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A new minimal element theorem and new generalizations of Ekeland's variational principle in complete lattice optimization problem

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Abstract

In this paper, we first introduce some types of set relations on the power set of n-dimensional Euclidean spaces which are proposed by Kuroiwa–Tanaka–Ha and Jahn–Ha. We also mention new types of cancellation laws of set relations. Second, we introduce a complete lattice-valued problem on the power set of n-dimensional Euclidean spaces proposed by Hamel et al. Applying nonlinear scalarizing technique in complete lattice, we present a new type of minimal element theorem and generalized Ekeland's variational principles in complete lattice optimization problem. We also present an existence theorem of minimal solutions related to the famous Takahashi's minimization theorem in complete lattice optimization problem.

Keywords Set order relations · Cancellation laws · Complete lattice · Minimal element theorem · Brézis–Browder's principle · Ekeland's variational principle · Takahashi's minimization theorem

Mathematics Subject Classification 06B23 · 06F30 · 49J53 · 58E17

1 Introduction

The set optimization problem formalized as follows:

$$(\mathrm{P}) \begin{cases} \text{Optimize} \quad F(x) \\ \text{Subject to} \quad x \in X, \end{cases}$$

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where X is a nonempty set, Y is a topological vector space ordered by a closed convex cone $C \subset Y$ and $F : X \to 2^Y$ is a set-valued map with domain X, that is, $F(x) \neq \emptyset$ for each $x \in X$.

The vector optimization problem is to take the union of all objective values and then search for (weakly, properly etc.) minimal points in this union with respect to the vector ordering. This approach has been applied as a leading idea since the late 1980 s, and supported by a number of researchers. This approach is called the vector approach to set optimization.

The situation changed in the case that set order relations were proposed by Kuroiwa–Tanaka–Ha [33, 35] around 2000. They introduced six types of set order relations on the power set of topological vector space applying a convex ordering cone C with nonempty interior. Its basic idea is to "lift" the concept of minimal (=non-dominated) image points from the elements of a vector space to those of the power set of the vector space: see [26]. Therefore, this approach is called the set relation approach to set optimization. Jahn [31] states in his book that the set relations approach 'opens a new and wide field of research' and the so called set relations 'turn out to be very promising in set optimization.' Since the lower type and upper type set order relations satisfy reflexivity and transitivity, many researchers recognize that the above two are specifically important in set optimization problem.

In the 2010 s, there is a big progress in set optimization problem. By the definitions of equivalent classes with respect to lower and upper type set order relations mentioned the above and hull operations, Hamel et al. [26] defines spaces of sets which enjoy lattice structure. They called the above one "complete lattice approach" to set optimization. The set order relations outwardly disappear and the subset or supset inclusions appears as a partial order. We will discuss the complete lattice optimization problem and introduce new concepts for this problem.

Recently, new types of cancellation laws of set order relations are proposed by Durea-Florea [16] and algebraic operations of set order relations [5] were investigated. In Sect. 4, we will also discuss cancellation laws of set order relations and the complete lattice.

The important applications of minimal points are of special interest in vector optimization problems, vector equilibrium problems, vector variational inequality, and vector complementarity problem. In 1976, Brézis and Browder established a famous minimal point theorem on a quasi-ordered set (so-called Brézis–Browder's principle) as follows:

Theorem 1.1 (Brézis–Browder [11]) *Let $(W, \preceq)$ be a quasi-ordered set (that is, $\preceq$ is a reflexive and transitive relation on W) and let $\phi : W \to \mathbb{R}$ be a function satisfying*

(A1) *ϕ is bounded below,*
(A2) *$w_1 \preceq w_2$ implies $\phi(w_1) \leq \phi(w_2)$,*
(A3) *for every $\preceq$-decreasing sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq W$ there exists some $w \in W$ such that $w \preceq w_n$ for all $n \in \mathbb{N}$.*

Then for every $w_0 \in W$ there exists some $\bar{w} \in W$ such that

(i) *$\bar{w} \preceq w_0$,*
(ii) *$\hat{w} \preceq \bar{w}$ implies $\phi(\hat{w}) = \phi(\bar{w})$.*

The Brézis–Browder's principle and various minimal/maximal point theorems have been generalized and improved in many various different directions; for more details, we refer the readers to the papers [1, 13–15, 19, 25, 26, 36, 37] and references therein.

In 1980 s, Gerstewitz [20] introduced a nonlinear scalarizing function for deriving separation theorems for nonconvex sets and scalarization methods in vector optimization. The readers can check a short history of Gerstewitz's scalarizing functions in Sect. 4.15 of [45]. Araya [6, 9] discussed generalizing Gerstewitz's function to set-valued version. The functions studied by Araya [6, 9] play the role of utility functions (for details, see Sec. 4 below). In this work, we will consider nonlinear scalarizing functions in complete lattices.

In this paper, we aim to obtain a new existence result of complete lattice optimization problem using the Brézis–Browder's principle. For this purpose, we establish some new concepts on complete lattice optimization problem to derive nonlinear scalarization technique in complete lattice which is a natural generalization of Gerstewitz's scalarizing function [20].

This paper is organized as follows. First, we introduce some types of set relations on the power set of n-dimensional Euclidean spaces which are proposed by Kuroiwa–Tanaka–Ha [35]. We also mention cancellation laws of set order relations. Second we introduce a complete lattice-valued problems on the power set of n-dimensional Euclidean spaces proposed by Hamel et al. [26]. Applying nonlinear scalarizing technique in complete lattice, we present a new type of minimal element theorem and generalized Ekeland's variational principle in complete lattice optimization problem. We also present an existence theorem of minimal solutions in complete lattice optimization problem.

2 Preliminaries

We first recall some notations, definitions and well-known results, which will be used in this paper. Let $\mathbb{R}^n$ be n-dimensional Euclidean space,

$$\mathbb{R}^n_+ := \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0\}$$

be its nonnegative orthant and $\mathbf{0}$ be the origin of $\mathbb{R}^n$, respectively.

For a set $A \subset \mathbb{R}^n$, $\mathrm{int}(A)$, $\mathrm{cl}(A)$ and $\mathrm{cor}(A)$ denote the topological interior, the topological closure, and algebraic interior respectively. A nonempty set A is called solid if $\mathrm{int}A \neq \emptyset$. The symbol $\mathcal{P}(\mathbb{R}^n)$ denote the family of nonempty subsets of $\mathbb{R}^n$ including the empty set $\emptyset$ and $\mathcal{V}$ denote the family of nonempty subsets of $\mathbb{R}^n$. The sum of two sets $V_1, V_2 \in \mathcal{V}$ and the product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ are defined by

(OP) $V_1 + V_2 := \{v_1 + v_2 \,|\, v_1 \in V_1, v_2 \in V_2\}, \qquad \alpha V := \{\alpha v \,|\, v \in V\}.$

In this paper, we assume that $C \subset \mathbb{R}^n$ is a solid pointed closed convex cone, that is, $\mathrm{int}C \neq \emptyset$, $C \cap (-C) = \{\mathbf{0}\}$, $\mathrm{cl}C = C$, $C + C \subset C$ and $t \cdot C \subset C$ for all $t \in [0, \infty)$.

Lemma 2.1 *For $C \subset \mathbb{R}^n$ a closed convex cone and $A, B, V \in \mathcal{V}$, the following statements hold:*

(i) $C + C = C$;
(ii) $C + \mathrm{int}(C) = \mathrm{int}(C)$;
(iii) $\mathrm{cl}(A) + \mathrm{cl}(B) \subset \mathrm{cl}(A + B)$;
(iv) $\mathrm{cl}(V + C) + C = \mathrm{cl}(V + C)$.

Definition 2.1 For $a, b \in \mathbb{R}^n$ and a solid convex cone $C \subset \mathbb{R}^n$, we define

$$a \leq_C b \quad \text{by} \quad b - a \in C \qquad\qquad a \leq_{\mathrm{int}C} b \quad \text{by} \quad b - a \in \mathrm{int}(C).$$

Proposition 2.1 *For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the following statements hold:*

(i) *$x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in \mathbb{R}^n$,*
(ii) *$x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$,*
(iii) *$\leq_C$ is reflexive and transitive. Moreover, if C is pointed, $\leq_C$ is antisymmetric and hence a partial order.*

We next introduce the concept of minimal elements in vector optimization problem, which are also known as Edgeworth-Pareto-minimal or efficient elements.

Definition 2.2 (*Optimality notions in vector optimization* [17]) Let Z denote a real vector space that is pre-ordered by some convex cone $C \subset Z$ and let A denote some nonempty subset of Z. We also suppose that $\mathrm{cor}(C) \neq \emptyset$.

- An element $\bar{z} \in A$ is called a *minimal* element of the set A, if

$$A \cap (\bar{z} - C) \subset \{\bar{z}\} + C.$$

  If C is pointed, then the above inclusions can be replaced by

$$A \cap (\bar{z} - C) = \{\bar{z}\}.$$

- An element $\bar{z} \in A$ is called a *weakly minimal* element of the set A, if

$$A \cap (\bar{z} - \mathrm{cor}(C)) = \emptyset.$$

Lemma 2.2 ([17]) *Let C have a nonempty algebraic interior and $C \neq Z$. Then every minimal element of the set A is also a weakly minimal element of the set A.*

3 Set optimization and complete lattice optimization problem

3.1 Preliminaries in set optimization

Definition 3.1 (*Kuroiwa–Tanaka–Ha* [35]) For $A, B \in \mathcal{V}$ and a solid closed convex cone $C \subset \mathbb{R}^n$, we define

- **(Lower type)** $A \leq_C^l B$ by $B \subset A + C$;
- **(Upper type)** $A \leq_C^u B$ by $A \subset B - C$.

Proposition 3.1 (see also [6, 9, 26]) *For $A, B, D \in \mathcal{V}$ and $\alpha \geq 0$, the following statements hold.*

(i) $\leq_C^l$ *and* $\leq_C^u$ *are reflexive and transitive.*

(ii) $A \leq_C^l B \iff -B \leq_C^u -A \iff B \leq_{-C}^l -A$.

(iii) $A \leq_C^l B \iff B + C \subset A + C$ *and* $A \leq_C^u B \iff A - C \subset B - C$.

(iv) $A \leq_C^l B$ *and* $A \leq_C^u B$ *are not comparable, that is,* $A \leq_C^l B$ *does not imply* $A \leq_C^u B$ *and* $A \leq_C^u B$ *does not imply* $A \leq_C^l B$.

(v) $A \leq_C^l B$ *implies* $A + D \leq_C^l B + D$ *and* $A \leq_C^u B$ *implies* $A + D \leq_C^u B + D$.

(vi) $A \leq_C^l B$ *implies* $\alpha A \leq_C^l \alpha B$ *and* $A \leq_C^u B$ *implies* $\alpha A \leq_C^u \alpha B$.

Definition 3.2 ([28, 40]) It is said that $A \in \mathcal{V}$ is

(i) C-*proper* (resp. $(-C)$-*proper*) if $A + C \neq \mathbb{R}^n$ (resp. $A - C \neq \mathbb{R}^n$).
(ii) C-*closed* (resp. $(-C)$-*closed*) if $A + C$ (resp. $A - C$) is a closed set,
(iii) C-*bounded* (resp. $(-C)$-*bounded*) if for each neighborhood U of zero in $\mathbb{R}^n$ there is some positive number $t > 0$ such that

$$A \subset tU + C \quad (\text{resp. } A \subset tU - C),$$

(iv) C-*compact* (resp. $(-C)$-*compact*) if any cover of A the form

$$\{U_\alpha + C \mid U_\alpha \text{ are open}\} \quad (\text{resp. } \{U_\alpha - C \mid U_\alpha \text{ are open}\})$$

admits a finite subcover,
(v) C-*convex* (resp. $(-C)$-*convex*) if $A + C$ (resp. $A - C$) is a convex set.

The symbol $\mathcal{V}_C$ denote the family of C-proper subsets of $\mathbb{R}^n$, $\mathcal{V}_{-C}$ denote the family of $(-C)$-proper subsets of $\mathbb{R}^n$, respectively. It is easy to see that every C-compact set is C-closed and C-bounded.

Introducing the equivalence relations

$$A \simeq_l B \iff A \leq_C^l B \quad \text{and} \quad B \leq_C^l A,$$
$$A \simeq_u B \iff A \leq_C^u B \quad \text{and} \quad B \leq_C^u A,$$

we can generate the set of equivalence classes which are denoted by $[\cdot]^l$ and $[\cdot]^u$, respectively. The followings are easily confirmed.

$$(\diamond) \quad A \in [B]^l \iff A + C = B + C, \quad A \in [B]^u \iff A - C = B - C.$$

Definition 3.3 (*l-minimal element, u-minimal element*) Let $\mathcal{S} \subset \mathcal{V}$. We say that $\bar{A} \in \mathcal{S}$ is a $l[u]$-minimal element if for any $A \in \mathcal{S}$,

$$A \preceq_C^{l[u]} \bar{A} \quad \text{implies} \quad \bar{A} \preceq_C^{l[u]} A.$$

The symbols $l[u]$-Min$(\mathcal{S}; C)$ denote the family of $l[u]$-minimal elements of $\mathcal{S}$.

3.2 Complete lattice optimization problem

In this section, we introduce the concept of lattice which is an abstract structure studied in the mathematical subdisciplines of order theory and abstract algebra.

Definition 3.4 (*Join, meet* [12]) Let P be a nonempty partially ordered set and $x, y \in P$. We write $x \vee y$ (read as 'x join y') in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ (read as 'x meet y') in place of $\inf\{x, y\}$ when it exists. Similarly, we write $\bigvee_P S$ (the 'join of S') and $\bigwedge_P S$ (the 'meet of S') instead of $\sup S$ and $\inf S$, when these exist.

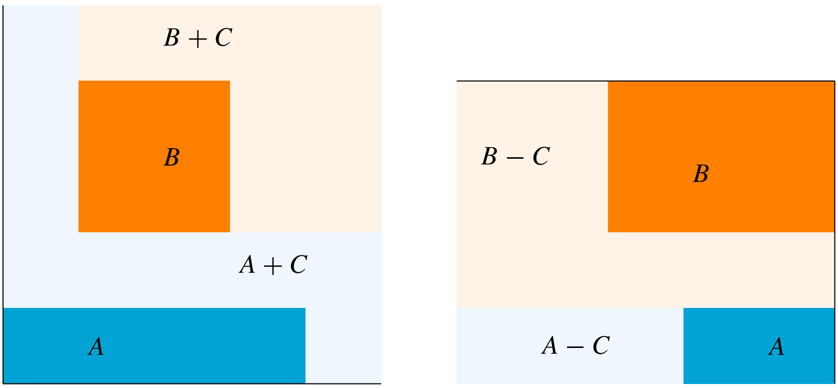
Definition 3.5 (*Lattice, complete lattice* [12]) Let P be a nonempty partially ordered set.

(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a lattice.
(ii) If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a complete lattice.

Proposition 3.2 ([12]) *Let L be a lattice. Then $\vee$ and $\wedge$ satisfy associative laws, commutative laws, idempotency laws and absorption laws.*

Next, we consider complete lattice-valued optimization problem on the power set of $\mathbb{R}^n$. We recall that the infimum of a subset $V \subseteq W$ of a partially ordered set $(W, \preceq)$ is an element $\bar{w} \in W$ satisfying $\bar{w} \preceq v$ for all $v \in V$ and $w \preceq \bar{w}$ whenever $w \preceq v$ for all $v \in V$. This means that the infimum is the greatest lower bound of V in W. The infimum of V is denoted by $\inf V$. Likewise, the supremum $\sup V$ is defined as the least upper bound of V (see also [26]). The property (iii) in Proposition 3.1 and $(\diamond)$ allow to define the following set

$$\mathcal{L} := \{A \in \mathcal{P}(\mathbb{R}^n) \,|\, A = A + C\}, \qquad \mathcal{U} := \{A \in \mathcal{P}(\mathbb{R}^n) \,|\, A = A - C\}.$$



Left: $A, B \in \mathcal{L}$ such that $A \supset B$. Right: $A, B \in \mathcal{U}$ such that $A \subset B$.

We can easily see that $(\mathcal{L}, \supseteq)$ and $(\mathcal{U}, \subseteq)$, are partially ordered set (that is, the above order relations satisfy the antisymmetric property).

Proposition 3.3 ([26]) *The pair $(\mathcal{L}, \supseteq)$ is a complete lattice. Moreover, for a subset $\mathcal{A} \subseteq \mathcal{L}$, the infimum and supremum of $\mathcal{A}$ are given by*

$$\inf \mathcal{A} = \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$$

where it is understood that $\inf \mathcal{A} = \emptyset$ *and* $\sup \mathcal{A} = \mathbb{R}^n$ *whenever $\mathcal{A} = \emptyset$. The greatest (top) element of $\mathcal{L}$ with respect to $\supseteq$ is $\emptyset$, the least (bottom) element is $\mathbb{R}^n$.*

Proposition 3.4 ([26]) *The following statements hold.*

(i) *For $A, B, D, E \in \mathcal{L}$, $A \supseteq B$, $D \supseteq E$ implies $A + D \supseteq B + E$.*
(ii) *For $A, B \in \mathcal{L}$, $A \supseteq B$, $s \geq 0$ implies $sA \supseteq sB$.*
(iii) *$\mathcal{A} \subseteq \mathcal{L}$, $B \in \mathcal{L}$ implies* $\inf(\mathcal{A} + B) = (\inf \mathcal{A}) + B$ *and $\mathcal{A} \subseteq \mathcal{L}$, $B \in \mathcal{L}$ implies* $\sup(\mathcal{A} + B) \supseteq (\sup \mathcal{A}) + B$, *where* $\mathcal{A} + B = \{A + B \,|A \in \mathcal{A}\}$.

Inspired by Definition 3.2, we introduce the following new concepts.

Definition 3.6 It is said that $A \in \mathcal{L}$ (resp. $B \in \mathcal{U}$) is

(i) $\mathcal{L}$-proper (resp. $\mathcal{U}$-proper) if $A \neq \mathbb{R}^n$ (resp. $B \neq \mathbb{R}^n$).
(ii) $\mathcal{L}$-closed (resp. $\mathcal{U}$-closed) if A (resp. B) is a closed set,
(iii) $\mathcal{L}$-bounded (resp. *$\mathcal{U}$-bounded*) if for each neighborhood U_1 (resp. U_2) of zero in $\mathbb{R}^n$

$$U_1 = U_1 + C \quad (\text{resp. } U_2 = U_2 - C),$$

there is some positive number $t > 0$ such that $A \subset tU_1$ (resp. $B \subset tU_2$),
(iv) $\mathcal{L}$-compact (resp. *$\mathcal{U}$-compact*) if any cover of A the form

$$\{U_\alpha |\, U_\alpha \text{ are open and } U_\alpha + C = U_\alpha\}$$
$$(\text{resp. } \{U_\alpha |\, U_\alpha \text{ are open and } U_\alpha - C = U_\alpha\})$$

admits a finite subcover,
(v) $\mathcal{L}$-convex (resp. *$\mathcal{U}$-convex*) if A (resp. B) is a convex set.

The symbol $\mathcal{L}_C$ denotes the family of $\mathcal{L}$-proper subsets of Y and $\mathcal{U}_{-C}$ denotes the family of $\mathcal{U}$-proper subsets of Y, respectively.

Remark 3.1 We first remark that the following relationships:

(i) Every $\mathcal{L}$-compact set is $\mathcal{L}$-closed and $\mathcal{L}$-bounded.
(ii) Every $\mathcal{U}$-compact set is $\mathcal{U}$-closed and $\mathcal{U}$-bounded.

We now compare $\mathcal{L}$-compactness with the concept of compactness in lattice shown below.

Definition 3.7 (*Compactness in lattice* [12]) Let L be a complete lattice and let $k \in L$. k is said to be compact if for every subset $S \subseteq L$, there is some finite subset T of S such that

$$k \leq \bigvee S \implies k \leq \bigvee T.$$

The set of compact elements of L is denoted $K(L)$.

Remark 3.2 When U_β is a finite subset of U_α, (iv) of Definition 3.6 can be written as follows:

$$A \leq \bigvee U_\alpha \implies A \leq \bigvee U_\beta$$

It can be seen that the two concepts completely coincide. Furthermore, you will also see that "compactness in topological space" implies "compactness in complete lattice". In other words, we found that compactness in ordered space is an extension of topological compactness under a certain situation.

We conclude this subsection by introducing the solution concept in complete lattice-valued optimization problem. We set

$$\begin{aligned} \mathrm{cl}(\mathcal{L}) &:= \{A \in \mathcal{P}(\mathbb{R}^n) \,|\, A = \mathrm{cl}(A + C)\}, \\ \mathrm{clconv}(\mathcal{L}) &:= \{A \in \mathcal{P}(\mathbb{R}^n) \,|\, A = \mathrm{clconv}(A + C)\}. \end{aligned}$$

Definition 3.8 ([26]) Let $\mathcal{A} \subseteq \mathrm{cl}(\mathcal{L})$. An element $\bar{A} \in \mathcal{A}$ is called l-minimal for $\mathcal{A}$ if it satisfies

$$A \in \mathcal{A}, \quad A \supseteq \bar{A} \implies A = \bar{A}.$$

The set of all l-minimal elements of $\mathcal{A}$ is denoted by Min$\mathcal{A}$.

Let M be a nonempty set and $F : M \to \mathrm{cl}(\mathcal{L})$ a set-valued mapping. Similar to [46], we consider the following complete lattice-valued optimization problem:

(CLOP) Minimize $F(x)$ subject to $x \in M$.

Definition 3.9 (*Minimal solutions*) A point $x_0 \in M$ is said to be

(i) an $\mathcal{L}$-minimal solutions of (CLOP) if for any $x \in M$, $F(x) \subset F(x_0)$ implies $F(x) = F(x_0)$. The set of all $\mathcal{L}$-minimal solutions of (CLOP) is denoted by $\mathrm{Min}(F(M); \subset)$.

(ii) a weak $\mathcal{L}$-minimal solutions of (CLOP) if for any $x \in M$, $F(x) \subset \mathrm{int}(F(x_0))$ implies $F(x) = F(x_0)$. The set of all weak $\mathcal{L}$-minimal solutions of (CLOP) is denoted by $\mathrm{wMin}(F(M); \subset)$.

Remark 3.3 In [26], they introduced complete lattice optimization problem (CL) using the concept of the infimum and minimal elements. Given a set $\mathcal{A} \subseteq \mathrm{cl}(\mathcal{L})$ or $\mathcal{A} \subseteq \mathrm{clconv}(\mathcal{L})$, complete lattice optimization problem look for

(CL) a set $\mathcal{B} \subseteq \mathcal{A}$ such that

$$\inf \mathcal{B} = \inf \mathcal{A} \quad \text{and} \quad \mathcal{B} \subseteq \text{Min}\mathcal{A}.$$

In this paper, for simplicity, we adopt the definition of the minimal solution of (CLOP) using the definition 3.8 and 3.9. The concept of minimal solutions using (CL) is a subject for future research.

3.3 Cancellation laws in complete lattices

The Rådström cancellation law [43] is a well-known fundamental result.

Proposition 3.5 *([43]) Let X be a normed space over the real field $\mathbb{R}$. Suppose that $A, B, C \subset X$ are nonempty sets and B is closed and convex, C is bounded, and*

$$A + C \subset B + C.$$

Then $A \subset B$.

After, Prakash-Sertel [41, 42] generalized the above result. In [5], the author rewrote the following forms using set order relations.

Proposition 3.6 (Cancelation law [5]: see also [41–43]) *For A, $B \in \mathcal{V}$ and $C \subset \mathbb{R}^n$ a closed convex cone, the following statements hold.*

(i) *If $B \in \mathcal{V}$ is bounded, then $B \leq^l_C B + A$ implies $\mathbf{0} \leq^l_C A$.*
(ii) *If $B \in \mathcal{V}$ is bounded, then $B + A \leq^u_C B$ implies $A \leq^u_C \mathbf{0}$.*
(iii) *If $B \in \mathcal{V}$ is compact, then $B \leq^l_{\text{int}C} B + A$ implies $\mathbf{0} \leq^l_{\text{int}C} A$.*
(iv) *If $B \in \mathcal{V}$ is compact, then $B + A \leq^u_{\text{int}C} B$ implies $A \leq^u_{\text{int}C} \mathbf{0}$.*

Recently in [16], they gave a new cancellation law which is a generalization of [41–43].

Proposition 3.7 (Durea-Florea [16]) *Let X be a normed space over the real field $\mathbb{R}$ and $C \subset X$ be a pointed closed convex cone. Suppose that $A, B, D \subset X$ are nonempty sets such that D is C-bounded and*

$$A + D \subset B + D + C.$$

Then we have that

$$A \subset \text{cl conv}(B + C).$$

Following the same line as [16], we have the following form.

Proposition 3.8 *Let $C \subset \mathbb{R}^n$ be a pointed closed convex cone. Suppose that $A, B, D \subset X$ are nonempty sets such that D is $(-C)$-bounded and*

$$A + D \subset B + D - C.$$

Then we have that

$$A \subset \text{cl conv}(B - C).$$

Using Proposition 3.7 and 3.8, we obtain new cancellation laws which is more relaxed version of Proposition 3.6.

Proposition 3.9 *We assume that $A_1 \in \mathcal{V}$ is C-closed, C-convex, $B_1 \in \mathcal{V}$ and $D_1 \in \mathcal{V}$ is C-bounded. We also assume that $A_2 \in \mathcal{V}$, $B_2 \in \mathcal{V}$ is $(-C)$-closed, $(-C)$-convex and $D_2 \in \mathcal{V}$ is $(-C)$-bounded. Then the following statements hold.*

(i) *$A_1 \leq_C^l B_1$ is equivalent to $A_1 + D_1 \leq_C^l B_1 + D_1$ and*
(ii) *$A_2 \leq_C^u B_2$ is equivalent to $A_2 + D_2 \leq_C^u B_2 + D_2$.*

Remark 3.4 In Proposition 3.9, we have found that the concept of C-closedness, C-convexity and C-boundedness play an important role to obtain cancellation laws. Using [43], Nuriya-Kuroiwa [32, 34] introduced parametrized embedding functions on compact and convex subset to observe l-type solutions. Moreover in [5], we assumed C-convexity to establish algebraic operations on $\mathcal{V}$. It is a subject of next research is to investigate the relationships among embedding theorems, cancellation laws and algebraic operations of set order relations.

As a direct consequence of Proposition 3.9, we obtain the following cancellation laws in complete lattices.

Proposition 3.10 *The following statements hold.*

(i) *Let $A, B, D \in \mathcal{L}$ be such that $\mathcal{L}$-closed, $\mathcal{L}$-bounded and $\mathcal{L}$-convex. Then*

$$A \supseteq B \iff A + D \supseteq B + D.$$

(ii) *Let $A, B, D \in \mathcal{U}$ be such that $\mathcal{U}$-closed, $\mathcal{U}$-bounded and $\mathcal{U}$-convex. Then*

$$A \subseteq B \iff A + D \subseteq B + D.$$

4 Nonlinear scalarizations in complete lattices

In 1980 s, Gerstewitz [20] introduced a nonlinear scalarizing function for deriving separation theorems for nonconvex sets and scalarization methods in vector optimization. We first recall the following concepts.

Definition 4.1 (*Scalarization directions of sets* [10]) Let A be a nonempty subset in a real vector space Y. A vector $k \in Y \setminus \{0\}$ is called a scalarization direction of A if the following condition hold:

(a) $\forall t \geq 0$, $A + tk \subseteq A$, and
(b) $\forall y \in Y$, $\exists t \in \mathbb{R}$, $y + tk \notin A$.

The set of all scalarization direction of A is denoted by $\text{sd}(A)$.

We remark that if $A = C$ is a convex cone, then $\mathrm{sd}(C) = C \backslash (-C)$.

Definition 4.2 (*Nonlinear scalarization functionals* [10, 21, 22, 45]) Let A be a nonempty subset in a real vector space Y and $k \in \mathrm{sd}(A)$ be a scalarization direction of A. The functional $\varphi_{A,k} : Y \to [-\infty, \infty]$ defined by

$$\varphi_{A,k}(y) = \inf\{t \in \mathbb{R} \,|\, y \in A + tk\}$$

with $\inf \emptyset = \infty$ is called Gerstewitz's nonlinear (separating) scalarization functional generated by the set A and the scalarization direction k.

The readers can check a short history of Gerstewitz's scalarizing functions in Section 4.15 of [45]. In this paper, we simply discuss that $C \subset \mathbb{R}^n$ a solid closed convex cone. Moreover, the scalarizing function $\varphi_{A,k}$ has a dual form. Agreeing $\sup \emptyset = -\infty$, we define $\psi_{A,k} : Y \to [-\infty, \infty]$

$$\psi_{A,k}(y) = \sup\{t \in \mathbb{R} \,|\, y \in -A + tk\} \quad \big(\varphi_{A,k}(y) = -\psi_{A,k}(-y)\big).$$

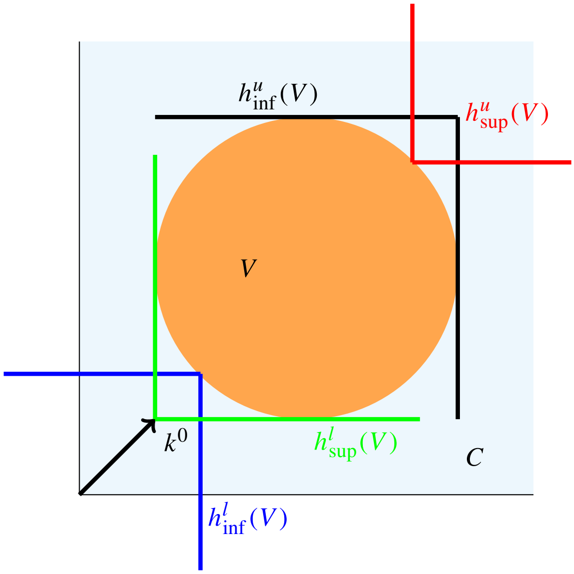
From the 2010 s, Araya discussed generalizing Gerstewitz's scalarization functionals to set-valued version: for more details, see [6, 9, 23]. Assume that $k^0 \in \mathrm{int}C$. Agreeing $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$, we defined $h^l_{\inf}(\cdot; k^0), h^u_{\inf}(\cdot; k^0) : \mathcal{V} \to [-\infty, \infty]$ and $h^l_{\sup}(\cdot; k^0), h^u_{\sup}(\cdot; k^0) : \mathcal{V} \to [-\infty, \infty]$ by

$$\begin{aligned}
h^l_{\inf}(V; k^0) &= \inf \left\{ t \in \mathbb{R} \,\middle|\, V \leq^l_C \{tk^0\} \right\} = \inf \left\{ t \in \mathbb{R} \,\middle|\, tk^0 \in V + C \right\}, \\
h^u_{\inf}(V; k^0) &= \inf \left\{ t \in \mathbb{R} \,\middle|\, V \leq^u_C \{tk^0\} \right\} = \inf \left\{ t \in \mathbb{R} \,\middle|\, V \subset tk^0 - C \right\}, \\
h^l_{\sup}(V; k^0) &= \sup \left\{ t \in \mathbb{R} \,\middle|\, \{tk^0\} \leq^l_C V \right\} = \sup \left\{ t \in \mathbb{R} \,\middle|\, V \subset tk^0 + C \right\}, \\
h^u_{\sup}(V; k^0) &= \sup \left\{ t \in \mathbb{R} \,\middle|\, \{tk^0\} \leq^u_C V \right\} = \sup \left\{ t \in \mathbb{R} \,\middle|\, tk^0 \in V - C \right\}.
\end{aligned}$$

The functions $h^l_{\inf}(\cdot; k^0)$, $h^u_{\inf}(\cdot; k^0)$, $h^l_{\sup}(\cdot; k^0)$ and $h^u_{\sup}(\cdot; k^0)$ play the role of utility functions.

Example 4.1 We set $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $k^0 = (1, 1)$ and

$$\begin{aligned}
&V = \{(x, y) \in \mathbb{R}^2 \,\big|\, (x-3)^2 + (y-3)^2 = 2^2\}, \\
&h^l_{\inf}(V; k^0) = 3 - \sqrt{2}, \qquad h^u_{\inf}(V; k^0) = 5, \\
&h^l_{\sup}(V; k^0) = 1, \qquad h^u_{\sup}(V; k^0) = 3 + \sqrt{2}.
\end{aligned}$$



We consider nonlinear scalarizing functions in complete lattices. Replacing $V \in \mathcal{V}$ with $V \in \mathcal{L}$ or $V \in \mathcal{U}$, that is, $h^l_{\inf}(\cdot; k^0), h^l_{\sup}(\cdot; k^0), : \mathcal{L} \to [-\infty, \infty]$ and $h^u_{\inf}(\cdot; k^0), h^u_{\sup}(\cdot; k^0) : \mathcal{U} \to [-\infty, \infty]$, we obtain the following form:

$$h^l_{\inf}(V; k^0) := \inf \left\{ t \in \mathbb{R} \middle| tk^0 \in V \right\},$$
$$h^u_{\inf}(V; k^0) := \inf \left\{ t \in \mathbb{R} \middle| V \subset tk^0 - C \right\},$$
$$h^l_{\sup}(V; k^0) := \sup \left\{ t \in \mathbb{R} \middle| V \subset tk^0 + C \right\},$$
$$h^u_{\sup}(V; k^0) := \sup \left\{ t \in \mathbb{R} \middle| tk^0 \in V \right\}.$$

We can confirm that the functions $h^l_{\inf}$ and $h^u_{\sup}$ are very similar to Minkowski functional.

Proposition 4.1 ([6, 9]) *The following statements hold:*

$$h^l_{\sup}(V; k^0) = -h^u_{\inf}(-V; k^0) \quad \text{and} \quad h^u_{\sup}(V; k^0) = -h^l_{\inf}(-V; k^0).$$

Definition 4.3 We say that the function

(i) $f_1 : \mathcal{L} \to [-\infty, \infty]$ is L-increasing if $V_1 \supset V_2$ implies $f_1(V_1) \leq f_1(V_2)$,
(ii) $f_2 : \mathrm{cl}(\mathcal{L}) \to [-\infty, \infty]$ is strictly L-increasing if $\mathrm{int}(V_1) \supset V_2$ implies $f_2(V_1) < f_2(V_2)$.

Remark 4.1 In this paper, we investigate l-infimum type scalarizing function based on the following two reasons:

- **(A)** It is suitable for minimization problem to adopt $\mathcal{L}$-valued complete lattices. Rockafellar-Wets [44] remarked that the second distributivity law does not extend to all of extended real field $\overline{\mathbb{R}}$. To solve the above problem, Löhne [38, 39] investigated the concept of conlinear spaces (semi-vector spaces: see also [5]). After, Hamel-Schrage [29] established an order theoretic and algebraic framework for the extended real numbers which includes extensions of the usual difference to expressions involving $-\infty$ and/or $+\infty$, so-called residuations. The authors of [44] consider that it is natural to associate minimization with inf-addition. From the above facts, Hamel et al. [26] also pointed out that associating $\leq_C^l$ with minimization and $\leq_C^u$ with maximization, the theory works for these cases: see also [26] (the footnote at bottom of page 77). The authors agree their opinions.
- **(B)** Gerstewitz's scalarizing functions φ_{C,k^0} [20] is suitable for minimization problem.

Replacing $V \in \mathcal{V}_C$ with $V \in \mathcal{L}_C$ and using Lemma 2.1 and [7], we obtain the following properties. The proofs of the following results are similar to Lemma 3.3 in [7], however, we give their proofs here for the sake of completeness and the reader's convenience.

Lemma 4.1 *Let* $k^0 \in \operatorname{int}C$. *The function* $h^l_{\inf}(\cdot; k^0) : \mathcal{L}_C \to (-\infty, \infty]$ *has the following properties:*

(i) $h^l_{\inf}(V; k^0) \leq t \iff tk^0 \in \operatorname{cl}(V)$;

(ii) $h^l_{\inf}(\cdot; k^0)$ *is L-increasing;*

(iii) $h^l_{\inf}(V + \lambda k^0; k^0) = h^l_{\inf}(V; k^0) + \lambda$ *for every* $\lambda \in \mathbb{R}$;

(iv) $h^l_{\inf}(\cdot; k^0)$ *is sublinear;*

(v) $h^l_{\inf}(\cdot; k^0)$ *is bounded from below;*

(vi) $h^l_{\inf}(V; k^0) < t \iff tk^0 \in \operatorname{int}(V)$;

(vii) $h^l_{\inf}(\cdot; k^0)$ *is strictly L-increasing.*

Proof We define

$$\Lambda^l_-(V; k^0) := \left\{ t \in \mathbb{R} \,\middle|\, tk^0 \in \operatorname{int}(V) \right\},$$
$$\Lambda^l(V; k^0) := \left\{ t \in \mathbb{R} \,\middle|\, tk^0 \in V \right\},$$
$$\Lambda^l_+(V; k^0) := \left\{ t \in \mathbb{R} \,\middle|\, tk^0 \in \operatorname{cl}(V) \right\}.$$

Then we have obviously that $\Lambda^l_-(V; k^0) \subset \Lambda^l(V; k^0) \subset \Lambda^l_+(V; k^0)$ and hence

$$\inf \Lambda^l_+(V; k^0) \leq \inf \Lambda^l(V; k^0) (= h^l_{\inf}(V; k^0)) \leq \inf \Lambda^l_-(V; k^0).$$

- **(i)** We assume $h^l_{\inf}(V; k^0) \leq t$ and let $t \in \mathbb{R}$ be fixed. Then by the definitions of $h^l_{\inf}$ and Λ^l being of epigraphical type (that is, $t \in \Lambda^l$ and $\hat{t} > t$ implies $\hat{t} \in \Lambda^l$, see [9]), we have

$$\left(t+\frac{1}{n}\right)k^0 \in V$$

for all $n \in \mathbb{N}$. Taking the limit when $n \to \infty$, we obtain $tk^0 \in \mathrm{cl}(V)$.
Conversely, by the definitions of $h^l_{\inf}(\cdot; k^0)$, we show

$$\inf \Lambda^l_+(V; k^0) = \inf \Lambda^l(V; k^0) = \inf \Lambda^l_-(V; k^0).$$

On the contrary, assume that $\inf \Lambda^l_+(V; k^0) < \inf \Lambda^l_-(V; k^0)$. Then there exists $t_1, t_2 \in \mathbb{R}$ such that $\inf \Lambda^l_+(V; k^0) \le t_1 < t_2 < \inf \Lambda^l_-(V; k^0)$. By $\inf \Lambda^l_+(V; k^0) \le t_1$ $[t_1k^0 \in \mathrm{cl}(V)]$ and using (iv) of Lemma 2.1, we have

$$(*) \quad t_1k^0 + C \subset \mathrm{cl}(V) + C = \mathrm{cl}(V).$$

On the other hand, we have

$$(**) \quad t_2k^0 \in t_2k^0 + C = t_1k^0 + C + (t_2 - t_1)k^0 \subset t_1k^0 + \mathrm{int}C = \mathrm{int}(t_1k^0 + C).$$

By $(*)$, we have the following inclusion

$$(***) \quad \mathrm{int}(t_1k^0 + C) \subset \mathrm{int}(\mathrm{cl}(V)) = \mathrm{int}(V).$$

By $(**)$ and $(***)$, we obtain $t_2k^0 \in \mathrm{int}(V)$, which contradicts the inequality $t_2 < \inf \Lambda^l_-(V; k^0)$.

(ii) Let $V_1, V_2 \in \mathcal{L}_C$ be such that $V_2 \subset V_1$. If $h^l_{\inf}(V_2; k^0) = \infty$, we have that condition (ii) clearly holds. Taking $h^l_{\inf}(V_2; k^0) \in \mathbb{R}$, we obtain

$$h^l_{\inf}(V_2; k^0)k^0 \subset \mathrm{cl}(V_2) \subset \mathrm{cl}(V_1).$$

Using (i) of Lemma 4.1, we have $h^l_{\inf}(V_1; k^0) \le h^l_{\inf}(V_2; k^0)$.

(iii) The conclusion follows immediately from the definition.

(iv) We prove sub-additivity. For any $V_1, V_2 \in \mathcal{L}_C$ by the definition of $h^l_{\inf}(\cdot; k^0)$ we have

$$h^l_{\inf}(V_1; k^0)k^0 \subset \mathrm{cl}(V_1) \quad \text{and} \quad h^l_{\inf}(V_2; k^0)k^0 \subset \mathrm{cl}(V_2).$$

If $h^l_{\inf}(V_1; k^0) = \infty$ or $h^l_{\inf}(V_2; k^0) = \infty$, we have that condition (v) clearly holds. By adding the above inclusions and using (iii) of Lemma 2.1, we obtain

$$\{h^l_{\inf}(V_1; k^0) + h^l_{\inf}(V_2; k^0)\}k^0 \subset \mathrm{cl}(V_1) + \mathrm{cl}(V_2) \subset \mathrm{cl}(V_1 + V_2).$$

Using conclusion (i), we obtain the sub-additivity of $h^l_{\inf}(\cdot; k^0)$. The positively homogeneity of $h^l_{\inf}(\cdot; k^0)$ is easy.

(v) If $V = \mathbb{R}^n$ for $V \in \mathcal{L}_C$, then we have $tk^0 \subset V = \mathbb{R}^n$ for all $t \in \mathbb{R}$, which is equivalent to $h^l_{\inf}(V; k^0) = -\infty$. Conversely, let $tk^0 \subset V$ for all $t \in \mathbb{R}$ and $V \in \mathcal{L}_C$. Then we have

$$tk^0 + C \subset V + C = V.$$

For $k^0 \in \mathrm{int}C$, it is known that

$$\bigcup_{t \in \mathbb{R}} (tk^0 + C) = \mathbb{R}^n$$

and hence $V = \mathbb{R}^n$.

(vi) Let $h^l_{\inf}(V; k^0) < t$. Then there exists $\hat{t} \in \mathbb{R}$ such that $h^l_{\inf}(V; k^0) \leq \hat{t} < t$. By using (i), we have

$$tk^0 = \hat{t}k^0 + (t - \hat{t})k^0 \in \mathrm{cl}(V) + (t - \hat{t})k^0 \subset \mathrm{int}(V).$$

Conversely, let $tk^0 \in \mathrm{int}(V)$. For $k^0 \in \mathrm{int}C$, it is known that

$$\mathrm{int}C = \bigcup_{\varepsilon > 0} (\varepsilon k^0 + \mathrm{int}C).$$

Therefore, we have

$$tk^0 \in \mathrm{int}(V) = \bigcup_{\varepsilon > 0} (\mathrm{int}(V) + \varepsilon k^0 + \mathrm{int}C + C)$$

and $\{\mathrm{int}(V) + \varepsilon k^0 + \mathrm{int}C + C\}_{\varepsilon > 0}$ is an open cover of $\{tk^0\}$. Since $\{tk^0\}$ is compact, we can find $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m > 0$ such that

$$tk^0 \in \bigcup_{i=1}^{m} (\mathrm{int}(V) + \varepsilon_i k^0 + \mathrm{int}C + C) = \mathrm{int}(V) + \varepsilon_0 k^0 + \mathrm{int}C \subset \mathrm{cl}(V) + \varepsilon_0 k^0,$$

where $\varepsilon_0 := \min\{\varepsilon_i | i = 1, 2 \cdots m\} > 0$. Then we have $(t - \varepsilon_0)k^0 \in \mathrm{cl}(V)$ and therefore $h^l_{\inf}(V; k^0) \leq t - \varepsilon_0 < t$.

(vii) Let $V_1, V_2 \in \mathrm{cl}(\mathcal{L}_C)$ such that L-closed and $V_2 \subset \mathrm{int}(V_1)$. Then we have

$$h^l_{\inf}(V_2; k^0)k^0 \subset \mathrm{cl}(V_2) = V_2 \subset \mathrm{int}(V_1).$$

Applying property (vi), we obtain the conclusion.

□

Corollary 4.1 *Suppose that $C \subset \mathbb{R}^n$ is a solid closed convex cone, $k^0 \in \mathrm{int}C$ and $V \in \mathrm{cl}(\mathcal{L}_C)$ a L-proper and L-closed set. Then we have*

$$\mathbf{0} \in \mathrm{int}(V) \Longleftrightarrow h^l_{\inf}(V; k^0) < 0.$$

Proof The proof of this Corollary is consequences of (vi) of Lemma 4.1. □

5 A minimal element theorem and generalized Ekeland's variational principle for complete lattices with set perturbation

The aim of this section is to present a minimal element theorem with set perturbation in complete lattice optimization problem using Brézis–Browder's principle, sublinear scalarizing functions for complete lattice. In [8], we defined the following new order relations on $X \times \mathcal{V}_C$. where X is a metric space. The idea of these relations depends on [19] and chapter 2 of [24].

$$(x_1, V_1) \preceq^l_{k^0, h^l_{\inf}} (x_2, V_2) \Longleftrightarrow \begin{cases} (x_1, V_1) \preceq^l_{k^0} (x_2, V_2) \\ h^l_{\inf}(V_1) < h^l_{\inf}(V_2) \end{cases} \text{ or } \quad \begin{cases} x_1 = x_2 \\ V_2 \in [V_1]^l, \end{cases}$$

where

$$(x_1, V_1) \preceq^l_{k^0} (x_2, V_2) \Longleftrightarrow V_1 + d(x_1, x_2)k^0 \leq^l_C V_2.$$

We see that $\preceq^l_{k^0, h^l_{\inf}}$ is reflexive and transitive on $X \times \mathcal{V}_C$.

Let $D_L \subset \mathbb{R}^n$ be a convex set. As a natural generalization of the above order relation, we define the following new order relation, on $X \times \mathcal{L}_C$, where X is a metric space:

$$(x_1, V_1) \preceq_{D_L} (x_2, V_2) \Longleftrightarrow V_2 + d(x_1, x_2)D_L \subset V_1.$$

Proposition 5.1 *Let $D_L \subset \mathbb{R}^n$ be a convex set. Then $\preceq_{D_L}$ is reflexive and transitive on $X \times \mathcal{L}_C$.*

Proof We can easily see that $\preceq_{D_L}$ is reflexive since $d(x_1, x_1) = 0$. We assume that $(x_1, V_1) \preceq_{D_L} (x_2, V_2)$ and $(x_2, V_2) \preceq_{D_L} (x_3, V_3)$. Then we have

$$V_2 + d(x_1, x_2)D_L \subset V_1 \quad \text{and} \quad V_3 + d(x_2, x_3)D_L \subset V_2.$$

Adding $d(x_1, x_2)D_L$ to the latter inclusion, we obtain

$$V_3 + d(x_1, x_2)D_L + d(x_2, x_3)D_L \subset V_2 + d(x_1, x_2)D_L \subset V_1.$$

Since $D_L \subset \mathbb{R}^n$ is a convex set, we have

$$d(x_1, x_2)D_L + d(x_2, x_3)D_L = \{d(x_1, x_2) + d(x_2, x_3)\}D_L.$$

Then we obtain

$$V_3 + \{d(x_1, x_2) + d(x_2, x_3)\}D_L \subset V_1.$$

By the triangle inequality and distance function is non-negative, we have

$$d(x_1, x_3)D_L \subset \{d(x_1, x_2) + d(x_2, x_3)\}D_L$$

and hence

$$V_3 + d(x_1, x_3)D_L \subset V_3 + \{d(x_1, x_2) + d(x_2, x_3)\}D_L \subset V_1,$$

which is a desired result. □

We also define new type order relations on $X \times \mathcal{L}_C$:

$$(x_1, V_1) \preceq_{D_L}^{h^l_{\inf}} (x_2, V_2) \Longleftrightarrow \begin{cases} (x_1, V_1) \preceq_{D_L} (x_2, V_2) \\ h^l_{\inf}(V_1; k^0) < h^l_{\inf}(V_2; k^0) \end{cases} \text{ or } \begin{cases} x_1 = x_2 \\ V_1 = V_2. \end{cases}$$

We also see that $\preceq_{D_L}^{h^l_{\inf}}$ is reflexive and transitive on $X \times \mathcal{L}_C$.

5.1 Existence results

Let P_X and P_Y be projections of $X \times Y$ onto X and Y, respectively, that is, for every $(x, y) \in X \times Y$

$$P_X(x, y) = x \qquad P_Y(x, y) = y.$$

Theorem 5.1 *Let X be a complete metric space, $C \subset \mathbb{R}^n$ a solid closed convex cone, $\mathcal{L}_C$ a family of L-proper and L-closed subsets of $\mathbb{R}^n$, $k^0 \in \text{int}C$, $D_L \in \text{clconv}(\mathcal{L}_C)$ a L-proper, L-closed and L-convex subset of $\mathbb{R}^n$ such that $\mathbf{0} \in \text{int}(D_L)$ and $\mathcal{A} \subset X \times \mathcal{L}_C$ a nonempty set. We assume the following conditions:*

(i) *$\mathcal{A}$ is bounded below (there exists $\tilde{V} \in \mathcal{L}_C$ such that $\tilde{V} \supset P_{\mathcal{L}_C}(\mathcal{A})$);*
(ii) *For all $\preceq_{D_L}$-decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $x_n \to x \in X$, there exists $(x, V) \in \mathcal{A}$ such that $(x, V) \preceq_{D_L} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then for every $(x_0, V_0) \in \mathcal{A}$ there exists $(\bar{x}, \bar{V}) \in \mathcal{A}$ such that

(a) *$(\bar{x}, \bar{V}) \preceq_{D_L} (x_0, V_0)$, and*
(b) *If $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq_{D_L} (\bar{x}, \bar{V})$ then $\hat{x} = \bar{x}$.*

Moreover, if we replace $\preceq_{D_L}$ with $\preceq_{D_L}^{h^l_{\inf}}$, conclusion (b) can be replaced to

(b') *If $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq_{D_L}^{h^l_{\inf}} (\bar{x}, \bar{V})$ then $\hat{x} = \bar{x}$ and $\hat{V} = \bar{V}$.*

Proof Let

$$\mathcal{A}_0 := \{(x, V) \in \mathcal{A} \mid (x, V) \preceq_{D_L} (x_0, V_0)\}.$$

We apply the Brézis–Browder's principle to the quasi-ordered set $(\mathcal{A}_0, \preceq_{D_L})$ and the following functional

$$\phi : \mathcal{A}_0 \to \mathbb{R}, \qquad \phi(x, V) := h^l_{\inf}(V; k^0).$$

We show that ϕ satisfies the assumptions of Theorem 1.1. By (ii) and (v) of Lemma 4.1, we have for $\tilde{V} \in \mathcal{L}_C$

$$-\infty < h^l_{\inf}(\tilde{V}; k^0) \leq h^l_{\inf}(P_{\mathcal{L}_C}(\mathcal{A}); k^0)$$

for all $x \in X$. Then, we have that $h^l_{\inf}(P_{\mathcal{L}_C}(\mathcal{A}); k^0)$ is bounded from below on X, that is, (A1) holds. By condition (ii) and (iv) of Lemma 4.1, we have that

$$(x_1, V_1) \preceq_{D_L} (x_2, V_2) \quad \left(\Longleftrightarrow V_2 + d(x_1, x_2)D_L \subset V_1 \right)$$

implies

$$\begin{aligned} &h^l_{\inf}(V_1; k^0) \leq h^l_{\inf}(V_2 + d(x_1, x_2)D_L; k^0) \\ &\leq h^l_{\inf}(V_2; k^0) + h^l_{\inf}(d(x_1, x_2)D_L; k^0) = h^l_{\inf}(V_2; k^0) + d(x_1, x_2)h^l_{\inf}(D_L; k^0). \end{aligned}$$

By Corollary 4.1, we have $h^l_{\inf}(D_L) < 0$ and hence

$$h^l_{\inf}(V_1; k^0) \leq h^l_{\inf}(V_2; k^0),$$

that is, (A2) holds. We easily see that condition (ii) implies (A3). Therefore, by Theorem 1.1, for every $(x_0, V_0) \in \mathcal{A}_0$ there exists $(\bar{x}, \bar{V}) \in \mathcal{A}_0$ such that

(1) $(\bar{x}, \bar{V}) \preceq_{D_L} (x_0, V_0)$,
(2) $(\hat{x}, \hat{V}) \preceq_{D_L} (\bar{x}, \bar{V})$ implies $\phi(\hat{x}, \hat{V}) = \phi(\bar{x}, \bar{V})$.

Condition (1) implies conclusion (a). Since $(\hat{x}, \hat{V}) \in \mathcal{A}_0$, by condition (ii) and (iv) of Lemma 4.1, we have that $(\hat{x}, \hat{V}) \preceq_{D_L} (\bar{x}, \bar{V})$ implies

$$h^l_{\inf}(\hat{V}; k^0) \leq h^l_{\inf}(\bar{V} + d(\hat{x}, \bar{x})D_L; k^0) \leq h^l_{\inf}(\bar{V}; k^0) + d(\hat{x}, \bar{x})h^l_{\inf}(D_L; k^0).$$

Now we have $h^l_{\inf}(\hat{V}; k^0) = \phi(\hat{x}, \hat{V}) = \phi(\bar{x}, \bar{V}) = h^l_{\inf}(\bar{V}; k^0)$, we obtain

$$d(\hat{x}, \bar{x})h^l_{\inf}(D_L; k^0) \geq 0.$$

Then, by the assumption and Corollary 4.1, we have $d(\hat{x}, \bar{x}) = 0$ and hence $\hat{x} = \bar{x}$, that is, conclusion (b) holds. To prove (b'), let

$$\begin{aligned} &\mathcal{B}_0 := \left\{ (x, V) \in \mathcal{A} \mid (x, V) \preceq^{h^l_{\inf}}_{D_L} (x_0, V_0) \right\}, \\ &\phi : \mathcal{B}_0 \to \mathbb{R}, \qquad \phi(x, V) := h^l_{\inf}(V; k^0). \end{aligned}$$

Similarly, we can also show that ϕ satisfies the assumptions of Theorem 1.1 and we obtain conclusion (b'). □

In 1972, Ekeland [18] presented the following variational principle, which provides powerful tools in modern variational analysis. In fact, the celebrated Ekeland's variational principle is a direct consequence of the Brézis–Browder's principle.

Theorem 5.2 (Ekeland [18]) *Let (X, d) be a complete metric space and $f : X \to (-\infty, \infty]$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Let $\varepsilon > 0$ and $u \in X$ satisfy*

$$f(u) \leq \inf_{x \in X} f(x) + \varepsilon.$$

Then there exists $v \in X$ such that

(i) $f(v) \leq f(u)$,
(ii) $d(u, v) \leq 1$, *and*
(iii) *for each $w \neq v$, $f(v) - \varepsilon d(v, w) < f(w)$.*

Using scalarizing functions $h^l_{\inf}(\cdot; k^0)$ and applying Theorem 5.1, we obtain the following new strong form and weak form of Ekeland's variational principle for complete lattices with set perturbation. We consider the following conditions:

(H) X is a complete metric space, $C \subset \mathbb{R}^n$ is a solid closed convex cone, $k^0 \in \text{int}C$, $D_L \in clconv(\mathcal{L}_C)$ is a L-proper, L-closed and L-convex set such that $\mathbf{0} \in \text{int}(D_L)$, $F : X \to \mathcal{L}_C$ is a L-proper and L-closed valued function. We also assume that

(i) F is bounded below (there exists $\tilde{V} \in \mathcal{L}_C$ such that $\tilde{V} \supset F(x)$ for all $x \in X$);
(ii) $\{\hat{x} \in X | (\hat{x}, F(\hat{x})) \preceq_{D_L} (x, F(x))\}$ is closed for all $x \in X$.

Theorem 5.3 (Strong form of generalized Ekeland's variational principle) *We suppose condition* (H). *Then for any $x_0 \in X$ with $F(x_0) + \text{int}(D_L) \not\subset F(x)$ for all $x \in X$, there exists $\bar{x} \in X$ such that*

(a) $F(\bar{x}) \supset F(x_0)$,
(b) $d(\bar{x}, x_0) \leq 1$ *and*
(c) $F(\bar{x}) + d(\bar{x}, x)D_L \not\subset F(x)$ *for all $x \in X$ with $x \neq \bar{x}$.*

Proof Let $\mathcal{A} = \text{gr}F := \{(x, F(x)) | x \in X\} \subset X \times \mathcal{L}_C$. Of course, $P_{\mathcal{L}_C}(\mathcal{A}) = F(X)$. Let us show that condition (ii) in Theorem 5.1 holds. Let $\{(x_n, V_n)\}_{n \in \mathbb{N}} \subset \mathcal{A}$ be a $\preceq_{D_L}$-decreasing sequence with $x_n \to x \in X$. Of course, $V_n = F(x_n)$. For all $n, p \in \mathbb{N}$, we have that

$$x_{n+p} \in \mathcal{A}_n := \{x \in X | F(x_n) + d(x, x_n)D_L \subset F(x)\}$$

By condition (ii), $\mathcal{A}_n$ contains a limit x of the sequence $(x_{n+p})_{p \in \mathbb{N}}$. Therefore, $(x, F(x)) \preceq_{D_L} (x_n, F(x_n))$ for every n. Therefore, all the assumptions of Theorem 5.1 are satisfied. Let $x_0 \in X$ satisfying $F(x_0) + \text{int}(D_L) \not\subset F(x)$ for all $x \in X$. Applying Theorem 5.1, there exists $\bar{x} \in X$ such that $(\bar{x}, F(\bar{x})) \in \text{gr}F$ satisfies

(1) $(\bar{x}, F(\bar{x})) \preceq_{D_L} (x_0, F(x_0))$,
(2) $(x, F(x)) \not\preceq_{D_L} (\bar{x}, F(\bar{x}))$ for all $x \neq \bar{x}$.

Condition (2) is condition (c). By condition (1), we have that

$$F(x_0) \subset F(x_0) + d(\bar{x}, x_0)D_L \subset F(\bar{x}),$$

that is, (a) holds. To prove condition (b), we suppose that $d(\bar{x}, x_0) > 1$. Then we obtain

$$d(\bar{x}, x_0)D_L \supset \mathrm{int}(D_L).$$

Moreover, we have

$$F(x_0) + \mathrm{int}(D_L) \subset F(x_0) + d(\bar{x}, x_0)D_L \subset F(\bar{x}),$$

a contradiction. Therefore we show that $d(\bar{x}, x_0) \leq 1$. □

Theorem 5.4 (Weak form of generalized Ekeland's variational principle) *We suppose condition* (H). *Then for any $x_0 \in X$, there exists $\bar{x} \in X$ such that*

(a) $F(\bar{x}) \supset F(x_0)$,
(b) $F(\bar{x}) + d(\bar{x}, x)D_L \not\subset F(x)$ *for all $x \in X$ with $x \neq \bar{x}$.*

Proof Let $\mathcal{A} = \mathrm{gr}F := \{(x, F(x)) | x \in X\} \subset X \times \mathcal{L}_C$. So $P_{\mathcal{L}_C}(\mathcal{A}) = F(X)$. Following the same argument as in the proof of Theorem 5.3, we can show that $(x, F(x)) \preceq_{D_L} (x_n, F(x_n))$ for every n. Therefore condition (ii) in Theorem 5.1 holds and hence all the assumptions of Theorem 5.1 are satisfied. Let $x_0 \in X$. Applying Theorem 5.1, there exists $\bar{x} \in X$ such that $(\bar{x}, F(\bar{x})) \in \mathrm{gr}F$ satisfies

(1) $(\bar{x}, F(\bar{x})) \preceq_{D_L} (x_0, F(x_0))$,
(2) $(x, F(x)) \not\preceq_{D_L} (\bar{x}, F(\bar{x}))$ for all $x \neq \bar{x}$.

Condition (2) is condition (b). By condition (1), we obtain

$$F(x_0) \subset F(x_0) + d(\bar{x}, x_0)D_L \subset F(\bar{x}),$$

which means that (a) holds. The proof is completed. □

By applying Theorem 5.4, we establish an existence theorem of minimal solutions related to the famous Takahashi's minimization theorem.

Theorem 5.5 (Generalized Takahashi's minimization theorem) *We suppose condition* (H). *Moreover, we assume*

(T) *For any $x \in X$ with $F(x) \notin \mathrm{Min}(F(X), \subset)$ there exists $y = y(x) \in X$ with $y \neq x$ such that*

$$F(x) + d(x, y)D_L \subset F(y).$$

Then there exists $p \in X$ *such that* $F(p) \in \mathrm{Min}(F(X), \subset)$.

Proof Let $x_0 \in X$. By Theorem 5.4, there exists $\bar{x} \in X$ such that

(a) $F(\bar{x}) \supset F(x_0)$,

(b) $F(\bar{x}) + d(\bar{x}, x)D_L \not\subset F(x)$ for all $x \in X$ with $x \neq \bar{x}$.

We verify $F(\bar{x}) \in \mathrm{Min}(F(X), \subset)$. Suppose that $F(\bar{x}) \notin \mathrm{Min}(F(X), \subset)$. By condition (T), there exists $y = y(\bar{x}) \in X$ with $y \neq \bar{x}$ such that

$$F(\bar{x}) + d(\bar{x}, y)D_L \subset F(y),$$

which contradicts to (b). Hence $F(\bar{x}) \in \mathrm{Min}(F(X), \subset)$. □

5.2 Some remarks on existence results

We obtained a minimal element theorem and Ekeland's variational principles (EVP) for set-valued map via nonlinear scalarizing technique. Setting $k^0 \in \mathrm{int}C$ and $D_L := \{-k^0\}$, we obtain l-type Ekeland's variational principle for set-valued map (see [8]). In [2, 3], they obtained set-valued EVP with respect to the weighted set relation that is roughly speaking a convex combination of l-type and u-type set order relations. On the other hand, we dealt with a special class of l-type set relation in this paper: see also Remark 4.1. More generalized result of [2, 3] is seen in [4]. However, there are still many open problems.

(a) In this paper, we dealt with complete lattice optimization problem, which is a set optimization problem with lattice structure. So, our results are expected to be applicable, for example, to Boolean algebra. Furthermore, our theory may have the potential to bridge discrete optimization problem and continuous optimization problem.

(b) In [27], they made a comprehensive research on minimal element theorems. Moreover, they proposed generalized Brézis–Browder's principle. Therefore, our existence theorem could be obtained more relaxed form. It is a subject of the next research that generalizations of our existence results and comparisons among existence results related to minimal element theorem.

(c) In [46], they proposed some existence results for weak minimal solutions of nonconvex set optimization problem whose image spaces have no topology. Combining the results in [46] and Theorem 5.5 may enable us to remove the topology of the image space $\mathbb{R}^n$ of set-valued map $F : X \to \mathcal{L}_C$.

6 Conclusions

In this paper, we establish new cancellation laws of set order relations. Moreover, we introduce new concepts on complete lattice optimization problem. Applying nonlinear scalarizing technique in complete lattice, we present a new type of minimal element theorem and generalized Ekeland's variational principles in complete lattice optimization problem. Moreover, we proposed an existence theorem of minimal solutions related to Takahashi's minimization theorem.

We have found that the family of C-closed, bounded and convex subset of $\mathbb{R}^n$ allow cancellation laws and algebraic operations on some complete lattice. This fact may bring a new insight to the complete lattice optimization problems and new existence results are expected.

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Author contributions Araya wrote the main manuscript text. Du obtained Theorem 5.3, 5.4, 5.5 and reviewed the manuscript.

Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no Conflict of interest.

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