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Second-order optimality conditions for set-valued optimization problems under the set criterion

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Abstract
This paper investigates second-order optimality conditions for general constrained set-valued optimization problems in normed vector spaces under the set criterion. To this aim we introduce several new concepts of second-order directional derivatives for set-valued maps by means of excess from a set to another one, and discuss some of their properties. By virtue of these directional derivatives and by adopting the notion of set criterion intoduced by Kuroiwa, we obtain second-order necessary and sufficient optimality conditions in the primal form. Moreover, under some additional assumptions we obtain dual second-order necessary optimality conditions in terms of Lagrange–Fritz–John and in terms of Lagrange–Karush–Kuhn–Tucker multipliers.

Keywords Second-order directional derivatives of set-valued maps · Optimality conditions · Set-valued optimization problems

Mathematics Subject Classification 54C60 · 90C46 · 46G05

1 Introduction

Recently, the first order optimality conditions for set-valued optimization problems under set criterion (i.e., the solution concepts of the problem are based on feasible points whose image sets are nondominated with respect to certain binary relations on the power set of the objective space (see [13–15])) have been widely investigated, and a lot of notions about directional derivatives for set-valued mappings have been proposed and applied to set up the optimality conditions; for example we cite [2–5, 8, 9, 11, 13–15, 20]. In [2], necessary optimality conditions have been obtained by using continuous selections of the objective mapping and their directional derivatives. In [3, 20], necessary and sufficient optimality conditions have been proved by using different graphical or epigraphical derivatives of set-valued mappings. Ha [8] introduced a

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Hausdorff-type distance relative to an ordering cone between two sets in a Banach space and used it to define a directional derivative for set-valued mappings and studied necessary and sufficient optimality conditions in the primal form. Based on a special concept of the difference of sets, Jahn [11] introduced a directional derivative of a set-valued map and applied it to formulate necessary and sufficient optimality conditions for set optimization problems with set less order relation. Han et al. [9] employed the oriented distance function of Hiriart-Urruty to define the Dini directional derivatives for set-valued mappings, and when the data of the problem are convex they derive necessary and sufficient optimality conditions in terms of this derivative. Burlica et al. [4] present a main concept of directional derivative for set-valued maps defined by means of excess from a set to another one, and established first order necessary and sufficient optimality conditions in the primal form.

The purpose of this paper is to investigate second order optimality conditions for set-valued optimization problems in the sense of set criterion. To this aim, inspired by [4], we introduce several second-order directional derivatives for set-valued maps by means of excess from a set to another one. By using these directional derivatives and by adopting the notion of set criterion introduced by Kuroiwa [13–15], the second-order necessary and sufficient optimality conditions are given in the primal form. Moreover, under some additional assumptions we obtain dual second-order necessary optimality conditions in terms of Lagrange–Fritz–John and in terms of Lagrange–Karush–Kuhn–Tucker multipliers. Since the set criterion of solution can be viewed as a weaker version of Pareto efficient concept (see Remark 1), our optimality results are sharper than those of [10, 16, 19] where the Pareto efficient notion was used. To our knowledge, until now there is no study on second-order optimality conditions under the set criterion. Therefore, this paper constitues an attempt in this field.

The outlines of the paper are as follows: Preliminaries results are described in Sect. 2. Second order directional derivatives of set-valued maps are introduced in Sect. 3. Primal second-order necessary and sufficient optimality conditions for the unconstrained and the constrained problems are given in Sect. 4. Dual second-order necessary optimality conditions are established in Sect. 5.

2 Preliminaries

Throughout this paper, let X, Y and Z be real normed vector spaces and 0_X, 0_Y, 0_Z, 0 be the origins of X, Y, Z, $\mathbb{R}$, respectively. Moreover, We denote by $\|.\|_X$, $\|.\|_Y$, $\|.\|_Z$ the norms on X, Y, Z, respectively. Let C be a pointed ($C \cap -C = \{0_Y\}$) closed and convex cone with nonempty interior introducing a partial order in Y. We denote by $\mathbb{B}_X$, $\mathbb{B}_Y$ and $\mathbb{B}_Z$ the closed unit balls of X, Y and Z, respectively. As usual, we denote by $int A$, $cl A$, ∂A, the interior, the closure, and the boundary of a subset $A \subset X$. Throughout the paper, Y^* and Z^* will denote the continuous duals of Y and Z, respectively, and we write $\langle ., . \rangle$ the canonical bilinear form with respect to the duality (Y^*, Y). Moreover, we denote by $\mathbb{S}_X$ the unit sphere centred at the origin of X and $v^{\perp}$ denotes the orthogonal subspace to $v \in X$.

Let $A_1,\ A_2$ be nonempty subsets of Y and $\alpha \in \mathbb{R}$. The next operations and rules will be used:

$$A_1 + A_2 = \{y_1 + y_2 : y_1 \in A_1, y_2 \in A_2\}, \quad \alpha A_1 = \{\alpha y : y \in A_1\},$$
$$\varnothing + A_1 = A_1 + \varnothing = \varnothing, \quad \alpha\varnothing = \varnothing.$$

Consider a set-valued map $F : X \rightrightarrows Y$. In the sequel we denote the domain $(dom F)$ and the graph $(gr F)$ of F respectively by

$$dom F := \{x \in X : F(x) \neq \varnothing\} \text{ and } gr F := \{(x, y) \in X \times Y : y \in F(x)\}.$$

If V is a nonempty subset of X, then $F(V) := \underset{x \in V}{\cup} F(x)$

Recall that F is C-convex if for each $\lambda \in [0, 1]$ and for each $(x_1, x_2) \in X \times Y$ one has

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C.$$

In the sequel, the epigraphical of F with respect to C is the set-valued map $Epi_C F : X \rightrightarrows Y$ defined by $Epi_C F(x) := F(x)+C$. Moreover, let $G : X \rightrightarrows Z$ be a set-valued mapping. In the sequel, the set-valued map $(F, G) : X \rightrightarrows Y \times Z$ is defined by

$$(F, G)(x) := F(x) \times G(x)\ \forall x \in X.$$

A set-valued map $F : X \rightrightarrows Y$ is Lipschitzian at $\overline{x} \in X$, if there exist a neighborhood V of $\overline{x}$ and $k > 0$ such that for all $x_1,\ x_2 \in V$,

$$F(x_1) \subset F(x_2) + k\|x_1 - x_2\|_X \mathbb{B}_Y.$$

Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a function with $dom f := \{y \in Y : f(y) < +\infty\}$, and $G : X \rightrightarrows Y$ be a set-valued mapping. The composite set-valued map $f \circ G : X \rightrightarrows \mathbb{R} \cup \{+\infty\}$ is defined by $(f \circ G)(x) := f(G(x))$ if $x \in dom(G)$ and $(f \circ G)(x) := \{+\infty\}$ otherwise. It is immediate that

$$dom(f \circ G) = \{x \in X : (f \circ G)(x) \neq \{+\infty\}\}.$$

For a closed cone S of Y, S° will be the negative polar of S, that is

$$S^\circ := \{y^* \in Y^* : \langle y^*, y\rangle \leq 0\ \ \forall y \in S\}.$$

Let B be a nonempty convex subset of Y. The normal cone $N(B, b)$ to B at $b \in B$ is

$$N(B, b) := \{y^* \in Y^* : \langle y^*, y - b\rangle \leq 0\ \ \forall y \in B\}.$$

Let A be a nonempty subset of Y. The element $\overline{y} \in A$ is said to be a weak Pareto minimal point of A with respect to C if

$$(A - \overline{y}) \cap -intC = \varnothing.$$

We shall denote by $WMin(A, C)$ the set of all weak Pareto minimal point of A with respect to C. If $A = \varnothing$, we put $WMin(A, C) = \varnothing$.

Definition 1 Let B be a nonempty subset of X and $F : X \rightrightarrows Y$ be a set valued mapping. It said that $\overline{x} \in B$ is,

(i) a local weak Pareto minimal point of F on B with respect to C, if there exists a neighborhood V of $\overline{x}$ such that

$$F(\overline{x}) \cap WMin(F(V \cap B), C) \neq \varnothing.$$

(ii) a local minimal point of F on B in set criterion (or local weak l-minimal point of F on B) with respect to C, if there exists a neighborhood V of $\overline{x}$ such that

$$F(\overline{x}) \not\subset F(x) + intC, \ \forall x \in V \cap B.$$

(iii) In (i) (resp. (ii)), if $V := X$, then $\overline{x}$ is called a global weak Pereto minimal point of F (resp. a global weak l-minimal point of F) on B with respect to C.

Remark 1 It is easy to see that if $\overline{x}$ is a local weak Pareto minimal point of F on B with respect to C, then it is a local weak l-minimal point of F on B with respect to C. The converse is not true, see for instance [1].

First of all, we recall some standard notions used in this paper.

Definition 2 (see [19]) Let S be a nonempty subset of X, $\overline{x} \in clS$ and $v \in X$.

(1) The first order contingent cone to S at $\overline{x}$ is

$$K(S, \overline{x}) := \{v \in X : \exists (t_n) \downarrow 0, \exists (v_n) \to v, \ \overline{x} + t_n v_n \in S, \forall n \in \mathbb{N}\}.$$

(2) The second-order contingent set of S at x with respect to v is

$$K^2(S, \overline{x}, v) := \{w \in X : \exists (t_n) \downarrow 0, \exists (w_n) \to w, \ \overline{x} + t_n v + t_n^2 w_n \in S, \ \forall n \in \mathbb{N}\}.$$

(3) The asymptotic second order contingent cone to S at $\overline{x}$ with respect to v is the set $K''(S, \overline{x}, v)$ of elements $w \in X$ such that there exist $(t_n, r_n) \to (0^+, 0^+)$ and $(w_n) \to w$ with $\frac{t_n}{r_n} \to 0$ and $\overline{x} + t_n v + t_n r_n w_n \in S$ for all $n \in \mathbb{N}$.

(4) The interior tangent cone to S at $\overline{x}$ is the set $I(S, \overline{x})$ of elements $v \in X$ such that there exists $\delta > 0$ with $\overline{x} + tu \in S$ for all $u \in v + \delta\mathbb{B}_X$ and $t \in (0, \delta]$.

Definition 3 (see Definition 5.21 of [18]) Let S be a nonempty subset of X and $\overline{x} \in cl(S)$. The Clarke tangent cone (or circa-tangent cone) to S at $\overline{x}$ is the set $T_C(S, \overline{x})$ of $v \in X$ such that for all sequences $(x_n) \subset S$ and $(t_n) \subset \mathbb{R}_+ \backslash \{0\}$ with $(x_n) \to \overline{x}$ and $(t_n) \to 0$ there exists a sequence (v_n) with limit v such that $x_n + t_n v_n \in S$ for all $n \in \mathbb{N}$.

Remark 2 Let S be a nonempty subset of X, $\overline{x} \in clS$ and $v \in X$.

(a) It is well known that the cone $I(S, \overline{x})$ is open and convex.

(b) The cone $K''(S, \overline{x}, v)$ and the set $K^2(S, \overline{x}, v)$ are closed. Moreover are convex whenever S is convex.

(c) Note that if $v \notin K(S, \overline{x})$ then $K^2(S, \overline{x}, v) = K''(S, \overline{x}, v) = \varnothing$.
If $v = 0$ then $K^2(S, \overline{x}, v) = K''(S, \overline{x}, v) = K(S, \overline{x})$.

(d) (see [6, Proposition 2.3]) If S is convex and $intS \neq \varnothing$, then

$$intK(S, \overline{x}) = I(intS, \overline{x}).$$

(e) The cone $T_C(S, \overline{x})$ is closed and convex. It is immediate that, $0_X \in T_C(S, \overline{x})$ and $T_C(S, \overline{x}) \subset K(S, \overline{x}) \subset cl(cone(S - \overline{x}))$. Moreover, if S is convex, then one obviously has (see also [12, Theorem 4.4.1 (3)], [7, Proposition 2.4 (i)] and [18, Proposition 5.26])

$$T_C(S, \overline{x}) = K(S, \overline{x}) = cl(cone(S - \overline{x})).$$

Indeed, we present a proof for the reader's convenience. Let $v \in \mathbb{R}_+(S - \overline{x}) \backslash \{0_X\}$. Then there exists $s > 0$ such that $\overline{x} + sv \in S$. Now, let $(x_n)_{n \in \mathbb{N}} \subset S$ with $\lim_{n \to +\infty} x_n = \overline{x}$, and let $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+ \backslash \{0\}$ with $\lim_{n \to +\infty} t_n = 0$. As $\lim_{n \to +\infty} \frac{t_n}{s} = 0$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{t_n}{s} \in (0, 1]$ for all $n \geq n_0$. By convexity of S, it follows that

$$x_n + t_n(\frac{\overline{x} - x_n}{s} + v) \in S, \ \forall n \geq n_0.$$

Now, put $v_n := \frac{\overline{x} - x_n}{s} + v$ for $n \geq n_0$ and $v_n := 0$ for $n < n_0$. Obviously, one has $(v_n) \to v$ as $n \to +\infty$. Moreover, one has

$$x_n + t_n v_n \in S, \ \forall n \in \mathbb{N}.$$

Therefore, $v \in T_C(S, \overline{x})$, and so $\mathbb{R}_+(S - \overline{x}) \backslash \{0_X\} \subset T_C(S, \overline{x})$, which implies that $cl(\mathbb{R}_+(S - \overline{x})) \subset T_C(S, \overline{x})$ since $T_C(S, \overline{x})$ is closed and $0_X \in T_C(S, \overline{x})$. Thus the proof is complete since $T_C(S, \overline{x}) \subset K(S, \overline{x}) \subset cl(cone(S - \overline{x}))$.

(f) Let S_1 and S_2 be nonempty subsets of X, and $\overline{x}_i \in S_i$ for $i := 1, 2$. It is immediate that the following relation (E) holds (see also [12, Theorem 4.2.10 (13)] and [18, Proposition 5.28])

$$(E): \quad T_C(S_1 \times S_2, (\overline{x}_1, \overline{x}_2)) = T_C(S_1, \overline{x}_1) \times T_C(S_2, \overline{x}_2).$$

Moreover, if S_1 and S_2 are convex then by (e) and relation (E), it follows that

$$K(S_1 \times S_2, (\overline{x}_1, \overline{x}_2)) = K(S_1, \overline{x}_1) \times K(S_2, \overline{x}_2).$$

Here, for $x \in X$ we set $d(x, C) := inf\{\| x - a \|_X \colon a \in C\}$ to denote the distance function to a nonempty set $C \subset X$ with $d(x, \emptyset) := +\infty$.

Now, let $A, \ B$ be nonemty subsets of Y. The excess $e(A, B)$ from A to B is defined by

$$e(A, B) := \sup_{a \in A} d(a, B) \text{with} e(\emptyset, B) := 0, \ e(\emptyset, \emptyset) := 0 \text{and} e(A, \emptyset) := +\infty.$$

Clearly, for $\rho > 0$ one has

$$e(A, B) < \rho \Rightarrow A \subset B + \rho \mathbb{B}_Y,$$
$$A \subset B + \rho \mathbb{B}_Y \Rightarrow e(A, B) \leq \rho.$$

Definition 4 (see [4]) Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty closed values, and $x, v \in X$.

(1) The (H_-)-directional derivative of F at x in the direction v the set, $D_{H_-}F(x)(v)$ of all $y \in Y$ such that for all $(t_n) \downarrow 0$ and $(v_n) \to v$

$$\lim_{n \to +\infty} \frac{1}{t_n} e(F(x) + t_n y, F(x + t_n v_n)) = 0.$$

(2) The (H_+)-directional derivative of F at x in the direction v the set, $D_{H_+}F(x)(v)$ of all $y \in Y$ such that for all $(t_n) \downarrow 0$ and $(v_n) \to u$

$$\lim_{n \to +\infty} \frac{1}{t_n} e(F(x + t_n v_n), F(x) + t_n y) = 0.$$

Inspired by Definition 4, we introduce the following definition.

Definition 5 Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values, and $x, \ v \in X$.

(1) The lower (H_-)-directional derivative of F at x in the direction v, is the set $D_{H_{-l}}F(x)(v)$ of all $y \in Y$ such that

$$\liminf_{t \downarrow 0, u \to v} \frac{1}{t} e(F(x) + ty, F(x + tu) = 0.$$

(2) The lower (H_+)-directional derivative of F at x in the direction v, is the set $D_{H_{+l}}F(x)(v)$ of all $y \in Y$ such that

$$\liminf_{t\downarrow 0, u\to v} \frac{1}{t} e(F(x+tu), F(x)+ty) = 0.$$

Remark 3 Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values, and $x,\ v \in X$.

(1) (see [4]) Let $\sigma \in \{-,\ +\}$. So, $D_{H_\sigma}F(x)(v)$ is closed, and if $\alpha > 0$ and $v' \in D_{H_\sigma}F(x)(v)$, then $\alpha v' \in D_{H_\sigma}F(x)(\alpha v)$. Moreover, if F is Lipschitzian at x, then $0 \in D_{H_\sigma}F(x)(0)$.
(2) Let $\theta \in \{-l,\ +l\}$. If $\alpha > 0$ and $v' \in D_{H_\theta}F(x)(v)$, then $\alpha v' \in D_{H_\theta}F(x)(\alpha v)$. Moreover, $gr\,D_{H_\theta}F(x)$ is closed. Indeed, we only prove the conclusion for $\theta := -l$ since the case when $\theta := +l$ is similar. As the part $\alpha v' \in D_{H_{-l}}F(x)(\alpha v)$ is obvious, we only check that $gr\,D_{H_{-l}}F(x)$ is closed. Let $(u_n) \to u$ and $(y_n) \to y$ with $y_n \in D_{H_{-l}}F(x)(u_n)$ for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists $(w_{n,m}) \to u_n$ and $(t_{n,m}) \downarrow 0$ such that

$$\lim_{m\to+\infty} \frac{1}{t_{n,m}} e(F(x)+ty_n, F(x+t_{n,m}w_{n,m})) = 0.$$

So, for each $n \in \mathbb{N}$, there exists $m_n \geq n$ such that for all $n \in \mathbb{N}$,

$$F(x) + t_{n,m_n}y_n \subset F(x+t_{n,m_n}w_{n,m_n}) + t_{n,m_n}\frac{1}{2^n}\mathbb{B}_Y$$

and

$$\|w_{n,m_n} - u_n\|_X \leq \frac{1}{2^n}.$$

Since $(u_n) \to u$ as $n \to +\infty$, then $(w_{n,m_n}) \to u$ as $n \to +\infty$. Now, let $\varepsilon > 0$. Then, there exists $n_0(\varepsilon) \geq 0$ such that for all $n \geq n_0(\varepsilon)$,

$$\|y_n - y\|_Y \leq \frac{1}{2}\varepsilon$$

and

$$\frac{1}{2^n} \leq \frac{\varepsilon}{2}.$$

Therefore, for all $n \geq n_0(\varepsilon)$,

$$F(x) + t_{n,m_n}y \subset F(x+t_{n,m_n}w_{n,m_n}) + t_{n,m_n}\varepsilon\mathbb{B}_Y,$$

and this completes the proof.

(3) $0 \in D_{H_\theta} F(x)(0)$, where $\theta \in \{-l,\ +l\}$.

(4) It is easy to see that

$$D_{H_-} F(x)(v) \subset D_{H_{-l}} F(x)(v) \quad \text{and} \quad D_{H_+} F(x)(v) \subset D_{H_{+l}} F(x)(v).$$

The inclusions in (4) of Remark 3 are strict. We illustrate that by the following example.

Example 1 Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set valued map defined by $F(x) = \{x\}$ if $x \geq 0$ and $F(x) = \{-x + 2\}$ if $x < 0$. Then one obviously has $D_{H_{-l}} F(0)(0) = D_{H_{+l}} F(0)(0) = \{0\}$. But $D_{H_-} F(0)(0) = D_{H_+} F(0)(0) = \varnothing$.

3 Second-order directional derivatives of set-valued maps

In this section, inspired by Definition 4, we introduce some new second-order directional derivatives for set-valued maps, which will be used in next sections.

Definition 6 Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values, and $(x, v) \in X \times X$, $v' \in Y$.

(1) The second-order lower (H_-)-directional derivative of F at x with respect to (v, v') in the direction $w \in X$, is the set $D^2_{H_{-l}} F(x, v, v')(w)$ of all w' such that

$$\liminf_{t \downarrow 0, u \to w} \frac{1}{t^2} e(F(x) + tv' + t^2 w', F(x + tv + t^2 u)) = 0.$$

(2) The second-order lower (H_+)-directional derivative of F at x with respect to (v, v') in the direction $w \in X$ is the set $D^2_{H_{+l}} F(x, v, v')(w)$ of all $w' \in Y$ such that

$$\liminf_{t \downarrow 0, u \to w} \frac{1}{t^2} e(F(x + tv + t^2 u), F(x) + tv' + t^2 w') = 0.$$

(3) The second-order (H_-)-directional derivative of F at x with respect to (v, v') in the direction $w \in X$ is the set $D^2_{H_-} F(x, v, v')(w)$ of all $w' \in Y$ such that

$$\lim_{t \downarrow 0, u \to w} \frac{1}{t^2} e(F(x) + tv' + t^2 w', F(x + tv + t^2 u)) = 0.$$

(4) The second-order (H_+)-directional derivative of F at x with respect to (v, v') in the direction $w \in X$ is the set $D^2_{H_+} F(x, v, v')(w)$ of all $w' \in Y$ such that

$$\lim_{t \downarrow 0, u \to w} \frac{1}{t^2} e(F(x + tv + t^2 u), F(x) + tv' + t^2 w') = 0.$$

(5) the asymptotic second-order lower (H_-)-directional derivative of F at x with respect to (v, v') in the direction $w \in X$, is the set $D''_{H_{-l}} F(x, v, v')(w)$ of all $w' \in Y$ such that

$$\liminf_{t\downarrow 0, s\downarrow 0, t/s\to 0, u\to w} \frac{1}{ts} e(F(x) + tv' + tsw', F(x + tv + tsu)) = 0.$$

(6) the asymptotic second-order lower (H_+)-directional derivative of F at x with respect to (v, v') in the direction $w \in X$, is the set $D''_{H_{+l}} F(x, v, v')(w)$ of all $w' \in Y$ such that

$$\liminf_{t\downarrow 0, s\downarrow 0, t/s\to 0, u\to w} \frac{1}{ts} e(F(x + tv + tsu), F(x) + tv' + tsw') = 0.$$

(7) the asymptotic second-order (H_-)-directional derivative of F at x with respect to (v, v') in the direction $w \in X$ is the set $D''_{H_-} F(x, v, v')(w)$ of all $w' \in Y$ such that

$$\lim_{t\downarrow 0, s\downarrow 0, t/s\to 0, u\to w} \frac{1}{ts} e(F(x) + tv' + tsw', F(x + tv + tsu)) = 0.$$

(8) The asymptotic second-order (H_+)-directional derivative of F at x with respect to (v, v') in the direction $w \in X$, is the set $D''_{H_+} F(x, v, v')(w)$ of all $w' \in Y$ such that

$$\lim_{t\downarrow 0, s\downarrow 0, t/s\to 0, u\to w} \frac{1}{ts} e(F(x + tv + tsu), F(x) + tv' + tsw') = 0.$$

Remark 4 Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values, and $(x, v, w) \in X \times X \times X$ and $v' \in Y$. From Definition 6, we have the following results:

(a) Let $\sigma \in \{-l, +l\}$. If $w' \in D^2_{H_\sigma} F(\bar{x}, v, v')(w)$ (resp. $D''_{H_\sigma} F(\bar{x}, v, v')(w)$), then $v' \in D_{H_\sigma} F(\bar{x})(v)$. Note that if $v' \notin D_{H_\sigma} F(\bar{x})(v)$, then

$$D^2_{H_\sigma} F(\bar{x}, v, v')(w) = D''_{H_\sigma} F(\bar{x}, v, v')(w) = \emptyset.$$

(b) Suppose that F is Lipschitzian at $\bar{x}$. If $w' \in D^2_{H_\theta} F(\bar{x}, v, v')(w)$ (resp. $D''_{H_\theta} F(\bar{x}, v, v')(w)$), then $v' \in D_{H_\theta} F(\bar{x})(v)$, where $\theta \in \{-, +\}$. Indeed, we only prove the case where $\theta := -$, since the other case is similar. Let $\epsilon > 0$. Then there exists $\delta > 0$ and a neighborhood U_0 of w such that for all $t \in (0, \delta]$ and $u \in U_0$,

$$F(\bar{x}) + tv' + t^2 w' \subset F(\bar{x} + tv + t^2 u) + t^2 \frac{\varepsilon}{3} \mathbb{B}_Y.$$

Since F is Lipschitzian at $\overline{x}$, then there exist $k > 0$ and a neighborhood V of $\overline{x}$ such that for all $x_1, x_2 \in V$,

$$F(x_1) \subset F(x_2) + k\|x_1 - x_2\|_X \mathbb{B}_Y.$$

Since $\lim_{t\downarrow 0, u\to w} \overline{x} + tv + t^2u = \overline{x} \in V$, $\lim_{t\downarrow 0, z\to v} \overline{x} + tz = \overline{x} \in V$ and $\lim_{t\downarrow 0} tw' = 0_Y$, then there exist $\delta_1 > 0$ with $\delta_1 \leq \delta$, and neighborhoods U_1 and U_2 of w and v, respectively, with $U_1 \subset U_0$ such that for all $t \in (0, \delta_1]$, $u \in U_1$ and $z \in U_2$ one has

$$\overline{x} + tv + t^2u \in V, \overline{x} + tz \in V, \|tw'\|_Y \leq \frac{\varepsilon}{3} \text{ and } k\|v + tu - z\|_X \leq \frac{\varepsilon}{3}.$$

Therefore, we conclude that

$$F(\overline{x}) + tv' \subset F(\overline{x} + tz) + t\varepsilon\mathbb{B}_Y$$

for all $t \in (0, \delta_1]$ and $z \in U_2$. Thus the proof is complete.

(c) For $\sigma \in \{-, \ -l, \}$ one has

$$D^2_{H_\sigma} F(\overline{x}, v, v') + C \subset D^2_{H_\sigma}(Epi_C F)(\overline{x}, v, v'),$$
$$D''_{H_\sigma} F(\overline{x}, v, v') + C \subset D''_{H_\sigma}(Epi_C F)(\overline{x}, v, v').$$

For $\sigma \in \{+, \ +l\}$ one has

$$D^2_{H_\sigma} F(\overline{x}, v, v') - C \subset D^2_{H_\sigma}(Epi_{(-C)} F)(\overline{x}, v, v'),$$
$$D''_{H_\sigma} F(\overline{x}, v, v') - C \subset D''_{H_\sigma}(Epi_{(-C)} F)(\overline{x}, v, v').$$

The proof of the following proposition is immediate.

Proposition 1 *Let $F, \ G : X \rightrightarrows Y$ and $\Gamma : X \rightrightarrows Z$ be set-valued maps with nonempty values, $(x, v, w) \in X \times X \times X$ and $(v', v'') \in Y \times Z$. The following statements hold:*

(1) Let $\sigma \in \{-, \ +, \ -l, \ +l\}$. Then

$$D^2_{H_\sigma} F(x, 0_X, 0_Y)(w) = D''_{H_\sigma} F(x, 0_X, 0_Y)(w) = D_{H_\sigma} F(x)(w),$$

(2) $D^2_{H_-} F(x, v, v') \subset D^2_{H_{-l}} F(x, v, v')(w)$,

(3) $D^2_{H_+} F(x, v, v') \subset D^2_{H_{+l}} F(x, v, v')(w)$,

(4) $D''_{H_-} F(x, v, v')(w) \subset D''_{H_{-l}} F(x, (v, v'))(w)$,

(5) $D''_{H_+} F(x, v, v')(w) \subset D''_{H_{+l}} F(x, v, v')(w)$,

(6) $D^2_{H_\sigma} F(x, v, v')(w)$ is closed for $\sigma \in \{-, \ +\}$. Moreover, for $\sigma \in \{-l, \ +l\}$, $gr D^2_{H_\sigma} F(x, v, v')$ is closed,

(7) For $\sigma \in \{-, \ +\}$, the set $D''_{H_\sigma} F(x, v, v')(w)$ is closed, and if $\alpha > 0$ and $w' \in D''_{H_\sigma} F(x, v, v')(w)$ then $\alpha w' \in D''_{H_\sigma} F(x, v, v')(\alpha w)$. Moreover, for $\sigma \in \{-l, \ +l\}$, $gr D''_{H_\sigma} F(x, v, v')$ is a closed cone,

(8) *For* $\sigma \in \{-, +\}$, *one has*

$$\begin{aligned}
&D^2_{H_\sigma} F(x, v, v') + D^2_{H_\sigma} G(x, v, v') \subset D^2_{H_\sigma}(F + G)(x, v, v')(w),\\
&D''_{H_\sigma} F(x, v, v')(w) + D''_{H_\sigma} G(x, v, v')(w) \subset D''_{H_\sigma}(F + G)(x, v, v')(w),\\
&D^2_{H_\sigma}(F, \Gamma)(x, v, v', v'')(w) = D^2_{H_\sigma} F(x, v, v')(w) \times D^2_{H_\sigma} \Gamma(x, v, v'')(w),\\
&D''_{H_\sigma}(F, \Gamma)(x, v, v', v'')(w) = D''_{H_\sigma} F(x, v, v')(w) \times D''_{H_\sigma} \Gamma(x, v, v'')(w).
\end{aligned}$$

(9) *For* $\sigma \in \{-l, +l\}$, *one has*

$$\begin{aligned}
&D^2_{H_\sigma}(F, \Gamma)(x, v, v', v'')(w) \subset D^2_{H_\sigma} F(x, v, v')(w) \times D^2_{H_\sigma} \Gamma(x, v, v'')(w),\\
&D''_{H_\sigma}(F, \Gamma)(x, v, v', v'')(w) \subset D''_{H_\sigma} F(x, v, v')(w) \times D''_{H_\sigma} \Gamma(x, v, v'')(w).
\end{aligned}$$

Again, consider Example 1 above. We have

$$D^2_{H_{-l}} F(0_X, 0, 0)(0) = D^2_{H_{+l}} F(0, 0, 0)(0) = \{0\}$$

and

$$D''_{H_{-l}} F(0, 0, 0)(0) = D''_{H_{+l}} F(0, 0, 0)(0_X) = \{0\}.$$

However, $D^2_{H_-} F(0, (0, 0))(0)$, $D^2_{H_+} F(0, 0, 0)(0)$, $D''_{H_-} F(0, 0, 0)(0)$ and $D''_{H_+} F(0, 0, 0)(0)$ are empty. Therefore, the inclusions in (2), (3), (4) and (5) of Proposition 1 are strict.

Definition 7 (see [6, 19]) Let $f : X \longrightarrow Y$ be a vector valued function and $\overline{x} \in X$.

(1) f is said to be first-order Hadamard directional differentiable at $\overline{x}$, if for all $v \in X$ the following limit exists

$$df(\overline{x}, v) := \lim_{t \downarrow 0, u \to v} t^{-1}[f(\overline{x} + tu) - f(\overline{x})].$$

(2) f is said to be second-order Hadamard directional differentiable at $\overline{x}$, if for all $v \in X$ one has $df(\overline{x}, v)$ exists and for all $w \in X$ the following limit exists

$$d^2 f(\overline{x}, v, w) := \lim_{t \downarrow 0, u \to w} t^{-2}[f(\overline{x} + tv + t^2 u) - tdf(\overline{x}, v) - f(\overline{x})].$$

(3) f is said to be asymptotic second-order Hadamard directional differentiable at $\overline{x}$, if for all $v \in X$ one has $df(\overline{x}, v)$ exists and for all $w \in X$ the following limit exists

$$d^2_0 f(\overline{x}, v, w) := \lim_{t \downarrow 0, r \downarrow 0, t/r \to 0, u \to w} (tr)^{-1}[f(\overline{x} + tv + tru) - tdf(\overline{x}, v) - f(\overline{x})].$$

The results of the Proposition 2 bellow are inspired from [4]. The results in [4] are established in terms of the first order (H_-) (resp. (H_+)) directional derivative of $F := f + C$ in the case where f is Fréchet différentiable at $\overline{x}$ under the closdness of $C + \mathbb{B}_{\mathbb{Y}}$.

Proposition 2 *Let $v, w \in X$ and $f : X \to Y$ be a second-order Hadamard differentiable mapping at $\overline{x}$. Then*

$$d^2 f(\overline{x}, v, w) + C \subset D^2_{H_-}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w) \tag{1}$$

and

$$d^2 f(\overline{x}, v, w) - C \subset D^2_{H_+}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w). \tag{2}$$

Moreover, if $C + \mathbb{B}_Y$ is closed, then

$$\begin{aligned} d^2 f(\overline{x}, v, w) + C &= D^2_{H_-}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w) \\ &= D^2_{H_{-l}}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w) \end{aligned} \tag{3}$$

and

$$\begin{aligned} d^2 f(\overline{x}, v, w) - C &= D^2_{H_+}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w) \\ &= D^2_{H_{+l}}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w). \end{aligned} \tag{4}$$

Proof It is easy to check (1) and (2). Suppose that $C + \mathbb{B}_Y$ is closed. We only prove (3) since (4) is similar. By Proposition 1, we start by proving that

$$D^2_{H_{-l}}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w) \subset d^2 f(\overline{x}, v, w) + C. \tag{5}$$

Let $w' \in D^2_{H_{-l}}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w)$. Then, there exists sequences $(t_n) \to 0^+$, $(w_n) \to w$ such that

$$\lim_{n\to\infty} t_n^{-2} e((Epi_C f)(\overline{x}) + t_n df(\overline{x}, v) + t_n^2 w', (Epi_C f)(\overline{x} + t_n v + t_n^2 w_n)) = 0.$$

Let $\varepsilon > 0$. There exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} &(Epi_C f)(\overline{x}) + t_n df(\overline{x}, v) + t_n^2 w' \\ &\quad \subset (Epi_C f)(\overline{x} + t_n v + t_n^2 w_n) + t_n^2 \varepsilon \mathbb{B}_Y, \ \forall n \geq n_0(\varepsilon). \end{aligned}$$

So, for all $n \geq n_0(\varepsilon)$ one has

$$w' - t_n^{-2}[f(\overline{x} + t_n v + t_n^2 w_n) - t_n df(\overline{x}, v) - f(\overline{x})] \in C + \varepsilon \mathbb{B}_Y.$$

Since $C + \mathbb{B}_Y$ is closed, then

$$w' - d^2 f(\overline{x}, v, w) \in C + \varepsilon \mathbb{B}_Y, \forall \varepsilon > 0.$$

As C is closed, it follows that

$$w' \in d^2 f(\overline{x}, v, w) + C,$$

and this completes the proof of the inclusion (5). From (1) and the statement (2) of Proposition 1, we conclude (3). □

The proof of the following proposition is similar to that of Proposition 2.

Proposition 3 *Let $v, w \in X$ and $f : X \to Y$ be a asymptotic second-order Hadamard differentiable mapping at $\overline{x}$. Then*

$$d_0^2 f(\overline{x}, v, w) + C \subset D''_{H_-}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w)$$

and

$$d_0^2 f(\overline{x}, v, w) - C \subset D''_{H_+}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w).$$

Moreover, if $C + \mathbb{B}_Y$ is closed, then

$$\begin{aligned} d_0^2 f(\overline{x}, v, w) + C &= D''_{H_-}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w) \\ &= D''_{H_{-l}}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w) \end{aligned}$$

and

$$\begin{aligned} d_0^2 f(\overline{x}, v, w) - C &= D''_{H_+}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w) \\ &= D''_{H_{+l}}(Epi_C f)(\overline{x}, v, df(\overline{x}, v))(w). \end{aligned}$$

4 Primal second-order optimality conditions

The purpose of this section is to establish second-order necessary and sufficient conditions for local weak l-minimal solutions of set-valued optimization problems in the primal form. We start with the unconstrained case.

Proposition 4 *Let $\overline{x} \in dom F$ be a local weak l-minimal point of a set-valued map $F : X \rightrightarrows Y$ with respect to C. Then, for all $v \in X$ and $v' \in D_{H_{-l}}(Epi_C F)(\overline{x})(v) \cap -\partial C$, the following results hold.*

(i) $D_{H_{-l}}(Epi_C F)(\overline{x})(X) \cap -int C = \varnothing$,
(ii) $D^2_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(X) \cap int K(-C, v') = \varnothing$,
(iii) $D''_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(X) \cap int K(-C, v') = \varnothing$.

Proof (i) is a direct consequence of (ii), Proposition 1, (3) of Remark 3 and $int K(-C, 0_Y) = -int C$. We only prove (ii) since (iii) is similar. Suppose one the contrary that there is $w \in X$ and $w' \in D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(w)$ with $w' \in$

$intK(-C, v')$. Since $intC \neq \emptyset$, then by (d) of Remark 2 on has $intK(-C, v') = I(-intC, v')$. Hence, there exists $\delta > 0$ such that

$$v' + tz \in -intC,\ \forall t \in (0, \delta]\ and\ z \in w' + \delta \mathbb{B}_Y. \tag{6}$$

By definition of $D^2_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(w)$, there exists sequences $(t_n) \downarrow 0$ and $(w_n) \to w$, and $n_0(\delta) \geq 0$ such that

$$F(\overline{x}) + t_n v' + t_n^2 w' \subset F(\overline{x} + t_n v + t_n^2 w_n) + C + t_n^2 \delta \mathbb{B}_Y,\ \forall n \geq n_0(\delta).$$

By (6), there exists $n_1(\delta) \geq n_0(\delta)$ such that $t_n \in (0, \delta]$ for all $n \geq n_0(\delta)$. Consequently for each $n \geq n_1(\delta)$ on has

$$t_n v' + t_n^2 w' + \delta t_n^2 \mathbb{B}_Y \subset -intC. \tag{7}$$

As $\mathbb{B}_Y = -\mathbb{B}_Y$, we conclude that

$$F(\overline{x}) \subset F(\overline{x} + t_n v + t_n^2 w_n) + intC,\ \forall n \geq n_1(\delta),$$

which contradicts the fact that $\overline{x}$ is a local weak l-minimal point of F on X. □

Remark 5 Part (ii) and (iii) of Proposition 4 are valid for all $v \in X$ and $v' \in Y$, but is only meaningful for $v \in X$ and $v' \in D_{H_{-l}}(Epi_C F)(\overline{x})(v) \cap -\partial C$, since if $v' \notin D_{H_{-l}}(Epi_C F)(\overline{x})(v)$ then

$$D^2_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(X) = D''_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(X) = \emptyset$$

(see Remark 4) and if $v' \notin -C$ then $K(-C, v') = \emptyset$. Finally, if $v' \in -intC$ then by (i) of Proposition 4 one has $v' \notin D_{H_{-l}}(Epi_C F)(\overline{x})(v)$.

The following example illustate Proposition 4.

Example 2 Let $C := \mathbb{R}_+ \times \mathbb{R}_+$ and $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be a set-valued map defined by

$$F(x) := \{y := (y_1, y_2) \in \mathbb{R}^2 : y_1 \geq -x_1,\ y_2 \geq x_1^2 + |x_2|\},\ \forall x := (x_1, x_2) \in \mathbb{R}^2,$$

where $\mathbb{R}_+ := \{x \in \mathbb{R}\ \ x \geq 0\}$. Take $\overline{x} := (0, 0) \in \mathbb{R}^2$. It is easy to verify that $\overline{x}$ is a local weak l-minimal point of F on $\mathbb{R}^2$. It follows from Definition 5 that the set

$$\begin{aligned} A &:= \{v' := (v_1', v_2') \in \mathbb{R}^2\ :\ v := (v_1, v_2) \in \mathbb{R}^2, v' \in D_{H_{-l}}(Epi_C F)(\overline{x})(v) \cap -\partial C\} \\ &= \{v' := (v_1', v_2') \in \mathbb{R}^2\ :\ v := (v_1, v_2) \in \mathbb{R}^2,\ 0 \geq v_1' \geq -v_1,\ 0 \geq v_2' \geq |v_2|\} \\ &= \{v' := (v_1', v_2') \in \mathbb{R}^2\ :\ v_1 \geq 0,\ v_2 = 0,\ 0 \geq v_1' \geq -v_1,\ v_2' = 0\}. \end{aligned}$$

Thus, for every $v := (v_1, v_2) \in \mathbb{R}^2$ and every $v' := (v_1', v_1') \in A$, we have $v_1 \geq 0,\ v_2 = 0$ and $0 \geq v_1' \geq -v_1,\ v_2' = 0$. Moreover, for every $v := (v_1, v_2) \in \mathbb{R}^2$,

$v' := (v'_1, v'_2) \in A$ and every $w := (w_1, w_2) \in \mathbb{R}^2$, it follows from Definition 6 that

$$D^2_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(w) = L_1 \text{ and } D''_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(w) = L_2,$$

where

$$L_1 := \begin{cases} \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \geq -w_1,\;\; w'_2 \geq v_1^2 + |w_2|\} \, if \;\, v'_1 = -v_1 \\ \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \in \mathbb{R},\;\; w'_2 \geq v_1^2 + |w_2|\},\; if \;\, v'_1 > -v_1, \end{cases}$$

and

$$L_2 := \begin{cases} \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \geq -w_1,\;\; w'_2 \geq |w_2|\} \, if \;\, v'_1 = -v_1 \\ \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \in \mathbb{R},\;\; w'_2 \geq |w_2|\},\; if \;\, v'_1 > -v_1. \end{cases}$$

Thus, together with

$$K(-C, v') = \begin{cases} -(\mathbb{R}_+ \times \mathbb{R}_+) \; if \;\, v'_1 = 0 \\ \mathbb{R} \times (-\mathbb{R}_+) \; if \;\, v'_1 < 0, \end{cases}$$

we have

$$D^2_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(X) \cap int K(-C, v') = \emptyset$$

and

$$D''_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(X) \cap int K(-C, v') = \emptyset.$$

Thus, Proposition 4 is fulfilled.

Now, we establish optimality conditions under an arbitrary constrained set. The verification of the following result is similar to that of Proposition 4. We present a proof for the reader's convenience.

Proposition 5 *Let $\overline{x} \in B \subset X$ be a local weak l-minimal point on B of a set-valued map $F : X \rightrightarrows Y$ with respect to C. Assume that F is Lipschitzian at $\overline{x}$. Then, for all $v \in K(B, \overline{x})$ and $v' \in D_{H_-}(Epi_C F)(\overline{x})(v) \cap -\partial C$, the following results hold.*

(i) ([4]) $D_{H_-}(Epi_C F)(\overline{x})(K(B, \overline{x})) \cap -intC = \emptyset$,
(ii) $D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(K^2(B, \overline{x}, v)) \cap int K(-C, v') = \emptyset$,
(iii) $D''_{H_-}(Epi_C F)(\overline{x}, v, v')(K''(B, \overline{x}, v)) \cap int K(-C, v') = \emptyset$.

Proof (i) is a direct consequence of (ii), Proposition 1, (1) of Remark 3, $int K(-C, 0_Y) = -intC$ and (c) of Remark 2. We only prove that (ii) holds since (iii) is similar. Suppose on the contrary that there is $w \in K^2(B, \overline{x}, v))$ and $w' \in D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(w)$ such that $w' \in int K(-C, v')$. By definition of $K^2(B, \overline{x}, v)$, there exists sequences $(t_n) \downarrow 0$ and $(w_n) \to w$ such that

$$\overline{x} + t_n v + t_n w_n \in B, \forall n \in \mathbb{N}.$$

Since $intC \neq \emptyset$, then by (d) of Remark 2 one has $intK(-C, v') = I(-intC, v')$. Hence, there exists $\delta > 0$ such that

$$v' + tz \in -intC, \ \forall t \in (0, \delta] \ and \ z \in w' + \delta\mathbb{B}_Y.$$

So, there exists $n_0(\delta) > 0$ such that $t_n \in (0, \delta]$ for all $n \geq n_0(\delta)$. Consequently for each $n \geq n_0(\delta)$ on has

$$t_n v' + t_n^2 w' + \delta t_n^2 \mathbb{B}_Y \subset -intC. \tag{8}$$

By definition of $D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(w)$, there exists $n_1(\varepsilon) \geq n_0(\delta)$ such that

$$F(\overline{x}) + t_n v' + t_n^2 w' \subset F(\overline{x} + t_n v + t_n w_n) + C + t_n^2 \delta\mathbb{B}_Y, \ \forall n \geq n_1(\delta).$$

By (8), we conclude that

$$F(\overline{x}) \subset F(\overline{x} + t_n v + t_n w_n) + intC, \ \forall n \geq n_1(\delta),$$

which contradicts that $\overline{x}$ is a local weak l-minimal point of F on B. Thus the proof is complete. □

Remark 6 Part (ii) and (iii) of Proposition 5 are valid for all $v \in X$ and $v' \in Y$, but is only meaningful for $v \in K(B, \overline{x})$ and $v' \in D_{H_-}(Epi_C F)(\overline{x})(v) \cap -\partial C$, since if $v \notin K(B, \overline{x})$ then $K^2(B, \overline{x}, v) = K''(B, \overline{x}, v) = \emptyset$ (see Remark 2), and if $v' \notin D_{H_-}(Epi_C F)(\overline{x})(v)$ then

$$D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(X) = D''_{H_-}(Epi_C F)(\overline{x}, v, v')(X) = \emptyset$$

(see Remark 4 (b)) and if $v' \notin -C$ then $K(-C, v') = \emptyset$. Finally, if $v' \in -intC$ then by (i) of Proposition 5 one has $v \notin K(B, \overline{x})$.

The following example illustrate Proposition 5.

Example 3 Let F and C be as in Example 2, and $B := [0, 1] \times [-1, 1]$. Take $\overline{x} := (0, 0)$. It is easy to very that F is Lipshitzian at $\overline{x}$, and $\overline{x}$ is a local weak l-minimal point of F on B. It follows from Definitions 2 and 5 that $K(B, \overline{x}) = \mathbb{R}_+ \times \mathbb{R}$ and

$$\begin{aligned} A_1 := \{&v' := (v'_1, v'_2) \in \mathbb{R}^2 \ : \ v := (v_1, v_2) \in K(B, \overline{x}), \\ &v' \in D_{H_-}(Epi_C F)(\overline{x})(v) \cap -\partial C\} \\ = \{&v' := (v'_1, v'_2) \in \mathbb{R}^2 \ : \ v_1 \geq 0, \ v_2 = 0, \ 0 \geq v'_1 \geq -v_1, \ v'_2 = 0\}. \end{aligned}$$

Thus, for every $v := (v_1, v_2) \in K(B, \overline{x})$ and every $v' = (v'_1, v'_2) \in A_1$, we have $v_1 \geq 0, \ v_2 = 0$ and $0 \geq v'_1 \geq -v_1, \ v'_2 = 0$. Moreover, for every $v := (v_1, v_2) \in K(B, \overline{x})$ and every $v' := (v'_1, v'_2) \in A_1$, it follows from Definition 2 that

$$K^2(B, \overline{x}, v) = K''(B, \overline{x}, v) = \begin{cases} \mathbb{R}_+ \times \mathbb{R}, \ if \ v_1 = 0. \\ \mathbb{R} \times \mathbb{R}, \ if \ v_1 > 0. \end{cases}$$

Also, for every for every $v := (v_1, v_2) \in K(B, \overline{x})$, $v' := (v'_1, v'_2) \in A_1$ and every $w := (w_1, w_2) \in K^2(B, \overline{x}, v)$, it follows from Definition 6 that

$$D^2_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(w) = L_3 \text{ and } D''_{H_{-l}}(Epi_C F)(\overline{x}, v, v')(w) = L_4,$$

where

$$L_3 := \begin{cases} \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \geq -w_1,\ w'_2 \geq v_1^2 + |w_2|\},\ if\ v'_1 = -v_1. \\ \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \in \mathbb{R},\ w'_2 \geq v_1^2 + |w_2|\},\ if\ v'_1 > -v_1 \end{cases}$$

and

$$L_4 := \begin{cases} \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \geq -w_1,\ w'_2 \geq |w_2|\},\ if\ v'_1 = -v_1 \\ \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \in \mathbb{R},\ w'_2 \geq |w_2|\},\ if\ v'_1 > -v_1. \end{cases}$$

Thus, together with

$$K(-C, v') = \begin{cases} -(\mathbb{R}_+ \times \mathbb{R}_+)\ if\ v'_1 = 0 \\ \mathbb{R} \times (-\mathbb{R}_+)\ if\ v'_1 < 0, \end{cases}$$

we have

$$D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(K^2(B, \overline{x}, v)) \cap int K(-C, v') = \emptyset$$

and

$$D''_{H_-}(Epi_C F)(\overline{x}, v, v')(K''(B, \overline{x}, v)) \cap int K(-C, v') = \emptyset.$$

Thus, Proposition 5 is fulfilled.

Now, we state second order sufficient optimality conditions. To this end, we start by the following recall.

Definition 8 Let B be a nonempty subset of X, $F : X \rightrightarrows Y$ be a set-valued mapping, and $\overline{x} \in B$. It is said that $\overline{x}$ is a strict local weak l-minimal point of F on B with respect to C if there exists a neighborhood V of $\overline{x}$ such that

$$F(\overline{x}) \not\subset F(x) + intC,\ \forall x \in V \cap B \backslash \{\overline{x}\}.$$

Remark 7 It is easy to see that every local l-weak minimal point is a strict local l-weak minimal point. Reciprocally, if $WMin(F(\overline{x}), C) \neq \emptyset$ and $\overline{x}$ is a strict local weak l-minimal point of F on B, then one obviously has $\overline{x}$ is a local weak l-minimal point of F on B.

Proposition 6 *Let B be a subset of a finite dimensional space X and let $F : X \rightrightarrows Y$ be a set-valued map, $\overline{x} \in B$ and $WMin(F(\overline{x}), C) \neq \emptyset$. Suppose that for every $v \in K(B, \overline{x}) \backslash \{0_X\}$ and every $v' \in D_{H_{+l}}(Epi_C F)(\overline{x})(v) \cap \partial C$, the following conditions hold.*

(i) $D^2_{H_+}(Epi_C F)(\overline{x}, v, v')(w) \cap int K(C, v') \neq \varnothing \ \forall\, w \in K^2(B, \overline{x}, v) \cap v^\perp$,
(ii) $D''_{H_+}(Epi_C F)(\overline{x}, v, v')(w) \cap int K(C, v') \neq \varnothing \ \ \forall w \in K''(B, \overline{x}, v) \cap v^\perp \backslash \{0_X\}$.

Then $\overline{x}$ is a strict local weak l-minimal point of F on B with respect to C.

Proof Suppose on the contray that there exists a sequence $(x_n) \to \overline{x}$ such that for all $n \in \mathbb{N}$ one has $x_n \in B \backslash \{\overline{x}\}$ and

$$F(\overline{x}) \subset F(x_n) + intC. \tag{9}$$

Setting $t_n := \|x_n - \overline{x}\|_X$ we may assume that $v_n := t_n^{-1}(x_n - \overline{x})$ converges to some unit vector $v \in K(B, \overline{x})$. Consider now, the sequence (w_n) such that

$$x_n = \overline{x} + t_n v + t_n^2 w_n, \ \forall n \in \mathbb{N}. \tag{10}$$

We have two cases. The case (a): The sequence (w_n) is bounded. Observe that $v + t_n w_n \in \mathbb{S}_X$ for all $n \in \mathbb{N}$. Since $K(S_X, v) = v^\perp$, so by passing to a subsequence, we may suppose that (w_n) converges to some $w \in K^2(B, \overline{x}, v)) \cap v^\perp$. Take $v' \in D_{H_{+l}}(Epi_C F)(\overline{x})(v) \cap \partial C$ and

$$w' \in D^2_{H_+}(Epi_C F)(\overline{x}, v, v')(w) \cap int K(C, v').$$

Since $intC \neq \varnothing$, then by (d) of Remark 2,

$$int K(C, v') = I(intC, v').$$

Hence, there exists $\delta > 0$ such that

$$v' + tz \in intC \tag{11}$$

for all $t \in (0, \delta]$ and $z \in w' + \delta \mathbb{B}_Y$. So, there exists $n_0(\delta) > 0$ such that

$$t_n \in (0, \delta] \ \forall n \geq n_0(\delta). \tag{12}$$

Consequently for each $n \geq n_0(\delta)$ one has

$$t_n v' + t_n^2 w' + \delta t_n^2 \mathbb{B}_Y \subset intC. \tag{13}$$

By definition of $D^2_{H_+}(Epi_C F)(\overline{x}, v, v')(w)$, there exists $n_1(\delta) \geq n_0(\delta)$ such that

$$F(\overline{x} + t_n v + t_n^2 w_n) \subset F(\overline{x}) + t_n v' + t_n^2 w' + C + t_n^2 \delta \mathbb{B}_Y, \ \forall n \geq n_1(\delta).$$

The latter with (9), (10), (13) and $\mathbb{B}_Y = -\mathbb{B}_Y$, we get

$$F(\overline{x}) \subset F(\overline{x}) + intC,$$

which contradicts that $W.Min(F(\overline{x}), C) \neq \emptyset$. The case (b): The sequence (w_n) is not bounded. We may suppose that $(\|w_n\|_X) \to +\infty$ (taking a subsequence if necessary). Then the sequence (x_n) can be written as

$$x_n = \overline{x} + t_n v + t_n^2 \|w_n\|_X (w_n \|w_n\|_X^{-1}). \tag{14}$$

Observe that, $r_n := t_n \|w_n\| = \|v_n - v\| \downarrow 0$, $\frac{t_n}{r_n} = \frac{1}{\|w_n\|_X} \to 0$ and

$$v + r_n (w_n \|w_n\|_X^{-1}) = v + t_n \|w_n\|_X (w_n \|w_n\|_X^{-1}) \in S_X \text{ for all } n \in \mathbb{N}.$$

without loss of generality we may assume that $(w_n | w_n\|_X^{-1}) \to w$. Then

$$w \in K''(B, \overline{x}, v) \cap v^{\perp} \backslash \{0_X\}.$$

By (11) and (12) we have

$$t_n v' + t_n^2 \|w_n\|_X w' + \delta t_n^2 \|w_n\|_X \mathbb{B}_Y \subset intC \ \ \forall n \geq n_0(\delta). \tag{15}$$

By definition of $D''_{H_+}(Epi_C F)(\overline{x}, v, v')(w)$, there exists $n_1(\delta) \geq n_0(\delta)$ such that

$$\begin{aligned} &F(\overline{x} + t_n v + t_n^2 \|w_n\|_X (w_n \|w_n\|_X^{-1})) \\ &\quad \subset F(\overline{x}) + t_n v' + t_n^2 \|w_n\|_X w' + C + t_n^2 \|w_n\|_X \delta \mathbb{B}_Y, \ \forall n \geq n_1(\delta). \end{aligned}$$

The latter with (9), (10), (14) and (15) we get

$$F(\overline{x}) \subset F(\overline{x}) + intC,$$

which contradicts that $WMin(F(\overline{x}), C) \neq \emptyset$. □

The following example illustrate Proposition 6.

Example 4 Let $C := \mathbb{R}_+ \times \mathbb{R}_+$, $B := [0, 1] \times [-1, 1]$ and $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be a set valued map defined by

$$F(x) := \{y := (y_1, y_2) \in \mathbb{R}^2 : y_1 \geq x_2^2 + x_1, \ \ y_2 \geq x_1^2 + |x_2|\}, \forall x := (x_1, x_2) \in \mathbb{R}^2.$$

Take $\overline{x} := (0, 0)$. It is easy to verify that $WMin(F(\overline{x}), C) \neq \emptyset$ and $\overline{x}$ is a strict local weak l-minimal point of F on B. It follows from Definitions 2 and 5 that

$$K(B, \overline{x}) = \mathbb{R}_+ \times \mathbb{R}$$

and

$$\begin{aligned} A_2 &:= \{v' := (v_1', v_2') \in \mathbb{R}^2 : v := (v_1, v_2) \in K(B, \overline{x}), \ \ v' \in D_{H_{+l}}(Epi_C F)(\overline{x})(v) \cap \partial C\} \\ &= \{v' := (v_1', v_2') \in \mathbb{R}^2 : v := (v_1, v_2) \in K(B, \overline{x}), 0 \leq v_1' \leq v_1, \ 0 \leq v_2' \leq |v_2|, v' \in \partial C\} \\ &= \{v' := (v_1', v_2') \in \mathbb{R}^2 : v_1 \geq 0, v_2 \in \mathbb{R}, \ 0 \leq v_1' \leq v_1, \ 0 \leq v_2' \leq |v_2|, \ v' \in \partial C\}. \end{aligned}$$

Thus, for every $v := (v_1, v_2) \in K(B, \overline{x})\backslash\{(0,0)\}$, and every $v' := (v'_1, v'_2) \in A_2$, we have $(v_1, v_2) \in (\mathbb{R}_+ \times \mathbb{R})\backslash\{(0,0)\}$, and $0 \le v'_1 \le v_1,\ 0 \le v'_2 \le |v_2|,\ v' \in \partial C$. Moreover, for every $v := (v_1, v_2) \in K(B, \overline{x})\backslash\{(0,0)\}$ and every $v' := (v'_1, v'_2) \in A_2$, it follows from Definition 2 that

$$K^2(B, \overline{x}, v) = K''(B, \overline{x}, v) = \begin{cases} \mathbb{R}_+ \times \mathbb{R},\ if\ v_1 = 0. \\ \mathbb{R} \times \mathbb{R},\ if\ v_1 > 0 \end{cases}$$

and

$$v^\perp = \begin{cases} \mathbb{R} \times \{0\},\ if\ v_1 = 0,\ v_2 \neq 0. \\ \{0\} \times \mathbb{R},\ if\ v_1 > 0,\ v_2 = 0. \\ \{(0,0)\},\ if\ v_1 > 0,\ v_2 \neq 0. \end{cases}$$

So,

$$K^2(B, \overline{x}, v) \cap v^\perp = \begin{cases} \mathbb{R}_+ \times \{0\},\ if\ v_1 = 0,\ v_2 \neq 0. \\ \{0\} \times \mathbb{R},\ if\ v_1 > 0,\ v_2 = 0, \\ \{(0,0)\}, if\ v_1 > 0,\ v_2 \neq 0 \end{cases}$$

and

$$K^2(B, \overline{x}, v) \cap (v^\perp\backslash\{(0,0)\}) = \begin{cases} (\mathbb{R}_+\backslash\{0\}) \times \{0\},\ if\ v_1 = 0,\ v_2 \neq 0. \\ \{0\} \times (\mathbb{R}\backslash\{0\}),\ if\ v_1 > 0,\ v_2 = 0. \end{cases}$$

Also, for every $v := (v_1, v_2) \in K(B, \overline{x})\backslash\{(0,0)\}$, $v' := (v'_1, v'_2) \in A_2$ and every $w := (w_1, w_2) \in K^2(B, \overline{x}, v) \cap v^\perp$, it follows from Definition 6 that

$$D^2_{H_+}(Epi_C F)(\overline{x}, v, v')(w) = L6,$$

where

$$L_6 := \begin{cases} \{w' := (w'_1, w'_2) \in \mathbb{R}^2 : w'_1 \le w_1 + v_2^2,\ w'_2 \le |w_2| + v_1^2\},\ if\ v'_1 = v_1,\ v'_2 = |v_2|. \\ \{w' := (w'_1, w'_2) \in \mathbb{R}^2 : w'_1 \le w_1 + v_2^2,\ w'_2 \in \mathbb{R}\}, \quad if\ v'_1 = v_1,\ v'_2 < |v_2|. \\ \{w' := (w'_1, w'_2) \in \mathbb{R}^2 : w'_2 \le |w_2| + v_1^2\}, \quad if\ v'_1 < v_1,\ v'_2 = |v_2|. \\ \mathbb{R} \times \mathbb{R}, \quad if\ v'_1 < v_1,\ v'_2 < |v_2|. \end{cases}$$

Further, for every $v := (v_1, v_2) \in K(B, \overline{x})\backslash\{(0,0)\}$, $v' := (v'_1, v'_2) \in A_2$ and every $w := (w_1, w_2) \in K^2(B, \overline{x}, v) \cap (v^\perp\backslash\{(0,0)\})$, it follows from Definition 6 that

$$D''_{H_+}(Epi_C F)(\overline{x}, v, v')(w) = L7,$$

where

$$L_7 := \begin{cases} \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \leq w_1,\; w'_2 \leq |w_2|\}, \; if \; v'_1 = v_1,\; v'_2 = |v_2|. \\ \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_1 \leq w_1,\; w'_2 \in \mathbb{R}\}, \quad if \; v'_1 = v_1,\; v'_2 < |v_2|. \\ \{w' := (w'_1, w'_2) \in \mathbb{R}^2 \;\; w'_2 \leq |w_2|\}, \qquad if \; v'_1 < v_1,\; v'_2 = |v_2|. \\ \mathbb{R} \times \mathbb{R}, \qquad if \; v'_1 < v_1,\; v'_2 < |v_2|. \end{cases}$$

Thus, together with

$$K(C, v') = \begin{cases} \mathbb{R}_+ \times \mathbb{R}_+, \; if \; v'_1 = v'_2 = 0. \\ \mathbb{R}_+ \times \mathbb{R}, \; if \; v'_1 = 0,\; v'_2 > 0. \\ \mathbb{R} \times \mathbb{R}_+, \; if \; v'_1 > 0,\; v'_2 = 0, \end{cases}$$

we have

$$D^2_{H_+}(Epi_C F)(\overline{x}, v, v')(K^2(B, \overline{x}, v) \cap v^\perp) \cap int K(C, v') \neq \varnothing$$

and

$$D''_{H_+}(Epi_C F)(\overline{x}, v, v')(K''(B, \overline{x}, v) \cap (v^\perp \backslash \{(0, 0)\})) \cap int K(C, v') \neq \varnothing.$$

Thus, Proposition 6 is fulfilled.

5 Dual second-order optimality conditions

From the results established in Sect. 4, we can derive dual second-order optimality conditions in terms of Lagrange–Fritz–John and in terms of Lagrange–Karush–Kuhn–Tucker multipliers. To this end, we start by the following lemmas.

Lemma 1 *Let S be a nonempty and convex subset of X, $F : X \rightrightarrows Y$ be a set-valued mapping, $\overline{x} \in dom F$, $(v, v') \in X \times Y$. If F is C-convex and Lipschitzian at $\overline{x}$, then $D_{H_-}(Epi_C F)(\overline{x})(S)$, $D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(S)$ and $D''_{H_-}(Epi_C F)(\overline{x}, v, v')(S)$ are convex.*

Proof We only prove that $D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(S)$ is convex since the cases of $D_{H_-}(Epi_C F)(\overline{x})(S)$ and $D''_{H_-}(Epi_C F)(\overline{x}, v, v')(S)$ are similar. Let $w'_i \in D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(S)$ for $i \in \{1,\; 2\}$. Then, there exists $w_i \in S$ for $i \in \{1,\; 2\}$ such that $w'_i \in D_{H_-}(Epi_C F)(\overline{x}, v, v')(w_i)$. Let $\varepsilon > 0$. Then there exist $\delta > 0$ and a neighborhood U_i of w_i with $i \in \{1,\; 2\}$ such that for all $t \in (0, \delta]$, $u_i \in U_i$ and $i \in \{1,\; 2\}$,

$$F(\overline{x}) + C + tv' + t^2 w'_i \subset F(\overline{x} + tv + t^2 u_i) + C + t^2 \frac{\varepsilon}{4} \mathbb{B}_Y.$$

Let $\lambda \in [0, 1]$. As F is C-convex and $F(\overline{x}) \subset \lambda F(\overline{x}) + (1-\lambda)F(\overline{x})$, then

$$\begin{aligned} &F(\overline{x}) + C + tv' + t^2(\lambda w_1' + (1-\lambda)w_2') \\ &\quad \subset F(\overline{x} + tv + t^2(\lambda u_1 + (1-\lambda)u_2) + C + t^2\frac{\varepsilon}{2}\mathbb{B}_Y. \end{aligned}$$

Since F is Lipschitzian at $\overline{x}$, then there exist a neighborhood U of $\overline{x}$ and $k > 0$ such that that for all $x_1,\ x_2 \in U$,

$$F(x_1) \subset F(x_2) + k\|x_1 - x_2\|_X \mathbb{B}_Y.$$

As $\displaystyle\lim_{t\downarrow 0, u\to \lambda w_1 + (1-\lambda)w_2} (\overline{x} + tv + t^2 u) = \overline{x} \in U$,

$$\lim_{t\downarrow 0, u_1\to w_1, u_2\to w_2, u\to \lambda w_1+(1-\lambda)w_2} k\|\lambda u_1 + (1-\lambda)u_2 - u\|_X = 0$$

and

$$\lim_{t\downarrow 0, u_1\to w_1, u_2\to w_2} (\overline{x} + tv + t^2(\lambda u_1 + (1-\lambda)u_2)) = \overline{x} \in U,$$

then there exist $\eta > 0$ with $\eta \leq \delta$, and neighborhoods U_1', U_2' and U_3' of w_1, w_2 and $\lambda w_1 + (1-\lambda)w_2$, respectively, with $U_1' \subset U_1$ and $U_2' \subset U_2$ such that for all $t \in (0, \eta]$, $u_1 \in U_1'$, $u_2 \in U_2'$ and $u \in U_3'$ one has

$$k\|\lambda u_1 + (1-\lambda)u_2 - u\|_X < \frac{\varepsilon}{2},$$

$$\overline{x} + tv + t^2(\lambda u_1 + (1-\lambda)u_2) \in U$$

and

$$\overline{x} + tv + t^2 u \in U.$$

Therefore, we conclude that

$$F(\overline{x}) + C + tv' + t^2(\lambda w_1' + (1-\lambda)w_2') \subset F(\overline{x} + tv + t^2 u) + C + t^2\varepsilon\mathbb{B}_Y$$

for all $t \in (0, \eta]$ and $u \in U_3'$. So,

$$\lambda w_1' + (1-\lambda)w_2' \in D_{H-}(Epi_C F)(\overline{x}, v, v')(\lambda w_1 + (1-\lambda)w_2).$$

Thus the proof is complete. □

In what follows the subset $B \subset X$ is defined by:

$$B := \{x \in S : G(x) \cap -Q \neq \emptyset\},$$

where S is a nonempty subset of X and $G : X \rightrightarrows Z$ is a set-valued mapping, and $Q \subset Z$ denotes the pointed, closed and convex cone with nonempty interior.

Lemma 2 *Let $\overline{x} \in B$ be a local weak l-minimal point on B of a set-valued map $F : X \rightrightarrows Y$ with respect to C. Then $\overline{x}$ is a local weak l-minimal point of (F, G) on S with respect to $C \times Q$.*

Proof Suppose on the contrary that there exists a sequence $(x_n) \subset S$ such that $(x_n) \to \overline{x}$ and

$$(F, G)(\overline{x}) \subset (F, G)(x_n) + int(C \times Q),\ \forall n \in \mathbb{N}.$$

Then for all $n \in \mathbb{N}$ one has

$$F(\overline{x}) \subset F(x_n) + intC$$

and

$$G(\overline{x}) \subset G(x_n) + intQ.$$

Take $\overline{z} \in G(\overline{x}) \cap -Q$. Then there exist $y_n \in G(x_n)$ and $a_n \in intQ$ such that $\overline{z} = y_n + a_n$.This means that $y_n = \overline{z} - a_n \in -Q - intQ \subset -Q$, and so $x_n \in B$, which contradicts that $\overline{x}$ is a local weak l-minimal point of F on B with respect to C.

□

Remark 8 (see [1]) Let B be a nonempty subset of Y, $F : X \rightrightarrows Y$ be a set valued mapping which is C-convex, and $\overline{x} \in B$. Then $\overline{x}$ is a local weak l-minimal point of F on B with respect to C if and only if it is a global weak l-minimal point of F on B with respect to C.

Now, we are able to establish second-order necessary optimality conditions in terms of Lagrange–Fritz–John multipliers.

Proposition 7 *Let $\overline{x} \in B$ be a local weak l-minimal point on B of a set-valued map $F : X \rightrightarrows Y$ with respect to C, and $v \in K(S, \overline{x})$, $v' \in D_{H\_}(Epi_C F)(\overline{x})(v) \cap -\partial C$ and $v'' \in D_{H\_}(Epi_Q G)(\overline{x})(v) \cap -\partial Q$. Suppose that*

(i) $D^2_{H\_}(Epi_{C\times Q}(F, G))(\overline{x}, v, v', v'')(K^2(S, \overline{x}, v))$ is convex,
(ii) F and G are Lipschitzian at $\overline{x}$.

Then there exists $(y^*, z^*) \in (-C)^\circ \times (-Q)^\circ \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that $\langle y^*, v'\rangle = 0$, $\langle z^*, v''\rangle = 0$ and

$$\langle y^*, w'\rangle + \langle z^*, w''\rangle \geq 0$$

for all $w \in K^2(S, \overline{x}, v)$, $w' \in D^2_{H\_} F(\overline{x}, v, v')(w)$ and $w'' \in D^2_{H\_} G(\overline{x}, v, v'')(w)$.

Proof From Lemma 2 and (ii) of Proposition 5, it follows that

$$cl(D^2_{H\_}(Epi_{C\times Q}(F,G))(\overline{x},v,v',v'')(K^2(S,\overline{x},v)))$$
$$\cap\, int K((-C)\times(-Q),(v',v''))=\varnothing.$$

By the standard Hahn-Banach separation theorem, there exists $(y^*,z^*)\in(Y^*\times Z^*)\backslash\{(0_{Y^*},0_{Z^*})\}$ such that

$$\langle(y^*,z^*),(w',w'')\rangle\geq\langle(y^*,z^*),(y,z)\rangle$$

for all $(w',w'')\in cl(D^2_{H\_}(Epi_{C\times Q}(F,G))(\overline{x},v,v',v'')(K^2(S,\overline{x},v)))$ and $(y,z)\in int K(-C\times(-Q),(v',v''))$. Since $K(-C\times(-Q),(v',v''))$ is closed and convex with nonempty interior, then

$$K(-C\times(-Q),(v',v''))=cl(int K(-C\times(-Q),(v',v''))).$$

By (c) of Remark 4 and (8) of Proposition 1, we get

$$\langle y^*,w'+d_1\rangle+\langle z^*,w''+d_2\rangle\geq\langle y^*,y\rangle+\langle z^*,z\rangle \tag{16}$$

for all $w\in K^2(S,\overline{x},v)$, $w'\in D^2_{H\_}F(\overline{x},v,v')(w)$, $w''\in D^2_{H\_}G(\overline{x},v,v'')(w)$, $(d_1,d_2)\in C\times Q$ and $(y,z)\in K(-C\times(-Q),(v',v''))$. Since $(-C)$ and $(-Q)$ are convex, then by (f) of Remark 2 one has

$$K((-C)\times(-Q),(v',v''))=K(-C,v')\times K(-Q,v'').$$

This with (16) imply

$$\langle y^*,w'\rangle+\langle z^*,w''\rangle\geq\langle y^*,y\rangle+\langle z^*,z\rangle \tag{17}$$

for all $w\in K^2(S,\overline{x},v)$, $w'\in D^2_{H\_}F(\overline{x},v,v')(w)$, $w''\in D^2_{H\_}G(\overline{x},v,v'')(w)$ and $(y,z)\in K(-C,v')\times K(-Q,v'')$. Hence

$$y^*\in(-C)^\circ\cap N(-C,v')\ and\ z^*\in(-Q)^\circ\cap N(-Q,v'').$$

Indeed, let $w_0\in K^2(S,\overline{x},v)$, $w'_0\in D^2_{H\_}F(\overline{x},v,v')(w_0)$, $w''_0\in D^2_{H\_}G(\overline{x},v,v'')(w_0)$, $(y,z)\in K(-C,v')\times K(-Q,v'')$ and $\lambda>0$. As $K(-C,v')\times K(-Q,v'')$ is a cone, then $\lambda(y,z)\in K(-C,v')\times K(-Q,v'')$. By (17) one has

$$\frac{1}{\lambda}(\langle y^*,w'_0\rangle+\langle z^*,w''_0\rangle)\geq\langle y^*,y\rangle+\langle z^*,z\rangle$$

Taking the limit as $\lambda\to+\infty$, we get

$$\langle y^*,y\rangle+\langle z^*,z\rangle\leq 0,\ \ \forall(y,z)\in K(-C,v')\times K(-Q,v'').$$

Since $0_Y \in K(-C, v')$ and $0_Z \in K(-Q, v'')$, then one obviously has $y^* \in (K(-C, v'))^\circ$ and $z^* \in (K(-Q, v''))^\circ$. By convexity of $(-C)$ and $(-Q)$, we have by (e) of Remark 2 that

$$K(-C, v') = cl(\mathbb{R}_+(-C - v')) \text{ and } K(-Q, v'') = cl(\mathbb{R}_+(-Q - v'')).$$

So, we get

$$y^* \in N(-C, v')) \text{ and } z^* \in N(-Q, v'').$$

As $v' \in (-C)$ and $v'' \in (-Q)$, it follows that $y^* \in (-C)^\circ \cap N(-C, v')$ and $z^* \in (-Q)^\circ \cap N(-Q, v'')$. Now, as $v' \in -C$ and $v'' \in -Q$, we get $\langle y^*, v' \rangle = 0$ and $\langle z^*, v'' \rangle = 0$. Since $0_Y \in K(-C, v')$ and $0_Z \in K(-Q, v'')$, we conclude from (17) the proof of Proposition 7. □

The proof of the following proposition runs in analogous way as in Proposition 7, when we apply Proposition 5 (iii) instead of Proposition 5(ii).

Proposition 8 *Let $\overline{x} \in B$ be a local weak l-minimal point on B of a set-valued map $F : X \rightrightarrows Y$ with respect to C, and $v \in K(S, \overline{x})$, $v' \in D_{H\_}(Epi_C F)(\overline{x})(v) \cap -\partial C$, $v'' \in D_{H\_}(Epi_Q G)(\overline{x})(v) \cap -\partial Q$ and the assumption (ii) of Proposition 7 be satisfied. Suppose that $D''_{H\_}(Epi_{C\times Q}(F, G))(\overline{x}, v, v', v'')(K''(S, \overline{x}, v))$ is convex. Then there exists $(y^*, z^*) \in (-C)^\circ \times (-Q)^\circ \backslash \{(0_{Y^*}, 0_{Z^*})\}$ such that $\langle y^*, v' \rangle = 0$, $\langle z^*, v'' \rangle = 0$ and*

$$\langle y^*, w' \rangle + \langle z^*, w'' \rangle \geq 0$$

for all $w \in K''(S, \overline{x}, v)$, $w' \in D''_{H\_} F(\overline{x}, v, v')(w)$ and $w'' \in D''_{H\_} G(\overline{x}, v, v'')(w)$.

The following corollary is a direct consequence of Propositions 1, 7 and Remark 2 (c).

Corollary 1 *Let $\overline{x} \in B$ be a local weak l-minimal point on B of a set-valued map $F : X \rightrightarrows Y$ with respect to C. Suppose that $D_{H-}(Epi_{C\times Q}(F, G))(\overline{x})(K(S, \overline{x}))$ is convex and the assumptions (ii) of Proposition 7 is satisfied. Then there exists $(y^*, z^*) \in (-C)^\circ \times (-Q)^\circ \backslash \{(0_{Y^*}, 0_{Z^*})\}$ such that*

$$\langle y^*, w' \rangle + \langle z^*, w'' \rangle \geq 0$$

for all $w \in K(S, \overline{x})$, $w' \in D_{H\_} F(\overline{x})(w)$ and $w'' \in D_{H\_} G(\overline{x})(w)$.

In next, we establish second-order necessary optimality conditions in terms of Lagrange–Karush–Kuhn–Tucker multipliers. To this end, we start by the following recall.

Definition 9 Let U be a nonempty subset of X. The core of U, is the set $core(U)$ of elements $a \in U$ such that for all $h \in X$, there exists $\alpha > 0$ such that $a + th \in U$ for all $t \in [-\alpha, \alpha]$.

Let $L(Z, Y)$ be the set of all continuous linear operators from Z into Y.

Proposition 9 *Let the assumption of Proposition 7 be satisfied. Moreover, suppose that there exists* $w_0 \in K^2(S, \overline{x}, v)$ *such that* $0_Z \in core(D^2_{H\_} G(\overline{x}, v, v'')(w_0) + Q)$. *Then, the following statements hold:*

(a) There exists $(y^*, z^*) \in (-C)^\circ \times (-Q)^\circ$ *with* $y^* \neq 0_{Y^*}$ *such that* $\langle y^*, v' \rangle = 0$, $\langle z^*, v'' \rangle = 0$ *and*

$$\langle y^*, w' \rangle + \langle z^*, w'' \rangle \geq 0$$

for all $w \in K^2(S, \overline{x}, v)$, $w' \in D^2_{H\_} F(\overline{x}, v, v')(w)$ *and* $w'' \in D^2_{H\_} G(\overline{x}, v, v'')(w)$.

(b) There exists $T \in L(Z, Y)$ *such that* $Tv'' = 0_Y$, $T(Q) \subset C$ *and*

$$(D^2_{H\_} F(\overline{x}, v, v') + (T \circ D^2_{H\_} G(\overline{x}, v, v'')))(w) \cap int K(-C, v') \\ = \varnothing, \forall w \in K^2(S, \overline{x}, v).$$

Proof (a) By Proposition 7, there exists $(y^*, z^*) \in (-C)^\circ \times (-Q)^\circ \backslash \{(0_{Y^*}, 0_{Z^*})\}$ such that $\langle y^*, v' \rangle = 0$, $\langle z^*, v'' \rangle = 0$ and

$$\langle y^*, w' \rangle + \langle z^*, w'' \rangle \geq 0 \tag{18}$$

for all $w \in K^2(S, \overline{x}, v)$, $w' \in D^2_{H\_} F(\overline{x}, v, v')(w)$ and $w'' \in D^2_{H\_} G(\overline{x}, v, v'')(w)$. Suppose that $y^* = 0_Y$. From (18) one has $\langle z^*, w'' + d \rangle \geq \langle z^*, d \rangle \geq 0$ for all $w'' \in D^2_{H\_} G(\overline{x}, v, v'')(w_0)$ and $d \in Q$. By our assumption, for all $h \in Z$, there exists $\alpha > 0$ such that

$$\alpha h \in D^2_{H\_} G(\overline{x}, v, v'')(w_0) + Q.$$

Then $\langle z^*, h \rangle \geq 0$ for all $h \in Z$. Thus $z^* = 0_{Z^*}$, which is a contradiction. (b) Since $intC \neq \varnothing$, then there exists $e \in intC$ such that $\langle y^*, e \rangle = 1$. We define a continuous linear operator $T : Z \to Y$ by $Tz = \langle z^*, z \rangle e$. Clearly, $Tv'' = 0_Y$, $T(Q) \subset C$ and $y^* \circ T = z^*$. Suppose on the contrary that there exists $w \in K^2(S, \overline{x}, v)$ and $w' \in int K(-C, v')$ such that

$$w' \in D^2_{H\_} F(\overline{x}, v, v')(w) + T \circ D^2_{H\_} G(\overline{x}, v, v'')(w).$$

Since $y^* \in (-C)^\circ$ and $\langle y^*, v' \rangle = 0$, then $y^* \in N(-C, v')$. Therefore, $\langle y^*, w' \rangle < 0$, which contradicts (18). Thus the proof is complete. □

The following example illustrate Proposition 9 (b).

Example 5 Let $S := [-1, 1] \times [-1, 1]$, $C := \mathbb{R}_+ \times \mathbb{R}_+$, $Q := \mathbb{R}_+$ and the set-valued maps $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$,

$$F(x) := \{y := (y_1, y_2) \in \mathbb{R}^2 : y_1 \geq x_1,\ y_2 \geq x_2\}, \forall x := (x_1, x_2) \in \mathbb{R}^2$$

and $G : \mathbb{R}^2 \to \mathbb{R}$,

$$G(x) := \{y \in \mathbb{R} : y \geq -x_1\}, \forall x := (x_1, x_2) \in \mathbb{R}^2.$$

Then $B := \{x \in S \;\; G(x) \cap (-\mathbb{R}_+) \neq \varnothing\} = [0, 1] \times [-1, 1]$. Take $\overline{x} := (0, 0)$. It is easy to verify that $\overline{x}$ is a local weak l-minimal point of F on B, and F and G are Lipschitzian at $\overline{x}$. It follows from Definitions 2 and 5 that

$$K(S, \overline{x}) = \mathbb{R} \times \mathbb{R}$$

and

$$\begin{aligned} A &:= \{(v', v'') := ((v_1', v_2'), v'') \in \mathbb{R}^3 \;:\; v := (v_1, v_2) \in K(S, \overline{x}),\; v' \in A_1,\; v'' \in A_2\} \\ &= \{(v_1', v_2', v'') \in \mathbb{R}^3 \;:\; v_1 \leq v_1' \leq 0,\; v_2 \leq v_2' \leq 0,\; 0 \geq v'' \geq -v_1\} \\ &= \{(v_1', v_2', v'') \in \mathbb{R}^3 \;: v_1 = 0,\; v_2 \leq 0,\; v_1' = 0,\; v_2 \leq v_2' \leq 0, v'' = 0\}, \end{aligned}$$

where $A_1 := D_{H_-}(Epi_C F)(\overline{x})(v) \cap -\partial C$ and $A_2 := D_{H_-}(Epi_Q G)(\overline{x})(v) \cap -\partial Q$.

Thus, for every $(v_1, v_2) \in K(S, \overline{x})$ and every $(v_1', v_2', v'') \in A$, we have $v_1 = 0$, $v_2 \leq 0$ and $v_1' = 0$, $v_2 \leq v_2' \leq 0$, $v'' = 0$. Moreover, for every $v := (v_1, v_2) \in K(S, \overline{x})$ and every $(v_1', v_2', v'') \in A$, it follows from Definition 2 that

$$K^2(S, \overline{x}, v) = \mathbb{R}^2.$$

Also, it follows from Lemma 1 that $D^2_{H-}(Epi_{C\times Q}(F, G))(\overline{x}, v, v', v'')(K^2(S, \overline{x}, v))$ is convex since F and G are C-convex and Q-convex, respectively. Further, for every $v := (v_1, v_2) \in K(S, \overline{x})$, $(v', v'') := ((v_1', v_2'), v'') \in A$ and every $w := (w_1, w_2) \in K^2(S, \overline{x}, v)$, we have

$$D^2_{H_-}(Epi_C F)(\overline{x}, v, v')(w) = L_1 \text{ and } D^2_{H_-}(Epi_Q G)(\overline{x}, v, v'')(w) = L_2,$$

where

$$L_1 := \begin{cases} \{w' := (w_1', w_2') \in \mathbb{R}^2 \;\; w_1' \geq w_1,\; w_2' \geq w_2\}\; if\; v_2' = v_2 \\ \{w' := (w_1', w_2') \in \mathbb{R}^2 \;\; w_1' \geq w_1,\; w_2' \in \mathbb{R}\},\; if\; v_2' > v_2, \end{cases}$$
$$L_2 := \{w'' \in \mathbb{R} \;\; w'' \geq -w_1\}.$$

Thus, together with

$$T(z) := (z, z), \forall z \in \mathbb{R}$$

and

$$K(-C, v') = \begin{cases} -(\mathbb{R}_+ \times \mathbb{R}_+)\; if\; v_2' = 0 \\ (-\mathbb{R}_+) \times \mathbb{R}\; if\; v_2' < 0, \end{cases}$$

we have $Tv'' = (0,0)$, $T(Q) \subset C$ and

$$(D^2_{H_-}(Epi_C F)(\overline{x}, v, v') + (T \circ D^2_{H_-}(Epi_Q G)(\overline{x}, v, v'')))(K^2(S, \overline{x}, v)) \\ \cap\, int K(-C, v') = \varnothing.$$

Thus, Proposition 9 is fulfilled.

From Proposition 9, we derive the following result closes to the one established in [7, Theorem 5.2]. The proof in [7, Theorem 5.2] deponds heavily on the us of Farkas' lemma.

Corollary 2 *Let $f : X \longrightarrow Y$ and $g : X \longrightarrow Z$ be single vector-valued mappings, which are twice Fréchet differentiables at $\overline{x} \in B := S \cap g^{-1}(-Q)$, where $g^{-1}(-Q) := \{x \in X\ g(x) \in -Q\}$, Y and Z are finite-dimensional, and S is a nonempty and convex subset of X. Let $v \in K(S, \overline{x})$, $\nabla f(\overline{x})v \in -\partial C$ and $\nabla g(\overline{x})v \in -\partial Q$. Suppose that $\overline{x}$ is a local weak Pareto minimal point of f on B, $K^2(S, \overline{x}, v) \neq \varnothing$ and $0_Z \in core(\nabla g(\overline{x})(K(K(S, \overline{x}), v)) + Q)$. Then there exists $(y^*, z^*) \in (-C)^\circ \times (-Q)^\circ$ with $y^* \neq 0_{Y^*}$ such that*

(a) $\langle y^*, \nabla f(\overline{x})v\rangle = 0$, $\langle z^*, \nabla g(\overline{x})v\rangle = 0$,
(b) $-y^* \circ \nabla f(\overline{x}) - z^* \circ \nabla g(\overline{x}) \in N(S, \overline{x})$,
(c) $\langle y^*, \nabla f(\overline{x})w + \frac{1}{2}\nabla^2 f(\overline{x})(v, v)\rangle + \langle z^*, \nabla g(\overline{x})w + \frac{1}{2}\nabla^2 g(\overline{x})(v, v)\rangle \geq 0$ *for all* $w \in K^2(S, \overline{x}, v)$.

Proof Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ be set-valued maps defined by

$$F(x) := \{f(x)\} \text{ and } G(x) := \{g(x)\}, \text{ for all } x \in X.$$

Let $v \in K(S, \overline{x})$. From [4], we get $D_{H_-}(Epi_C F)(\overline{x})(v) = \nabla f(\overline{x})v + C$ and $D_{H_-}(Epi_Q G)(\overline{x})(v) = \nabla g(\overline{x})v + Q$, where $\nabla f(\overline{x})$ is the first order Fréchet derivative of f at $\overline{x}$. Moreover, by computing, we get

$$D^2_{H-}(Epi_{C\times Q}(F, G))(\overline{x}, v, \nabla f(\overline{x})v, \nabla g(\overline{x})v)(K^2(S, \overline{x}, v)) \\ = \nabla(f, g)(\overline{x})(K^2(S, \overline{x}, v)) + \frac{1}{2}\nabla^2(f, g)(\overline{x})(v, v) + C \times Q,$$

where $\nabla^2(f, g)(\overline{x})$ is the second order Fréchet derivative of (f, g) at $\overline{x}$. Therefore, we have

$$v' := \nabla f(\overline{x})v \in D_{H_-}(Epi_C F)(\overline{x})(v) \cap -\partial C, \\ v'' := \nabla g(\overline{x})v \in D_{H_-}(Epi_Q G)(\overline{x})(v) \cap -\partial Q$$

and $D^2_{H-}(Epi_{C\times Q}(F, G))(\overline{x}, v, \nabla f(\overline{x})v, \nabla g(\overline{x})v)(K^2(S, \overline{x}, v))$ is convex. Further, as $\nabla g(\overline{x})(K(K(S, \overline{x}), v)) + Q$ is convex, it follows by relation (6,1) of [12] that $0_Y \in core(\nabla g(\overline{x})(K(K(S, \overline{x}), v)) + Q)$ is equivalent to

$$Z = \nabla g(\overline{x})(K(K(S, \overline{x}), v)) + Q.$$

Since S is convex, then (see [7, Proposition 2.5]) one has

$$(R): \quad K^2(S, \overline{x}, v) + K(K(S, \overline{x}), v) \subset K^2(S, \overline{x}, v).$$

As $K^2(S, \overline{x}, v) \neq \emptyset$, it follows that

$$Z = \nabla g(\overline{x})(K^2(S, \overline{x}, v)) + Q.$$

which implies that

$$Z = \nabla g(\overline{x})(K^2(S, \overline{x}, v)) + \frac{1}{2}\nabla^2 g(\overline{x})(v, v) + Q.$$

As

$$D^2_{H\_} G(\overline{x}, v, \nabla g(\overline{x})v)(K^2(S, \overline{x}, v)) = \nabla g(\overline{x})(K^2(S, \overline{x}, v)) + \frac{1}{2}\nabla^2 g(\overline{x})(v, v),$$

then we conclude that

$$0_Z \in core(D^2_{H\_} G(\overline{x}, v, \nabla g(\overline{x})v)(K^2(S, \overline{x}, v)) + Q).$$

Therefore, the proof follows from Proposition 9 and relation (R). □

Remark 9 (a) In Proposition 9, the operator $T := 0_{L(Z,Y)}$ in general does not verify the statement (b) of this proposition since $\overline{x}$ is a local weak l-minimal point of F on B but it is not necessarily a local weak l-minimal point of F on S (see Example 5).

(b) In Propsitions 7 and 9, the vector $v \in K(S, \overline{x})$ such that $v' \in D_{H\_}(Epi_C F)(\overline{x})(v) \cap -\partial C$ and $v'' \in D_{H\_}(Epi_Q G)(\overline{x})(v) \cap -\partial Q$ is called the critical direction (see for instance [7, 17]). Therefore, in order to establish second order necessary optimality conditions, the vectors v' and v'' are chosen appropriately as a function of vector v (see Corollary 2 where $v \in K(S, \overline{x})$, $v' := \nabla f(\overline{x})v$ and $v'' := \nabla g(\overline{x})v$). Consequently, the dependence of the multipliers of these vectors is not a restriction (for more details see [7, Theorem 5.2] and [17, Theorem 4.4]).

The proof of the following proposition runs in analogous way as in Proposition 9, when we apply Proposition 8 instead of Proposition 7.

Proposition 10 *Let the assumption of Proposition 8 be satisfied. Moreover, suppose that there exists $w_0 \in K''(S, \overline{x}, v)$ such that $0_Z \in core(D''_{H\_} G(\overline{x}, v, v'')(w_0) + Q)$. Then, the following statements hold:*

(a) There exists $(y^*, z^*) \in (-C)^\circ \times (-Q)^\circ$ with $y^* \neq 0_{Y^*}$ such that $\langle y^*, v' \rangle = 0$, $\langle z^*, v'' \rangle = 0$ and

$$\langle y^*, w' \rangle + \langle z^*, w'' \rangle \geq 0$$

for all $w \in K''(S, \overline{x}, v)$, $w' \in D''_{H\_} F(\overline{x}, v, v')(w)$ and $w'' \in D''_{H\_} G(\overline{x}, v, v'')(w)$.

(b) *There exists* $T \in L(Z, Y)$ *such that* $Tv'' = 0_Y$, $T(Q) \subset C$ *and*

$$(D''_{H\_} F(\overline{x}, v, v') + (T \circ D''_{H\_} G(\overline{x}, v, v'')))(w) \cap intK(-C, v') = \emptyset, \forall w \in K''(S, \overline{x}, v).$$

The following result is a direct consequence of Propositions 1, 9 and Remark 1.

Corollary 3 *Let the assumptions of Corollary 1 be satisfied. Moreover, suppose that there exists* $w_0 \in K(S, \overline{x})$ *such that* $0_Z \in core(D_{H\_} G(\overline{x})(w_0) + Q)$. *Then, the following statements hold:*

(a) *There exists* $(y^*, z^*) \in (-C)^\circ \times (-Q)^\circ$ *with* $y^* \neq 0_{Y^*}$ *such that*

$$\langle y^*, w' \rangle + \langle z^*, w'' \rangle \geq 0$$

for all $w \in K(S, \overline{x})$, $w' \in D'_{H\_} F(\overline{x})(w)$ *and* $w'' \in D_{H\_} G(\overline{x})(w)$.

(b *There exists* $T \in L(Z, Y)$ *such that* $T(Q) \subset C$ *and*

$$(D_{H\_} F(\overline{x})(w) + (T \circ D_{H\_} G(\overline{x}))(w)) \cap -intC = \emptyset, \ \forall w \in K(S, \overline{x}).$$

6 Conclusion

Inspired by [4], we propose several new concepts of second-order directional derivatives for set-valued maps by means of excess from a set to another one, and discuss some of their properties. By using these directional derivatives and by adopting the notion of set criterion introduced by Kuroiwa [13–15], we obtain second-order necessary and sufficient optimality conditions in the primal form. Moreover, under some additional assumptions, we obtain dual second-order necessary optimality conditions in terms of Lagrange–Fritz–John and in terms of Lagrange–Karush–Kuhn–Tucker multipliers. The case of sufficient second order optimality conditions in terms of Lagrange–Karush–Kuhn–Tucker multipliers will be treated elsewhere.

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Declarations

Conflict of interest The author has not any Conflict of interest.

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