On approximate solutions for robust semi-infinite multi-objective convex symmetric cone optimization

Baha Alzalg[1,2] · **Amira Achouak Oulha**[1,3]

Abstract

We present approximate solutions for the robust semi-infinite multi-objective convex symmetric cone programming problem. By using the robust optimization approach, we establish an approximate optimality theorem and approximate duality theorems for approximate solutions in convex symmetric cone optimization problem involving infinitely many constraints to be satisfied and multiple objectives to be optimized simultaneously under the robust characteristic cone constraint qualification. We also give an example to illustrate the obtained results in an important special case, namely the robust semi-infinite multi-objective convex second-order cone program.

Keywords Robust symmetric cone optimization · Semi-infinite programming · Multi-objective programming · Approximate optimality conditions · Approximate duality theorems

Mathematics Subject Classification 90C20 · 90C25 · 90C29 · 90C34 · 90C46

1 Introduction

Robust optimization [1–4] has become a very active methodological approach that is established to deal with optimization problems under uncertainty. The uncertainty means that the entering parameters of those problems are not acknowledged precisely on the time while an answer must be determined. In recent years, many authors

✉ Baha Alzalg
b.alzalg@ju.edu.jo

Amira Achouak Oulha
amy9170464@ju.edu.jo

1 Department of Mathematics, The University of Jordan, Amman 11942, Jordan

2 Department of Computer Science and Engineering, The Ohio State University, Columbus, OH 43210, USA

3 Department of Mathematics, University of Bachir El Ibrahimi, 34000 Bordj Bou Arreridj, Algeria

have established approximate optimality conditions and duality theorems for approximate solutions (ϵ-solutions) for different classes of optimization problems [5–21]. More specifically, Jeyakumar and Li established strong duality for robust semidefinite linear programming [8], Lee and Jiao established quasi approximate solutions for robust convex programming [9], Lee and Lee established approximate solutions for robust convex programming [11], robust fractional programming [12], robust convex semidefinite programming [14], and robust semi-infinite programming [15]. Lee and Lee also established optimality conditions and duality theorems for robust semi-infinite multi-objective programming [13].

Convex symmetric cone optimization [22–26] problems are a class of convex optimization problems in which we minimize a convex function over the intersection of an affine set with the Cartesian product of symmetric cones. Well-known examples of symmetric cones are the nonnegative orthant cone, the second-order cone, the cone of symmetric positive semidefinite matrices, the cone of complex hermitian positive semidefinite matrices, and the cone of quaternion Hermitian positive semidefinite matrices. Therefore, well-studied special cases of symmetric programming are linear programming, second-order cone programming [27–30] and semidefinite programming [31–33]. Other special cases are optimization problems over complex Hermitian positive semidefinite matrices, and optimization problems over quaternion Hermitian positive semidefinite matrices [22, 23, 25].

To illustrate the modeling potential of conic programming and the extensive applicability of symmetric cone programming, we mention that all convex programming problems can be formulated as conic programs [34], and that almost all real-world applications of conic programming are associated with symmetric cones [27, 28, 31, 32, 34]. There is a strong relationship between symmetric cone programming and Euclidean Jordan algebras. When we optimize over symmetric cones, the importance of Euclidean Jordan algebras stems from the fact that a cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra. Some Jordan algebraic notations, are listed in Table 1. These notations will be used in the sequel. Readers who are unfamiliar with the theory of Jordan algebra are encouraged to read [22, Section 2].

Despite the genuine need for establishing an approximate optimality theorem and approximate duality theorems for convex symmetric cone programming problems, there are no symmetric conicity analogs of these approximate theorems. Inspired by this gap in the literature, in this paper, we establish ϵ-solutions for the robust convex symmetric cone programming problem. Our setting is general in the sense that our convex symmetric cone optimization problem involves infinitely many constraints and multiple objective functions. That is, we establish an ϵ-optimality theorem and ϵ-duality theorems for robust semi-infinite multi-objective convex symmetric cone programming. We also apply our results to an important special case, namely the robust semi-infinite multi-objective convex second-order cone program.

Table 1 Some Jordan algebraic notations will be used throughout the paper

Notation	Denotation
$\mathcal{J}^n$	Euclidean Jordan algebra with dimension n (we drop n if it is known from the context)
rank$(\mathcal{J})$	The rank of $\mathcal{J}$
e	The identity element of $\mathcal{J}$
$x \circ y$	The Jordan product that maps (x, y) from $\mathcal{J} \times \mathcal{J}$ to $\mathcal{J}$
x^2	$x \circ x$ (in general, for $n \geq 2$, $x^n := x^{n-1} \circ x$)
$\mathcal{K}_{\mathcal{J}}$	The cone of squares of a Euclidean Jordan algebra $\mathcal{J}$ defined as $\mathcal{K}_{\mathcal{J}} := \{x^2 : x \in \mathcal{J}\}$
int $\mathcal{K}_{\mathcal{J}}$	The interior of the cone $\mathcal{K}_{\mathcal{J}}$
$\lambda_1, \ldots, \lambda_r$	The eigenvalues of an element x in a rank-r algebra $\mathcal{J}$ (the roots of its characteristic polynomial)
trace(x)	The trace of an element x in $\mathcal{J}$ $\left(\text{trace}(x) := \sqrt{\sum_{i=1}^r \lambda_i}\right)$
$x \bullet y$	The Frobenius inner product of $x, y \in \mathcal{J}$ defined as $x \bullet y := \text{trace}(x \circ y)$
$\|x\|$	The Frobenius norm of an element $x \in \mathcal{J}$ $\left(\|x\| := \sqrt{x \bullet y} = \sqrt{\sum_{i=1}^r \lambda_i^2}\right)$
$x \succeq_{\mathcal{K}_{\mathcal{J}}} 0$ $(x \succ_{\mathcal{K}_{\mathcal{J}}} 0)$	x is an element in $\mathcal{K}_{\mathcal{J}}$ (int $\mathcal{K}_{\mathcal{J}}$)
	We also write $x \succeq 0$ $(x \succ 0)$ if $\mathcal{J}$ is known from the context
$y \preceq_{\mathcal{K}_{\mathcal{J}}} x$ $(y \prec_{\mathcal{K}_{\mathcal{J}}} x)$	Same as $x \succeq_{\mathcal{K}_{\mathcal{J}}} y$ $(x \succ_{\mathcal{K}_{\mathcal{J}}} y)$

The semi-infinite multi-objective symmetric programming problem is defined as

$$\text{(SIMSP)} \begin{cases} \min & \left(f_1(x), f_2(x), \ldots, f_K(x) \right) \\ \text{s.t.} & a_0^{(t)} + \sum_{i=1}^m x_i a_i^{(t)} \succeq 0, \quad t \in T, \end{cases}$$

where $x \in \mathbb{R}^m$, $f_k : \mathbb{R}^m \rightarrow \mathbb{R}$, $k = 1, \ldots, K$, for $a_i^{(t)} \in \mathcal{J}$ for $i = 0, 1, \ldots, m$, $t \in T$, $\mathcal{J}$ is a Jordan algebra with dimension n and rank r, and T is an arbitrary index set that can be infinite.

The semi-infinite multi-objective symmetric programming problem with uncertain data in the constraints is defined as

$$\text{(USIMSP)} \begin{cases} \min & \left(f_1(x), f_2(x), \ldots, f_K(x) \right) \\ \text{s.t.} & a_0^{(t)} + \sum_{i=1}^m x_i a_i^{(t)} \succeq 0, \quad t \in T, \end{cases}$$

where, for each $i = 0, 1, \ldots, m$ and $t \in T$, the element $a_i^{(t)}$ belongs to an uncertainty set $\mathcal{V}_i^{(t)} \subseteq \mathcal{J}$.

The robust counterpart of USIMSP is defined as

$$
\text{(RSIMSP)} \quad
\begin{cases}
\min & \left(f_1(x), f_2(x), \ldots, f_K(x) \right) \\[2mm]
\text{s.t.} & a_0^{(t)} + \sum_{i=1}^{m} x_i a_i^{(t)} \succeq 0, \quad \forall a_i^{(t)} \in \mathcal{V}_i^{(t)}, \quad i = 0, 1, \ldots, m, \ t \in T.
\end{cases}
$$

Hence, the robust feasible set $\mathcal{F}_{\mathcal{P}}$ of RSIMSP is given as

$$
\mathcal{F}_{\mathcal{P}} = \left\{ x \in \mathbb{R}^m : a_0^{(t)} + \sum_{i=1}^{m} x_i a_i^{(t)} \succeq 0, \ \forall a_i^{(t)} \in \mathcal{V}_i^{(t)}, \ t \in T, \ i \in I \right\},
$$

where $I := \{0, 1, 2, \ldots, m\}$. We make the following assumptions throughout this paper.

Assumption 1.1 For each $t \in T$ and $i \in I$, $\mathcal{V}_i^{(t)} \subseteq \mathcal{J}$ is compact and convex.

Assumption 1.2 The robust feasiblity set $\mathcal{F}_{\mathcal{P}}$ has a nonempty interior.

Assumption 1.1 is necessary to give a characterization and prove properties of the robust characteristic cone which will be given in the next section. Assumption 1.2 is called the Slater condition and is necessary to prove and apply the robust version of Farkas' lemma.

We use $\mathbb{R}_+^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \ldots, n\}$ to denote the nth dimensional nonnegative orthant cone of $\mathbb{R}^n$. Its interior, $\text{int } \mathbb{R}_+^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \ldots, n\}$, is denoted by $\mathbb{R}_{++}^n$. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \mathbb{R}_+^K$, we say that $\bar{x} \in \mathcal{F}_{\mathcal{P}}$ is an ϵ-solution of RSIMSP if for any $x \in \mathcal{F}_{\mathcal{P}}$ we have that $f_k(x) \geq f_k(\bar{x}) - \epsilon_k$ for each $k = 1, \ldots, K$. Our focus in this paper is to present the approximate solutions (ϵ-solutions) for RSIMSP.

The next pages of the paper are structured as follows: Sect. 2 defines the robust characteristic cone and proves its convexity. The ϵ-optimality condition theorem and the ϵ-duality theorems for robust semi-infinite multi-objective symmetric programming are established in Sects. 3 and 4, respectively. In Sect. 5, we apply our results to an important special case, robust semi-infinite multi-objective second-order cone program.

2 The robust characteristic cone

In this section, we define the robust characteristic cone for our setting and and prove that it is closed and convex. First, we present some preliminaries.

Let $\bar{\mathbb{R}} := [-\infty, +\infty]$ and $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a function. We say f is proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. A proper

function f is said to be convex if for all $\mu \in [0, 1]$, we have

$$f((1 - \mu)x + \mu y) \leq (1 - \mu)f(x) + \mu f(y)$$

for all $x, y \in \mathbb{R}^n$. The domain of f is defined to be the set $\mathrm{dom}\, f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The epigraph of f is defined to be the set $\mathrm{epi}\, f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$.

The subdifferential of f at $x \in \mathbb{R}^n$ is defined as

$$\partial f(x) = \begin{cases} \{x^\star \in \mathbb{R}^n : \langle x^\star, y - x \rangle \leq f(y) - f(x), \ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \mathrm{dom}\, f, \\ \varnothing, & \text{otherwise.} \end{cases}$$

In general, for any $\epsilon \geq 0$, the ϵ-subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^\star \in \mathbb{R}^n : \langle x^\star, y - x \rangle \leq f(y) - f(x) + \epsilon, \ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \mathrm{dom}\, f, \\ \varnothing, & \text{otherwise.} \end{cases}$$

A function f is said to be a lower semi-continuous function if $\liminf_{y \to x} f(y) \geq f(x)$ for all $x \in \mathbb{R}^n$. The conjugate function of any proper convex function g on $\mathbb{R}^n$ is the function $g^\star : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined as

$$g^\star(x^\star) = \sup \left\{ \langle x^\star, x \rangle - g(x) : x \in \mathbb{R}^n \right\}$$

for any $x^\star \in \mathbb{R}^n$. The following proposition is due to Jeyakumar et al. [35].

Proposition 2.1 *Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. If one of the functions f and g is continuous, then*

$$\mathrm{epi}\, (f + g)^\star = \mathrm{epi}\, f^\star + \mathrm{epi}\, g^\star.$$

For a given set $A \subset \mathbb{R}^n$, we write $\mathrm{cl}\, A$ and $\mathrm{co}\, A$ to denote the closure of A and the convex hull generated by A, respectively. The indicator function δ_A is defined as

$$\delta_A(x) = \begin{cases} 0, & x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

In convex programming, a constrained minimization problem over a closed convex subset C of $\mathbb{R}^n$ can be reformulated as an unconstrained minimization problem by replacing its objective function, say f, with the function $(f + \delta_C)(x)$. The following proposition is due to Hiriart-Urruty and Lemarechal [36].

Proposition 2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, C be a closed convex subset of $\mathbb{R}^n$, and $\epsilon \geq 0$. Then*

$$\partial_\epsilon (f + \delta_C)(\bar{x}) = \bigcup_{\substack{\epsilon_0 \geq 0, \ \epsilon_1 \geq 0, \\ \epsilon_0 + \epsilon_1 = \epsilon}} \left\{ \partial_{\epsilon_0} f(\bar{x}) + \partial_{\epsilon_1} \delta_C(\bar{x}) \right\}.$$

We have also the following proposition [37, 38].

Proposition 2.3 *Let I is an arbitrary index set, and $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous convex function for $i \in I$. Assume that there exists $x_0 \in \mathbb{R}^n$ such that $\sup_{i \in I} g_i(x_0) \leq +\infty$. Then*

$$
epi \left(\sup_{i \in I} g_i \right)^\star = cl \left(co \bigcup_{i \in I} epi \, g_i^\star \right).
$$

Let $\mathcal{D}$ be the robust characteristic cone defined as

$$
\mathcal{D} := \bigcup_{a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, i \in I, \, t \in T}
\left\{ \sum_{t \in T} \left(z^{(t)} \bullet a_1^{(t)}, \; \ldots \;, \; z^{(t)} \bullet a_m^{(t)}, \; -z^{(t)} \bullet a_0^{(t)} - r^{(t)} \right) : z^{(t)} \geq 0, r^{(t)} \geq 0 \right\}.
$$

The following lemma shows that $\mathcal{D}$ is indeed a cone in $\mathbb{R}^{m+1}$ under Assumption 1.1.

Lemma 2.1 *The set $\mathcal{D} \subset \mathbb{R}^{m+1}$ is a cone.*

Proof It s clear that $0 \in \mathcal{D}$. To prove the desired result, we need to show that for each $x \in \mathcal{D}$ and $\lambda \in \mathbb{R}_{++}$, we have $\lambda x \in \mathcal{D}$. Since $x \in \mathcal{D}$, there exist $a_i = (a_i^{(t)})_{t \in T}, i \in I, z = (z^{(t)})_{t \in T}$, and $r \in \mathbb{R}_+^{(T)}$, where $a_i^{(t)} \in \mathcal{V}_i^{(t)}$ and $z^{(t)} \succeq 0$, for all $t \in T$ and $i \in I$, such that $x_i = \sum_{t \in T} (z^{(t)} \bullet a_i^{(t)})$ for $i = 1, \ldots, m$, and $x_{m+1} = -\sum_{t \in T} (z^{(t)} \bullet a_0^{(t)} + r^{(t)})$. It follows that $\lambda x_i = \sum_{t \in T} (\lambda z^{(t)} \bullet a_i^{(t)})$ and $\lambda x_{m+1} = -\sum_{t \in T} (\lambda z^{(t)} \bullet a_0^{(t)} + \bar{r})$. Note that $\lambda r^{(t)} \geq 0$, and that $\lambda z^{(t)} \succeq 0$ because $\mathcal{K}_\mathcal{J}$ is a cone. Thus, $\lambda x \in \mathcal{D}$. $\square$

The following lemma is due to [21, Lemma 4.2].

Proposition 2.4 *Let $\epsilon \in \mathbb{R}_+$, then $\bar{x}$ is an ϵ-solution of RSIMSP if*

$$
\sum_{k=1}^{K} f_k(x) \geq \sum_{k=1}^{K} f_k(\bar{x}) - \epsilon
$$

for any $x \in \mathcal{F}_\mathcal{P} \cap \{x \in \mathbb{R}^m : f_k(x) \geq f_k(\bar{x}), k = 1, \ldots, K\}$.

We say that RSIMSP satisfies the convexity condition if for every $t \in T$ we have

$$
\mathcal{V}_i^{(t)} = \left\{ a_0^{(t)} + \sum_{j=1}^{l} u_{ij}^{(t)} a_j^{(t)} : \left(u_{i_1}^{(t)}, u_{i_2}^{(t)}, \ldots, u_{i_l}^{(t)} \right) \in U_i^{(t)} \right\},
$$

where $U_i^{(t)}$ is compact convex subset of $\mathbb{R}^l$, $a_0^{(t)} \in \mathcal{J}$ and $a_j^{(t)} \succeq 0$, for each $t \in T, i \in I$, and $j = 1, 2, \ldots, l$. We point out that if the above convexity condition holds, then

the uncertainty sets V_i^t includes the box uncertainty sets of linear programming and the spectrahedral uncertainty sets of semidefinite programming as special cases.

The closeness and convexity of the robust characteristic cone $\mathcal{D}$ are necessary for the robust characteristic cone constraint qualification to hold. The following lemma proves the convexity of $\mathcal{D}$ under the convexity condition of RSIMSP (see also [39, Proposition 1]).

Lemma 2.2 *If RSIMSP satisfies the convexity condition, the robust characteristic cone $\mathcal{D}$ is convex.*

Proof Let $x, y \in \mathcal{D}$ and let $\lambda \in [0, 1]$. We want to show that $\lambda x + (1 - \lambda)y \in \mathcal{D}$. From the definition of $\mathcal{D}$, there exist $h_i = (h_i^{(t)})_{t \in T}, c_i = (c_i^{(t)})_{t \in T}, i \in I, b = (b^{(t)})_{t \in T}, n = (n^{(t)})_{t \in T}$, and $r, s \in \mathbb{R}_+^{(T)}$, where $h_i^{(t)}, c_i^{(t)} \in V_t$, and $b^{(t)}, n^{(t)} \geq 0$, for $i \in I, t \in T$, such that

$$
x_i = \sum_{t \in T} \left(b^{(t)} \bullet h_i^{(t)} \right), \ i \in I - \{0\}, \ x_{m+1} = -\sum_{t \in T} \left(b^{(t)} \bullet h_0^{(t)} + r^{(t)} \right),
$$
$$
y_i = \sum_{t \in T} \left(n^{(t)} \bullet c_i^{(t)} \right), \ i \in I - \{0\}, \ y_{m+1} = -\sum_{t \in T} \left(n^{(t)} \bullet c_0^{(t)} + s^{(t)} \right).
$$

Since $h_i^{(t)}, c_i^{(t)} \in V_t$, there exist $(u_{i_1}^{(t)}, u_{i_2}^{(t)}, \ldots, u_{i_l}^{(t)}) \in U_i^{(t)}$ and $(v_{i_1}^{(t)}, v_{i_2}^{(t)}, \ldots, v_{i_l}^{(t)}) \in U_i^{(t)}$ such that, for each $i \in I$, we have

$$
h_i^{(t)} = a_0^{(t)} + \sum_{j=1}^{l} u_{i_j}^{(t)} a_j^{(t)}, \quad \text{and} \quad c_i^{(t)} = a_0^{(t)} + \sum_{j=1}^{l} v_{i_j}^{(t)} a_j^{(t)}.
$$

Fixing an $i \in I - \{0\}$, we have that

$$
\lambda x_i + (1 - \lambda)y_i = \lambda \sum_{t \in T} \left(b^{(t)} \bullet h_i^{(t)} \right) + (1 - \lambda) \sum_{t \in T} \left(n^{(t)} \bullet c_i^{(t)} \right)
$$
$$
= \sum_{t \in T} \left(\lambda b^{(t)} \bullet \left(a_0^{(t)} + \sum_{j=1}^{l} u_{i_j}^{(t)} a_j^{(t)} \right) + (1 - \lambda) n^{(t)} \right.
$$
$$
\left. \bullet \left(a_0^{(t)} + \sum_{j=1}^{l} v_{i_j}^{(t)} a_j^{(t)} \right) \right)
$$
$$
= \sum_{t \in T} \left(a_j^{(0)} \bullet \left(\lambda b^{(t)} + (1 - \lambda)n^{(t)} \right) \right.
$$
$$
\left. + \sum_{j=1}^{l} \left(\lambda u_{i_j}^{(t)} b^{(t)} \bullet a_j^{(t)} + (1 - \lambda) v_{i_j}^{(t)} n^{(t)} \bullet a_j^{(t)} \right) \right).
$$

For $i \in I$, $t \in T$ and $j = 1, 2, \ldots, l$, we define $w_{i_j}^{(t)}$ as

$$
w_{i_j}^{(t)} = \begin{cases} \dfrac{\lambda u_{i_j}^{(t)} \, b^{(t)} \bullet a_j^{(t)} + (1 - \lambda) v_{i_j}^{(t)} \, n^{(t)} \bullet a_j^{(t)}}{\left(\lambda b^{(t)} + (1 - \lambda) n^{(t)}\right) \bullet a_j^{(t)}}, & \text{if } \left(\lambda b^{(t)} + (1 - \lambda) n^{(t)}\right) \bullet a_j^{(t)} \neq 0, \\[4mm] u_{i_j}^{(t)}, & \text{if } \left(\lambda b^{(t)} + (1 - \lambda) n^{(t)}\right) \bullet a_j^{(t)} = 0. \end{cases}
$$

By the convexity of $U_i^{(t)}$, it is clear that $(w_{i_1}^{(t)}, w_{i_2}^{(t)}, \ldots, w_{i_l}^{(t)}) \in U_i^{(t)}$ for $i \in I$. In addition, for each $i \in I - \{0\}$, $t \in T$ and $j = 1, 2, \ldots, l$, we have that

$$
w_{i_j}^{(t)} \left(\lambda b^{(t)} + (1 - \lambda) n^{(t)}\right) \bullet a_j^{(t)} = u_{i_j}^{(t)} \left(\lambda b^{(t)} \bullet a_j^{(t)}\right) + v_{i_j}^{(t)} \left((1 - \lambda) n^{(t)} \bullet a_j^{(t)}\right). \tag{1}
$$

Note that if $(\lambda b^{(t)} + (1 - \lambda) n^{(t)}) \bullet a_j^{(t)} \neq 0$, the equality in (1) follows trivially. If $(\lambda b^{(t)} + (1 - \lambda) n^{(t)}) \bullet a_j^{(t)} = 0$, then $\lambda b^{(t)} \bullet a_j^{(t)} = (1 - \lambda) n^{(t)} \bullet a_j^{(t)} = 0$ because $b^{(t)}, n^{(t)}, a_j^{(t)} \succeq 0$, for all $i \in I - \{0\}, t \in T$ and $j = 1, 2, \ldots, l$, and hence the equality in (1) follows in this case as well. It follows immediately that

$$
\lambda x_i + (1 - \lambda) y_i = \sum_{t \in T} \left(\left(\lambda b^{(t)} + (1 - \lambda) n^{(t)}\right) \bullet a_0^{(t)} + \sum_{j=1}^{l} \left(w_{i_j}^{(t)} \left(\lambda b^{(t)} + (1 - \lambda) n^{(t)}\right) \bullet a_j^{(t)}\right) \right)
$$

$$
= \sum_{t \in T} \left(\left(\lambda b^{(t)} + (1 - \lambda) n^{(t)}\right) \bullet \left(a_0^{(t)} + \sum_{j=1}^{l} w_{i_j}^{(t)} a_j^{(t)}\right) \right),
$$

for $i \in I - \{0\}$. Similarly it is also seen that

$$
\lambda x_{m+1} + (1 - \lambda) y_{m+1} = -\sum_{t \in T} \left(\left(\lambda b^{(t)} + (1 - \lambda) n^{(t)}\right) \bullet \left(a_0^{(t)} + \sum_{j=1}^{l} w_{0_j}^{(t)} a_j^{(t)}\right) \right.
$$

$$
\left. + \left(\lambda r^{(t)} + (1 - \lambda) s^{(t)}\right) \right).
$$

Note that $\lambda r^{(t)} + (1 - \lambda) s^{(t)} \geq 0$, $\lambda b^{(t)} + (1 - \lambda) n^{(t)} \succeq 0$, and $(w_{i_1}^{(t)}, w_{i_2}^{(t)}, \ldots, w_{i_l}^{(t)}) \in U_i^{(t)}$, for each $i \in I$ and $t \in T$. This implies that $\lambda x + (1 - \lambda) y \in \mathcal{D}$, and therefore $\mathcal{D}$ is convex. The proof is complete. $\qquad \square$

The following lemma proves the closeness of $\mathcal{D}$ under Assumptions 1.1 and 1.2 (see also [8, Corollary 2.1]).

Lemma 2.3 *If the Slater condition holds, the robust characteristic cone $\mathcal{D}$ is closed.*

Proof Let $\{x(k)\}_{k=1}^{\infty} := \{(x_1(k), x_2(k), \ldots, x_{m+1}(k))\}_{k=1}^{\infty}$ be a sequence in $\mathcal{D}$ that is convergent to the point $x := (x_1, x_2, \ldots, x_{m+1}) \in \mathbb{R}^{m+1}$. To show that $\mathcal{D}$ is closed, we want to show that $x \in \mathcal{D}$. From the definition of $\mathcal{D}$, there exist $a_i(k) = (a_i^{(t)}(k))_{t \in T}$, $i \in$

I, $z(k) = (z^{(t)}(k))_{t \in T}$, and $r(k) \in \mathbb{R}_+^{(T)}$, where $a_i^{(t)}(k) \in \mathcal{V}_i^{(t)}$ and $z^{(t)}(k) \succeq 0$, for all $t \in T$ and $i \in I$, such that

$$x_i(k) = \sum_{t \in T} \left(z^{(t)}(k) \bullet a_i^{(t)}(k) \right), \quad i = 1, 2, \ldots, m, \tag{2}$$

and

$$x_{m+1}(k) = -\sum_{t \in T} \left(z^{(t)}(k) \bullet a_0^{(t)}(k) + r^{(t)}(k) \right). \tag{3}$$

Based on Assumption 1.1, the sequence $\{a_i^{(t)}(k)\}_{k=1}^\infty$ has a convergent subsequence. Therefore, after passing to a convergent subsequence, if necessary, we may assume that $a_i^{(t)}(k) \to a_i^{(t)} \in \mathcal{V}_i^{(t)}$. Now, we show that $\{\|(z^{(t)}(k)\|\}$ is a bounded sequence by contradiction. Suppose on the contrary that $\|(z^{(t)}(k)\| \to +\infty$. Then, $z^{(t)}(k)/\|z^{(t)}(k)\| \to z^{(t)} \in \mathcal{K}_{\mathcal{J}} - \{0\}$. Dividing both sides of (2) and (3) by $\|z^{(t)}(k)\|$ and applying the limit, we obtain

$$\sum_{t \in T} \left(z^{(t)} \bullet a_i^{(t)} \right) = 0, \ i = 1, 2, \ldots, m, \quad \text{and} \quad -\sum_{t \in T} \left(z^{(t)} \bullet a_0^{(t)} \right)$$

$$= \sum_{t \in T} \lim_{k \to \infty} \frac{r^{(t)}(k)}{\|z^{(t)}(k)\|} \geq 0.$$

Based on Assumption 1.2, there exists $x^{(0)} \in \mathbb{R}^m$ such that $a_0^{(t)} + \sum_{i=1}^m x_i^{(0)} a_i^{(t)} \succ 0$, for all $a_i^{(t)} \in \mathcal{V}_i^{(t)}$, and

$$z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^m x_i^{(0)} a_i^{(t)} \right) = z^{(t)} \bullet a_0^{(t)} + \sum_{i=1}^m x_i^{(0)} \left(z^{(t)} \bullet a_i^{(t)} \right) \leq 0.$$

This is on one side of the coin, but on the other side, since $z^{(t)} \in \mathcal{K}_{\mathcal{J}} - \{0\}$, we have $z^{(t)} \bullet (a_0^{(t)} + \sum_{i=1}^m x_i^{(0)} a_i^{(t)}) > 0$, which is a contradiction. Hence, the sequence $\{\|z^{(t)}(k)\|\}_{k=1}^\infty$ is bounded. From (3), the sequence $\{r^{(t)}(k)\}_{k=1}^\infty$ is also bounded. Therefore, after passing to a subsequence, if necessary, we can assume that $z^{(t)}(k) \to \bar{z}^{(t)}$ and $r^{(t)}(k) \to \bar{r}^{(t)}$. By applying the limit in (2) and (3), we have

$$x_i = \sum_{t \in T} \left(\bar{z}^{(t)} \bullet a_i^{(t)} \right), \ i = 1, 2, \ldots, m, \quad \text{and} \quad x_{m+1} = -\sum_{t \in T} \left(\bar{z}^{(t)} \bullet a_0^{(t)} + \bar{r}^{(t)} \right).$$

Thus, $x \in \mathcal{D}$. This completes the proof. $\qquad \square$

3 ϵ-Optimality theorem

In this section, we establish the ϵ-optimality theorem for our problem. First, we prove two intermediate lemmas. The next lemma is the robust version of Farkas' lemma for our setting, and is based on Assumptions 1.2 and 1.1.

Lemma 3.1 *Let* $(c_k, \alpha_k) \in \mathbb{R}^m \times \mathbb{R}$ *for* $k = 1, \ldots, K$, *then*

$$\mathcal{F_P} \subset \{x \in \mathbb{R}^m : \langle c_k, x \rangle \geq \alpha_k, \ k = 1, \ldots, K\} \iff \sum_{K=1}^{K} (c_k, \alpha_k) \in cl\ co\ \mathcal{D}.$$

Proof Assume that $\mathcal{F_P} \subset \{x \in \mathbb{R}^m : \langle c_k, x \rangle \geq \alpha_k, \ k = 1, \ldots, K\}$, and let $\phi_k(x) = \langle c_k, x \rangle - \alpha_k$ for $k = 1, 2, \ldots, K$. Then $\mathcal{F_P} \subset \{x \in \mathbb{R}^m : \phi_k(x) \geq 0, \ k = 1, \ldots, K\}$. It follows that $\sum_{k=1}^{K} \phi_k(x) + \delta_{\mathcal{F_P}}(x) \geq 0$ for all $x \in \mathbb{R}^m$. Note that ϕ_k is continuous for $k = 1, 2, \ldots, K$. Therefore, using Proposition 2.1, we have

$$(0, 0) \in epi \left(\sum_{k=1}^{K} \phi_k + \delta_{\mathcal{F_P}} \right)^{\star} = \sum_{k=1}^{K} epi\ \phi_k^{\star} + epi\ \delta_{\mathcal{F_P}}^{\star}$$

$$= \sum_{k=1}^{K} (c_k, \alpha_k) + \{0\} \times \mathbb{R}_+ + epi\ \delta_{\mathcal{F_P}}^{\star}.$$

Thus,

$$\sum_{k=1}^{K} (c_k, \alpha_k) \in -epi\ \delta_{\mathcal{F_P}}^{\star} - \{0\} \times \mathbb{R}_+. \tag{4}$$

The desired result is obtained by showing that $\sum_{k=1}^{K} (c_k, \alpha_k) \in cl\ co\ \mathcal{D}$. To prove this, in light of (4), it is enough to show that epi $\delta_{\mathcal{F_P}}^{\star} = -cl\ co\ \mathcal{D}$.

Note that, for each $z^{(t)} \succeq 0, a_i^{(t)} \in \mathcal{V}_i^{(t)}, i \in I, \ t \in T$ and $\xi \in \mathbb{R}^m$, we have

$$\left(-z^{(t)} \bullet a_0^{(t)} - \left\langle \cdot, \left(z^{(t)} \bullet a_1^{(t)}, \ldots, z^{(t)} \bullet a_m^{(t)} \right) \right\rangle \right)^{\star}(\xi)$$

$$= \sup_{x \in \mathbb{R}^m} \left\{ \langle \xi, x \rangle - \left(-z^{(t)} \bullet a_0^{(t)} - \left\langle x, \left(z^{(t)} \bullet a_1^{(t)}, \ldots, z^{(t)} \bullet a_m^{(t)} \right) \right\rangle \right) \right\}$$

$$= \sup_{x \in \mathbb{R}^m} \left\{ \sum_{i=1}^{m} \xi_i x_i + \sum_{i=1}^{m} \left(x_i\ z^{(t)} \bullet a_i^{(t)} \right) \right\} + z^{(t)} \bullet a_0^{(t)}$$

$$= \sup_{x \in \mathbb{R}^m} \left\{ \sum_{i=1}^{m} \left(x_i \left(\xi_i + z^{(t)} \bullet a_i^{(t)} \right) \right) \right\} + z^{(t)} \bullet a_0^{(t)}$$

$$= \begin{cases} z^{(t)} \bullet a_i^{(t)}, & \text{if } \xi_i = -z^{(t)} \bullet a_i^{(t)}, \ i = 1, \ldots, m, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5}$$

Note also that, for any $x \in \mathbb{R}^m$, we have

$$\delta_{\mathcal{F}_{\mathcal{P}}}(x) = \sup_{\substack{a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, i \in I, \\ z^{(t)} \succeq 0, \, t \in T}} \left(-z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^m x_i a_i^{(t)} \right) \right).$$

It follows that

$$\text{epi } \delta^\star_{\mathcal{F}_{\mathcal{P}}} = \text{epi} \left(\sup_{\substack{a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, i \in I, \, t \in T \\ z^{(t)} \in \succeq 0, \, t \in T}} \sum_{t \in T} \left(-z^{(t)} \bullet a_0^{(t)} - \left\langle \cdot, \left(z^{(t)} \bullet a_1^{(t)}, \ldots, z^{(t)} \bullet a_m^{(t)} \right) \right\rangle \right) \right)^\star$$

$$= \text{cl} \left(\text{co} \bigcup_{\substack{a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, i \in I, \\ z^{(t)} \succeq 0, \, t \in T}} \text{epi} \sum_{t \in T} \left(-z^{(t)} \bullet a_0^{(t)} - \left\langle \cdot, \left(z^{(t)} \bullet a_1^{(t)}, \ldots, z^{(t)} \bullet a_m^{(t)} \right) \right\rangle \right) \right)^\star$$

$$\text{cl} \left(\text{co} \bigcup_{\substack{a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, i \in I, \, t \in T}} \left\{ \sum_{t \in T} \left(-z^{(t)} \bullet a_1^{(t)}, \ldots, -z^{(t)} \bullet a_m^{(t)}, z^{(t)} \bullet a_0^{(t)} + r \right) : \right.$$

$$\left. \left. z^{(t)} \succeq 0, \, r^{(t)} \geq 0 \right\} \right) = -\text{cl co } \mathcal{D},$$

where the second equality follows from Propositions 2.1 and 2.3 and the third equality follows from (5). The proof is complete. □

The following lemma is also based on Assumption 1.1.

Lemma 3.2 *Let $\bar{x} \in \mathcal{F}_{\mathcal{P}}$ and $\epsilon \geq 0$. Let also $f_k : \mathbb{R}^m \to \mathbb{R}$ be a convex function for $k = 1, 2, \ldots, K$. Then $\bar{x}$ is an ϵ-solution of RSIMSP if and only if for each $k = 1, 2, \ldots, K$ and $t \in T$, there exist $\epsilon_k, \epsilon_t \geq 0$ and $\xi_k \in \partial_{\epsilon_k} f_k(\bar{x})$ such that $\sum_{k=1}^K \epsilon_k + \sum_{t \in T} \epsilon_t = \epsilon$ and*

$$\left(\sum_{k=1}^K \xi_k, \left\langle \sum_{k=1}^K \xi_k, \bar{x} \right\rangle - \sum_{t \in T} \epsilon_t \right) \in cl \, co \, \mathcal{D}.$$

Proof Assume that $\bar{x}$ is an ϵ-solution of RSIMSP, where $\epsilon \geq 0$. Then, for any $x \in \mathcal{F}_{\mathcal{P}}$, $\sum_{k=1}^K f_k(x) \geq \sum_{k=1}^K f_k(\bar{x}) - \epsilon$. It follows that, for any $x \in \mathbb{R}^m$, we have

$$\sum_{k=1}^K f_k(x) + \delta_{\mathcal{F}_{\mathcal{P}}}(x) \geq \sum_{k=1}^K f_k(\bar{x}) + \delta_{\mathcal{F}_{\mathcal{P}}}(\bar{x}) - \epsilon.$$

Then, according to the definition of ϵ-subdifferentiability, we deduce that $0 \in \partial_\epsilon (\sum_{k=1}^K f_k + \delta_{\mathcal{F}_\mathcal{P}})(\bar{x})$, which in view of Proposition 2.2 is equivalent to

$$0 \in \sum_{k=1}^K \partial_{\epsilon_k} f_k(\bar{x}) + \sum_{t \in T} \partial_{\epsilon_t} \delta_{\mathcal{F}_\mathcal{P}}(\bar{x}).$$

Therefore, for each $k = 1, \ldots, K$ and $t \in T$, there exist $\epsilon_k, \epsilon_t \geq 0$, $\xi_k \in \partial_{\epsilon_k} f_k(\bar{x})$, and $-\xi_t \in \partial_{\epsilon_t} \delta_{\mathcal{F}_\mathcal{P}}(\bar{x})$ such that

$$\sum_{k=1}^k \epsilon_k + \sum_{t \in T} \epsilon_t = \epsilon \quad \text{and} \quad \sum_{k=1}^k \xi_k - \sum_{t \in T} \xi_t = 0.$$

Equivalently, for each $k = 1, \ldots, K$ and $t \in T$, there exist $\epsilon_k, \epsilon_t \geq 0$ and $\xi_k \in \partial_{\epsilon_k} f_k(\bar{x})$ such that $\langle \xi_k, x \rangle \geq \langle \xi_k, \bar{x} \rangle - \epsilon_t$ for any $x \in \mathcal{F}_\mathcal{P}$ and $t \in T$, and hence

$$\left\langle \sum_{k=1}^K \xi_k, x \right\rangle \geq \left\langle \sum_{k=1}^K \xi_k, \bar{x} \right\rangle - \sum_{t \in T} \epsilon_t$$

for any $x \in \mathcal{F}_\mathcal{P}$. By Lemma 3.1, we conclude that for each $k = 1, \ldots, K$ and $t \in T$, there exist $\epsilon_k, \epsilon_t \geq 0$ and $\xi_k \in \partial_{\epsilon_k} \delta_{\mathcal{F}_\mathcal{P}}(\bar{x})$ such that

$$\left(\sum_{k=1}^K \xi_k, \left\langle \sum_{k=1}^K \xi_k, \bar{x} \right\rangle - \sum_{t \in T} \epsilon_t \right) \in \text{cl co } \mathcal{D}.$$

The proof is complete.   □

In light of Lemmas 3.1 and 3.2, we can obtain the ϵ-optimality theorem under the robust characteristic cone constraint qualification.

Theorem 3.1 *(Approximate optimality theorem) Consider the RSIMSP problem, and let $\bar{x} \in \mathcal{F}_\mathcal{P}$. Then $\bar{x}$ is an ϵ-solution of RSIMSP if and only if there exist $(\epsilon_t)_{t \in T} \in \mathbb{R}_+^{(T)}, \epsilon_k \in \mathbb{R}_+, \xi_k \in \partial_{\epsilon_k} f_k(\bar{x}), k = 1, 2, \ldots, K, \bar{a}_i = (\bar{a}_i^{(t)})_{t \in T}, i \in I, \text{ and } \bar{z} = (\bar{z}^{(t)})_{t \in T}, \text{ where } \bar{a}_i^{(t)} \in \mathcal{V}_i^{(t)} \text{ and } \bar{z}^{(t)} \succeq 0, \text{ for all } t \in T \text{ and } i \in I, \text{ such that } \sum_{k=1}^K \epsilon_k + \sum_{t \in T} \epsilon_t = \epsilon, \text{ and}*

$$\sum_{k=1}^K \xi_k = \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \bar{a}_1^{(t)}, \bar{z}^{(t)} \bullet \bar{a}_2^{(t)}, \ldots, \bar{z}^{(t)} \bullet \bar{a}_m^{(t)} \right), \tag{6}$$

$$\sum_{t \in T} \epsilon_t \geq \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\bar{a}_0^{(t)} + \sum_{i=1}^m \bar{x}_i \bar{a}_i^{(t)} \right) \right) \geq 0. \tag{7}$$

Proof Assume that $\bar{x}$ is an ϵ-solution of RSIMSP, where $\epsilon \geq 0$. By Lemma 3.2, for each $t \in T$ and $k = 1, 2, \ldots, K$, there exist $\epsilon_k, \epsilon_t \geq 0$ and $\xi_k \in \partial_{\epsilon_k} f_k(\bar{x})$ such that $\sum_{k=1}^{K} \epsilon_k + \sum_{t \in T} \epsilon_t = \epsilon$ and

$$
\left(\sum_{k=1}^{K} \xi_k, \left\langle \sum_{k=1}^{K} \xi_k, \bar{x} \right\rangle - \sum_{t \in T} \epsilon_t \right) \in \bigcup_{a_i^{(t)} \in \mathcal{V}_i^{(t)},\ i \in I,\ t \in T}
$$

$$
\times \left\{ \sum_{t \in T} \left(z^{(t)} \bullet a_1^{(t)}, \ldots, z^{(t)} \bullet a_m^{(t)}, -z^{(t)} \bullet a_0^{(t)} - r \right) : z^{(t)} \succeq 0,\ r^{(t)} \geq 0 \right\}.
$$

It follows that

$$
\sum_{k=1}^{K} \xi_k = \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \bar{a}_1^{(t)}, \bar{z}^{(t)} \bullet \bar{a}_2^{(t)}, \ldots, \bar{z}^{(t)} \bullet \bar{a}_m^{(t)} \right),
$$

$$
\left\langle \sum_{k=1}^{K} \xi_k, \bar{x} \right\rangle - \sum_{t \in T} \epsilon_t = -\sum_{t \in T} \left(\bar{z}^{(t)} \bullet \bar{a}_0^{(t)} + \bar{r}^{(t)} \right),
$$

for some $(\bar{r}^{(t)})_{t \in T} \in \mathbb{R}_+^{(T)}$, $\bar{a}_i = (\bar{a}_i^{(t)})_{t \in T}$, $i \in I$, and $\bar{z} = (\bar{z}^{(t)})_{t \in T}$, with $\bar{a}_i^{(t)} \in \mathcal{V}_i^{(t)}$ and $\bar{z}^{(t)} \succeq 0$, for all $t \in T$ and $i \in I$.

By combining (3) and (3), we get

$$
\sum_{t \in T} \epsilon_t \geq \sum_{t \in T} \left(\epsilon_t - \bar{r}^{(t)} \right) = \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \bar{a}_0^{(t)} \right) + \left\langle \sum_{k=1}^{K} \xi_k, \bar{x} \right\rangle
$$

$$
= \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\bar{a}_0^{(t)} + \sum_{i=1}^{m} \bar{x}_i \bar{a}_i^{(t)} \right) \right) \geq 0.
$$

This proves the first direction.

For the second, assume that there exist $(\epsilon_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, $\epsilon_k \in \mathbb{R}_+$, $\xi_k \in \partial_{\epsilon_k} f_k(\bar{x})$, $k = 1, 2, \ldots, K$, $\bar{a}_i = (\bar{a}_i^{(t)})_{t \in T}$, $i \in I$, and $\bar{z} = (\bar{z}^{(t)})_{t \in T}$, where $\bar{a}_i^{(t)} \in \mathcal{V}_i^{(t)}$ and $\bar{z}^{(t)} \succeq 0$, for all $t \in T$ and $i \in I$, such that the equality $\sum_{k=1}^{K} \epsilon_k + \sum_{t \in T} \epsilon_t = \epsilon$ holds, and that (6) and (7) hold. By the definition of the ϵ-subdifferentiality of f, for any $x \in \mathbb{R}^m$, we have that

$$
\sum_{k=1}^{K} f_k(x) - \sum_{k=1}^{K} f_k(\bar{x}) \geq \left\langle \sum_{k=1}^{K} \xi_k, x - \bar{x} \right\rangle - \sum_{k=1}^{K} \epsilon_k
$$

$$
= \left\langle \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \bar{a}_1^{(t)}, \ldots, \bar{z}^{(t)} \bullet \bar{a}_m^{(t)} \right), x - \bar{x} \right\rangle - \sum_{k=1}^{K} \epsilon_k
$$

$$= \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\sum_{i=1}^{m} x_i \bar{a}_i^{(t)} \right) - \bar{z}^{(t)} \bullet \left(\sum_{i=1}^{m} \bar{x}_i \bar{a}_i^{(t)} \right) \right) - \sum_{k=1}^{K} \epsilon_k$$

$$\geq \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\bar{a}_0^{(t)} + \sum_{i=1}^{m} \bar{x}_i \bar{a}_i^{(t)} \right) \right) - \sum_{k=1}^{K} \epsilon_k - \sum_{t \in T} \epsilon_t$$

$$= \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\bar{a}_0^{(t)} + \sum_{i=1}^{m} \bar{x}_i \bar{a}_i^{(t)} \right) \right) - \epsilon \geq -\epsilon,$$

where the first equality is obtained from (6), the second and third inequalities follow from (7). Therefore, $\sum_{k=1}^{K} f_k(x) \geq \sum_{k=1}^{K} f_k(\bar{x}) - \epsilon$, for any $x \in \mathcal{F}_{\mathcal{P}}$. Thus, from Proposition 2.4, $\bar{x}$ is an ϵ-solution of RSIMSP. This proves the second direction. The result is established. $\qquad\square$

In multi-objective optimization problems, a solution is called Pareto optimal if none of the objective values can be improved without degrading some of the other objective values. The following remark is an immediate corollary of Theorem 3.1.

Remark 3.1 If the vectors $(\epsilon_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ and $(\epsilon_k)_{1 \leq k \leq K} \in \mathbb{R}_+^K$ in Theorem 3.1 tend to the null vector, we obtain (exact) optimality conditions for Pareto solutions.

4 ϵ-Duality theorems

The (Wolfe-type) dual problem associated with the RSIMSP problem is the problem

(RSIMSD)

$$\left[\begin{array}{l} \max \left(f_1(y) - \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} y_i a_i^{(t)} \right) \right), \dots, f_K(y) \right. \\[2ex] \qquad \left. - \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} y_i a_i^{(t)} \right) \right) \right) \\[2ex] \text{s.t.} \quad 0 \in \sum_{k=1}^{K} \partial_{\epsilon_k} f_k(y) - \sum_{t \in T} \left(z^{(t)} \bullet a_1^{(t)}, z^{(t)} \bullet a_2^{(t)}, \dots, z^{(t)} \bullet a_m^{(t)} \right), \\[2ex] \qquad \sum_{k=1}^{K} \epsilon_k \leq \epsilon, \ \epsilon_k \geq 0, \ z^{(t)} \succeq 0, \ a_i^{(t)} \in \mathcal{V}_i^{(t)}, \ t \in T, \ i \in I, \ k = 1, 2, \dots, K. \end{array} \right.$$

Note that RSIMSD has the feasibility set

$$\mathcal{F}_{\mathcal{D}} := \left\{ (y, a_0, \dots, a_m, z) : a_i = \left(a_i^{(t)} \right)_{t \in T}, z = \left(z^{(t)} \right)_{t \in T}, \ 0 \in \sum_{k=1}^{K} \partial_{\epsilon_k} f_k(y) \right.$$

$$\left. - \sum_{t \in T} \left(z^{(t)} \bullet a_1^{(t)}, \dots, z^{(t)} \bullet a_m^{(t)} \right), \right.$$

$$\sum_{k=1}^{K} \epsilon_k \leq \epsilon, \quad \epsilon_k \geq 0, \quad y \in \mathbb{R}^m, \quad a_i^{(t)} \in \mathcal{V}_i^{(t)}, z^{(t)} \succeq 0,$$

$$t \in T, i \in I, k = 1, 2, \ldots, K \Bigg\}.$$

Let $\epsilon \geq 0$. The point $(\bar{x}, \bar{a}_0, \ldots, \bar{a}_m, \bar{z})$ is called an ϵ-solution of RSIMSD if for any $(y, a_0, \ldots, a_m, z) \in \mathcal{F}_\mathcal{D}$ we have

$$\sum_{k=1}^{K} f_k(\bar{x}) - \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\bar{a}_0^{(t)} + \sum_{i=1}^{m} \bar{x}_i \bar{a}_i^{(t)} \right) \right)$$

$$\geq \sum_{k=1}^{K} f_k(y) - \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} y_i a_i^{(t)} \right) \right) - \epsilon.$$

Now, we are ready to establish the ϵ-weak duality theorem, which holds between the RSIMSP problem and its dual, the RSIMSD problem.

Theorem 4.1 *(Approximate weak duality theorem) For any feasible solution x of RSIMSP and any feasible solution $(y, a_0, a_1, \ldots, a_m, z)$ of RSIMSD, we have*

$$\sum_{k=1}^{K} f_k(x) \geq \sum_{k=1}^{K} f_k(y) - \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} y_i a_i^{(t)} \right) \right) - \epsilon.$$

Proof Let x and $(y, a_0, a_1, \ldots, a_m, z)$ be feasible solutions of RSIMSP and RSIMSD, respectively. Then $a_i = (a_i^{(t)})_{t \in T}$ and $z = (z^{(t)})_{t \in T}$, where $a_i^{(t)} \in \mathcal{V}_i^{(t)}$ and $z^{(t)} \succeq 0$, for all $t \in T$ and $i \in I$, and the inequality $\sum_{t \in T} (z^{(t)} \bullet (a_0^{(t)} + \sum_{i=1}^{m} x_i a_i^{(t)})) \geq 0$ holds. It follows that, for each $k = 1, 2, \ldots, K$, there exist $\epsilon_k \geq 0, \xi_k \in \partial_{\epsilon_k} f_k(x)$ such that $\sum_{k=1}^{K} \xi_k = \sum_{t \in T} \left(z^{(t)} \bullet a_1^{(t)}, z^{(t)} \bullet a_2^{(t)}, \ldots, z^{(t)} \bullet a_m^{(t)} \right)$ and $\sum_{k=1}^{K} \epsilon_k \leq \epsilon$. Then, by the definition of the ϵ-subdifferentiability, we have that

$$\sum_{k=1}^{K} f_k(x) - \left(\sum_{k=1}^{K} f_k(y) - \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} y_i a_i^{(t)} \right) \right) \right)$$

$$\geq \left\langle \sum_{k=1}^{K} \xi_k, x - y \right\rangle - \sum_{k=1}^{K} \epsilon_k + \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} y_i a_i^{(t)} \right) \right)$$

$$= \left\langle \sum_{t \in T} \left(z^{(t)} \bullet a_1^{(t)}, z^{(t)} \bullet a_2^{(t)}, \ldots, z^{(t)} \bullet a_m^{(t)} \right), x - y \right\rangle - \sum_{k=1}^{K} \epsilon_k$$

$$+ \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} y_i a_i^{(t)} \right) \right)$$

$$= \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} x_i a_i^{(t)} \right) \right) - \sum_{k=1}^{K} \epsilon_k$$

$$\geq - \sum_{k=1}^{K} \epsilon_k \geq - \epsilon.$$

The proof is complete. $\square$

Now, we state and prove the ϵ-strong duality, which holds theorem between RSIMSP and RSIMSD under the robust characteristic cone constraint qualification.

Theorem 4.2 *(Approximate strong duality theorem) Assume that the robust characteristic cone $\mathcal{D}$ is closed and convex. If $\bar{x}$ is an ϵ-solution of RSIMSP, then there exist $a_i = (a_i^{(t)})_{t \in T}, i \in I$, and $z = (z^{(t)})_{t \in T}$, where $a_i^{(t)} \in \mathcal{V}_i^{(t)}$ and $z^{(t)} \succeq 0$, for all $t \in T$ and $i \in I$, such that $(\bar{x}, \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_m, \bar{z})$ is a 2ϵ-solution of RSIMSD.*

Proof Let $\bar{x}$ be an ϵ-solution of RSIMSP, then by using Theorem 3.1, there exist $(\epsilon_t)_{t \in T} \in \mathbb{R}_+^{(T)}, \epsilon_k \in \mathbb{R}_+, \xi_k \in \partial_{\epsilon_k} f_k(\bar{x}), k = 1, 2, \ldots, K, \bar{a}_i = (\bar{a}_i^{(t)})_{t \in T}, i \in I$, and $\bar{z} = (\bar{z}^{(t)})_{t \in T}$, where $\bar{a}_i^{(t)} \in \mathcal{V}_i^{(t)}$ and $\bar{z}^{(t)} \succeq 0$, for all $t \in T$ and $i \in I$, such that

$$\sum_{k=1}^{K} \epsilon_k + \sum_{t \in T} \epsilon_t = \epsilon, \quad \sum_{k=1}^{K} \xi_k = \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \bar{a}_1^{(t)}, \bar{z}^{(t)} \bullet \bar{a}_2^{(t)}, \ldots, \bar{z}^{(t)} \bullet \bar{a}_m^{(t)} \right),$$

$$\text{and} \quad \sum_{t \in T} \epsilon_t \geq \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\bar{a}_0^{(t)} + \sum_{i=1}^{m} \bar{x}_i \bar{a}_i^{(t)} \right) \right) \geq 0.$$

Therefore, the point $(\bar{x}, \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_m, \bar{z})$ is a feasible solution for RSIMSD. Then, using Theorem 4.1, for any feasible solution $(y, a_0, a_1, \ldots, a_m, z)$ of RSIMSD, we have that

$$\sum_{k=1}^{K} f_k(\bar{x}) - \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\bar{a}_0^{(t)} + \sum_{i=1}^{m} \bar{x}_i \bar{a}^{(t)} \right) \right)$$

$$- \left(\sum_{k=1}^{K} f_k(y) - \sum_{t \in T} \left(z^{(t)} \bullet \left(a_0^{(t)} + \sum_{i=1}^{m} y_i a_i^{(t)} \right) \right) \right)$$

$$\geq -\epsilon - \sum_{t \in T} \left(\bar{z}^{(t)} \bullet \left(\bar{a}_0^{(t)} + \sum_{i=1}^{m} \bar{x}_i \bar{a}_i^{(t)} \right) \right)$$

$$\geq -\epsilon - \sum_{t \in T} \epsilon_t$$

$$= -\epsilon - \epsilon + \sum_{k=1}^{K} \epsilon_k \geq -2\epsilon.$$

This means that $(\bar{x}, \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_m, \bar{z})$ is a 2ϵ-solution of RSIMSD. $\square$

Now, we give the ϵ-strong duality between RSIMSP and RSIMSD under the Slater condition and the weakened robust characteristic cone constraint qualification.

Corollary 4.1 *Assume that the robust Slater condition holds and that the robust characteristic cone $\mathcal{D}$ is convex. If $\bar{x}$ is an ϵ-solution of RSIMSP, then there exist $a_i = (a_i^{(t)})_{t \in T}, i \in I$, and $z = (z^{(t)})_{t \in T}$, where $a_i^{(t)} \in \mathcal{V}_i^{(t)}$ and $z^{(t)} \succeq 0$, for all $t \in T$ and $i \in I$, such that $(\bar{x}, \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_m, \bar{z})$ is a 2ϵ-solution of RSIMSD.*

Proof By Lemma 2.3, the robust characteristic cone $\mathcal{D}$ is closed. The result immediately follows from Theorem 4.2. □

In the remaining part of this paper, we shall demonstrate in an example that the approximate weak and strong duality of a second-order cone program hold true even though the Slater condition fails.

5 An illustrative example

Throughout this section, we use "," for adjoining vectors and matrices in a row, and use ";" for adjoining them in a column. So, for example, if a, and b are vectors, then $(a^\mathsf{T}, b^\mathsf{T})^\mathsf{T} = (a; b)$.

Let $\mathcal{E}^n$ be the n-dimensional real vector space $\mathbb{R} \times \mathbb{R}^{n-1}$ whose elements are indexed from 0. For each vector $x \in \mathcal{E}^n$, we write $\bar{x}$ for the sub-vector consisting of entries 1 through $n - 1$; therefore $x = (x_0; \bar{x})$.

The nth-dimensional second-order cone (also known as the quadratic or Lorentz cone) is defined as

$$\mathcal{E}_+^n := \left\{ (x_0; \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 \geq \|\bar{x}\| \right\},$$

where $\| \cdot \|$ denotes the Euclidean norm.
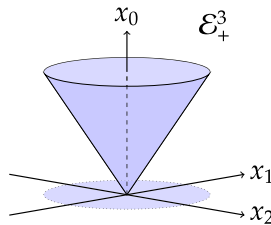
The cone $\mathcal{E}_+^n$ is closed, pointed (i.e., it does not contain a pair of opposite nonzero vectors) and convex with nonempty interior in $\mathbb{R}^n$. It is also known that $\mathcal{E}_+^n$ is self-dual (i.e., it equals its dual cone), and homogeneous (i.e., its automorphism group acts transitively on its interior). Therfore, the cone $\mathcal{E}_+^n$ is symmetric [22, 27]. The graph to the right shows the 3rd-dimensional second-order cone $\mathcal{E}_+^3$.

Table 2 lists some notions from the Euclidean Jordan algebra associated with the second-order cone.

Table 2 Some notions associated with the Jordan algebra of the second-order cone

Notion	Definition
Euclidean Jordan algebra	$\mathcal{E}^n := \left\{ (x_0; \bar{x}) : x_0 \in \mathbb{R}, \bar{x} \in \mathbb{R}^{n-1} \right\}$
Identity vector	$e := (1; 0) \in \mathcal{E}^n$
Jordan product $\circ : \mathcal{E}^n \times \mathcal{E}^n \to \mathcal{E}^n$	$x \circ y := \left(x^{\mathsf{T}} y; x_0 \bar{y} + y_0 \bar{x} \right)$
Square of x	$x^2 := x \circ x = \left(\|x\|^2; 2x_0 \bar{x} \right)$
Cone of squares	$\mathcal{K}_{\mathcal{E}^n} := \left\{ x^2 : x \in \mathcal{E}^n \right\} = \mathcal{E}_+^n$
Frobenius inner product $\bullet : \mathcal{E}^n \times \mathcal{E}^n \to \mathbb{R}$	$x \bullet y := x^{\mathsf{T}} y$
Frobenius norm	$\|x\| := \sqrt{x \bullet x} = \sqrt{x^{\mathsf{T}} x}$

In this section, we consider the robust semi-infinite multi-objective convex second-order cone programming problem:

$$\text{(RSIMSOCP)} \begin{cases} \min \ \left(x_1 + x_2^2; \ x_1 \right) \\ \text{s.t.} \ \ a_0^{(t)} + x_1 a_1^{(t)} + x_2 a_2^{(t)} \succeq 0, \ t \in [0, 1], \\ \qquad a_i^{(t)} \in \mathcal{V}_i^{(t)} \subseteq \mathcal{E}^3, \ t \in [0, 1], i = 0, 1, 2, \end{cases}$$

where $\mathcal{V}_0^{(t)}$, $\mathcal{V}_1^{(t)}$ and $\mathcal{V}_2^{(t)}$, $t \in [0, 1]$, are the uncertainty subsets:

$$\mathcal{V}_0^{(t)} := \left\{ \left(u_0^{(t)}; 0; u_0^{(t)} \right) : u_0^{(t)} \in [-t, 0] \right\},$$
$$\mathcal{V}_1^{(t)} := \left\{ \left(1; u_1^{(t)}; 1 \right) : u_1^{(t)} \in [-t, t] \right\},$$
$$\mathcal{V}_2^{(t)} := \left\{ \left(u_2^{(t)}; 0; u_2^{(t)} \right) : u_2^{(t)} = t \right\}.$$

Now, for any $t \in [0, 1]$, we have

$$a_0^{(t)} + x_1 a_1^{(t)} + x_2 a_2^{(t)} = \begin{bmatrix} u_0^{(t)} + x_1 + x_2 u_2^{(t)} \\ x_1 u_1^{(t)} \\ u_0^{(t)} + x_1 + x_2 u_2^{(t)} \end{bmatrix}.$$

One can see that $\mathcal{F}_\mathcal{P} = \{ (x_1; x_2) : x_1 = 0, \ x_2 \geq 1 \}$ is the set of all robust feasible solutions of RSIMSOCP. Let $\epsilon \geq 0$, then $S_{\mathcal{F}_\mathcal{P}} = \{ (0; x_2) : 1 \leq x_2 \leq \sqrt{1 + \epsilon} \}$ is the set of all ϵ-solutions of RSIMSOCP.

The robust characteristic cone is

$$\mathcal{D} = \bigcup_{a_i^{(t)} \in \mathcal{V}_i^{(t)}, i=0,1,2, \ t \in T} \left\{ \sum_{t \in T} \left(z^{(t)\mathsf{T}} a_1^{(t)}; z^{(t)\mathsf{T}} a_2^{(t)}; -z^{(t)\mathsf{T}} a_0^{(t)} - r^{(t)} \right) : z^{(t)} \in \mathcal{E}_+^3, \ r^{(t)} \geq 0 \right\}.$$

Note that $z^{(t)} \in \mathcal{E}_+^3$ means that $z_0^{(t)} \geq \left\| (z_1^{(t)}; z_2^{(t)}) \right\| = ((z_1^{(t)})^2 + (z_2^{(t)})^2)^{1/2}$. It follows that

$$\mathcal{D} = \bigcup_{\substack{u_0^{(t)} \in [-t, 0], \, u_2^{(t)} = t, \\ u_1^{(t)} \in [-t, t], \, t \in T}} \left\{ \sum_{t \in T} \left(z_0^{(t)} + z_1^{(t)} u_1^{(t)} + z_2^{(t)}; \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)}; \right. \right.$$

$$\left. - \left(z_0^{(t)} + z_2^{(t)} \right) u_0^{(t)} - r^{(t)} \right) : z_0^{(t)} \geq \sqrt{ \left(z_1^{(t)} \right)^2 + \left(z_2^{(t)} \right)^2 }, \, r^{(t)} \geq 0 \right\},$$

which is the set $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, hence $\mathcal{D}$ is closed and convex.

It is clear that $a_0^{(t)} + x_1 a_1^{(t)} + x_2 a_2^{(t)}$ is on the boundary of the second-order cone for any $(x_1; x_2) \in \mathcal{F}_\mathcal{P}$, hence the robust Slater condition fails.

We now formulate the Wolf dual problem, RSIMSOCD, of RSIMSOCP as follows

(RSIMSOCD)

$$\begin{cases}
\max & \left(y_1 + y_2^2 - \sum_{t \in T} \left(z^{(t)^\mathsf{T}} \left(a_0^{(t)} + y_1 a_1^{(t)} + y_2 a_2^{(t)} \right) \right), \right. \\
& \left. y_1 - \sum_{t \in T} \left(z^{(t)^\mathsf{T}} \left(a_0^{(t)} + y_1 a_1^{(t)} + y_2 a_2^{(t)} \right) \right) \right) \\
\text{s.t.} & 0 \in \partial_{\epsilon_1} f_1(y) + \partial_{\epsilon_2} f_2(y) - \sum_{t \in T} \left(z^{(t)^\mathsf{T}} a_1^{(t)}, z^{(t)^\mathsf{T}} a_2^{(t)} \right), \\
& \epsilon_1 + \epsilon_2 \leq \epsilon, \, \epsilon_1 \geq 0, \, \epsilon_2 \geq 0, \, z^{(t)} \in \mathcal{E}_+^3, \, a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, t \in T, \, i = 0, 1, 2.
\end{cases}$$

Let $\mathcal{U} = [-t, 0] \times [-t, t] \times \{t\}$. Then the feasible set $\mathcal{F}_\mathcal{D}$ is

$$\mathcal{F}_\mathcal{D} = \left\{ \left(y_1, y_2, a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, z^{(t)} \right) \in \mathbb{R}^2 \times \mathcal{U} \times \mathcal{E}_+^3 : (0, 0) \in \partial_{\epsilon_1} f_1(y_1, y_2) + \partial_{\epsilon_2} f_2(y_1, y_2) \right.$$

$$- \sum_{t \in T} \left(z^{(t)^\mathsf{T}} a_1^{(t)} + z^{(t)^\mathsf{T}} a_2^{(t)} \right),$$

$$\left. \epsilon_1 + \epsilon_2 \leq \epsilon, \epsilon_1 \geq 0, \epsilon_2 \geq 0, a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, t \in T, i = 0, 1, 2 \right\}$$

$$= \left\{ \left(y_1, y_2, a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, z^{(t)} \right) \in \mathbb{R}^2 \times \mathcal{U} \times \mathcal{E}_+^3 : (0, 0) \in \{2\} \right.$$

$$\times \left[2y_2 - 2\sqrt{\epsilon_1 + \epsilon_2}, 2y_2 + 2\sqrt{\epsilon_1 + \epsilon_2} \right]$$

$$- \sum_{t \in T} \left(\left(z_0^{(t)} + z_1^{(t)} u_1^{(t)} + z_2^{(t)} \right); \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} \right), \, \epsilon_1 + \epsilon_2 \leq \epsilon, \epsilon_1 \geq 0,$$

$$\left. \epsilon_2 \geq 0, a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, t \in T, \, i = 0, 1, 2 \right\}$$

$$= \left\{ \left(y_1, y_2, a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, z^{(t)} \right) \in \mathbb{R}^2 \times \mathcal{U} \times \mathcal{E}_+^3 : 2y_2 - 2\sqrt{\epsilon_1 + \epsilon_2} \right.$$

$$\leq \sum_{t \in T} \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} \leq 2y_2 + 2\sqrt{\epsilon_1 + \epsilon_2},$$

$$\left. \sum_{t \in T} \left(z_0^{(t)} + z_1^{(t)} u_1^{(t)} + z_2^{(t)} \right) = 2, \, \epsilon_1 + \epsilon_2 \leq \epsilon, \, \epsilon_1 \geq 0, \epsilon_2 \geq 0, \, a_i^{(t)} \in \mathcal{V}_i^{(t)}, \, t \in T, \, i = 0, 1, 2 \right\}.$$

Then, for any $(x_1, x_2) \in \mathcal{F}_P$ and any $(y_1, y_2, a_0^{(t)}, a_1^t, a_2^{(t)}, z^{(t)}) \in \mathcal{F}_D$, we have, with $x_1 = 0$ and $x_2 \geq y_2$, that

$$f_1(x_1, x_2) + f_2(x_1, x_2) - \left(f_1(y_1, y_2) + f_2(y_1, y_2) - \sum_{t \in T} \left(z^{(t)^\mathsf{T}} \left(a_0^{(t)} + y_1 a_1^{(t)} + y_2 a_2^{(t)} \right) \right) \right)$$

$$= x_2^2 - y_1 - y_2^2 - y_1 + \sum_{t \in T} \left(\left(z_0^{(t)} + z_2^{(t)} \right) u_0^{(t)} + \left(z_0^{(t)} + z_1^{(t)} u_1^{(t)} + z_2^{(t)} \right) y_1 + \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} y_2 \right)$$

$$\geq \left(2y_2 + 2\sqrt{\epsilon_1 + \epsilon_2} \right) (x_2 - y_2) - \epsilon_1 + \left(-2 + \sum_{t \in T} \left(z_0^{(t)} + z_1^{(t)} u_1^{(t)} + z_2^{(t)} \right) \right) y_1 - \epsilon_2$$

$$+ \sum_{t \in T} \left(\left(z_0^{(t)} + z_2^{(t)} \right) u_0^{(t)} + \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} y_2 \right)$$

$$\geq \sum_{t \in T} \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} (x_2 - y_2) - \epsilon_1 - \epsilon_2 + \sum_{t \in T} \left(\left(z_0^{(t)} + z_2^{(t)} \right) u_0^{(t)} + \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} y_2 \right)$$

$$= \sum_{t \in T} \left(z_0^{(t)} + z_2^{(t)} \right) \left(u_2^{(t)} x_2 + u_0^{(t)} \right) - \epsilon_1 - \epsilon_2$$

$$= \sum_{t \in T} \left(z^{(t)^\mathsf{T}} \left(a_0^{(t)} + 0.a_1^{(t)} + x_2 a_2^{(t)} \right) \right) - \epsilon_1 - \epsilon_2$$

$$\geq -\epsilon_1 - \epsilon_2 \geq -\epsilon.$$

Also, for any $(x_1, x_2) \in \mathcal{F}_P$ and any $(y_1, y_2, a_0^{(t)}, a_1^t, a_2^{(t)}, z^{(t)}) \in \mathcal{F}_D$, we have, with $x_1 = 0$ and $x_2 < y_2$, that

$$f_1(x_1, x_2) + f_2(x_1, x_2) - \left(f_1(y_1, y_2) + f_2(y_1, y_2) - \sum_{t \in T} \left(z^{(t)^\mathsf{T}} \left(a_0^{(t)} + y_1 a_1^{(t)} + y_2 a_2^{(t)} \right) \right) \right)$$

$$= x_2^2 - y_1 - y_2^2 - y_1 + \sum_{t \in T} \left(\left(z_0^{(t)} + z_2^{(t)} \right) u_0^{(t)} + \left(z_0^{(t)} + z_1^{(t)} u_1^{(t)} + z_2^{(t)} \right) y_1 + \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} y_2 \right)$$

$$\geq \left(2y_2 - 2\sqrt{\epsilon_1 + \epsilon_2} \right) (x_2 - y_2) - \epsilon_1 + \left(-2 + \sum_{t \in T} \left(z_0^{(t)} + z_1^{(t)} u_1^{(t)} + z_2^{(t)} \right) \right) y_1 - \epsilon_2$$

$$+ \sum_{t \in T} \left(\left(z_0^{(t)} + z_2^{(t)} \right) u_0^{(t)} + \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} y_2 \right)$$

$$\geq \sum_{t \in T} \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} (x_2 - y_2) - \epsilon_1 - \epsilon_2 + \sum_{t \in T} \left(\left(z_0^{(t)} + z_2^{(t)} \right) u_0^{(t)} + \left(z_0^{(t)} + z_2^{(t)} \right) u_2^{(t)} y_2 \right)$$

$$= \sum_{t \in T} \left(z_0^{(t)} + z_2^{(t)} \right) \left(u_2^{(t)} x_2 + u_0^{(t)} \right) - \epsilon_1 - \epsilon_2$$

$$= \sum_{t \in T} \left(z^{(t)^\mathsf{T}} \left(a_0^{(t)} + 0.a_1^{(t)} + x_2 a_2^{(t)} \right) \right) - \epsilon_1 - \epsilon_2$$

$$\geq -\epsilon_1 - \epsilon_2 \geq -\epsilon.$$

This implies that for any $(x_1, x_2) \in \mathcal{F}_P$ and any $(y_1, y_2, a_0^{(t)}, a_1^t, a_2^{(t)}, z^{(t)}) \in \mathcal{F}_D$, we have

$$f_1(x_1, x_2) + f_2(x_1, x_2) - \left(f_1(y_1, y_2) + f_2(y_1, y_2) - \sum_{t \in T} \left(z^{(t)^\mathsf{T}} \left(a_0^{(t)} + y_1 a_1^{(t)} + y_2 a_2^{(t)} \right) \right) \right)$$

$$\geq -\epsilon. \tag{8}$$

Therefore, the approximate weak duality theorem (Theorem 4.1) holds.

For the strong duality, let $(\bar{x}_1, \bar{x}_2) = (0, \sqrt{1+\epsilon}) \in S_{\mathcal{F}_\mathcal{P}}, \epsilon_1 + \epsilon_2 = (\sqrt{1+\epsilon} - 1)^2$. Let also

$$\bar{z}^{(t)} = \begin{bmatrix} 2\sqrt{1+\epsilon} - 2\sqrt{\epsilon_1 + \epsilon_2} \\ 0 \\ 0 \end{bmatrix}, \quad \bar{a}_0^{(t)} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad \bar{a}_1^{(t)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and}$$

$$\bar{a}_2^{(t)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad t \in T.$$

One can see that $(\bar{x}_1, \bar{x}_2, \bar{a}_0^{(t)}, \bar{a}_1^t, \bar{a}_2^{(t)}, \bar{z}^{(t)}) \in \mathcal{F}_\mathcal{D}$ for $t \in T$. Furthermore, for any $(y_1, y_2, a_0^{(t)}, a_1^t, a_2^{(t)}, z^{(t)}) \in \mathcal{F}_\mathcal{D}$, we have that

$$f_1(\bar{x}_1, \bar{x}_2) + f_2(\bar{x}_1, \bar{x}_2) - \sum_{t \in T} \left(\bar{z}^{(t)^\mathsf{T}} \left(\bar{a}_0^{(t)} + \bar{x}_1 \bar{a}_1^{(t)} + \bar{x}_2 \bar{a}_2^{(t)} \right) \right)$$

$$- f_1(y_1, y_2) - f_2(y_1, y_2) + \sum_{t \in T} \left(z^{(t)^\mathsf{T}} \left(a_0^{(t)} + y_1 a_1^{(t)} + y_2 a_2^{(t)} \right) \right)$$

$$\geq -\epsilon - \sum_{t \in T} \left(\bar{z}^{(t)^\mathsf{T}} \left(\bar{a}_0^{(t)} + \bar{x}_1 \bar{a}_1^{(t)} + \bar{x}_2 \bar{a}_2^{(t)} \right) \right)$$

$$= -\epsilon - \left(2\sqrt{1+\epsilon} - 2\sqrt{\epsilon_1 + \epsilon_2} \right) \left(-1 + \sqrt{1+\epsilon} \right)$$

$$= -\epsilon - \left(\left(\sqrt{1+\epsilon} - \sqrt{\epsilon_1 + \epsilon_2} - 1 \right)^2 - \epsilon_1 - \epsilon_2 + \epsilon \right)$$

$$= -\epsilon + \epsilon_1 + \epsilon_2 - \epsilon \geq -2\epsilon,$$

where we used (8) to obtain the first inequality. Thus, the approximate strong duality theorem (Theorem 4.2) also holds.

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