We continue to study the integral Eq. (1.2), where $0 < p < 1 \leq q$, $A : [0, +\infty) \to (0, +\infty)$ is continuous and monotone increasing, $H : [0, +\infty) \to [0, +\infty)$ is continuous and bounded (we will assume that there exists $0 < \overline{H} < +\infty$ such that $0 \leq H(x) \leq \overline{H}$ for $x \in [0, +\infty)$). $\varphi(u)$ is the same as Sect. 2. $f : [0, 1] \times (0, +\infty) \to [0, +\infty)$ is continuous and singular at $x = 0$.

We define a cone

$$P' = \{u \in E : u(t) \geqslant c(t)\|u\|,\ t \in [0, 1]\},$$

and a set

$$\Omega_\rho = \{u \in P' : \|u\| < \rho\},$$

where $c(t)$ will be given in $(H_1)'$.

$(H_1)'$ $G : [0, 1] \times [0, 1] \to [0, +\infty)$ is continuous and there exists a continuous function $c : [0, 1] \to [0, 1]$ with $0 < c(t) < 1$ for $t \in (0, 1)$ such that

$$G(t, s) \geqslant c(t)\mathcal{G}(s), \quad t, s \in [0, 1],$$

where $\mathcal{G}(s) = \max_{t\in[0,1]} G(t, s),\ s \in [0, 1]$;

$(H_2)'$ $\gamma : [0, 1] \to [0, 1]$ is continuous and $\gamma(t) \geq c(t)\|\gamma\|$ for $t \in [0, 1]$, where $c(t)$ is the same as we defined in $(H_1)'$;

$(H_3)'$ For any $0 < r < R < +\infty$,

$$\lim_{m\to\infty} \sup_{u\in\overline{\Omega_R}\setminus\Omega_r} \int_{e(m)} \mathcal{G}(s) f(s, u(s))ds = 0,$$

where $e(m) = [0, \frac{1}{m}] \cup [\frac{m-1}{m}, 1]$.

For a given $\theta \in (0, \frac{1}{2})$, denote $\eta = \min\{c(t) : \theta \leq t \leq 1 - \theta\}$.

Lemma 3.1 *Assume that* $(H_1)'$-$(H_3)'$ *hold. Then* $T : \overline{\Omega_R}\backslash\Omega_r \to P'$ *is completely continuous.*

Proof For any $u \in \overline{\Omega_R}\backslash\Omega_r$, we have $r \le \|u\| \le R$, and $0 \le \varphi(u) \le \|u\|\varphi(\mathbf{1}) \le R\varphi(\mathbf{1}) < +\infty$. For any $u \in \overline{\Omega_R}\backslash\Omega_r$, we have

$$\begin{aligned} 0 < M_1 &\triangleq (1-2\theta)((\eta r)^p + (\eta r)^q) \\ &\le \int_\theta^{1-\theta} (\eta\|u\|)^p + (\eta\|u\|)^q dr \\ &\le \int_0^1 (u^p(r) + u^q(r))dr \\ &\le \|u\|^p + \|u\|^q \le R^p + R^q \triangleq M_2. \end{aligned}$$

It follows from $(H_3)'$ that there exists a natural number $l > 0$ such that

$$\sup_{u\in\overline{\Omega_R}\backslash\Omega_r} \int_{e(l)} \mathcal{G}(s)f(s,u(s))ds < 1.$$

For $s \in [\frac{1}{l}, \frac{l-1}{l}]$, we have $\eta_1 r \le \eta_1\|u\| \le u(s) \le \|u\| \le R$, where $\eta_1 = \min\{c(t) : \frac{1}{l} \le t \le \frac{l-1}{l}\}$. Then for any $t \in [0,1]$, we have

$$\begin{aligned} Tu(t) &\le \overline{H} + (A(M_1))^{-1}\left[\int_{e(l)} \mathcal{G}(s)f(s,u(s))ds + f^M_{[\frac{1}{l},\frac{l-1}{l}]\times[\eta_1 r,R]}\int_{\frac{1}{l}}^{\frac{l-1}{l}} \mathcal{G}(s)ds\right] \\ &\le \overline{H} + (A(M_1))^{-1}\left(1 + f^M_{[\frac{1}{l},\frac{l-1}{l}]\times[\eta_1 r,R]}\int_0^1 \mathcal{G}(s)ds\right) < +\infty. \end{aligned}$$

The proof of $(Tu)(t) \geqslant c(t)\|Tu\|$ is similar to the proof of Lemma 2.3, so we omit it. Thus, $T(\overline{\Omega_R}\backslash\Omega_r) \subseteq P'$.

Suppose that $u_n, u_0 \in \overline{\Omega_R}\backslash\Omega_r$ and $\|u_n - u_0\| \to 0$ $(n \to \infty)$. Then

$$0 \le \varphi(u_n) \le R\varphi(\mathbf{1}) < +\infty, \quad 0 \le \varphi(u_0) \le R\varphi(\mathbf{1}) < +\infty,$$

and

$$\begin{aligned} 0 < M_1 &\le \int_0^1 ((u_n(r))^p + (u_n(r))^q)dr \le M_2, \\ 0 < M_1 &\le \int_0^1 ((u_0(r))^p + (u_0(r))^q)dr \le M_2. \end{aligned}$$

By $(H_3)'$, $\forall\, \epsilon > 0$, there exists a natural number $m_0 > 0$ such that

$$\sup_{u\in\overline{\Omega_R}\backslash\Omega_r} \int_{e(m_0)} \mathcal{G}(s)f(s,u(s))ds < \frac{\epsilon A(M_1)}{6}.$$

For $s \in [\frac{1}{m_0}, \frac{m_0-1}{m_0}]$, we have

$$\eta_2 r \le u_n(s) \le R, \quad \eta_2 r \le u_0(s) \le R,$$

where $\eta_2 = \min\{c(s) : \frac{1}{m_0} \le s \le \frac{m_0-1}{m_0}\}$. Since f is uniformly continuous on $[\frac{1}{m_0}, \frac{m_0-1}{m_0}] \times [\eta_2 r, R]$, then

$$\lim_{n\to\infty} |f(s, u_n(s)) - f(s, u_0(s))| = 0.$$

Applying Lebesgue dominated convergence theorem, we have

$$(A(M_1))^{-1} \int_{\frac{1}{m_0}}^{\frac{m_0-1}{m_0}} \mathcal{G}(s)|f(s, u_n(s)) - f(s, u_0(s))|ds \to 0 \ (n \to +\infty).$$

For the above $\epsilon > 0$, there exists a natural number N_1 such that if $n > N_1$, then

$$(A(M_1))^{-1} \int_{\frac{1}{m_0}}^{\frac{m_0-1}{m_0}} \mathcal{G}(s)|f(s, u_n(s)) - f(s, u_0(s))|ds < \frac{\epsilon}{3}.$$

Since H and φ are continuous, then there exists a natural number N_2 such that for $n > N_2$,

$$|H(\varphi(u_n)) - H(\varphi(u_0))| < \frac{\epsilon}{3}.$$

Therefore, for $n > N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}\|Tu_n - Tu_0\| \le& |H(\varphi(u_n)) - H(\varphi(u_0))| + 2(A(M_1))^{-1} \int_{e(m_0)} \mathcal{G}(s) f(s, u_0(s))ds \\ &+(A(M_1))^{-1} \int_{\frac{1}{m_0}}^{\frac{m_0-1}{m_0}} \mathcal{G}(s)|f(s, u_n(s)) - f(s, u_0(s))|ds < \epsilon,\end{aligned}$$

which implies T is continuous.

Assume that B is a bounded subset in $\overline{\Omega_R}\backslash\Omega_r$, from Ascoli–Arzela theorem and the Lebesgue dominated convergence theorem, it is easy to prove that $T(B)$ is uniformly bounded and equicontinuous. Thus, $T : \overline{\Omega_R}\backslash\Omega_r \to P'$ is completely continuous. □

The main tool is the Guo–Krasnoselskii fixed point theorem on cones.

Lemma 3.2 [23] *Let E be a Banach space and let P be a cone in E. Let Ω_1 and Ω_2 be two bounded open subsets in E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let the operator $A : P \cap (\overline{\Omega_2}\backslash\Omega_1) \to P$ be completely continuous. If the following conditions are satisfied:*

(i) $\|Au\| \le \|u\|$ *for any* $u \in P \cap \partial\Omega_1$, $\|Au\| \ge \|u\|$ *for any* $u \in P \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geq \|u\|$ *for any* $u \in P \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ *for any* $u \in P \cap \partial\Omega_2$,

then A *has at least one fixed point in* $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Denote

$$L = \frac{1}{A(2)} \int_0^1 \mathcal{G}(s)ds, \quad l = \frac{\eta^2}{A(2)} \int_\theta^{1-\theta} \mathcal{G}(s)ds,$$

where $\theta \in \left(0, \frac{1}{2}\right)$ *is given previously.*

Theorem 3.3 *Assume that* $(H_1)'$*-*$(H_3)'$ *hold. Further assume that the following conditions hold:*

$(H_4)'$

$$0 < l^{-1} < f^0 = \liminf_{x \to 0} \min_{t \in [0,1]} \frac{f(t,x)}{x} \leq \infty;$$

$(H_5)'$

$$0 \leq f^\infty = \limsup_{x \to +\infty} \max_{t \in [0,1]} \frac{f(t,x)}{x} \leq L^{-1}.$$

Then (1.2) has at least one positive solution.

Proof From Lemma 3.1 and the extension theorem of a completely continuous operator, for any $R > 0$, there exists $\widetilde{T} : \overline{\Omega}_R \to P'$, which is still completely continuous. Without loss of the generality, we still write it as T.

By $(H_4)'$, there exist $\epsilon_1 > 0$ and $0 < r \leq 1$ such that

$$f(t,x) \geq (l^{-1} + \epsilon_1)x, \quad 0 < x \leq r, \ 0 \leq t \leq 1.$$

For $u \in \partial\Omega_r$, we have

$$\int_0^1 (u^p(r) + u^q(r))dr \leq \|u\|^p + \|u\|^q = r^p + r^q \leq 2.$$

Thus,

$$0 < A\left(\int_0^1 (u^p(r) + u^q(r))dr\right) \leq A(2).$$

It follows that

$$\begin{aligned}(Tu)(t) &\geq \frac{1}{A(2)} \int_0^1 G(t,s) f(s,u(s))ds \\ &\geq \frac{1}{A(2)} \int_0^1 c(t)\mathcal{G}(s)(l^{-1} + \epsilon_1)u(s)ds\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{c(t)}{A(2)}(l^{-1}+\epsilon_1)\int_{\theta}^{1-\theta}\mathcal{G}(s)\eta\|u\|ds \\
&\geq \frac{\eta^2}{A(2)}(l^{-1}+\epsilon_1)\|u\|\int_{\theta}^{1-\theta}\mathcal{G}(s)ds > \|u\|, \quad t\in[\theta,1-\theta].
\end{aligned}
$$

Therefore, $\|Tu\| \geq \|u\|$ for $u \in \partial\Omega_r$.

On the other hand, by $(H_5)'$, there exist $0 < \epsilon_2 < L^{-1}$ and $R_1 > 1$ such that

$$
f(t,x) \leq (L^{-1}-\epsilon_2)x, \quad x \geq R_1,\ 0 \leq t \leq 1
$$

Let

$$
M_0 = \overline{H} + \sup_{u\in\partial\Omega_{R_1}} \int_0^1 \left(A\left(\int_0^1 (u^p(r)+u^q(r))dr\right)\right)^{-1} \mathcal{G}(s)f(s,u(s))ds.
$$

From Lemma 3.1, we know $M_0 < +\infty$. We choose $R > \max\{R_1, \frac{1}{\epsilon_2 L}M_0\}$. Let

$$
D(u) = \{t \in [0,1] : u(t) > R_1\}.
$$

For any $u \in \partial\Omega_R$ and $t \in D(u)$, we have $R_1 < u(t) \leq R$, which implies $f(t,u(t)) \leq (L^{-1}-\epsilon_2)u(t)$. For $u \in \partial\Omega_R$, there exists $t_0 \in [0,1]$ such that $\|u\| = u(t_0)$. Let $u_1(t) = \min\{u(t), R_1\}$. Then $u_1(t) \leq R_1$ for $t \in [0,1]$ and $u_1(t_0) = \min\{u(t_0), R_1\} = \min\{\|u\|, R_1\} = R_1$, which implies that $u_1 \in \partial\Omega_{R_1}$. Thus, for $t \in [0,1]$,

$$
\begin{aligned}
Tu(t) &\leq \overline{H} + \int_{D(u)} \left(A\left(\int_0^1 (u^p+u^q)(r)dr\right)\right)^{-1}\mathcal{G}(s)f(s,u(s))ds \\
&\quad + \int_{[0,1]\backslash D(u)} \left(A\left(\int_0^1 ((u_1)^p+(u_1)^q)(r)dr\right)\right)^{-1}\mathcal{G}(s)f(s,u_1(s))ds \\
&\leq \overline{H} + \frac{1}{A(2)}(L^{-1}-\epsilon_2)\int_{D(u)}\mathcal{G}(s)\|u\|ds \\
&\quad + \int_0^1 \left(A\left(\int_0^1 ((u_1)^p+(u_1)^q)(r)dr\right)\right)^{-1}\mathcal{G}(s)f(s,u_1(s))ds \\
&\leq (L^{-1}-\epsilon_2)L\|u\| + M_0 < \|u\|.
\end{aligned}
$$

Therefore, $\|Tu\| \leq \|u\|$ for $u \in \partial\Omega_R$.

By Lemma 3.2, we conclude that T has a fixed point $u \in \overline{\Omega_R}\backslash\Omega_r$, and Eq. (1.2) has at least one positive solution. □

Example 3.4 Let $p = \frac{1}{2},\ q = 2,\ A(x) = 2+x,\ H(x) = 3-e^{-x},\ \varphi(u) = \int_0^1 u(s)d(2s),\ f(t,x) = 2-t+|\ln x|$. Consider the following nonlocal problem:

$$
\begin{cases}
-\left(2+\int_0^1 (u^{\frac{1}{2}}(s)+u^2(s))ds\right)u''(t) = 2-t+|\ln u(t)|, \quad t\in(0,1), \\
u(0) = 3-e^{-\int_0^1 u(s)d(2s)}, \quad u(1) = 0.
\end{cases}
\tag{3.1}
$$

Obviously, $f(t, x)$ is singular at $x = 0$. It is easy to see that $\gamma(t) = 1 - t$, and

$$G(t, s) = \begin{cases} t(1-s),\ 0 \le t \le s \le 1, \\ s(1-t),\ 0 \le s \le t \le 1. \end{cases}$$

Let $\mathcal{G}(s) = s(1-s),\ c(t) = t(1-t)$. Then $(H_1)'$ and $(H_2)'$ are satisfied.

For any $0 < r < R < +\infty$ and $u \in \overline{\Omega_R}\backslash\Omega_r$, we have $0 < rc(t) \le u(t) \le R$ for $t \in [0, 1]$. Then $|\ln u(t)| \le |\ln R| + |\ln rc(t)|$. Due to $\int_0^1 |\ln c(s)|ds = \int_0^1 (|\ln s| + |\ln(1-s)|)ds = 2$, the absolute continuity of the integral yields that $\lim_{m\to\infty} \int_{e(m)} |\ln c(s)|ds = 0$. Thus,

$$\begin{aligned} &\lim_{m\to\infty} \sup_{u\in\overline{\Omega_R}\backslash\Omega_r} \int_{e(m)} \mathcal{G}(s) f(u(s))ds \\ &\le \lim_{m\to\infty} \int_{e(m)} (2 - s + |\ln R| + |\ln rc(s)|)ds \\ &= (3 + 2|\ln R| + 2|\ln r|) \lim_{m\to\infty} \frac{1}{m} + \lim_{m\to\infty} \int_{e(m)} |\ln c(s)|ds = 0. \end{aligned}$$

So assumption $(H_3)'$ is satisfied.

We choose $\theta = \frac{1}{4}$. Then $\eta = \frac{3}{16}$, $l = \frac{(\frac{3}{16})^2}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)ds = \frac{11}{98304}$, $L = \frac{1}{4}\int_0^1 s(1-s)ds = \frac{1}{24}$. Direct computation shows that

$$\liminf_{x\to 0} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \infty, \quad \limsup_{x\to+\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x} = 0.$$

Therefore the assumptions of Theorem 3.3 are satisfied, and nonlocal boundary value problem (3.1) has at least one positive solution.

4 The case *f* changes sign

In this section, we investigate nonlocal differential Eq. (1.1) subject to a specific nonlocal boundary condition

$$\alpha u(0) - \beta u'(0) = 0, \quad \delta u'(1) = \varphi(u), \tag{4.1}$$

where $\alpha > 0,\ \beta > 0,\ \delta > 0,\ \varphi(u)$ is the same as Sect. 2. Using the fixed point theorem in double cones, we obtain the existence of positive solutions. It is worth mentioning that the nonlinearity is allowed to change sign and tend to negative infinity.

$(H)''$ $f : [0, 1] \times [0, +\infty) \to \mathbb{R}$ is continuous, and $f(t, 0) \ge 0\ (\not\equiv 0)$ for $t \in [0, 1]$.

Let

$$\Delta = 1 - \int_0^1 \frac{\beta + \alpha t}{\alpha \delta} d\alpha(t), \quad H(t,s) = \frac{\beta}{\alpha} + \begin{cases} s, & 0 \le s \le t \le 1, \\ t, & 0 \le t \le s \le 1. \end{cases}$$

By using standard arguments we obtain the following lemma.

Lemma 4.1 *Suppose that* $\Delta \neq 0$. *For any* $g \in C([0,1])$, *the nonlocal problem*

$$\begin{cases} -A\left(\int_0^1 (u^p(s) + u^q(s))ds\right) u''(t) = g(t), & t \in (0,1), \\ \alpha u(0) - \beta u'(0) = 0, \quad \delta u'(1) = \varphi(u). \end{cases}$$

has a solution

$$u(t) = \int_0^1 \left(A\left(\int_0^1 (u^p + u^q)(r)dr\right)\right)^{-1} G(t,s) g(s) ds, \quad t \in [0,1],$$

where

$$G(t,s) = \frac{\beta + \alpha t}{\alpha \delta \Delta} \int_0^1 H(\tau, s) d\alpha(\tau) + H(t,s).$$

In the following we always assume that $\Delta > 0$. Obviously, $G : [0,1] \times [0,1] \to [0, +\infty)$ is continuous.

Define two cones K and K':

$$K = \{u \in E : u(t) \ge 0, \ t \in [0,1]\}, \quad K' = \{u \in K : u \text{ is concave on } [0,1]\}.$$

Define $\acute{\alpha} : K' \to [0, +\infty)$ by

$$\acute{\alpha}(x) = \min_{t \in [\theta, 1-\theta]} x(t), \quad \theta \in \left(0, \frac{1}{2}\right).$$

For $\rho > 0, \ a > 0, \ b > 0$, let

$$\widehat{V}_\rho = \left\{ u \in K : \int_0^1 (u^p(s) + u^q(s))ds < \rho \right\},$$
$$K_r = \{u \in K : \|u\| < r\}, \quad K'_r = \{u \in K' : \|u\| < r\},$$
$$K(b) = \{u \in K : \acute{\alpha}(u) < b\}, \quad K_a(b) = \{u \in K : a < \|u\|, \ \acute{\alpha}(u) < b\}.$$

From the definition of $\acute{\alpha}$, we immediately obtain the following properties.

Lemma 4.2 $\acute{\alpha}$ *is a continuous increasing function satisfying* $\acute{\alpha}(x) \le \|x\| \le M\acute{\alpha}(x)$, *where* $M \ge 1$ *is a constant.*

Define operators $S,\ T$ *and* $\tilde{T}: E \to E$ *as follows:*

$$Su(t) = \int_0^1 \left(A\left(\int_0^1 (u^p(r) + u^q(r))dr\right)\right)^{-1} G(t,s)f(s,u(s))ds, \quad t \in [0,1],$$

$$Tu(t) = \left(\int_0^1 \left(A\left(\int_0^1 (u^p(r) + u^q(r))dr\right)\right)^{-1} G(t,s)f(s,u(s))ds\right)^+, \quad t \in [0,1],$$

$$\tilde{T}u(t) = \int_0^1 \left(A\left(\int_0^1 (u^p(r) + u^q(r))dr\right)\right)^{-1} G(t,s)f^+(s,u(s))ds, \quad t \in [0,1],$$

where $f^+(t,x) = \max\{f(t,x), 0\}$. *We define* $\psi : E \to K$ *by* $(\psi u)(t) = \max\{u(t), 0\}$. *Then* $T = \psi \circ S$ *and* u *is a positive solution of problem (1.1) and (4.1) if and only if* u *is a positive fixed point of operator* S.

By the arguments similar to Lemma 3.2 *in* [11], *we have the following result.*

Lemma 4.3 *If* $S : \overline{\widehat{V}_{\rho_2}} \backslash \widehat{V}_{\rho_1} \to E$ *is completely continuous, then* $T = \psi \circ S : \overline{\widehat{V}_{\rho_2}} \backslash \widehat{V}_{\rho_1} \to K$ *is completely continuous.*

Lemma 4.4 *Assume that* $(H)''$ *and* (H_3) *hold. Then* $T : \overline{\widehat{V}_{\rho_2}} \backslash \widehat{V}_{\rho_1} \to K$ *and* $\tilde{T} : K' \cap \left(\overline{\widehat{V}_{\rho_2}} \backslash \widehat{V}_{\rho_1}\right) \to K'$ *are completely continuous.*

Proof From the continuity of f, it is easy to show that $S : \overline{\widehat{V}_{\rho_2}} \backslash \widehat{V}_{\rho_1} \to E$ is completely continuous. So $T : \overline{\widehat{V}_{\rho_2}} \backslash \widehat{V}_{\rho_1} \to K$ is completely continuous by Lemma 4.3. By standard arguments, $\tilde{T} : K' \cap \left(\overline{\widehat{V}_{\rho_2}} \backslash \widehat{V}_{\rho_1}\right) \to K'$ is completely continuous. □

Lemma 4.5 *For any* $\rho > 0$, $\overline{K'_{M_\rho}} \subseteq \overline{\widehat{V}_\rho} \bigcap K' \subseteq \overline{K'_{E_\rho}}$, *where* $M_\rho \in (0, \rho^{\frac{1}{q}})$ *is the unique positive solution of* $x^p + x^q = \rho$ *and* $E_\rho \in (0, \frac{1}{\theta}(\frac{\rho}{1-2\theta})^{\frac{1}{q}})$ *is the unique positive solution of* $(\theta x)^p + (\theta x)^q - \frac{\rho}{1-2\theta} = 0$.

Proof The proof is similar to Lemma 2.2, so we omit it. □

Lemma 4.6 [11] *Let* X *be a real Banach space with norm* $\|\cdot\|$, *and let* $K,\ K' \subset X$ *be two cones with* $K' \subseteq K$. *Suppose that* $T : K \to K$ *and* $\tilde{T} : K' \to K'$ *are two completely continuous operators and* $\acute{\alpha}(x) : K' \to [0, +\infty)$ *is a continuous increasing functional satisfying* $\acute{\alpha}(x) \le \|x\| \le M\acute{\alpha}(x)$ *for all* $x \in K'$ *and for some constant* $M \ge 1$. *Suppose that there exist two constants* $b > a > 0$ *such that*

(1) $\|\tilde{T}x\| < a$ *for* $x \in \partial K'_a$, *and* $\acute{\alpha}(\tilde{T}x) > b$ *for* $x \in \partial K'(b)$;
(2) $Tx = \tilde{T}x$ *for* $x \in K'_a(b) \bigcap \{x : \tilde{T}x = x\}$.

Then T *has a fixed point* y *in* K *satisfying* $\|y\| > a,\ \acute{\alpha}(y) < b$.

Theorem 4.7 *Assume that* $(H)''$ *and* (H_3) *hold, and* $\delta > \varphi(\mathbf{1})$. *Suppose that there exist constants* $d > 0,\ \theta \in (0, \frac{1}{2})$ *such that* $0 < \left(1 + \frac{\alpha}{\beta}\right) d < E_{\rho_1} < \theta M_{\rho_2} < M_{\rho_2}$, *and*

(1) $f(t,x) < \frac{E_{\rho_1}}{I}$ *for* $(t,x) \in [0,1] \times [0, E_{\rho_1}]$, *where* $I = \frac{1}{Q_1} \max_{t\in[0,1]} \int_0^1 G(t,s))ds$;
(2) $f(t,x) > \frac{\theta M_{\rho_2}}{J}$ *for* $(t,x) \in [\theta, 1-\theta] \times [\theta M_{\rho_2}, M_{\rho_2}]$, *where* $J = \frac{1}{Q_2} \min_{t\in[\theta,1-\theta]} \int_\theta^{1-\theta} G(t,s)ds$;
(3) $f(t,x) \geq 0$ *for* $(t,x) \in [0,1] \times [d, M_{\rho_2}]$.

Then boundary value problem (1.1) *and* (4.1) *has a positive solution.*

Proof For $u \in \partial K'_{E_{\rho_1}}$ and $t \in [0,1]$, we have $0 \leq u(t) \leq \|u\| = E_{\rho_1}$. Then

$$f^+(t, u(t)) \leq \frac{E_{\rho_1}}{I}.$$

Since $\min_{t\in[\theta,1-\theta]} u(t) \geq \theta\|u\|$ and $\partial K'_{E_{\rho_1}} \subseteq \overline{K'_{E_{\rho_1}}} \subseteq \overline{K'_{M_{\rho_2}}} \subseteq \widehat{V}_{\rho_2} \bigcap K'$, we have

$$\int_0^1 (u^p(r) + u^q(r))dr \geq \int_\theta^{1-\theta} [(\theta\|u\|)^p + (\theta\|u\|)^q]dr = \rho_1,$$

and

$$\int_0^1 (u^p(r) + u^q(r))dr \leq \int_0^1 (\|u\|^p + \|u\|^q)dr \leq (M_{\rho_2})^p + (M_{\rho_2})^q = \rho_2.$$

Then

$$\begin{aligned} \|\tilde{T}u\| &\leq \frac{1}{Q_1} \max_{t\in[0,1]} \int_0^1 G(t,s) f^+(s, u(s))ds \\ &\leq \frac{1}{Q_1} \max_{t\in[0,1]} \int_0^1 G(t,s) \frac{E_{\rho_1}}{I} ds = E_{\rho_1}, \quad u \in \partial K'_{E_{\rho_1}}. \end{aligned}$$

For $u \in \partial K'(\theta M_{\rho_2})$, we have $E_{\rho_1} < \theta M_{\rho_2} = \acute{\alpha}(u) \leq \|u\| \leq \frac{1}{\theta}\acute{\alpha}(u) = M_{\rho_2}$. By Lemma 4.5, we deduce that $\rho_1 < \int_0^1 (u^p(r)+u^q(r))dr \leq \rho_2$, and $\theta M_{\rho_2} \leq u(t) \leq M_{\rho_2}$ for $t \in [\theta, 1-\theta]$. Thus, we calculate

$$\begin{aligned} \acute{\alpha}(\tilde{T}u) &= \min_{t\in[\theta,1-\theta]} (\tilde{T}u)(t) > \min_{t\in[\theta,1-\theta]} \frac{1}{Q_2} \int_\theta^{1-\theta} G(t,s) \frac{\theta M_{\rho_2}}{J} ds \\ &= \theta M_{\rho_2}, \quad u \in \partial K'(\theta M_{\rho_2}). \end{aligned}$$

On the other hand, for $u \in K'_{E_{\rho_1}}(\theta M_{\rho_2}) \bigcap \{u : \tilde{T}u = u\}$, we have $\|u\| > E_{\rho_1} > \left(1 + \frac{\alpha}{\beta}\right) d$, $\acute{\alpha}(u) < \theta M_{\rho_2}$, $\alpha u(0) - \beta u'(0) = 0$ and $\delta u'(1) = \varphi(u)$. Since u is concave, $u(t)$ minimizes at $t = 0$ or $t = 1$. Due to $\varphi(u) = \int_0^1 u(s)d\alpha(s) \geq 0$, we have $u'(1) = \frac{1}{\delta}\varphi(u) \geq 0$. Therefore, $u(0) = \min_{0\leq t\leq 1} u(t)$ and $u(1) = \|u\|$.

We claim that $u(0) \geq d$. Otherwise, we assume $u(0) < d$. Then there exists $t_0 \in (0, 1)$ such that $u'(t_0) = u(1) - u(0) = \|u\| - u(0) > \frac{\alpha}{\beta}d$. Thanks to the fact that u is concave, $u'(0) \geq u'(t_0) > \frac{\alpha}{\beta}d$. It follows that

$$0 = \alpha u(0) - \beta u'(0) < \alpha d - \beta \frac{\alpha}{\beta} d = 0,$$

which is a contradiction. Then for $t \in [0, 1]$, we have $d \leq u(0) \leq u(t) \leq \|u\| \leq \frac{1}{\theta}\acute{\alpha}(u) < M_{\rho_2}$. By condition (3) of the theorem, we have $f^+(t, u(t)) = f(t, u(t))$. Thus, for $u \in K'_{E_{\rho_1}}(\theta M_{\rho_2}) \bigcap \{u : \tilde{T}u = u\}$, we obtain that $Tu = \tilde{T}u = u$. It follows from Lemma 4.6 that T has a fixed point u_0 in K satisfying $\|u_0\| > E_{\rho_1} > 0$ and $\acute{\alpha}(u_0) < \theta M_{\rho_2}$. Thus $E_{\rho_1} \leq \|u_0\| \leq M_{\rho_2}$, and $\rho_1 \leq \int_0^1 (u_0^p(r) + u_0^q(r))dr \leq \rho_2$.

We next show that u_0 is a positive solution of problem (1.1) and (4.1). It is sufficient to prove that u_0 is a fixed point of S. Arguing indirectly, we suppose that there exists $t_1 \in (0, 1)$ such that $u_0(t_1) \neq (Su_0)(t_1)$. From $u_0(t) = (Tu_0)(t) = \max\{(Su_0)(t), 0\}$, it follows that $(Su_0)(t_1) < 0$ and $u_0(t_1) = 0$. Assume that (t_2, t_3) contains t_1, and it is the maximum interval which satisfies $(Su_0)(t) < 0$ for any $t \in (t_2, t_3)$. By $(H)''$, we get $[t_2, t_3] \neq [0, 1]$. Thus $t_2 > 0$ or $t_3 < 1$.

For the case $t_3 < 1$, we have $(Su_0)(t) < 0$, $u_0(t) = 0$ for any $t \in (t_2, t_3)$, so $(Su_0)(t_3) = 0$, and $(Su_0)'(t_3) \geq 0$. Noting that

$$(Su_1)''(t) = -\left(A\left(\int_0^1 (u_0^p(r) + u_0^q(r))ds\right)\right)^{-1} f(t, 0) \leq 0, \quad t \in (t_2, t_3),$$

so $(Su_0)'(t) \geq (Su_0)'(t_3) \geq 0$ for $t \in [t_2, t_3]$. It follows that $(Su_0)(t) \leq (Su_0)(t_3) = 0$ for $t \in [t_2, t_3]$, which implies $t_2 = 0$, which contradicts $(Su_0)'(0) = \frac{\alpha}{\beta}(Su_0)(0) < 0$.

For the case $t_2 > 0$, we have $u_0(t) = 0$ for $t \in [t_2, t_3]$ and $(Su_0)(t_2) = 0$. Then $(Su_0)'(t_2) \leq 0$. From $(H_1)''$, we have

$$(Su_0)''(t) = -\left(A\left(\int_0^1 (u_0^p(r) + u_0^q(r))dr\right)\right)^{-1} f(t, 0) \leq 0, \quad t \in (t_2, t_3),$$

and

$$(Su_0)'(t) \leq (Su_0)'(t_2) \leq 0, \quad t \in [t_2, t_3].$$

Then $(Su_0)(t) \leq (Su_0)(t_2) = 0$ for $t \in [t_2, t_3]$. Therefore, $t_3 = 1$ and $(Su_0)'(1) < 0$.

We claim that $(Su_0)(t) \geq 0$ for $t \in [0, t_2]$. If otherwise, there exists $t_5 \in (0, t_2)$ such that $(Su_0)(t_5) < 0$ and $u_0(t_5) = 0$. Assume that $(t_6, t_7) \subseteq [0, t_2)$ contains t_5, and it is the maximum interval which satisfies $(Su_0)(t) < 0$ for any $t \in (t_6, t_7)$. Due to $t_7 < t_2 < 1$, we can obtain a contradiction by a similar proof to the case of $t_3 < 1$. Thus, $(Su_0)(t) \geq 0$ for $t \in [0, t_2]$. Explicit computations show that

$$0 > \delta(Su_0)'(1) = \varphi(Su_0) = \int_0^1 (Su_0)(t)d\alpha(t) \geq \int_{t_2}^1 (Su_0)(t)d\alpha(t),$$

and

$$|\delta(Su_0)'(1)| \leq \left|\int_{t_2}^{1}(Su_0)(t)d\alpha(t)\right| \leq \int_{t_2}^{1}|(Su_0)(t)|d\alpha(t).$$

In view of the fact that $(Su_0)'(t) \leq 0$ and $(Su_0)(t) \leq 0$ for $t \in [t_2, 1]$, we have

$$|(Su_0)'(1)| \geq \frac{|(Su_0)(t) - (Su_0)(t_2)|}{t - t_2}, \quad t \in (t_2, 1).$$

Thus, $|(Su_0)(t)| \leq (t - t_2)|(Su_0)'(1)| \leq |(Su_0)'(1)|$ for $t \in (t_2, 1)$. It follows that

$$|\delta(Su_0)'(1)| \leq \int_{t_2}^{1}|(Su_0)'(1)|d\alpha(t) \leq \varphi(\mathbf{1})|(Su_0)'(1)|.$$

Hence $\delta \leq \varphi(\mathbf{1})$, which is a contradiction to $\delta > \varphi(\mathbf{1})$. In a conclusion, u_0 is a fixed point of S, i.e., u_0 is a positive solution of problem (1.1) and (4.1). □

Example 4.8 Let $p = \frac{1}{2},\ q = 2,\ \alpha = \beta = \delta = 1,\ \varphi(u) = \int_0^1 u(s)\, d(\frac{1}{2}s)$. Then $\delta > \varphi(\mathbf{1})$. Choose $A(t) = \sin t,\ \rho_1 = 0.1$ and $\rho_2 = 2$. Then (H_3) holds and $E_{\rho_1} \approx 0.16,\ M_{\rho_2} = 1$. Define the function $f : [0, 1] \times [0, +\infty) \to \mathbb{R}$ by

$$f(t, x) = \frac{1}{1000}t + \begin{cases} \frac{1}{1000}x, & x \in [0, 0.2], \\ 15.98x^2 - 3.195x, & x \in (0.2, 1], \\ -x + 13.785, & x \in (1, +\infty). \end{cases}$$

It is easy to see that $(H)''$ holds. Consider the nonlocal problem

$$\begin{cases} -\sin\left(\int_0^1 (u^{\frac{1}{2}}(s) + u^2(s))ds\right)u''(t) = f(t, u(t)), \quad t \in (0, 1), \\ u(0) - u'(0) = 0, \quad u'(1) = \int_0^1 u(s)d\left(\frac{1}{2}s\right). \end{cases} \tag{4.2}$$

Let $d = \frac{1}{20} > 0,\ \theta = \frac{1}{4}$. Then $0 < \left(1 + \frac{\alpha}{\beta}\right)d < E_{\rho_1} < \theta M_{\rho_2} < M_{\rho_2}$. Simple calculation shows that $I = \frac{935}{9},\ J = \frac{895}{384}$,

$$\max_{(t,x)\in[0,1]\times[0,0.16]} f(t, x) \approx 0.00116 < \frac{0.16}{I} \approx 0.0015,$$

$$\min_{(t,x)\in[\frac{1}{4},\frac{3}{4}]\times[0.25,1]} f(t, x) \approx 0.2 > \frac{0.2}{J_1} \approx 0.11$$

and

$$\min_{(t,x)\in[0,1]\times[\frac{1}{20},1]} f(t, x) \geq 0.$$

So, all conditions of Theorem 4.7 are satisfied. Then nonlocal problem (4.2) has a positive solution.

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