Positivity (2022) 26: 73
https://doi.org/10.1007/s11117-022-00935-9

Positivity

A duality result in locally convex cones

Amir Dastouri[1] · **Asghar Ranjbari**[1]

Received: 11 July 2021 / Accepted: 29 July 2022 / Published online: 11 August 2022

Abstract
In this paper, we consider a locally convex cone $(\mathcal{P}, \mathcal{V})$ and verify the dual of $(Conv(\mathcal{P}), \overline{\mathcal{V}})$ the locally convex cone of the non-empty convex subsets of $\mathcal{P}$. Under some semilattice conditions, we characterize the dual of $Conv(\underbrace{\cdots}_{n\ times}(Conv(\mathcal{P}))$.

Keywords $\bigvee$ −semilattice cone · Locally convex cone · Convex set

Mathematics Subject Classification 46A03 · 46A20

1 Introduction

Duality theory is a powerfull technique to study a wide class of related problems in pure and applied mathematics. For example the Hahn-Banach extension and separation theorems studied by means of duals (see [8]). The collection of all non-empty convex subsets of a cone (or a vector space) is interesting in convexity and approximation theory (for example see [5]). This collection is a cone. We consider the non-empty convex subsets of a cone $\mathcal{P}$, denoted by $Conv(\mathcal{P})$, and verify the dual of it, when $\mathcal{P}$ is a locally convex cone. We note that some elements of the dual of $Conv(\mathcal{P})$ have already been introduced (see [6], I: Example 2.1(e) and Example 5.31 (b)). Firstly we review the structure of locally convex cones briefly:

A nonempty set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers is called a *cone* whenever the addition is associative and commutative, there is a neutral element $0 \in \mathcal{P}$ and for the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$,

✉ Asghar Ranjbari
ranjbari@tabrizu.ac.ir

Amir Dastouri
a.dastouri@tabrizu.ac.ir

[1] Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

$\alpha(a+b)=\alpha a+\alpha b$, $1a=a$ and $0a=0$ for all $a,b\in\mathcal{P}$ and nonnegative reals α and β.

The theory of locally convex cones as introduced and developed by K. Keimel and W. Roth in [4]. It uses an order theoretical concept or a convex quasi-uniform structure on a cone. In this paper, we use the former. For some recent researches see [1–3, 7].

A (*preordered cone*) is a cone $\mathcal{P}$ endowed with a preorder (reflexive transitive relation) $\le$ which is compatible with the addition and scalar multiplication, that is $x\le y$ implies $x+z\le y+z$ and $r\cdot x\le r\cdot y$ for all $x,y,z\in\mathcal{P}$ and $r\in\mathbb{R}_+=\{r\in\mathbb{R}\ :\ r\ge 0\}$. Every ordered vector space is an ordered cone. The cones $\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ and $\overline{\mathbb{R}}_+=\mathbb{R}_+\cup\{+\infty\}$, with the usual order and algebraic operations (specially $0\cdot(+\infty)=0$), are ordered cones that are not embeddable in vector spaces.

A subset $\mathcal{V}$ of a preordered cone $\mathcal{P}$ is called an *(abstract) 0-neighborhood system*, if

(v_1) $0<v$ for all $v\in\mathcal{V}$;
(v_2) for all $u,v\in\mathcal{V}$ there is a $w\in\mathcal{V}$ with $w\le u$ and $w\le v$;
(v_3) $u+v\in\mathcal{V}$ and $\alpha v\in\mathcal{V}$ whenever $u,v\in\mathcal{V}$ and $\alpha>0$.

Let $a\in\mathcal{P}$ and $v\in\mathcal{V}$. We define $v(a)=\{b\in\mathcal{P}\mid b\le a+v\}$, resp. $(a)v=\{b\in\mathcal{P}\mid a\le b+v\}$, to be a neighborhood of a in the *upper*, resp. *lower* topologies on $\mathcal{P}$. The common refinement of the upper and lower topologies is called *symmetric* topology. We denote the neighborhoods of a in the symmetric topology by $v(a)v$. The pair $(\mathcal{P},\mathcal{V})$ is called a *full locally convex cone* if the elements of $\mathcal{P}$ are *bounded below*, i.e. for every $a\in\mathcal{P}$ and $v\in\mathcal{V}$ we have $0\le a+\rho v$ for some $\rho>0$. Each subcone of $\mathcal{P}$, not necessarily containing $\mathcal{V}$, is called a *locally convex cone*.

We note that if $(\mathcal{Q},\mathcal{V})$ is a locally convex cone, $\mathcal{Q}\oplus(\mathcal{V}\cup\{0\})$ with the algebraic operation

$$(a,v_1)+(b,v_2)=(a+b,v_1+v_2),$$
$$\alpha(a,v_1)=(\alpha a,\alpha v_1),$$

and the preorder

$$\begin{cases}(a,0)\le(b,0)\Leftrightarrow a\le b\\ (0,v_1)\le(0,v_2)\Leftrightarrow v_1\le v_2\\ (a,0)\le(b,v_1)\Leftrightarrow a\le b+v_1,\end{cases}$$

for all $a,b\in\mathcal{Q}$, $v_1,v_2\in\mathcal{V}$ and $\alpha\in\mathbb{R}^+$, $(\mathcal{Q}\oplus(\mathcal{V}\cup\{0\}),\mathcal{V})$ is a full locally convex cone which $\mathcal{Q}$ and $\mathcal{V}$ can be embedded in $\mathcal{Q}\oplus(\mathcal{V}\cup\{0\})$ by the mappings $a\to(a,0)$ and $v\to(0,v)$ for all $a\in\mathcal{Q}$ and $v\in\mathcal{V}$.

For cones $\mathcal{P}$ and $\mathcal{Q}$ a mapping $t:\mathcal{P}\to\mathcal{Q}$ is called a *linear operator* if $t(a+b)=t(a)+t(b)$ and $t(\alpha a)=\alpha t(a)$ hold for $a,b\in\mathcal{P}$ and $\alpha\ge 0$.

A *linear functional* on a cone $\mathcal{P}$ is a linear mapping $\mu:\mathcal{P}\to\overline{\mathbb{R}}$.

Let $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ be two locally convex cones. The linear operator $t:(\mathcal{P},\mathcal{V})\to(\mathcal{Q},\mathcal{W})$ is called uniformly continuous or simply u-continuous if for every $w\in\mathcal{W}$ one can find a $v\in\mathcal{V}$ such that $a\le b+v$ implies $t(a)\le t(b)+w$. It is easy to see that the u-continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$.

According to the definition of u-continuity, a linear functional μ on $(\mathcal{P}, \mathcal{V})$ is u-continuous if there is a $v \in \mathcal{V}$ such that $a \leq b + v$ implies $\mu(a) \leq \mu(b) + 1$. The u-continuous linear functionals on a locally convex cone $(\mathcal{P}, \mathcal{V})$ (into $\overline{\mathbb{R}}$) form a cone with the usual addition and scalar multiplication of functions. This cone is called the *dual cone* of $\mathcal{P}$ and denoted by $\mathcal{P}^*$.

For a locally convex cone $(\mathcal{P}, \mathcal{V})$, the polar v° of $v \in \mathcal{V}$ consists of all linear functionals μ on $\mathcal{P}$ satisfying $\mu(a) \leq \mu(b) + 1$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. We have $\cup\{v^\circ : v \in \mathcal{V}\} = \mathcal{P}^*$. The cones $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} : a \geq 0\}$ with (abstract) 0-neighborhood $\mathcal{V} = \{\varepsilon > 0 : \varepsilon \in \mathbb{R}\}$ are locally convex cones. The dual cones of $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+$ under $\mathcal{V}$ consists of all nonnegative reals and the functional 0_∞ such that $0_\infty(a) = 0$ for all $a \in \mathbb{R}$ and $0_\infty(+\infty) = +\infty$.

2 Dual of the cone of non-empty convex sets of a locally convex cone

A subset A of a cone $\mathcal{P}$ is said convex, if $\lambda a + (1 - \lambda)b \in A$, whenever $a, b \in \mathcal{P}$ and $0 \leq \lambda \leq 1$. Let $\mathcal{P}$ be a preordered cone and $Conv(\mathcal{P})$ be the cone of all non-empty convex subsets of $\mathcal{P}$, endowed with the usual addition and multiplication of sets by non-negative scalars, that is $\alpha A = \{\alpha a \mid a \in A\}$ and $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$ for $A, B \in Conv(\mathcal{P})$ and $\alpha \geq 0$. We consider the order on $Conv(\mathcal{P})$ by

$$A \preceq B \quad if \quad A \subseteq\downarrow B,$$

where $\downarrow B = \{x \in \mathcal{P} | x \leq b \text{ for some } b \in B\}$ is the decreasing hull of the set B in $\mathcal{P}$. Note that $\downarrow B$ is again a convex subset of $\mathcal{P}$. The requirements for a preordered cone are easily checked. The neighborhood system in $Conv(\mathcal{P})$ is $\overline{\mathcal{V}} := \{\overline{v} = \{v\} \mid v \in \mathcal{V}\}$, that is

$$A \preceq B + \overline{v} \quad if \quad A \subseteq\downarrow (B + \{v\})$$

for $A, B \in Conv(\mathcal{P})$ and $\overline{v} \in \overline{\mathcal{V}}$. The cone $Conv(\mathcal{P})$ with (abstract) 0-neighborhood system $\overline{\mathcal{V}}$) is a locally convex cone. Via the embedding $x \to \{x\} : \mathcal{P} \to Conv(\mathcal{P})$ the preordered cone $\mathcal{P}$ itself may be considered as a subcone of $Conv(\mathcal{P})$ (see [6], I, Example 1.4 (c)).

Definition 1 We say that a preordered cone $\mathcal{P}$ is a $\bigvee$-semilattice cone if the order of $\mathcal{P}$ is antisymmetric and if
($\bigvee 1$) every non-empty subset $A \subseteq \mathcal{P}$ has a supremum $\sup A \in \mathcal{P}$ and $\sup(A + b) = \sup A + b$ hold for all $b \in \mathcal{P}$.

Moreover, if $\mathcal{P}$ with an abstract neighborhood system $\mathcal{V}$ is a locally convex cone and
($\bigvee 2$) for $\emptyset \neq A \subseteq \mathcal{P}$, $b \in \mathcal{P}$ and $v \in V$ such that $a \leq b + v$ for all $a \in A$, we have $\sup A \leq b + v$,
then $(\mathcal{P}, \mathcal{V})$ is said a $\bigvee$-semilattice locally convex cone.

In particular, every $\bigvee$-semilattice cone $\mathcal{P}$ contains a largest element, that is $+\infty = \sup \mathcal{P}$, which can be adjoined as a maximal element to any $\bigvee$-semilattice cone with the convention that $a + (+\infty) = +\infty$, $\alpha \cdot (+\infty) = +\infty$, $0 \cdot (+\infty) = 0$ and $a \leq +\infty$ for all $a \in \mathcal{P}$ and $\alpha > 0$.

Remark 1 We note that the condition ($\bigvee 2$) of definition 1 is necessary and the definition of supremum does not imply this condition in locally convex cones necessarily. We show this in the following example.

Example 1 Let $\mathbb{R}$ be as a cone and $\mathcal{V} = \{\bar{\epsilon} = (-\infty, \epsilon) \ : \ \epsilon \in \mathbb{R}_{>0}\}$.
Let

$$\mathcal{P} = \{(a, B) \ : \ a \in \mathbb{R} \text{ and } B \in \mathcal{V} \cup \{\{0\}\}\}.$$

We define

$$(a, B) + (c + D) = (a + c, B + D),$$

and

$$\lambda(a, B) = (\lambda a, \lambda B)$$

for all $(a, B), (c, D) \in \mathcal{P}$. Also, we define the preorder

$$(a, B) \leq (c, D) \Leftrightarrow \begin{cases} a \leq c & \text{if } B = D = \{0\} \\ a + B \subseteq c + D & \text{if } D \neq \{0\} \end{cases},$$

for all $(a, B), (c, D) \in \mathcal{P}$. Then $(\mathcal{P}, \mathcal{V})$ is a full locally convex cone. Now, we can embedded $\mathbb{R}$ in $\mathcal{P}$ by $a \to (a, \{0\})$ and we can consider $\mathbb{R}$ as a subcone of $\mathcal{P}$. We have

$$a \leq b + \bar{\epsilon} \Leftrightarrow (a, \{0\}) \leq (b, (-\infty, \epsilon)) \Leftrightarrow \{a\} \subseteq (-\infty, b + \epsilon) \Leftrightarrow a \in (-\infty, b + \epsilon).$$

Now, for the set $A = (0, 5) \subseteq \mathbb{R}$, by considering the embedding, we have $\bar{A} = \{(a, \{0\}) \ : \ a \in (0, 5)\}$. Let $b = 4$ and $\bar{1} = (-\infty, 1) \in \mathcal{V}$. Then

$$\begin{aligned} a \in (0, 5) \Leftrightarrow a \in (0, 4 + 1) \Rightarrow a \in (-\infty, 4 + 1) \\ \Rightarrow (a, \{0\}) \leq (4, \{0\}) + (0, (-\infty, 1)), \end{aligned}$$

for all $(a, \{0\}) \in \bar{A}$, i.e.

$$a \leq 4 + \bar{1},$$

for all $a \in A = (0, 5)$. On the other hand, $\sup A = 5$ (in $\mathbb{R}$) and we have

$$5 \notin (-\infty, 5) = (-\infty, 4 + 1) \Rightarrow (5, \{0\}) \not\leq (4, \{0\}) + (0, (-\infty, 1)),$$

i.e. $5 \not\leq 4 + \bar{1}$. Although, $\mathcal{P}$ is not a $\bigvee$-semilattice cone, $\mathbb{R}$ is a $\bigvee$-semilattice cone. Also, the locally convex cone $(\mathbb{R}, \mathcal{V})$ is not a $\bigvee$-semilattice locally convex cone.

Remark 2 We note that definition 1 is similar to the definition of “locally convex $\bigvee$-semilattice cone" in [6], I, 5.4. In this definition, the order do not coincide with the weak preorder necessarily.

We define $Conv^n(\mathcal{P}) := Conv(Conv^{n-1}(\mathcal{P}))$ for $n = 2, 3, \ldots$ and $Conv^1(\mathcal{P}) = Conv(\mathcal{P})$. Let

$$\{a\}^n := \underbrace{\{\cdots\{}_{n\ times}\ a\ \underbrace{\}\cdots\}}_{n\ times} \tag{1}$$

for all $a \in \mathcal{P}$. It is easy to see that $\{a\}^n \in Conv^n(\mathcal{P})$ for all $n \in \mathbb{N}$. This shows that $\mathcal{P}$ is embedded in $Conv^n(\mathcal{P})$ (the mapping $a \longrightarrow \{a\}^n$ is the embedding). The cone $Conv^n(\mathcal{P})$ with the (abstract) 0-neighborhood system $\overline{\mathcal{V}}^n$ is a locally convex cone, where $\overline{\mathcal{V}}^n := \{\bar{v}^n := \{v\}^n \mid v \in \mathcal{V}\}$.

Example 2 For the cone $\mathbb{R}$, we have $A^1 = [0, 1] \in Conv(\overline{\mathbb{R}})$, $A^2 = \{[0, a] \mid , a \in [0, 1]\}$ is an element of $Conv^2(\mathbb{R})$ and $A^3 = \{\{[0, a] \mid , a \in [0, b]\} \mid b \in [0, 1]\}$ is an element of $Conv^3(\mathbb{R})$.

For the element A^n of $Conv^n(\mathcal{P})$ we define

$$sup^s(A^n) := \sup\{sup^s(A^{n-1}) \mid A^{n-1} \in A^n\}$$

for $n = 2, 3, \ldots$ and $sup^s(A^1) = \sup A$. It is easy to see that $sup^s(A^n) \in \mathcal{P}$ for all $n \in \mathbb{N}$.

The following lemma is an special case of Lemma 5.5 of [6].

Lemma 1 *Let $\mathcal{P}$ be a $\bigvee -$semilattice cone and $\{A_i\}_{i\in I}$ be a collection of non-empty subsets of $\mathcal{P}$. Then*

$$\sup\left(\bigcup_{i\in I} A_i\right) = \sup\{\sup A_i \mid i \in I\}.$$

Proof Let $a \in \bigcup_{i\in I} A_i$ be arbitrary. Then there exists $i \in I$ such that $a \in A_i$. We have $a \leq \sup A_i$ and so $a \leq \sup\{\sup A_i \mid i \in I\}$. Then

$$\sup\left(\bigcup_{i\in I} A_i\right) \leq \sup\{\sup A_i \mid i \in I\}.$$

On the other hand, $\sup A_i \leq \sup(\bigcup_{i\in I} A_i)$ for all $i \in I$. This conclude that

$$\{\sup A_i \mid i \in I\} \leq \sup\left(\bigcup_{i\in I} A_i\right).$$

□

Remark 3 We note that $A^2 \in Conv^2(\mathcal{P})$ but the elements of A^2 belong to $Conv^1(\mathcal{P}) = Conv(\mathcal{P})$. This implies that the union of the elements of A^2 ($\bigcup_{A^1 \in A^2} A^1$) belongs to the power set of $\mathcal{P}$. Also, $A^3 \in Conv^3(\mathcal{P})$ and the elements of A^3 belong to $Conv^2(\mathcal{P})$. Then the union of the elements of A^3 ($\bigcup_{A^2 \in A^3} A^2$) belongs to the power set of $Conv^2(\mathcal{P})$ and the union of these sets ($\bigcup_{A^2 \in A^3} \bigcup_{A^1 \in A^2} A^1$) belongs again to the power set of $\mathcal{P}$. By continuing this process, we conclude that $A^n \in Conv^n(\mathcal{P})$ and the elements of A^n belong to $Conv^{n-1}(\mathcal{P})$. Then

$$\bigcup_{A^{n-1} \in A^n} \cdots \bigcup_{A^2 \in A^3} \bigcup_{A^1 \in A^2} A^1$$

belongs to the power set of $\mathcal{P}$. By Lemma 1, we have

$$sup^s(A^n) = \sup\left(\bigcup_{A^{n-1} \in A^n} \cdots \bigcup_{A^2 \in A^3} \bigcup_{A^1 \in A^2} A^1 \right).$$

Let $\mathcal{P}$ be a cone and $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ be a functional. We define

$$\overline{\mu}(A) := \sup\{\mu(a) \mid a \in A\}, \quad A \in Conv(\mathcal{P}),$$

moreover, if $\mathcal{P}$ is a $\bigvee$-semilattice cone, we define

$$\overline{\overline{\mu}}(A) := \mu(\sup A), \quad A \in Conv(\mathcal{P}).$$

Lemma 2 *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and $\mu \in \mathcal{P}^*$. Then $\overline{\mu} \in Conv(\mathcal{P})^*$. Moreover, if $(\mathcal{P}, \mathcal{V})$ is $\bigvee$-semilattice locally convex cone, then $\overline{\overline{\mu}} \in Conv(\mathcal{P})^*$.*

Proof We have

$$\begin{aligned} \overline{\mu}(\alpha A + B) &= \sup\{\mu(\alpha a + b) \mid a \in A, b \in B\} \\ &= \sup\{\alpha\mu(a) + \mu(b) \mid a \in A, b \in B\} \\ &= \alpha \sup\{\mu(a) \mid a \in A\} + \sup\{\mu(b) \mid b \in B\} \\ &= \alpha\overline{\mu}(A) + \overline{\mu}(B), \end{aligned}$$

for all $A, B \in Conv(\mathcal{P})$ and all $\alpha \geq 0$. So $\overline{\mu}$ is linear.

Now, if $(\mathcal{P}, \mathcal{V})$ is $\bigvee$-semilattice locally convex cone, then

$$\begin{aligned} \sup(A + B) &= \sup\left(\bigcup_{b \in B} (A + b) \right) \\ &= \sup\{\sup(A + b) \mid b \in B\} \quad \text{(by Lemma 1)} \\ &= \sup\{\sup A + b \mid b \in B\} \\ &= \sup(A) + \sup(B). \end{aligned}$$

This yields that $\mu(\sup(A+B)) = \mu(\sup(A)) + \mu(\sup(B))$ and then $\overline{\overline{\mu}}(A+B) = \overline{\overline{\mu}}(A) + \overline{\overline{\mu}}(B)$ for all $A, B \in Conv(\mathcal{P})$. Also,

$$\overline{\overline{\mu}}(\alpha A) = \mu(\sup(\alpha A)) = \mu(\alpha \sup A) = \alpha\mu(\sup A) = \alpha\overline{\overline{\mu}}(A),$$

for all $\alpha \geq 0$ and $A \in Conv(\mathcal{P})$. Therefore $\overline{\overline{\mu}}$ is linear.

Now, we show that $\overline{\mu}$ and $\overline{\overline{\mu}}$ are u-continuous extensions of μ to $Conv(\mathcal{P})$. Via of continuity of μ, there is a $v \in \mathcal{V}$ such that $a \leq b + v$ implies $\mu(a) \leq \mu(b) + 1$. Let $A \preceq B + \{v\}$. Then, for each $a \in A$ there exists $b \in B$ such that $a \leq b + v$. We have

$$\begin{aligned} \mu(a) \leq \mu(b) + 1 &\Rightarrow \mu(a) \leq \sup\{\mu(b) \mid b \in B\} + 1 \\ &\Rightarrow \sup\{\mu(a) \mid a \in A\} \leq \sup\{\mu(b) \mid b \in B\} + 1 \\ &\Rightarrow \overline{\mu}(A) \leq \overline{\mu}(B) + 1. \end{aligned}$$

This shows that $\overline{\mu}$ is u-continuous. Also if $(\mathcal{P}, \mathcal{V})$ is $\bigvee$-semilattice locally convex cone, we have

$$\begin{aligned} a \leq \sup(B) + v &\Rightarrow \sup(A) \leq \sup(B) + v \quad (\text{by } \bigvee 2) \\ &\Rightarrow \mu(\sup(A)) \leq \mu(\sup(B)) + 1 \\ &\Rightarrow \overline{\overline{\mu}}(A) \leq \overline{\overline{\mu}}(B) + 1. \end{aligned}$$

This yields that $\overline{\overline{\mu}}$ is u-continuous.

□

Proposition 1 *Let $\mathcal{P}$ be a preordered cone, μ be a monotone functional on $\mathcal{P}$ and $\widetilde{\mu}$ be a monotone extension of μ on $Conv(\mathcal{P})$. Then $\overline{\mu} \leq \widetilde{\mu}$. Furthermore, if $\mathcal{P}$ is a $\bigvee$-semilattice cone, then*

$$\overline{\mu} \leq \widetilde{\mu} \leq \overline{\overline{\mu}}. \tag{2}$$

Proof Let $\overline{\mu} \not\leq \widetilde{\mu}$. Then there exists $A \in Conv(\mathcal{P})$ such that $\overline{\mu}(A) \not\leq \widetilde{\mu}(A)$ i.e. $\widetilde{\mu}(A) < \overline{\mu}(A) = \sup\{\mu(a) \mid a \in A\}$. Then there exists $a \in A$ such that $\widetilde{\mu}(A) < \mu(a) = \widetilde{\mu}(\{a\})$ (by the supremum property). On the other hand, $\{a\} \preceq A$ and so $\widetilde{\mu}(\{a\}) \leq \widetilde{\mu}(A)$. This contradiction yields that $\overline{\mu} \leq \widetilde{\mu}$.

Now, let $\mathcal{P}$ be a $\bigvee$-semilattice cone. Let $A \in Conv(\mathcal{P})$ be arbitrary. We have $A \preceq \{\sup A\}$. Then $\widetilde{\mu}(A) \leq \widetilde{\mu}(\{supA\}) = \mu(supA) = \overline{\overline{\mu}}(A)$. □

Let $\mathcal{P}$ be a $\bigvee$-semilattice cone. We denote

$$\Omega(\mathcal{P}) := \{\mu \in \mathcal{L}(\mathcal{P}) \mid \mu \text{ is monotone and } \overline{\mu}(A) = \overline{\overline{\mu}}(A), \forall A \in Conv(\mathcal{P})\},$$

where $\mathcal{L}(\mathcal{P})$ is the cone of all linear functionals on $\mathcal{P}$.

Corollary 1 *Let $\mathcal{P}$ be a $\bigvee -$semilattice cone. Then the elements of $\Omega(\mathcal{P})$ have unique extensions to $Conv(\mathcal{P})$.*

By the assumptions of the Corollary 1, we conclude that the elements of $\Omega(\mathcal{P})$ have unique extensions to $Conv^n(\mathcal{P})$.

Proposition 2 *Let $\mathcal{P}$ be a $\bigvee$ −semilattice cone. Then*

$$sup^s(A^n) + sup^s(B^n) = sup^s(A^n + B^n),$$

for all $n \in \mathbb{N}$ and $A^n, B^n \in Conv^n(\mathcal{P})$.

Proof For $n = 1$, let $A^1 = A$ and $B^1 = B$ be elements of $Conv^1(\mathcal{P}) = Conv(\mathcal{P})$. We have

$$\begin{aligned} sup^s(A + B) &= \sup\left(\bigcup_{b\in B}(A + b)\right) \\ &= \sup\{\sup(A + b) \mid b \in B\} \quad \text{(by Lemma 1)} \\ &= \sup\{\sup A + b \mid b \in B\} \\ &= sup^s(A) + sup^s(B). \end{aligned}$$

Now, let

$$sup^s(A^{n-1}) + sup^s(B^{n-1}) = sup^s(A^{n-1} + B^{n-1}).$$

Then

$$\begin{aligned} sup^s(A^n + B^n) &= sup^s(\{A^{n-1} + B^{n-1} \mid A^{n-1} \in A^n,\ B^{n-1} \in B^n\}) \\ &= \sup(\{sup^s(A^{n-1} + B^{n-1}) \mid A^{n-1} \in A^n\ ,\ B^{n-1} \in B^n\}) \\ &= \sup(\{sup^s(A^{n-1}) + sup^s(B^{n-1}) \mid A^{n-1} \in A^n\ ,\ B^{n-1} \in B^n\}) \\ &= \sup(\{sup^s(A^{n-1}) \mid A^{n-1} \in A^n\}) \\ &\quad + \sup(\{sup^s(B^{n-1}) \mid B^{n-1} \in B^n\}) \\ &= sup^s(A^n) + sup^s(B^n). \end{aligned}$$

□

Let $coh(F)$ denote the convex hull of the set F, the smallest convex set containing F. We set

$$coh^s(A^n) := coh(\{coh^s(A^{n-1}) \mid A^{n-1} \in A^n\} \cup \{sup^s(A^n)\}^n)$$

for $n = 2, 3, \ldots$ and $coh^s(A^1) = coh(A \cup \{sup(A)\})$.

Proposition 3 *Let $\mathcal{P}$ be a $\bigvee$ −semilattice cone. Then*

$$\sup(coh^s(A^n)) = \sup(A^n),$$

all $A^n \in Conv^n(\mathcal{P})$ and $n \in \mathbb{N}$.

Proof First we show that $\sup(coh^s(A^1)) = \sup(A^1)$. Let $x \in coh^s(A^1)$ be arbitrary. Then there are $\lambda_1, \lambda_2, \dots, \lambda_k \geq 0$ and $a_1, a_2, \dots, a_k \in A^1 \cup \{\sup(A^1)\}$ such that $\sum_{i=1}^k \lambda_i = 1$ and $x = \sum_{i=1}^k \lambda_i a_i$. On the other hand, $\lambda_i a_i \leq \lambda_i \sup(A^1)$ for all $i = 1, 2, \dots, k$. We have

$$x = \sum_{i=1}^{k} \lambda_i a_i \leq \sum_{i=1}^{k} \lambda_i \sup A^1 = \sup(A^1).$$

This yields that

$$\sup(coh^s(A^1)) = \sup(A^1),$$

since $\sup(A^1) \in coh^s(A^1)$.

Now, let $\sup(coh^s(A^{n-1})) = \sup(A^{n-1})$ for all $A^{n-1} \in Conv^{n-1}(\mathcal{P})$. Consider $A^n \in Conv^n(\mathcal{P})$ and $\mathcal{X} \in coh^s(A^n)$. Then there are $\lambda_1, \lambda_2, \dots, \lambda_k \geq 0$ and $A_1^{n-1}, A_2^{n-1}, \dots, A_k^{n-1} \in A^n \cup \{\sup(A^n)\}^n$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\mathcal{X} = \sum_{i=1}^k \lambda_i coh^s(A_i^{n-1})$. On the other hand,

$$\lambda_i coh^s(A_i^{n-1}) \preceq \lambda_i \{sup^s(coh^s(A_i^{n-1}))\}^n = \{sup^s(A_i^{n-1})\}^n \preceq \lambda_i \{sup^s(A^n)\}^n,$$

for all $i = 1, 2, \dots, k$. So

$$\mathcal{X} = \sum_{i=1}^{k} \lambda_i coh^s(A_i^{n-1}) \preceq \sum_{i=1}^{k} \lambda_i \{sup^s(A^n)\}^n = \{sup^s(A^n)\}^n,$$

and so

$$sup^s(\mathcal{X}) \leq sup^s(A^n).$$

Since $\{sup^s(A^n)\}^n \in coh^s(A^n)$, we have

$$sup^s(coh^s(A^n)) = sup^s(A^n).$$

□

Remark 4 By Proposition 3 and by considering the construction of $coh^s(A^n)$, we have

$$\{sup^s(coh^s(A^n))\}^n \in coh^s(A^n)$$

for all $n \in \mathbb{N}$.

Example 3 For the cone $\overline{\mathbb{R}}$, we have $\{0\}, \{0, +\infty\} \in Conv(\overline{\mathbb{R}})$ and $A^2 = \{\{0\}, \{0, +\infty\}\}$ is an element of $Conv^2(\overline{\mathbb{R}})$. We have $sup^s(A^2) = \sup\{0, +\infty\} = +\infty$ and $coh^s(A^2) = \{\{0\}, \{0, +\infty\}, \{+\infty\}\}$.

For every positive integer n we introduce

$$Conv_s{}^n(\mathcal{P}) := \{coh^s(A^n) \mid A^n \in Conv^n(\mathcal{P})\}.$$

Theorem 1 *Let $\mathcal{P}$ be a $\bigvee-$semilattice cone. Then $Conv_s{}^n(\mathcal{P})$ is a subcone of $Conv^n(\mathcal{P})$ for all $n \in \mathbb{N}$.*

Proof Let $\mathcal{A}, \mathcal{B} \in Conv_s{}^1(\mathcal{P})$. Then there exist $A^1, B^1 \in Canv(\mathcal{P})$ such that

$$\mathcal{A} = coh^s(A^1) = coh(A^1 \cup \{sup(A^1)\})$$

and

$$\mathcal{B} = coh^s(B^1) = coh(B^1 \cup \{sup(B^1)\}).$$

We conclude that $\mathcal{A}, \mathcal{B} \in Canv(\mathcal{P})$. Put $\mathcal{A} + \mathcal{B} = \mathcal{C}$. We have

$$\sup(\mathcal{A}) + \sup(\mathcal{B}) = \sup(\mathcal{A} + \mathcal{B}) = \sup(\mathcal{C}),$$

by Proposition 2 (for case $n = 1$). Since $\mathcal{A}, \mathcal{B}$ contain their suprema, then $\mathcal{C}$ contains its supremum. Hence

$$\mathcal{C} = coh(\mathcal{C} \cup \{\sup(\mathcal{C})\}) = coh^s(\mathcal{C}),$$

which conclude that $\mathcal{C} \in Conv_s{}^1(\mathcal{P})$. On the other hand, for each $\alpha \geq 0$,

$$\alpha\mathcal{A} = \alpha coh^s(A^1) = coh^s(\alpha A^1) = coh(\alpha A^1 \cup \{sup(\alpha A^1)\}),$$

and so $\alpha\mathcal{A} \in Conv_s{}^1(\mathcal{P})$. Hence $Conv_s{}^1(\mathcal{P})$ is a subcone of $Conv(\mathcal{P})$. For completion of induction, first we show that $Conv_s{}^{n+1}(\mathcal{P}) \subseteq Conv^{n+1}(\mathcal{P})$. For this, let $\mathcal{A} \in Conv_s{}^{n+1}(\mathcal{P})$. There is $A^{n+1} \in Canv^{n+1}(\mathcal{P})$ such that $\mathcal{A} = coh^s(A^{n+1})$ and so

$$\mathcal{A} = coh^s(A^{n+1}) = coh(\{coh^s(A^n) \mid A^n \in A^{n+1}\} \cup \{sup^s(A^{n+1})\}^n).$$

Let $\mathcal{X} \in \mathcal{A}$ be arbitrary. There exist $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ and $A_1^n, A_2^n, \ldots, A_k^n \in A^{n+1} \cup \{\sup(A^{n+1})\}^n$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\mathcal{X} = \sum_{i=1}^k \lambda_i coh^s(A_i^n)$. On the other hand, $\lambda_i coh^s(A_i^n) \in Conv_s{}^n(\mathcal{P})$, for all $i = 1, 2, \ldots, k$. Hence

$$\mathcal{X} = \sum_{i=1}^{k} \lambda_i coh^s(A_i^n) \in Conv_s{}^n(\mathcal{P}),$$

and then $\mathcal{A} \in Canv^{n+1}(\mathcal{P})$.

Now, let $\mathcal{A}, \mathcal{B} \in {Conv_s}^{n+1}(\mathcal{P})$. Then for all $\mathcal{X} \in \mathcal{A} \subseteq Canv_s^n(\mathcal{P})$ and $\mathcal{Y} \in \mathcal{B} \subseteq Canv_s^n(\mathcal{P})$, we have $\mathcal{X} + \mathcal{Y} \in Canv_s^n(\mathcal{P})$. By Proposition 3 and Remark 4, we have $\{\sup^s(\mathcal{A})\}^n \in \mathcal{A}$ and $\{\sup^s(\mathcal{B})\}^n \in \mathcal{B}$. Also, by Proposition 2, we have

$$\{sup^s(\mathcal{A}+\mathcal{B})\}^{n+1} \in \mathcal{A}+\mathcal{B},$$

and then

$$\mathcal{A}+\mathcal{B} = coh(\{coh^s(\mathcal{Z}) \mid \mathcal{Z} \in \mathcal{A}+\mathcal{B}\} \cup \{sup^s(\mathcal{A}+\mathcal{B})\}^{n+1}).$$

Now, by considering the properties of *sup* and *coh* (convex hull of a set), we have $\alpha\mathcal{A} \in {Conv_s}^{n+1}(\mathcal{P})$ for all $\alpha \geq 0$ and $\mathcal{A} \in {Conv_s}^{n+1}(\mathcal{P})$. □

Now, we characterize the elements of ${Conv_s}^n(\mathcal{P})^*$. First we recall a theorem.

Theorem 2 ([4], II, 2.9) *Let $\mathcal{Q}$ be subcone of the locally convex cone $(\mathcal{P}, \mathcal{V})$. Then every u-continuous linear functional on $\mathcal{Q}$ can be extended to a u-continuous linear functional on $\mathcal{P}$.*

Theorem 3 *If $(\mathcal{P}, \mathcal{V})$ is a $\bigvee$-semilattice locally convex cone, then for all $n \in \mathbb{N}$, $(Conv^n(\mathcal{P}))^*$ and $\mathcal{P}^*$ coincide, in the sense that any vector of $\mathcal{P}^*$ has a unique extension to a vector of $(Conv^n(\mathcal{P}))^*$ and conversely any vector $(Conv^n(\mathcal{P}))^*$ can be restricted to a vector of $\mathcal{P}^*$.*

Proof By considering (1) we can embed $\mathcal{P}$ into ${Conv_s}^n(\mathcal{P})$. It is easy to see that the restriction of each element of ${Conv_s}^n(\mathcal{P})^*$ on $\mathcal{P}$ belongs to $\mathcal{P}^*$ and by Theorem 2, the extension of each element of $\mathcal{P}^*$ to ${Conv_s}^n(\mathcal{P})$ is an element of ${Conv_s}^n(\mathcal{P})^*$. So it is sufficient to show that each element of $\mathcal{P}^*$ has a unique extension in ${Conv_s}^n(\mathcal{P})^*$. Let $\mu \in \mathcal{P}^*$. Define $(\bar{\mu})^n$ as follows:

$$(\bar{\mu})^1(A) := \bar{\mu}(A) = \sup\{\mu(a) \mid a \in A\} \quad (A \in Conv_s(\mathcal{P})), \tag{3}$$

and

$$(\bar{\mu})^n(A^n) := \sup\{(\bar{\mu})^{n-1}(A^{n-1}) \mid A^{n-1} \in A^n\} \quad (A^n \in {Conv_s}^n(\mathcal{P})), \tag{4}$$

for $n = 2, 3, \ldots$. By Lemma 2, the functional $(\bar{\mu})^1$ is u-continuous and by repeating this process $(\bar{\mu})^n$ is u-continuous too. We have $(\bar{\mu})^1(A) = \mu(sup^s(A))$ and $(\bar{\mu})^n(A^n) = (\bar{\mu})^{n-1}(\{sup^s(A^n)\}^{n-1})$, since A contains $\sup A$. By Remark 4 and Proposition 2 the mapping $(\bar{\mu})^n$ is an extension of μ to ${Conv_s}^n(\mathcal{P})$. Let ϑ_n be another u-continuous extension of μ to ${Conv_s}^n(\mathcal{P})$ (which exists by Theorem 2). We show that $\vartheta_n = \bar{\mu}^n$.

Let $A^n \in Conv_s^n(\mathcal{P})$. Since $A^n \preceq \{sup^s(A^n)\}^n$ and $\{sup^s(A^n)\}^n \preceq A^n$, then $\vartheta_n(A^n) \leq \vartheta_n(\{sup^s(A^n)\}^n)$ and $\vartheta_n(\{sup^s(A^n)\}^n) \leq \vartheta_n(A_n)$ and so

$$\vartheta_n(A^n) = \vartheta_n(\{sup^s(A^n)\}^n) = \mu(\{sup^s(A^n)\}^n) = (\bar{\mu})^n(\{sup^s(A^n)\}^n) = (\bar{\mu})^n(A^n).$$

This completes the proof. □

In the following example we consider the locally convex cone $\overline{\mathbb{R}}$ and we characterize all elements of the dual of the locally convex cone $(Conv^n(\overline{\mathbb{R}}), \overline{\mathcal{V}}^n)$, where $\mathcal{V} = \{\epsilon > 0 \mid \epsilon \in \mathbb{R}\}$.

Example 4 We know that $\overline{\mathbb{R}}$ is a $\bigvee -$semilattice locally convex cone. It is easy to see that

$$\begin{aligned} Conv_s(\overline{\mathbb{R}}) &= \{[a,b], (c,d], (-\infty, d], \{e\}, A \cup \{+\infty\} \mid A \in Conv(\mathbb{R}), \\ &\qquad a, b, c, d, e \in \overline{\mathbb{R}} \text{ with } a < b \text{ and } c < d\} \\ &= Conv(\overline{\mathbb{R}}) \setminus \{(a,b), (-\infty, b), [c,d) \mid a, b, c, d \in \overline{\mathbb{R}}\}. \end{aligned}$$

According to Theorem 3, $(Conv^n(\mathbb{R}))^*$ and $\mathbb{R}^*$ coincide, in the sense that any vector of $\mathbb{R}^*$ has a unique extension to a vector of $(Conv^n(\mathbb{R}))^*$ and conversely any vector $(Conv^n(\mathbb{R}))^*$ can be restricted to a vector of $\mathbb{R}^*$ for all $n \in \mathbb{N}$.

Since $\Omega(\overline{\mathbb{R}}) = \overline{\mathbb{R}}^* \setminus \{0_\infty\} = \mathbb{R}^*$, every element of $\mathbb{R}^*$ has a unique extension in $(Conv^n(\overline{\mathbb{R}}))^*$ by Corollary 1. The element 0_∞ violates the Ω condition at just one point $+\infty$. So two different extensions $\overline{0_\infty}(A)$ and $\overline{\overline{0_\infty}}$ can be written for it in $Conv(\overline{\mathbb{R}})^*$ as the following:

$$\begin{aligned} \overline{0_\infty}(A) &= \sup\{0_\infty(a) | a \in A\} = 0, \\ \overline{\overline{0_\infty}}(A) &= 0_\infty(\sup A) = 0, \end{aligned}$$

for all $A \in Conv(\overline{\mathbb{R}})$ which $\sup(A) \neq +\infty$,

$$\begin{aligned} \overline{0_\infty}(A) &= \sup\{0_\infty(a) | a \in A\} = +\infty, \\ \overline{\overline{0_\infty}}(A) &= 0_\infty(\sup A) = +\infty, \end{aligned}$$

for $A \in Conv(\overline{\mathbb{R}})$ with $+\infty \in A$ and

$$\begin{aligned} \overline{0_\infty}(A) &= \sup\{0_\infty(a) | a \in A\} = 0, \\ \overline{\overline{0_\infty}}(A) &= 0_\infty(\sup A) = 0_\infty(\infty) = +\infty, \end{aligned}$$

for all $A \in \mathcal{Q}$, where $\mathcal{Q} := \{A \in Conv(\overline{\mathbb{R}}) \mid \sup(A) = +\infty \text{ and } +\infty \notin A\}$. Let γ be another extension of 0_∞ to $Conv(\overline{\mathbb{R}})$. Then $\gamma(A) = \overline{0_\infty}(A) = \overline{\overline{0_\infty}}(A) = 0$ for all $A \in Conv(\overline{\mathbb{R}})$ which $\sup(A) \neq +\infty$ and $\gamma(A) = \overline{0_\infty}(A) = \overline{\overline{0_\infty}}(A) = +\infty$ for $A \in Conv(\overline{\mathbb{R}})$ with $+\infty \in A$, by Theorem 3. Now, let $A, B \in \mathcal{Q}$. It is easy to see that $A \preceq B$ and $B \preceq A$ and then $\gamma(A) = \gamma(B)$. In particular, $\gamma(A) = \gamma(\alpha A) = \alpha\gamma(A)$ since $\alpha A \in \mathcal{Q}$ for all positive reals α. By the above consideration $\gamma = \overline{0_\infty} = 0$ or $\gamma = \overline{\overline{0_\infty}} = +\infty$ on $\mathcal{Q}$. Therefore $\overline{0_\infty}$ and $\overline{\overline{0_\infty}}$ are only extensions of 0_∞ on $Conv(\overline{\mathbb{R}})$. This yields that $(Conv(\overline{\mathbb{R}}))^* \setminus \{\overline{0_\infty}, \overline{\overline{0_\infty}}\}$ and $\overline{\mathbb{R}}^*$ coincide.

Now, we show that the extensions of the mappings $\overline{0_\infty}$ and $\overline{\overline{0_\infty}}$ to the cone $Conv^n(\overline{\mathbb{R}})$ are unique: Let $\overline{0_\infty}^n$ and $\overline{\overline{0_\infty}}^n$ be the extensions of $\overline{0_\infty}$ and $\overline{\overline{0_\infty}}$ on $Conv^n(\overline{\mathbb{R}})$, respectively. Let $A \in Conv^n(\overline{\mathbb{R}})\setminus Conv^n(\mathbb{R})$. Then $\{+\infty\}^n \preceq A$ and $A \preceq \{+\infty\}^n$. These yield that

$$\overline{0_\infty}^n(A) = \overline{0_\infty}^n(\{\infty\}^n) = 0_\infty(+\infty) = +\infty,$$
$$\overline{\overline{0_\infty}}^n(A) = \overline{\overline{0_\infty}}^n(\{+\infty\}^n) = 0_\infty(+\infty) = +\infty.$$

On the other hand, if $A \in Conv^n(\mathbb{R})$, then $A \preceq \{(0, +\infty)\}^{n-1}$ and so $\overline{0_\infty}^n(A) \leq 0$. Also there exists $a \in \mathbb{R}$ such that $\{a\}^n \preceq A$. Then $0 = \overline{0_\infty}^n(\{a\}^n) \leq \overline{0_\infty}^n(A)$. We conclude that $\overline{0_\infty}^n(A) = 0$ for all $A \in Conv^n(\mathbb{R})$.

If there is $b \in \mathbb{R}$ such that $A \preceq \{b\}^n$, then $\overline{\overline{0_\infty}}(A) \leq 0$ and so $\overline{\overline{0_\infty}}(A) = 0$ by the similar way which applied for $\overline{0_\infty}^n(A)$. Otherwise $\{b\}^n \preceq A$ for all $b \in \mathbb{R}$. Then $\{(0, +\infty)\}^{n-1} \preceq A$ and so $+\infty = \overline{\overline{0_\infty}}(\{(0, +\infty)\}^{n-1}) \leq \overline{\overline{0_\infty}}(A)$. This yields that $\overline{\overline{0_\infty}}(A) = +\infty$. We conclude that the elements of $(Conv^n(\overline{\mathbb{R}}))^*$ are all non-negative reals, $\overline{0_\infty}^n$ and $\overline{\overline{0_\infty}}^n$ for all $n \in \mathbb{N}$. Also we have showed that the cones $(Conv(\overline{\mathbb{R}}))^*$ and $(Conv^n(\overline{\mathbb{R}}))^*$ coincide.

We conclude that $(Conv^n(\overline{\mathbb{R}}))^* \setminus \{\overline{0_\infty}^n, \overline{\overline{0_\infty}}^n\}$ and $\overline{\mathbb{R}}^*$ coincide.

References

1. Ayaseh, D., Ranjbari, A.: Order bornological locally convex lattice cones. Vladikavkaz Math. J. **19**(3), 21–30 (2017)
2. Ayaseh, D., Ranjbari, A.: Bornological Convergence in Locally Convex Cones. Mediterr. J. Math. **13**(4), 1921–1931 (2016)
3. Dastouri, A., Ranjbari, A.: Some Notes on Barreledness in Locally Convex Cones, Bull. Iran. Math. Soc. https://doi.org/10.1007/s41980-020-00519-x
4. Keimel, K., Roth, W.: Ordered cones and approximation. Lecture Notes in Mathematics, vol. 1517. Springer-Verlag, Berlin (1992)
5. Keimel, K., Roth, W.: A Korovkin type approximation theorem for set-valued functions. Proc. Amer. Math. Soc. **104**, 819–824 (1988)
6. Roth, W.: Operator-valued measures and integrals for cone-valued functions. Lecture Notes in Mathematics, vol. 1964. Springer-Verlag, Berlin (2009)
7. Roth, W.: Korovkin theory for cone-valued functions. Positivity **21**(3), 449–472 (2017)
8. Roth, W.: Hahn-Banach type theorems for locally convex cones. J. Austral. Math. Soc. Ser. A **68**(1), 104–125 (2000)